Heavy-ion reactions at low energies

1. Introduction (experimental and theoretical aspects)
2. Deep-inelastic collisions (properties and description)
3. Nucleus-nucleus interaction (methods of calculation)
4. Pecularities of fusion reactions (adiabatic and diabatic treatments)

1st nuclear reaction with *p* beam: 1931

p, α, *d* beams for study of nuclear structure

50-60th years – ion sourses

heavy ion wave length < 0.1 fm (classical particles)

50th : linear accelerators in USA

60th : ciclotron in Dubna

70-80th : linear accelerator at GSI, ciclotron at GANIL and ...

beam of light nuclei:

nuclear reactions / processes: elastic, inealastic scattering, nuclear transfer reactions, formation and decay of compound nucleus

beam of heavy ions:

the Coulomb fission of heavy nuclei and excitation of high-spin states, population of highly-deformed nuclear states, multinucleon transfer reactions, compound nucleus formation

Problems of synthesis of superheavy nuclei

Production of exotic nuclei, new isotopes

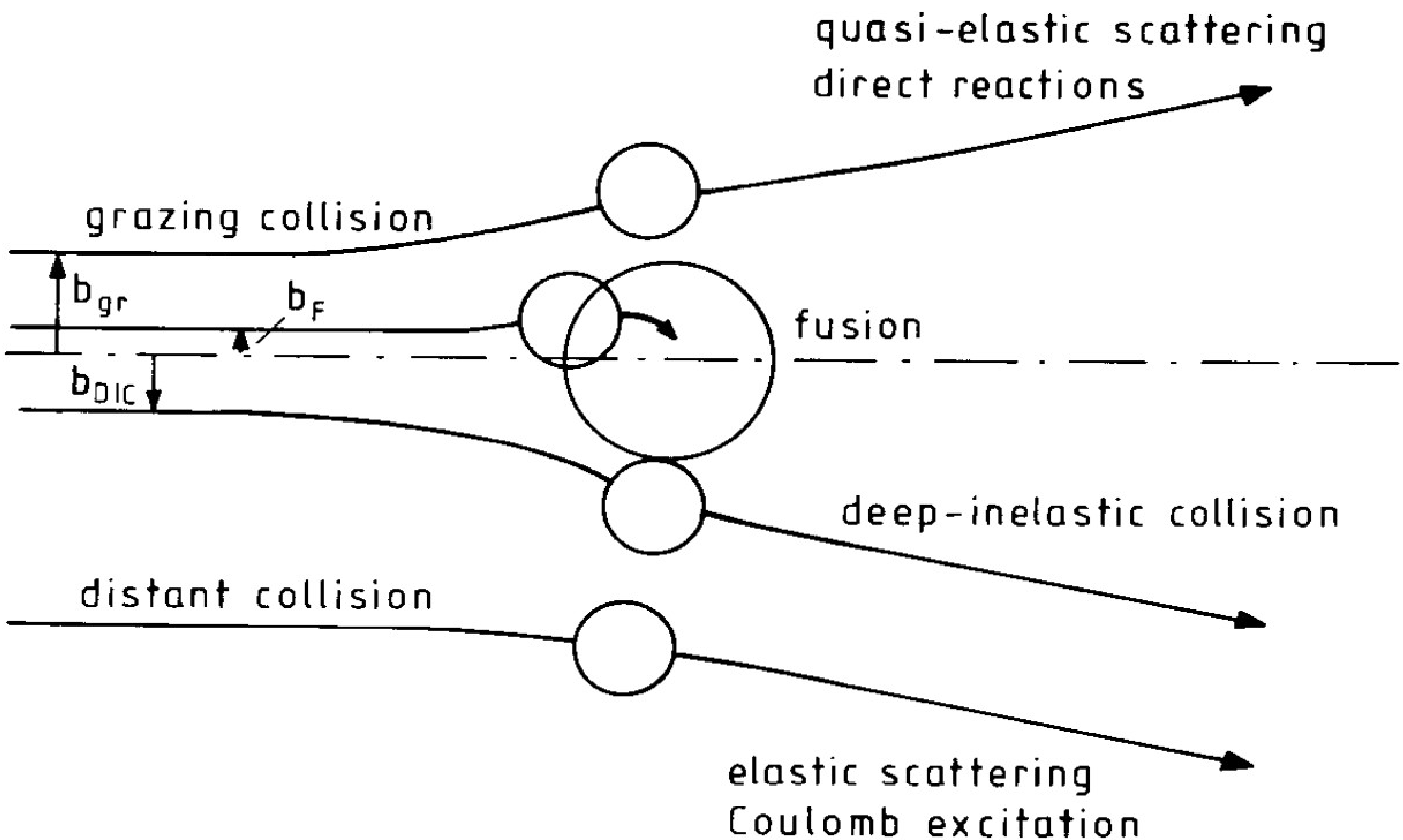
Study of various decay modes including fission, emission of delayed proton, *p* and 2*p* radioactivity

High-spin states

Highly excited compound nuclei

Sub-barrier processes

Cluster or molecule states



Classification of reactions by impact parameter.

impact parameter $b=l/k$

Reaction cross section

$$\sigma_{\mathrm{r}} = \sum_{l} \sigma_{\mathrm{r}}(l),$$

$$\sigma_{\mathrm{r}}(l) = \frac{\pi}{k^2}(2l+1)T_l$$

For large angular momenta

$$\sigma_{\mathrm{r}} = \int_0^\infty dl\, \sigma_{\mathrm{r}}(l)$$

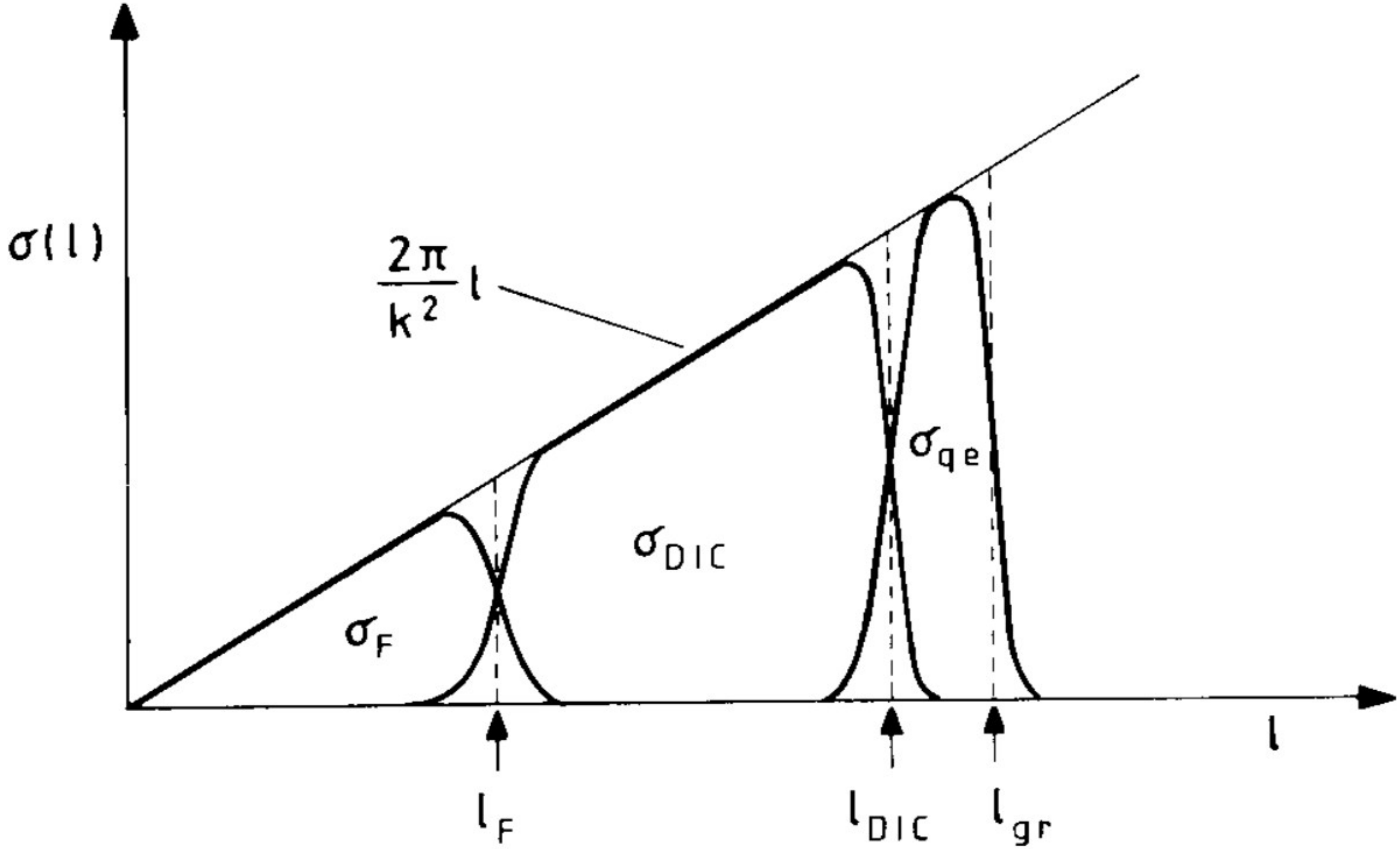
$$\sigma_{\mathrm{r}}(l) = \frac{2\pi}{k^2}\, l\, T(l).$$

$$T(l) = \begin{cases} 1 & \text{for } l < l_{\text{gr}}, \\ 0 & \text{for } l > l_{\text{gr}}, \end{cases} \qquad l_{\text{gr}} = k b_{\text{gr}}$$

$$\sigma_{\text{r}}(l) = \frac{2\pi}{k^2} l$$

between the values $l = 0$ and $l = l_{\text{gr}}$.

$$\begin{array}{lllll} 0 & < & l & < l_{\text{F}} & \text{fusion,} \\ l_{\text{F}} & < & l & < l_{\text{DIC}} & \text{deep-inelastic collisions,} \\ l_{\text{DIC}} & < & l & < l_{\text{gr}} & \text{quasi-elastic collisions.} \end{array}$$

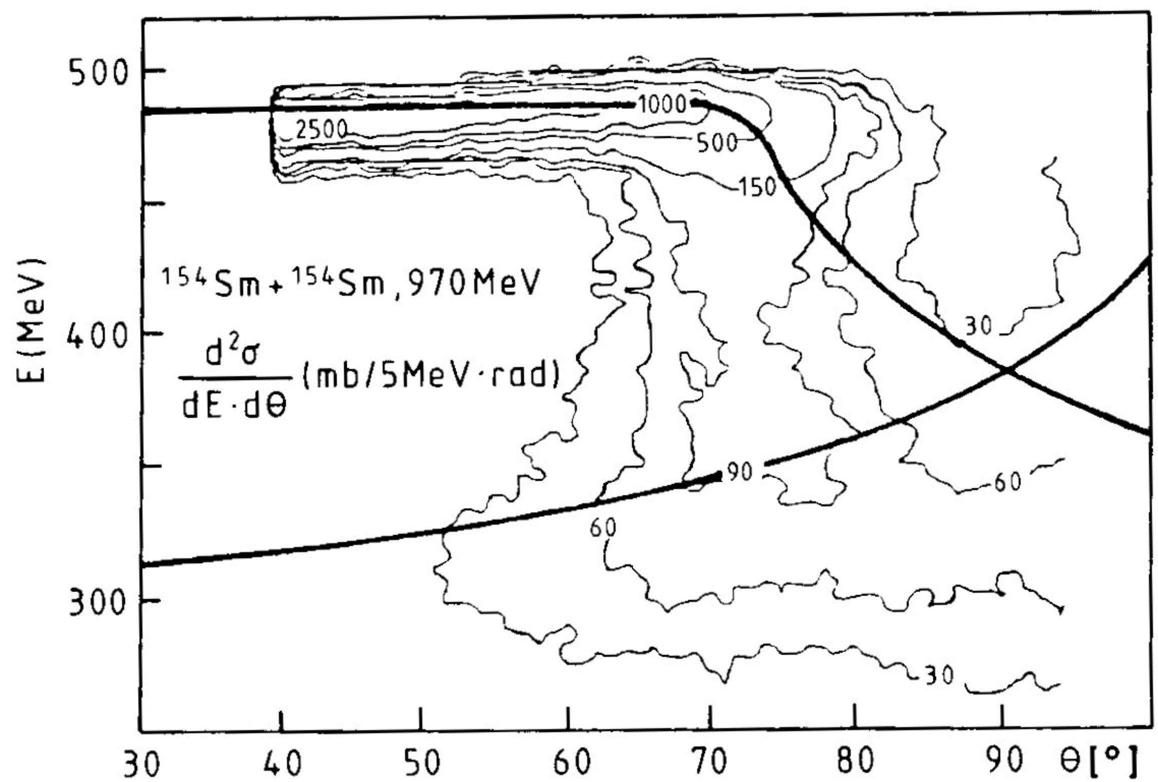


$$\sigma_{\mathrm{F}} = \frac{2\pi}{k^2}\int_0^{l_{\mathrm{F}}} l\,dl = \frac{\pi}{k^2} l_{\mathrm{F}}^2,$$

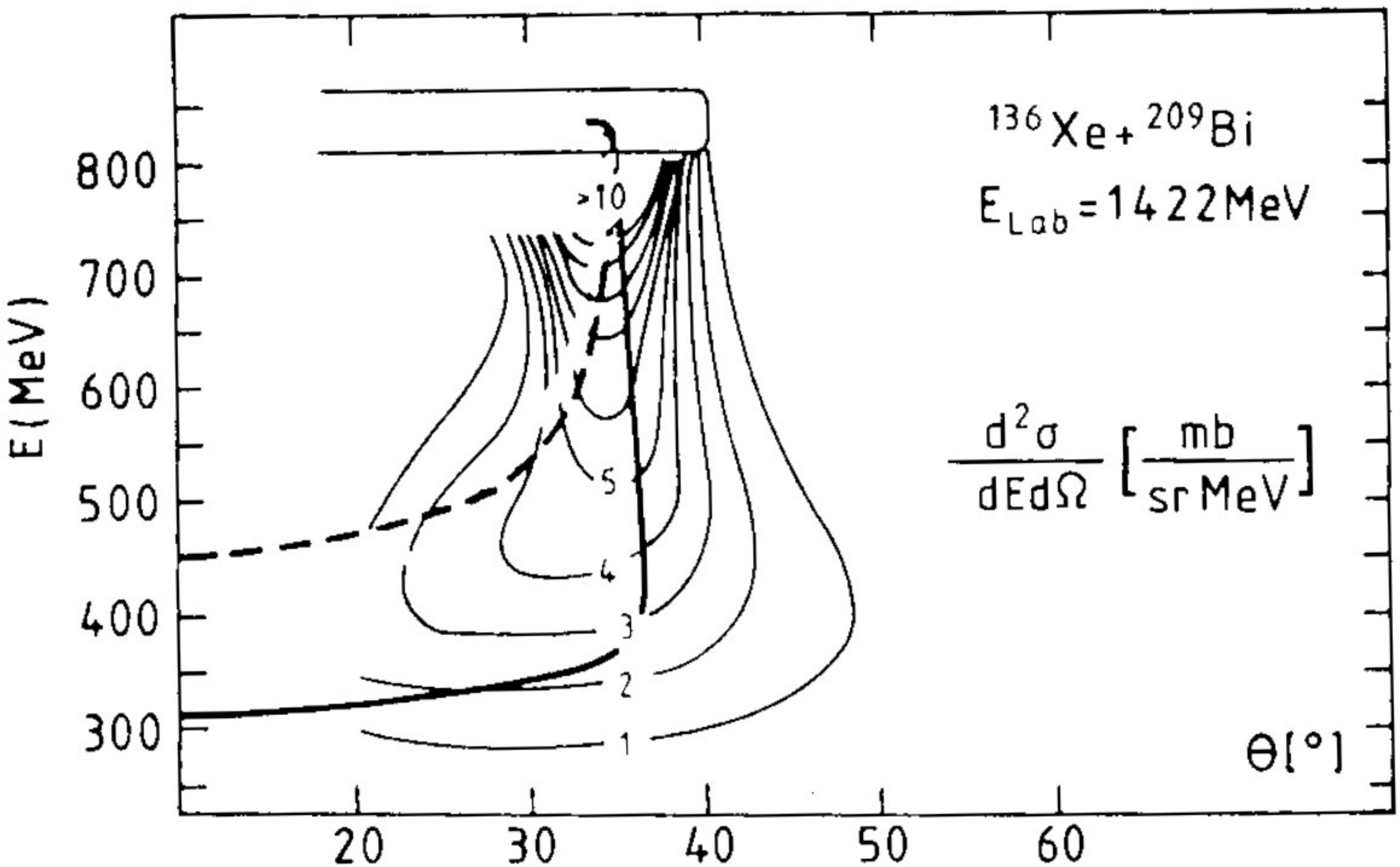
$$\sigma_{\mathrm{DIC}} = \frac{2\pi}{k^2}\int_{l_{\mathrm{F}}}^{l_{\mathrm{DIC}}} l\,dl = \frac{\pi}{k^2}(l_{\mathrm{DIC}}^2 - l_{\mathrm{F}}^2)$$

Measurments of γ-multiplicities show that the γ-rays emitted after a DIC carry angular momentum, which is taken out of the relative motion of the collision partners. This shows that there is considerable transfer of angular momentum from the relative motion to the internal system.

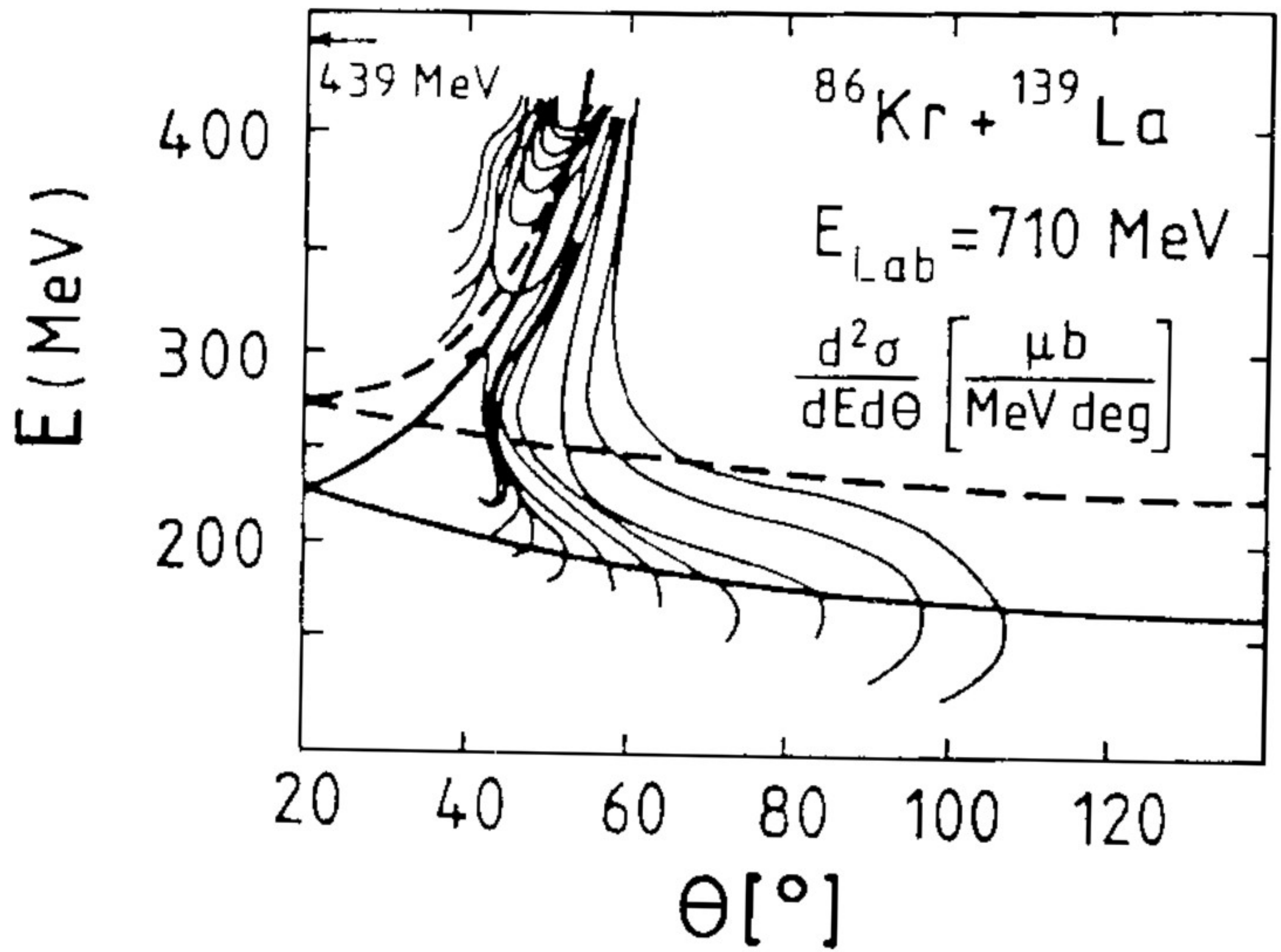
- Coulomb-like collisions. Collision partners are higly charged and the incident energy is relatively low. The Coulomb repulsion dominates and the projectile is strongly reflected to large, backward angles.



- Focussing collisions. Higher energies or lighter nuclei. Scattering into a narrow angular region.



- Orbiting collisions. The attractive nuclear force dominates over the Coulomb force. This pulls the trajectory of the projectile around the target into the region of negative scattering angles.



End of 60th – discovery of new type of nuclear reactions – DIC

Mechanism: dynamic & statistic pecularities
formation of DNS – result of nucler viscosity and microscopic effects

nuclear molecule $\longleftrightarrow$ DNS

quasistationary states dynamics

Study of DIC

identification of the products

scattering chamber

radiochemistry

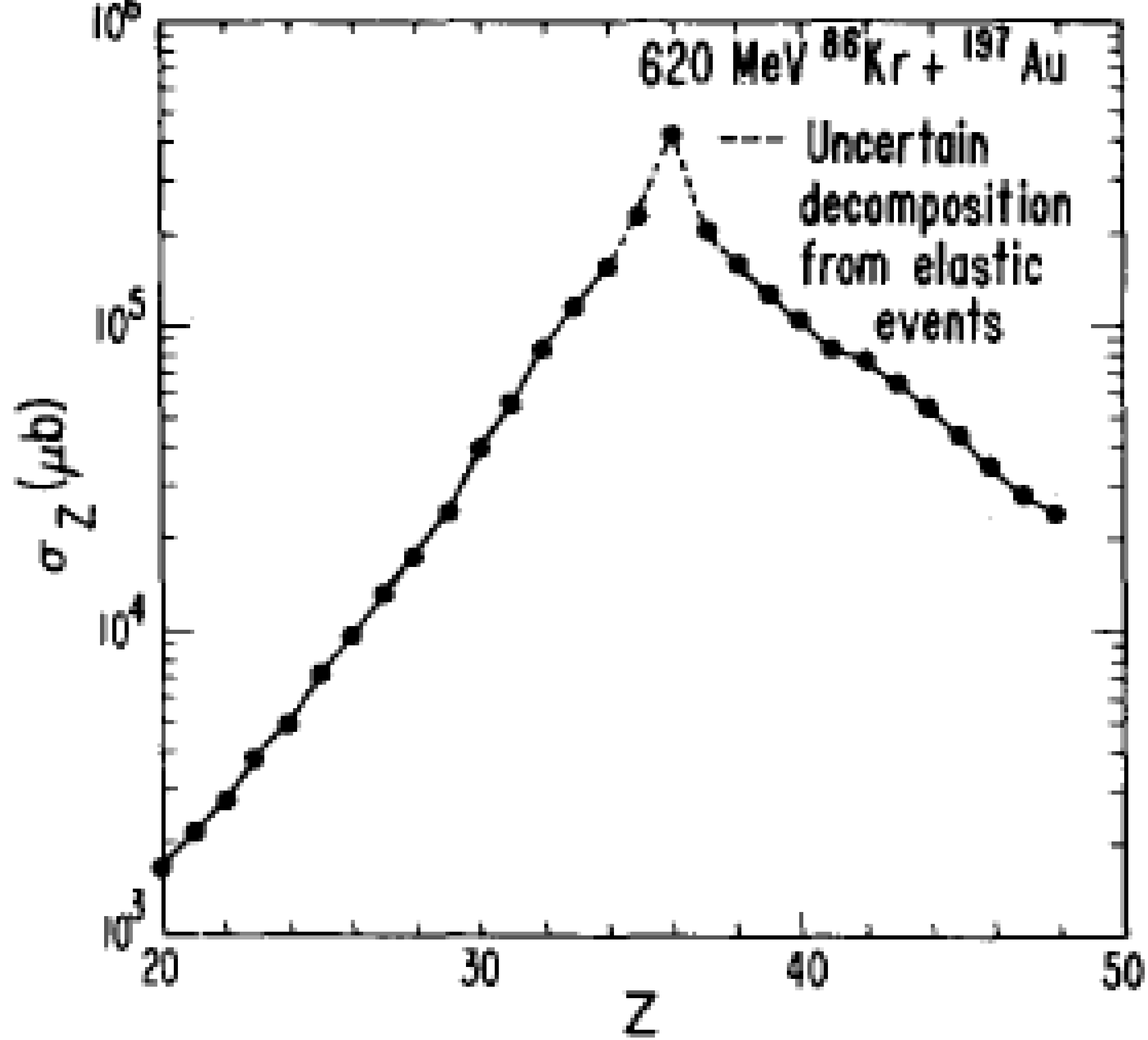
ΔE-E detectors

time of flight

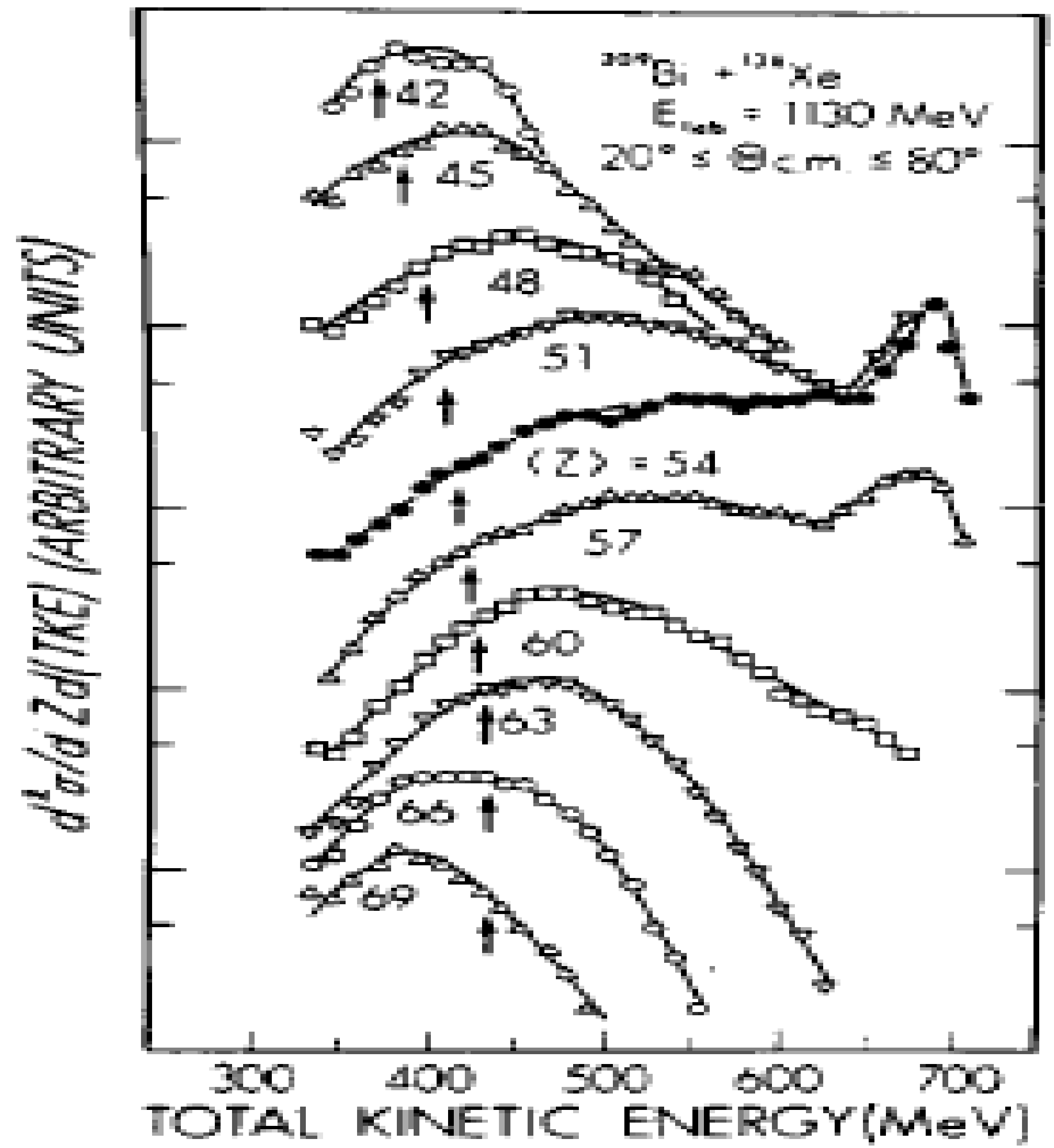
magnetic spectrometra

two-shoulder detectors

detectors for *n*, *p*, α, and γ



620 MeV ^{86}Kr + ^{197}Au
--- Uncertain decomposition from elastic events
σ_Z (μb)
Z
10^6
10^5
10^4
10^3
20
30
40
50



Bi + Xe
E = 1130 MeV
20° ≤ θc.m. ≤ 80°
42
45
48
51
⟨Z⟩ = 54
57
60
63
66
69
d²σ/dZd(TKE) (ARBITRARY UNITS)
300
400
500
600
700
TOTAL KINETIC ENERGY(MeV)

Characteristics of DIC

- total dissipation of kinetic energy $\longrightarrow$ energy distribution has maxima at V_b for the fragments, independent on $E_{c.m.}$
- angular distributions have maxima at forward angles decrease of anisotropy with increasing number of transfered nucleon
- large variation of mass (charge) distributions (max. at A_p (A_t) and Z_p (Z_t))
- N/Z ratio
- sharing of excitation energy and angular momentum

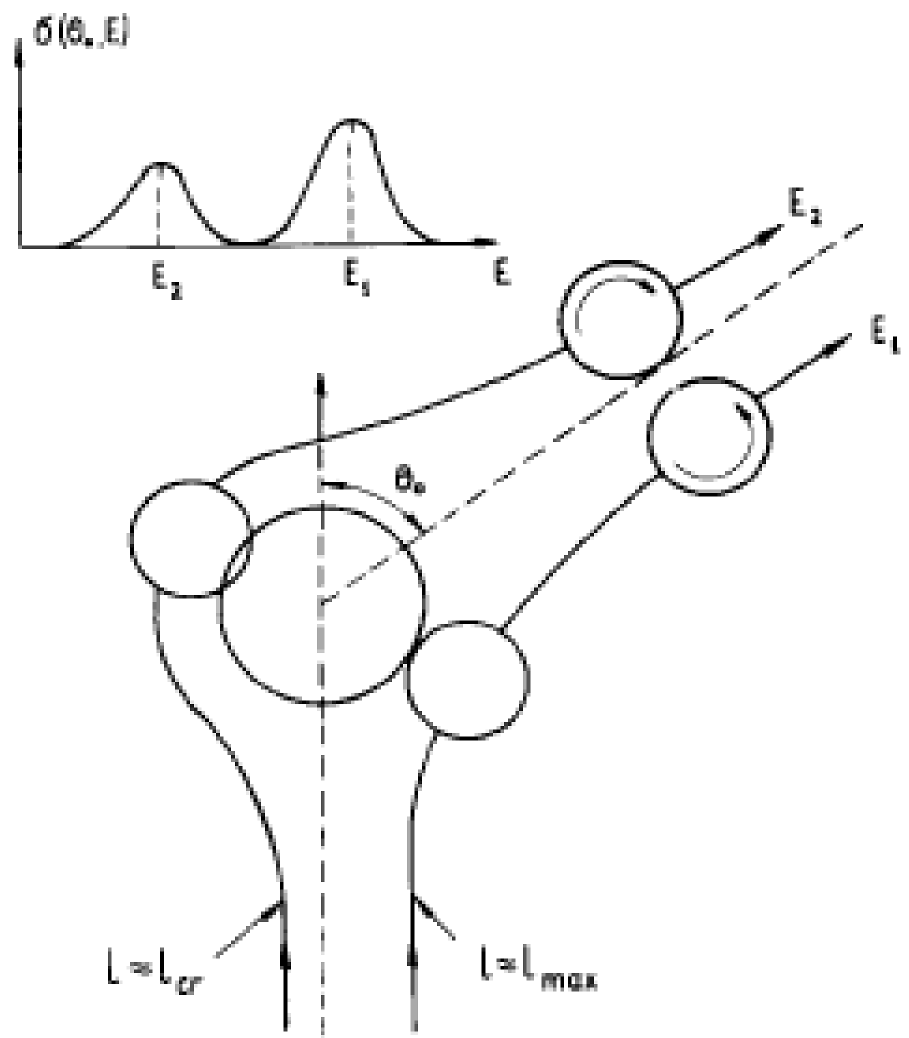
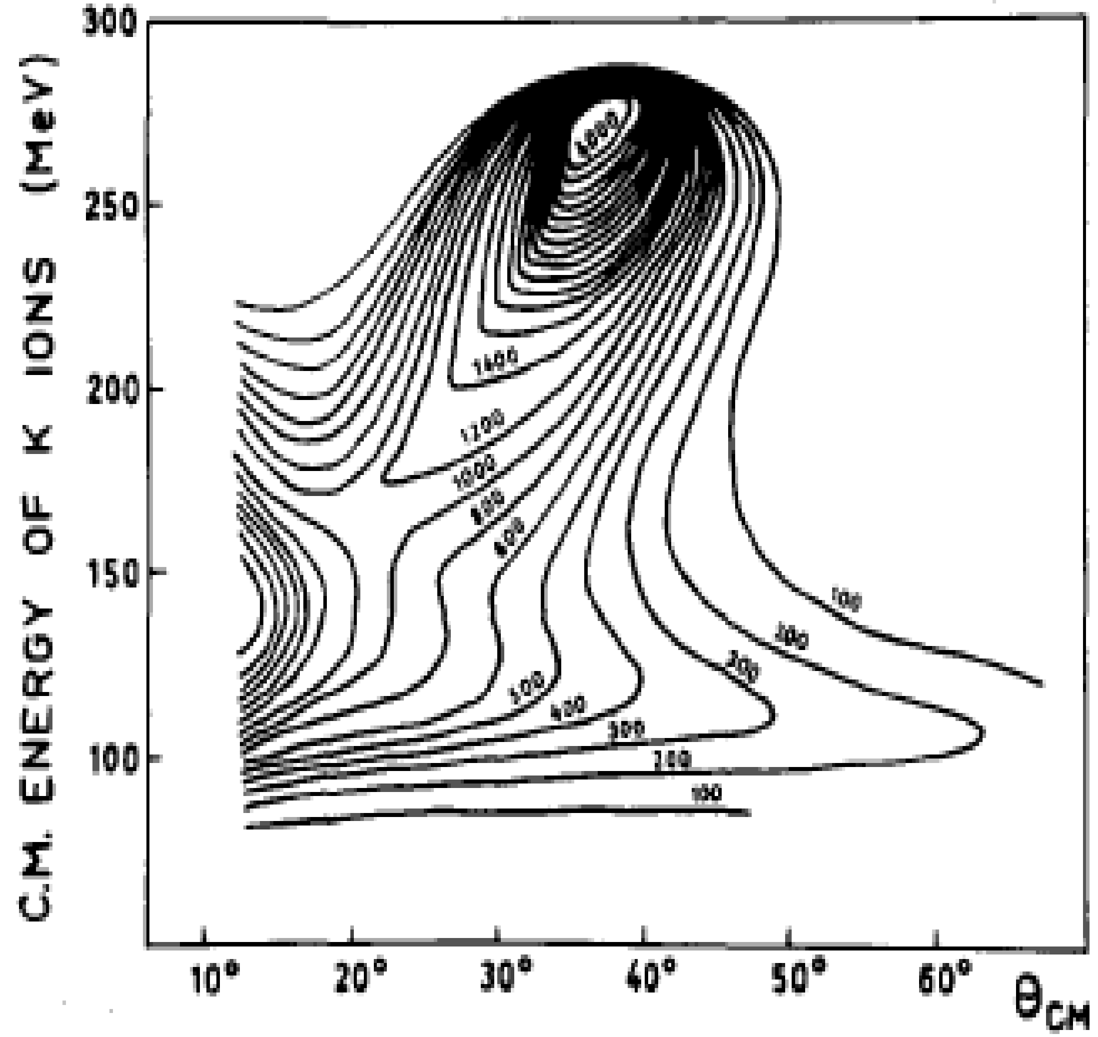


Illustration of the formation of two peaks in the energy spectrum



Contour diagram representing the transfer reaction data for $^{232}Th(^{40}Ar,K)$ at 388 MeV

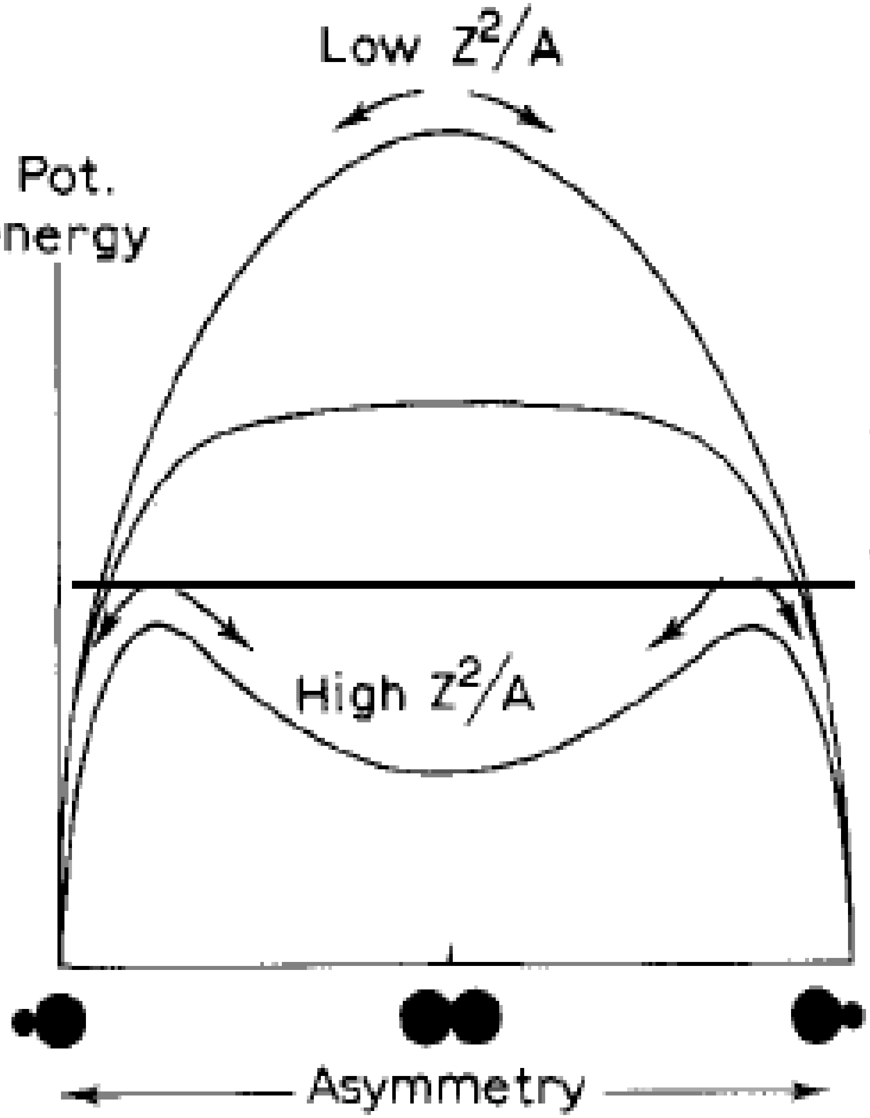


Illustration of the dependence of the potential energy of the system of two touching nuclear drops on mass asymmetry and parameter $(Z_1 + Z_2)^2/(A_1 + A_2)$.

Set of coordinates for the description of DNS evolution:

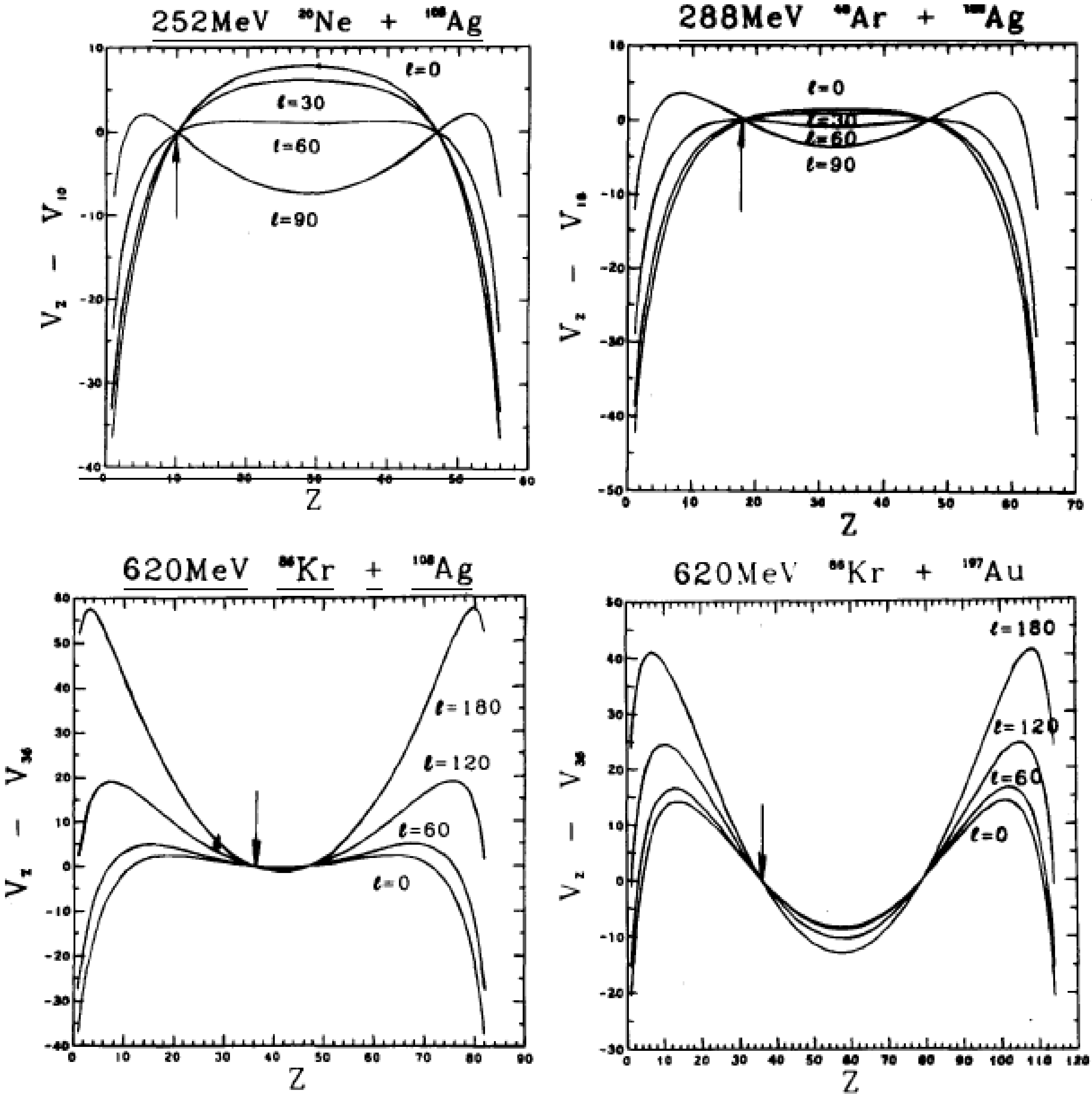
$$\eta_Z=(Z_1-Z_2)/(Z_1+Z_2)\ ,\ \ \eta=(A_1-A_2)/(A_1+A_2)\ ,\ \ R$$

The potential energy of DNS:

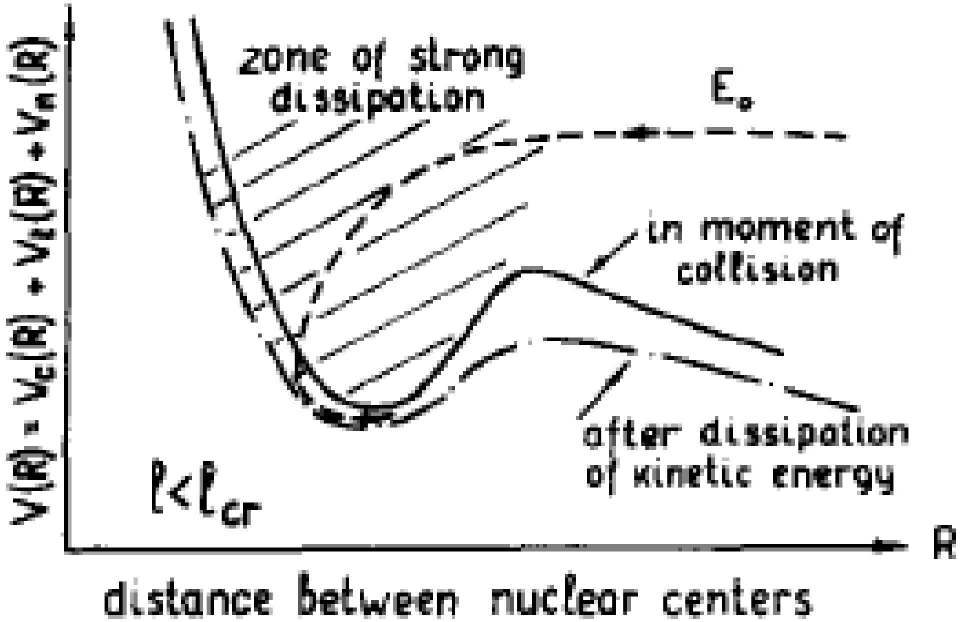
$$U(R,\eta,\eta_Z,\beta_{1,}\beta_{2,}J)=B_1+B_2+V(R,\eta,\eta_Z,\beta_{1,}\beta_{2,}J)$$

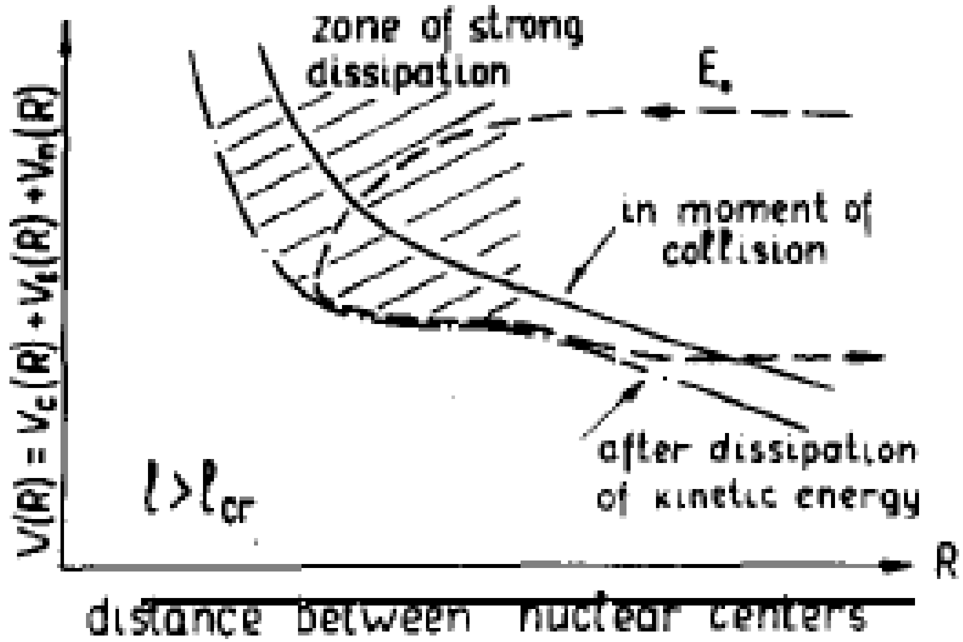
The nucleus-nucleus potential:

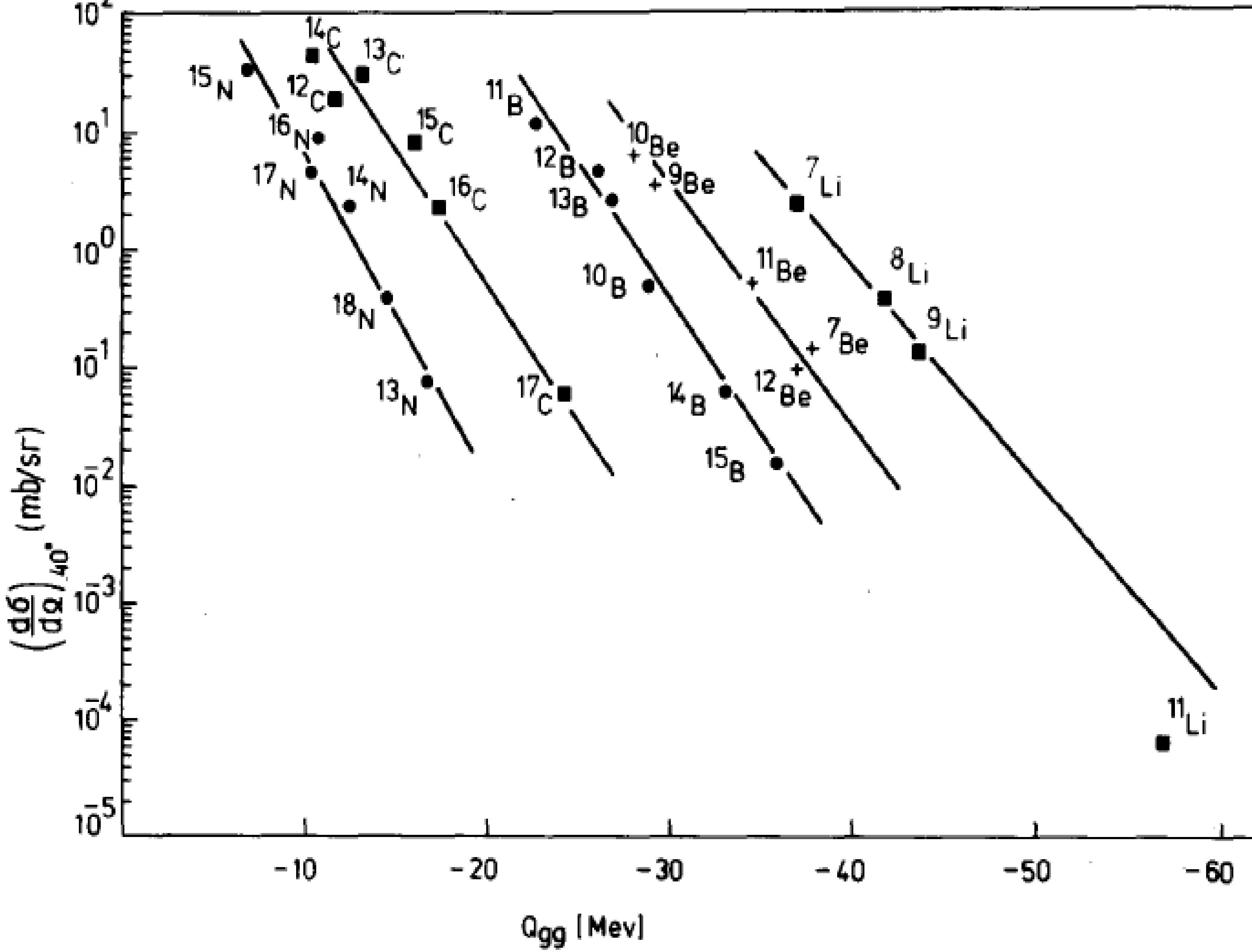
$$V(R,\eta,\eta_Z,\beta_{1,}\beta_{2,}J)=V_C(R,\eta_Z,\beta_{1,}\beta_2)+V_N(R,\eta,\beta_{1,}\beta_2)+V_{rot}(\eta,\beta_{1,}\beta_{2,}J)$$



252MeV 20Ne + 108Ag
ℓ=0
ℓ=30
ℓ=60
ℓ=90
Vz − V10
Z
288MeV 40Ar + 108Ag
ℓ=0
ℓ=30
ℓ=60
ℓ=90
Vz − V18
Z
620MeV 86Kr + 108Ag
ℓ=180
ℓ=120
ℓ=60
ℓ=0
Vz − V36
Z
620MeV 86Kr + 197Au
ℓ=180
ℓ=120
ℓ=60
ℓ=0
Vz − V36
Z


zone of strong dissipation
E₀
in moment of collision
after dissipation of kinetic energy
l<l_cr
V(R) = V_c(R) + V_l(R) + V_n(R)
R
distance between nuclear centers


zone of strong dissipation
E₀
in moment of collision
after dissipation of kinetic energy
l>l_cr
V(R) = V_c(R) + V_l(R) + V_n(R)
R
distance between nuclear centers



$Q_{gg} = (M_1 + M_2) - (M_3 + M_4)$

^{232}Th + ^{16}O 137 MeV

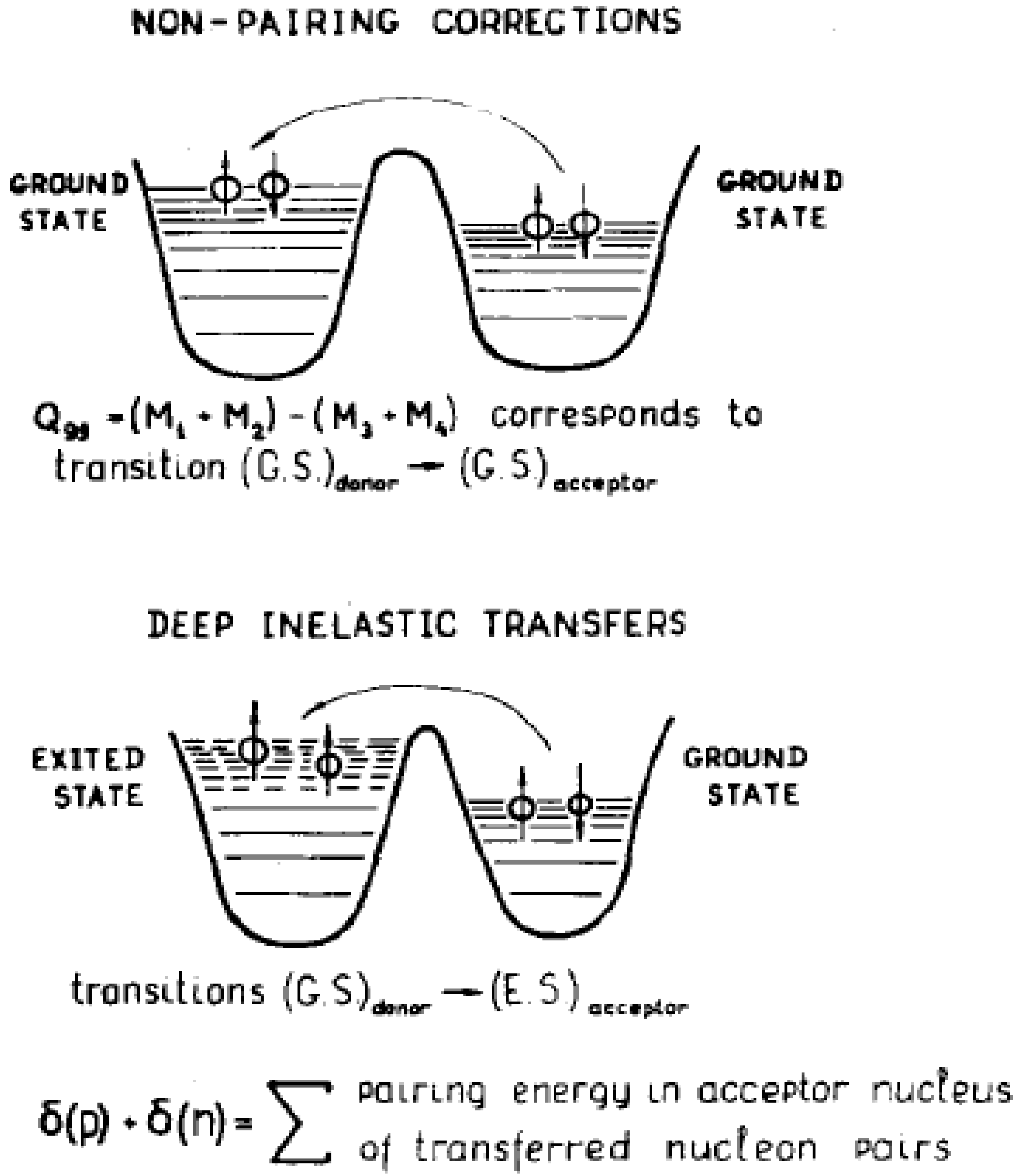
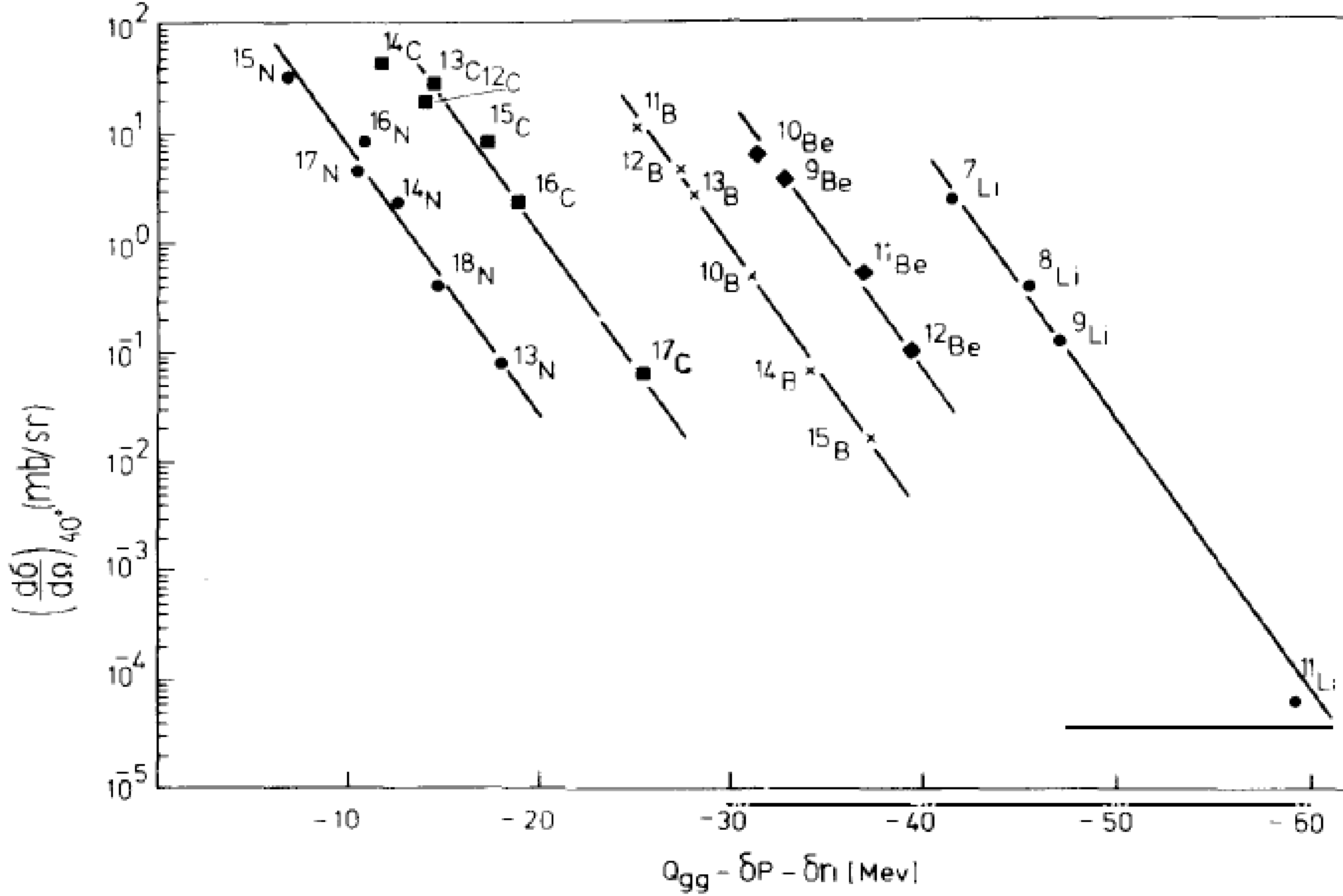
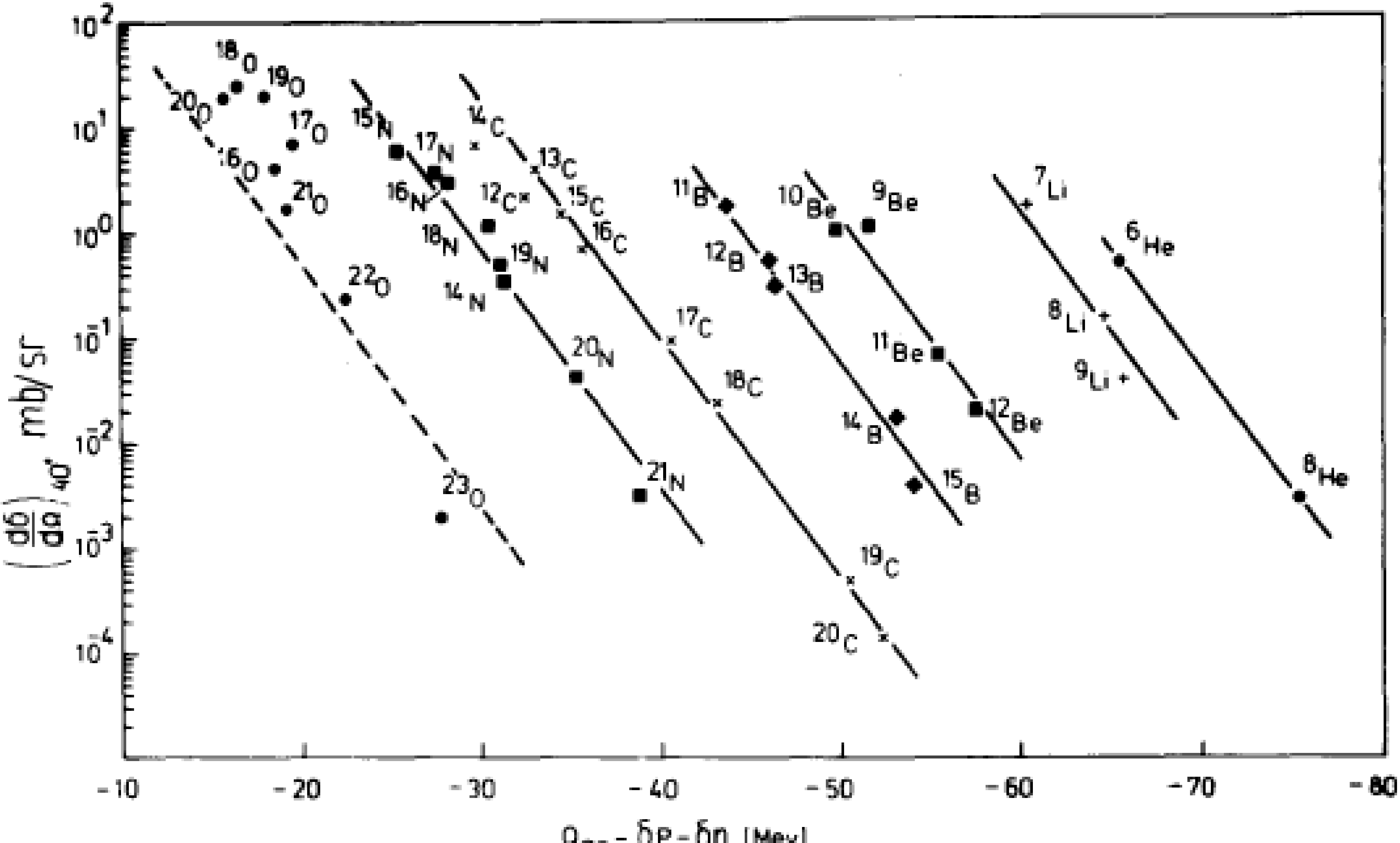


Illustration of the necessity of introducing corrections for non-pairing.



10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\left(\frac{d\sigma}{d\Omega}\right)_{40°}$ (mb/sr)
-10
-20
-30
-40
-50
-60
$Q_{gg} - \delta P - \delta n$ [Mev]
^{15}N
^{16}N
^{17}N
^{14}N
^{18}N
^{13}N
^{14}C
^{13}C
^{12}C
^{15}C
^{16}C
^{17}C
^{11}B
^{12}B
^{13}B
^{10}B
^{14}B
^{15}B
^{10}Be
^{9}Be
^{11}Be
^{12}Be
^{7}Li
^{8}Li
^{9}Li
^{11}Li

$^{232}Th + ^{16}O$



The Q_{gg} systematics for the reaction $^{232}Th + ^{22}Ne$, corrected for non-pairing.

$$\sigma \backsim \exp\{(Q_{gg} + \Delta E_c)/T\}$$

Langevin description

Two colliding nuclei with reduced mass *M* move in the field of the interaction potential *V(R)*, where *R* is the collective coordinate. Lagrangian

$$\mathcal{L}_0(R, \dot{R}) = \frac{1}{2} M \dot{R}^2 - V(R).$$

The internal motion is described by a set of harmonic oscillators of mass m_i and frequency ω_i with internal coordinate q_i.

The internal Lagrangian:

$$\mathcal{L}_{\mathrm{intl}}(q_i, \dot{q}_i) = \sum_i \frac{m_i}{2} (\dot{q}_i^2 - \omega_i^2 q_i^2).$$

The interaction between the collective motion and the internal subsystem is assumed to be separable and linear in coordinate. This drastic assumtion allows us to do analytic.

The full Lagrangian

$$\mathcal{L}(R, \dot{R}; q_i, \dot{q}_i) = \mathcal{L}_0(R, \dot{R}) + \mathcal{L}_{\rm intl}(q_i, \dot{q}_i) + \sum_i f_i(R) q_i,$$

where $f_i(R)$ is the form factor of the coupling, it vanishes at R beyond which the reaction partners cease to interest and has the same range as the potential $V(R)$. The equations of motion:

$$M\ddot{R} = -\frac{dV(R)}{dR} + \sum_i q_i \frac{df_i(R)}{dR},$$

$$m_i \ddot{q}_i = -m_i \omega_i^2 q_i + f_i(R).$$

In order to get the equation in *R* alone, we must eliminate the internal coordinates. So,

$$q_i(t) = q_i^0(t) + \int_{t_0}^{t} ds \frac{f_i(R(s))}{m_i \omega_i} \sin \omega_i (t - s),$$

where the first term is the solution of the homogeneous part with $f_i=0$, and has the form

$$q_i^0(t) = q_{i0} \cos \omega_i (t - t_0) + \frac{p_{i0}}{m_i \omega_i} \sin \omega_i (t - t_0);$$

q_{i0} and p_{i0} are the values of the coordinates and momentum of the oscillators of the bath at an initial time t_0. The second term incorporates the effect of coupling.

Substituting the solution for internal coordinates, we obtain the differential equation for *R*

$$M\ddot{R} = -\frac{dV(R)}{dR} + \sum_i \frac{1}{m_i\omega_i}\int_{t_0}^{t} ds\ f_i(R(s))\sin\omega_i(t-s)\,\frac{df_i(R)}{dR}$$

$$+\sum_i q_i^0(t)\frac{df_i(R)}{dR};$$

Integrating by parts in the second term,

$$\int_{t_0}^{t} ds\ f_i(R(s))\sin\omega_i(t-s) = \frac{f_i(R(s))}{\omega_i}\cos\omega_i(t-s)\bigg|_{s=t_0}^{s=t}$$

$$-\int_{t_0}^{t} ds\ \frac{df_i(R(s))}{dR}\dot{R}(s)\frac{1}{\omega_i}\cos\omega_i(t-s),$$

The surface term contributes only at the upper limit *s*=*t*

$$\left.\frac{f_i(R(s))}{\omega_i}\cos\omega_i(t-s)\right|_{s=t_0}^{s=t}=\frac{f_i(R)}{\omega_i},$$

As a result

$$M\ddot{R}=\widetilde{F}(R)+F_{\mathrm{frict}}(R,\dot{R})+F_{\mathrm{L}}(R,t).$$

The renormalized conservative force

$$\widetilde{F}(R)=-\frac{d\widetilde{V}(R)}{dR}$$

The renormalized potential

$$\widetilde{V}(R)=V(R)-\sum_i\frac{1}{2m_i\omega_i^2}[f_i(R)]^2.$$

We have defined the friction force

$$F_{\text{frict}}(R, \dot{R}) = -\sum_i \frac{1}{m_i \omega_i^2} \int_{t_0}^{t} ds \frac{df_i(R(t))}{dR} \cos \omega_i (t-s) \frac{df_i(R(s))}{dR} \dot{R}(s)$$

and the Langevin force

$$F_{\text{L}}(R, t) = \sum_i q_i^0(t) \frac{df_i(R)}{dR}.$$

The renormalization term can be taken away by writing the full Lagrangian in Caderia and Leggett form

$$\mathcal{L}(R, \dot{R}; q_i, \dot{q}_i) = \mathcal{L}_0(R, \dot{R}) + \sum_i \frac{m_i}{2} \dot{q}_i^2 - \sum_i \frac{m_i \omega_i^2}{2} \left(q_i - \frac{f_i(R)}{m_i \omega_i^2} \right)^2$$

The equation for *R* derived from such a Lagrangian contains only the original potential *V(R)*.

The friction force (non-Markovian)

$$F_{\text{frict}}(R, \dot{R}) = -\int_{t_0}^{t} ds\, \gamma(t, s)\dot{R}(s).$$

Here, we introduce the friction kernel (assume, for simplicity, m_i=m and $f_i(R)$=$f(R)$)

$$\gamma(t, s) = f'(R(t))\, f'(R(s)) \sum_i \frac{1}{m\omega_i^2} \cos \omega_i (t - s),$$

where $f'(R) = df(R)/dR$. The sum over i is a sum of many terms with varying signs which effectively vanishes except when all the cosines have nearly vanishing arguments, i.e. $|t - s| \leq \epsilon$; the small time interval ϵ is the memory time determining the retardation of the friction force, i.e. Its length of memory.Therefore,

$$\sum_i \frac{1}{m\omega_i^2} \cos \omega_i (t - s) \approx 2\gamma_0\, \delta_\epsilon (t - s),$$

where $\delta_\epsilon (t - s)$ is a 'smeared-out δ-function with a range ϵ.

Integrating over *t*, we get

$$2\gamma_0 = \int_{-\infty}^{\infty} dt \sum_i \frac{1}{m\omega_i^2} \cos \omega_i t ,$$

where the factor 2 is introduced for convenience. The friction kernel then becomes

$$\gamma(t, s) = 2\gamma(R)\, \delta_\epsilon(t - s)$$

with the friction coefficient

$$\gamma(R) = \gamma_0 \, [f'(R)]^2 .$$

The dependence of *R(t)* on *t* is assumed to be weak, so that we can set *R(s)=R(t)* for $|s-t| \leq \epsilon$.

Let us introduce the spectral density $g(\omega)$ of the intrinsic excitations, which allows us

$$\sum_i \cdots \rightarrow \int_0^{\infty} d\omega \, g(\omega) \cdots .$$

Then

$$\int_0^\infty d\omega\, g(\omega) \frac{1}{m\omega^2} \cos\omega(t-s) \approx 2\gamma_0\, \delta_\epsilon(t-s),$$

where

$$2\gamma_0 = \int_{-\infty}^{\infty} dt \int_0^\infty d\omega\, g(\omega) \frac{1}{m\omega^2} \cos\omega t\,.$$

$$F_{\text{frict}}(R, \dot{R}) = -\gamma(R)\, \dot{R}$$

Energy loss

$$\dot{E}(t) = F_{\text{frict}}(R, \dot{R})\, \dot{R} = -\gamma(R)\, \dot{R}^2,$$

Langevin force

For simplicity, we assume the same form factors.

$$F_{\mathrm{L}}(R, t) = f'(R)\,\xi(t),$$

where

$$\xi(t) = \sum_i q_i^0(t)$$

The oscillators are assumed to represent a 'heat bath' (Brownian motion). Owing to the implicit interactions of the oscillators of the bath, the coordinates q_{i0} and momenta p_{i0} are treated as random variables whose distributions has mean value zero,

$$\langle q_{i0} \rangle = 0, \qquad \langle p_{i0} \rangle = 0,$$

where $\langle ... \rangle$ denotes the average over the ensemble of these variables. They are regarded as uncorrelated,

$$\langle q_{i0}\, q_{j0} \rangle = \delta_{ij} \langle q_{i0}^2 \rangle,$$
$$\langle p_{i0}\, p_{j0} \rangle = \delta_{ij} \langle p_{i0}^2 \rangle,$$
$$\langle q_{i0}\, p_{j0} \rangle = 0,$$

where the quantities $\langle q_{i0}^2 \rangle$ and $\langle q_{i0}^2 \rangle$ are the mean-square elongation and momentum of the i-th oscillator, respectively.

$$\langle \xi(t) \rangle = 0,$$

$$\langle \xi(t)\xi(t') \rangle = \sum_i \langle q_{i0}^2 \rangle \cos\omega_i (t - t_0) \cos\omega_i (t' - t_0)$$
$$+ \sum_i \frac{1}{m_i^2 \omega_i^2} \langle p_{i0}^2 \rangle \sin\omega_i (t - t_0) \sin\omega_i (t' - t_0).$$

$$2\cos\omega_i(t-t_0)\cos\omega_i(t'-t_0) = \cos\omega_i(t-t') + \cos\omega_i(t+t'+2t_0),$$
$$2\sin\omega_i(t-t_0)\sin\omega_i(t'-t_0) = \cos\omega_i(t-t') - \cos\omega_i(t+t'+2t_0),$$

Thus,

$$\langle \xi(t)\xi(t')\rangle \approx \sum_i \frac{\langle \epsilon_{i0}\rangle}{m_i\omega_i^2} \cos\omega_i(t-t'),$$

where $\langle \epsilon_{i0}\rangle$ is the mean energy of the i-th oscillator,

$$\langle \epsilon_{i0}\rangle = \frac{\langle p_{i0}^2\rangle}{2m_i} + \tfrac{1}{2} m_i\omega_i^2 \langle q_{i0}^2\rangle .$$

We assume that the heat bath is in equilibrium and can be characterized by a temperature *T*.

$$\langle \epsilon_{i0}\rangle = k_{\rm B} T ,$$

Then

$$\langle \xi(t)\xi(t')\rangle \approx k_B T \sum_i \frac{1}{m_i \omega_i^2} \cos \omega_i (t - t').$$

$m_i = m$

$$\langle \xi(t)\xi(t')\rangle = 2d_0\, \delta_\epsilon (t - t')$$

where the *correlation strength* d_0 is given by

$$d_0 = \gamma_0\, k_B T.$$

The normalized time-dependent variable

$$\Gamma(t) = \frac{1}{\sqrt{d_0}}\, \xi(t),$$

with Gaussian distribution.

$$\langle \Gamma(t) \rangle = 0,$$
$$\langle \Gamma(t)\Gamma(t') \rangle = 2\delta_\epsilon(t - t'),$$

The average of the Langevin force is

$$\langle F_L(R(t), t) \rangle = 0.$$

Its correlation function is

$$\langle F_{\mathrm{L}}(R(t), t) F_{\mathrm{L}}(R(t'), t') \rangle = 2D(R)\, \delta_\epsilon(t - t'),$$

where we have introduced the *fluctuation strength coefficient*

$$D(R) = d_0[f'(R)]^2,$$

$$F_{\mathrm{L}}(R, t) = \sqrt{D(R)}\, \Gamma(t).$$

Fluctuation-dissipation theorem

$$D(R) = \gamma(R)\, k_{\mathrm{B}} T,$$

connects the fluctuation strength coefficient *D* of the Langevin force with the friction coefficient γ. It is a consequence of the fact that the friction and Langevin forces have their origin in the coupling between the collective motion and the bath.

At low temperatures

$$\langle \epsilon_i \rangle = \tfrac{1}{2} \hbar \omega_i \coth\left(\frac{\hbar \omega_i}{2 k_{\mathrm{B}} T} \right)$$

Langevin equations, their applicability to DIC

$$M\ddot{R} = \widetilde{F}(R) + F_{\text{frict}}(R, \dot{R}) + F_{\text{L}}(R, t)$$

$$F_{\text{frict}}(R, \dot{R}) = -\gamma(R)\,\dot{R},$$

$$F_{\text{L}}(R, t) = \sqrt{D(R)}\,\Gamma(t),$$

$$\dot{R} = \frac{P}{M},$$

$$\dot{P} = \widetilde{F}(R) - \gamma(R)\frac{P}{M} + \sqrt{D(R)}\,\Gamma(t).$$

Generalization to the multidimensional case.

The internal system equlibrates quickly, its equilibration time is smaller than the correlation time ϵ and also smaller than the time scale of collective motion.

Fokker-Planck equation for distribution function

$$\frac{\partial}{\partial t} d(x;t) = -\sum_i \frac{\partial}{\partial x_i} v_i(x) d(x;t) + \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij}(x) d(x;t),$$

$$v_R = \frac{P}{M},$$

$$v_P = \widetilde{F}(R) - \gamma(R)\frac{P}{M},$$

$$D_{RR} = 0,$$

$$D_{PP} = D(R),$$

$$D_{RP} = 0.$$

$$\frac{\partial}{\partial t} d(R,P;t) = \left[-\frac{\partial}{\partial R}\frac{P}{M} - \frac{\partial}{\partial P}\left(\widetilde{F}(R) - \gamma(R)\frac{P}{M}\right) + \frac{\partial^2}{\partial P^2} D(R) \right] d(R,P;t).$$

Simple examples
1-dim., const. coefficients

$$\frac{\partial}{\partial t} d(Z;t) = \left(-v \frac{\partial}{\partial Z} + D \frac{\partial^2}{\partial Z^2} \right) d(Z;t).$$

Introducing the new variable $X = Z - vt$ in the place of Z, we obtain the equation

$$\frac{\partial}{\partial t} d(X;t) = D \frac{\partial^2}{\partial X^2} d(X;t).$$

With the initial condition $d(X;0) = \delta(X)$ it has the solution

$$d(X;t) = \frac{1}{\sqrt{4\pi Dt}} e^{-X^2/4Dt},$$

or

$$d(Z;t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(Z-vt)^2/4Dt}.$$

$$\langle 1 \rangle = \int dZ\, d(Z;t) = 1.$$

The mean value and the variance are found from the first and second moments of the distribution function.

$$\bar{Z}(t) = \langle Z \rangle = \int dZ\, Z\, d(Z;t) = vt,$$

$$\sigma_{ZZ}^2 = \langle [Z - \bar{Z}(t)]^2 \rangle = \int dZ\, (Z - vt)^2 d(Z;t) = 2Dt.$$

1-dim., variable drift coefficients

$$\frac{\partial}{\partial t} d(Z, t) = \left(-\frac{\partial}{\partial Z} v(Z) + D \frac{\partial^2}{\partial Z^2} \right) d(Z, t).$$

The equilibrium solution ($t \longrightarrow \infty$) has the form of Boltzmann distribution

$$d(Z) \propto e^{-U(Z)/k_B T},$$

where T is the temperature of the system.

$$\frac{\partial}{\partial Z} \left(-v(Z) + D \frac{\partial}{\partial Z} \right) d(Z, t) = 0,$$

$$v(Z) = \frac{1}{d(Z)} \frac{d}{dZ} D\, d(Z),$$

$$v(Z) = -\frac{D}{k_B T}\frac{\partial U(Z)}{\partial Z}.$$

In the first approximation

$$U(Z) = \frac{C}{Z_{tot}^2}(Z - Z_s)^2.$$

Here Z_{tot} is the total charge of projectile and target, and $Z_s = Z_{tot}/2$. The factor C is the stiffness of the driving potential.

$$\frac{\partial}{\partial t}d(Z,t) = \left(\frac{\partial}{\partial Z}\frac{2CD}{k_B T Z_{tot}^2}(Z - Z_s) + D\frac{\partial^2}{\partial Z^2}\right)d(Z,t).$$

$$d(Z,t) = \frac{1}{\sqrt{2\pi\sigma_{ZZ}^2(t)}}\exp\left(-\frac{[Z - \bar{Z}(t)]^2}{2\sigma_{ZZ}^2(t)}\right),$$

for an initial projectile charge $Z(0) = Z_{\text{proj}}$,

$$\bar{Z}(t) = Z_{\text{s}} - (Z_{\text{proj}} - Z_{\text{s}}) \exp\left(- \frac{2CD}{k_{\text{B}} T Z_{\text{tot}}^2} t \right),$$

$$\sigma_{ZZ}^2(t) = \frac{k_{\text{B}} T Z_{\text{tot}}^2}{2C} \left[1 - \exp\left(- \frac{4CD}{k_{\text{B}} T Z_{\text{tot}}^2} t \right) \right].$$

For large interaction times the system evolves towards symmetry,

$$\bar{Z}(t) \to Z_{\text{s}} = \frac{Z_{\text{tot}}}{2} \ \text{ for } \ t \to \infty.$$

$$\sigma_{ZZ}^2(t) \to \frac{k_{\text{B}} T Z_{\text{tot}}^2}{2C} \ \text{ for } \ t \to \infty.$$

General case

d(R,P;t)

$$\bar{R}(t) = \langle R(t) \rangle = \iint dR\,dP\, R\,d(R, P; t),$$

$$\bar{P}(t) = \langle P(t) \rangle = \iint dR\,dP\, P\,d(R, P; t),$$

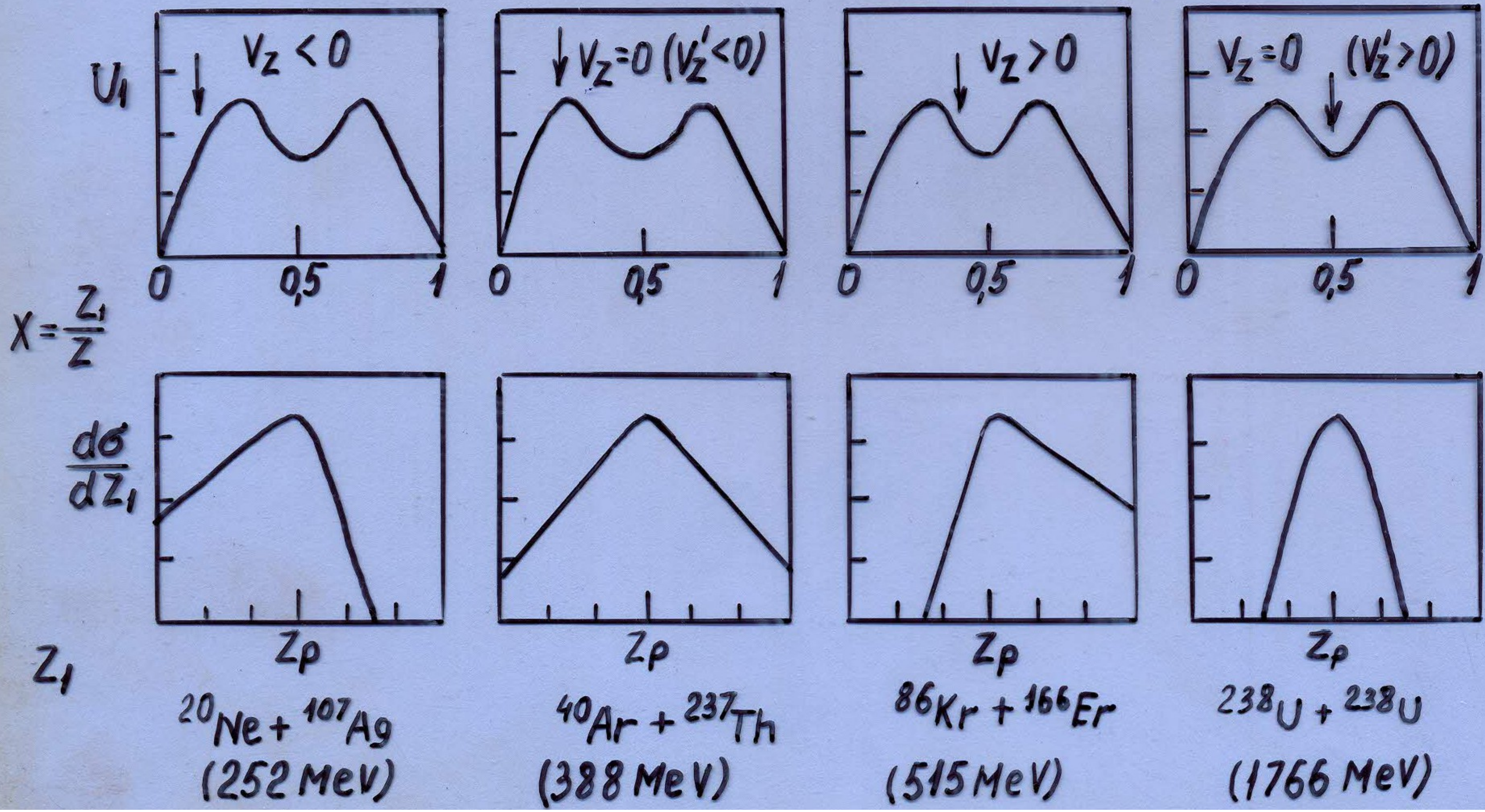
$$\sigma^2_{RR}(t) = \langle [R - \bar{R}(t)]^2 \rangle = \langle R^2 \rangle - \bar{R}^2(t),$$

$$\sigma^2_{PP}(t) = \langle [P - \bar{P}(t)]^2 \rangle = \langle P^2 \rangle - \bar{P}^2(t),$$

$$\sigma^2_{RP}(t) = \langle [R - \bar{R}(t)][P - \bar{P}(t)] \rangle = \langle RP \rangle - \bar{R}(t)\bar{P}(t).$$

$$\frac{\partial}{\partial t} P(Z_1, t) = -\frac{\partial}{\partial Z_1}(V_Z P) + \frac{\partial^2}{\partial Z_1^2}(D_Z P)$$

$V_Z \sim \frac{\partial U_e}{\partial Z_1}$ $\qquad$ $\langle Z_1 \rangle = Z_p + V_Z \cdot t$ $\qquad$ $\sigma_Z^2 = 2 D_Z \cdot t$

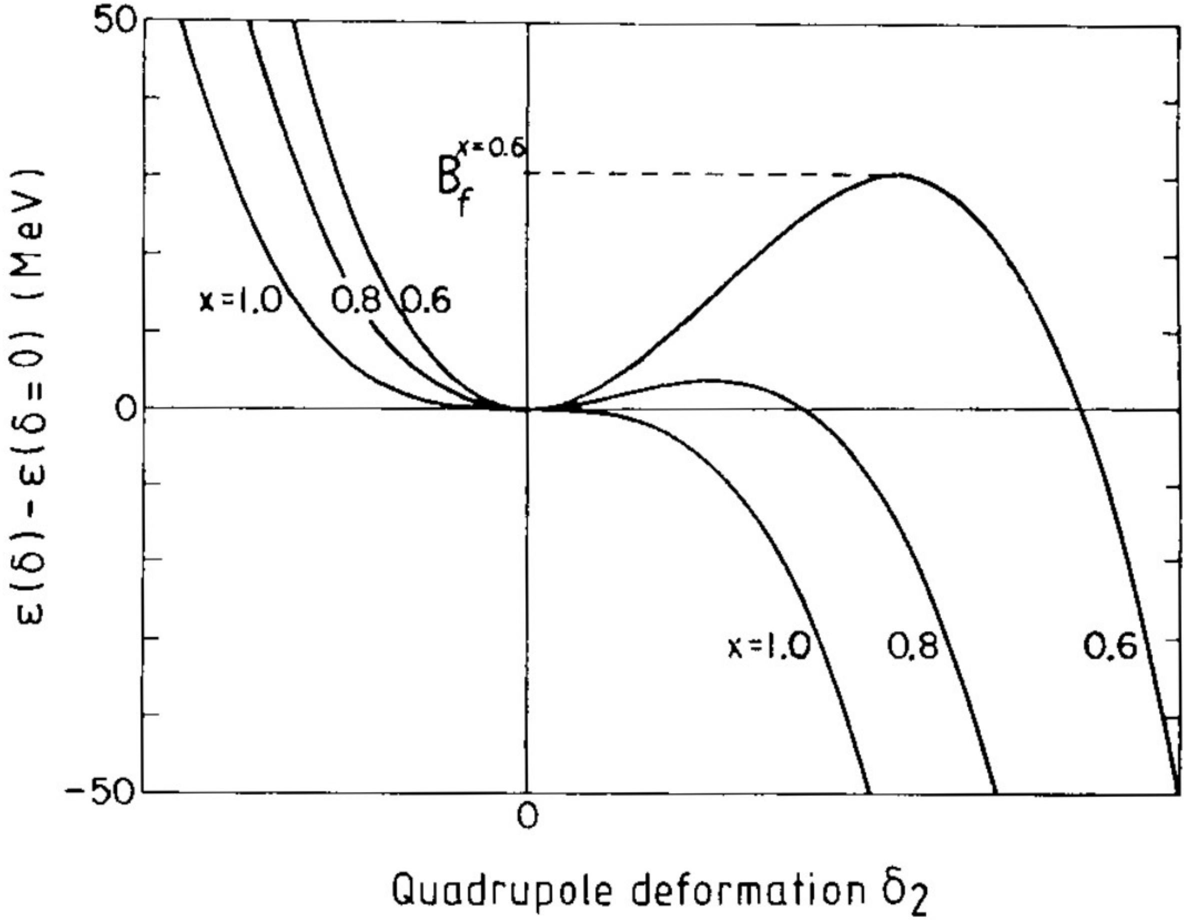


Fusion

stability of the formed compound nuclei
fisility parameter

$$x \approx \frac{1}{50}\frac{Z^2}{A} \approx \frac{Z}{120}.$$

x>1 – unstable
x<1 - stable



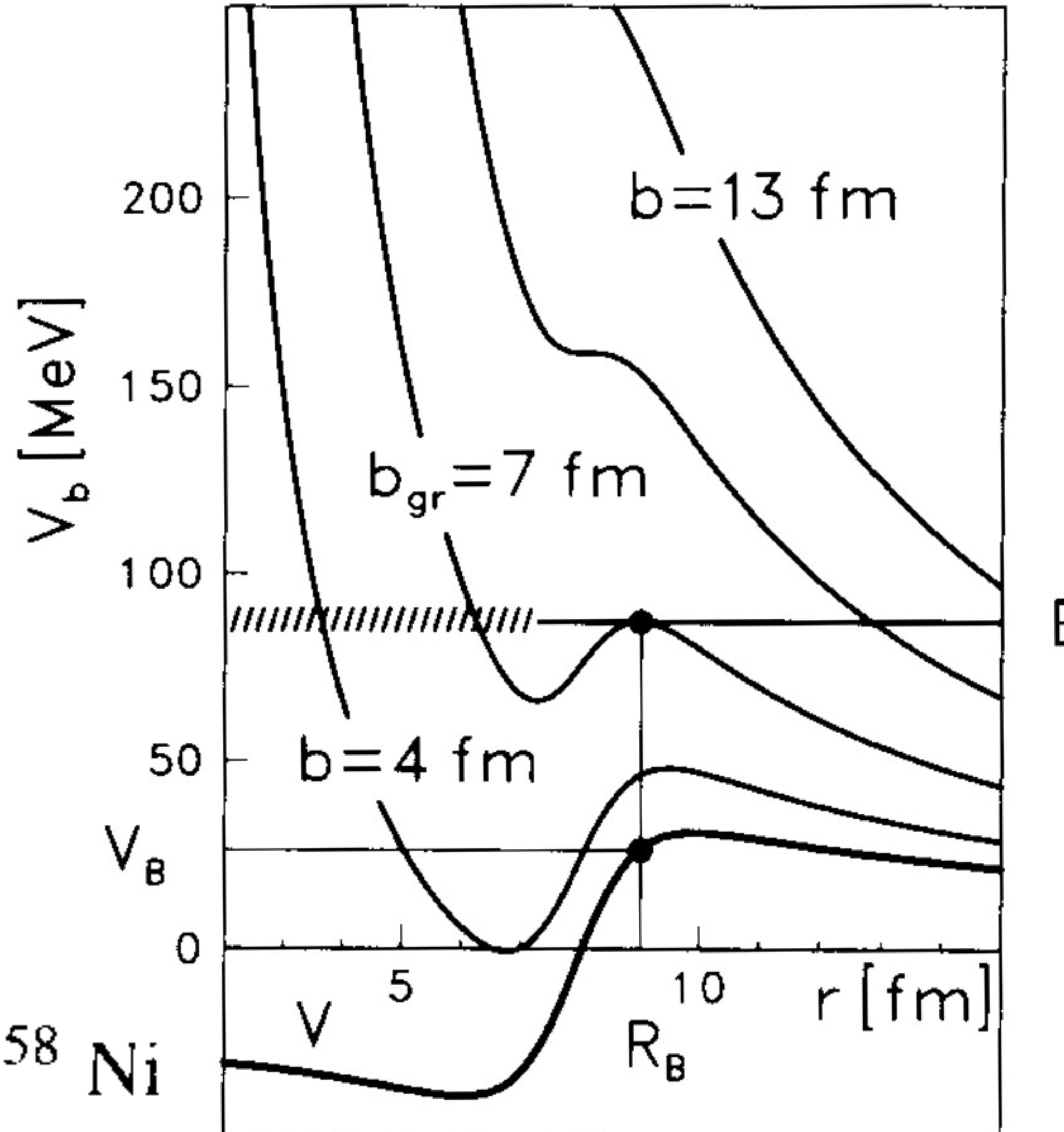
The projectile moves in the field of the Coulomb-plus-nuclear potential *V(r)*. For a given impact parameter *b* the radial motion is governed by the potential

$$V_b(r) = V(r) + E\frac{b^2}{r^2},$$

$$V_B = V(R_B) = V_{b=0}(R_B),$$

$$V_B + E\frac{b_{gr}^2}{R_B^2} = E,$$

$$b_{gr} = R_B\sqrt{1 - \frac{V_B}{E}}.$$

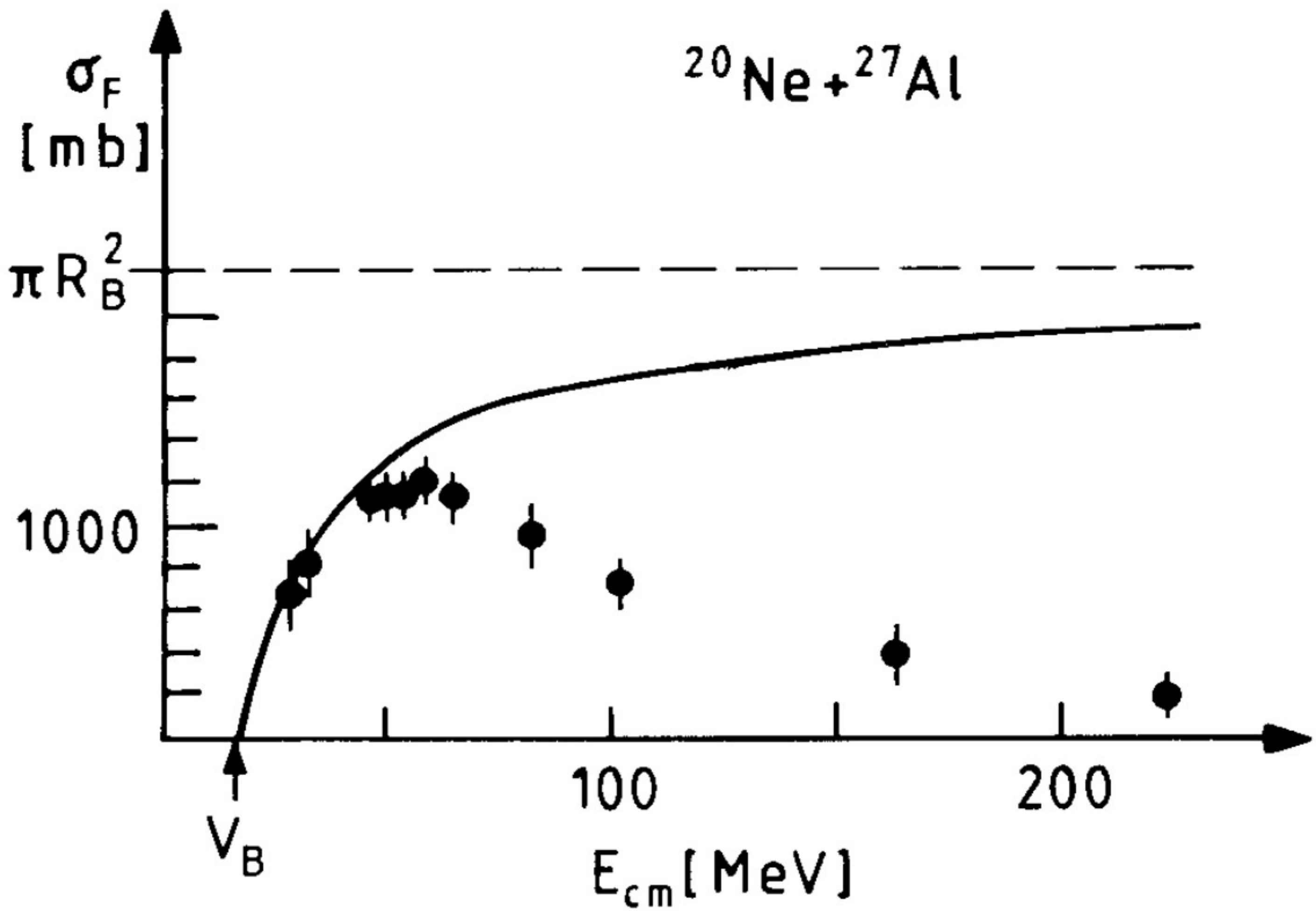


$^{18}O + ^{58}Ni$

Total fusion cross section

$$\sigma_F = \pi b_{gr}^2 .$$

$$\sigma_F(E) = \pi R_B^2 \left(1 - \frac{V_B}{E}\right) .$$



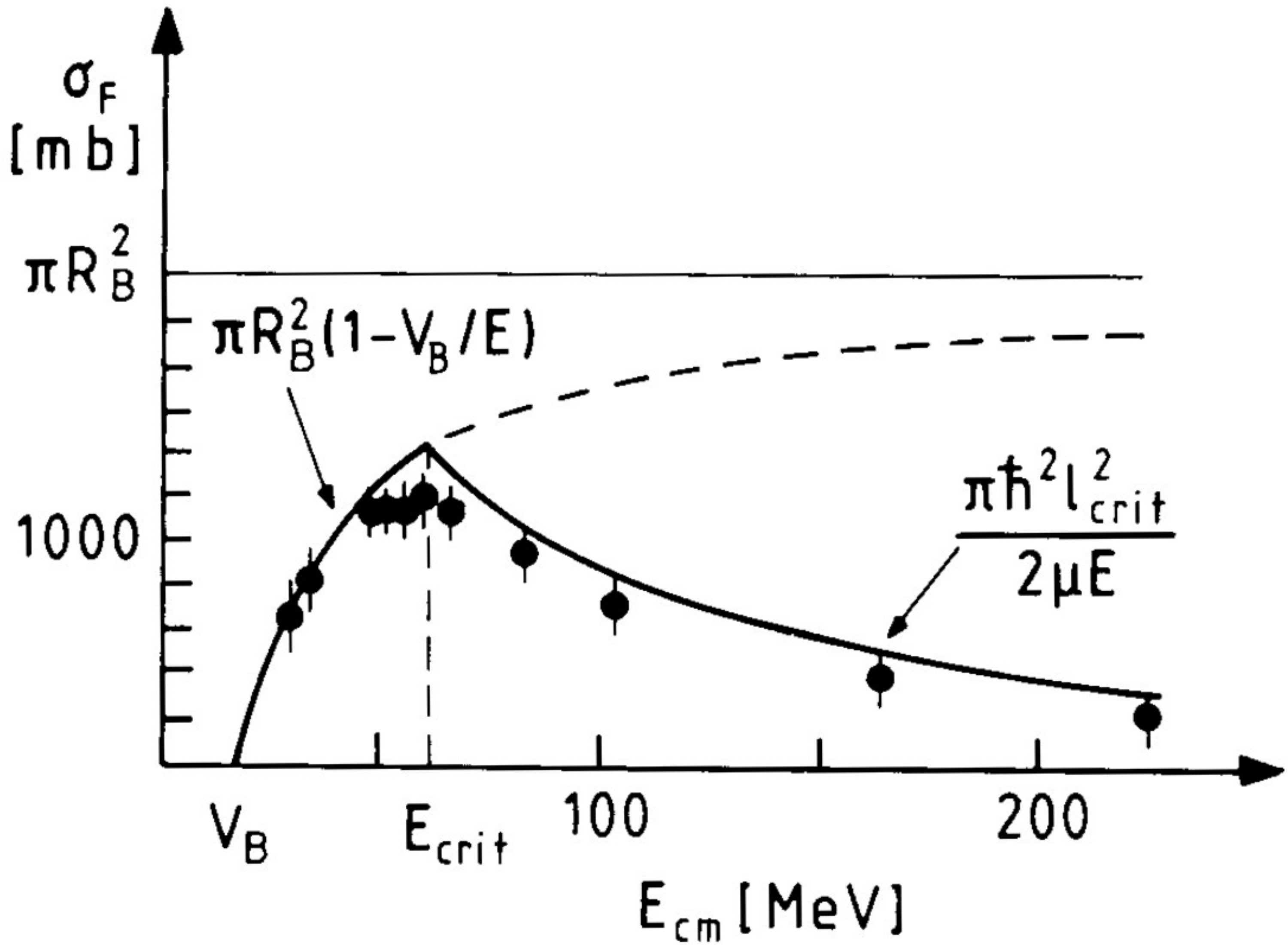
Limitation by angular momentum

The compound nucleus becomes unstable against fission above the certain value of angular momentum $l_{crit}=l_{crit}^{f}$, $b_{\mathrm{crit}} = l_{\mathrm{crit}}/k$.

$$\sigma_{\mathrm{F}} = \begin{cases} \pi b_{\mathrm{gr}}^2 & \text{for } b_{\mathrm{gr}} < b_{\mathrm{crit}}, \\ \pi b_{\mathrm{crit}}^2 & \text{for } b_{\mathrm{gr}} > b_{\mathrm{crit}}. \end{cases}$$

$$\sigma_{\mathrm{F}} = \begin{cases} \pi R_{\mathrm{B}}^2 (1 - V_{\mathrm{B}}/E) & \text{for } E < E_{\mathrm{crit}}, \\ \pi \hbar^2 l_{\mathrm{crit}}^2 / 2\mu E & \text{for } E > E_{\mathrm{crit}}. \end{cases}$$

^{20}Ne + ^{27}Al



σ_F
[mb]
πR_B^2
πR_B^2(1−V_B/E)
1000
πħ^2l_crit^2 / 2μE
V_B
E_crit
100
200
E_cm [MeV]

Sub-barrier fusion

transmission coefficient in the WKB approximation

$$T = \exp\left(-\frac{2}{\hbar}\int_b^a |p(x')|dx'\right),$$

$$p(x) = \sqrt{2\mu[E - V(x)]},$$

For the parabolic barrier, Hill-Wheeler formula

$$T = T(E) = \frac{1}{1 + \exp[2\pi(V_B - E)/\hbar\omega]}.$$

$$T_l(E) = \frac{1}{1 + \exp\{2\pi[V_B + \hbar^2 l(l+1)/2\mu R_B^2 - E]/\hbar\omega_B\}},$$

$$\omega_B^2 = \left| \frac{1}{\mu} \frac{d^2}{dr^2} \left(V(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2} \right) \right|_{R_B}$$

$$\sigma_F(E) = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) T_l(E)$$
$$\approx \frac{2\pi}{k^2} \int_0^{\infty} \frac{l\,dl}{1 + \exp\{2\pi[V_B + \hbar^2 l^2/2\mu R_B^2 - E]/\hbar\omega_B\}}.$$

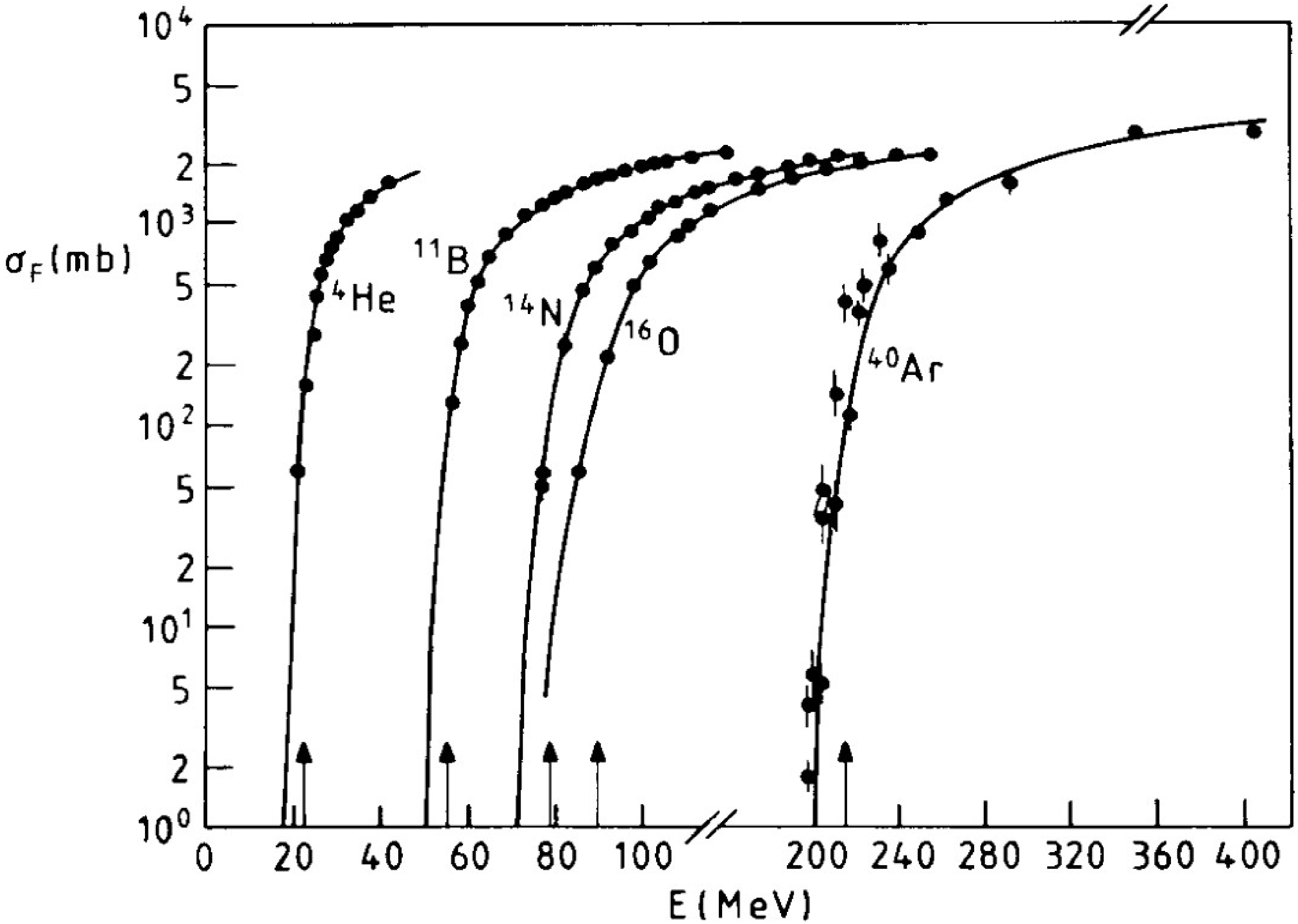
With the substitutions $y = l^2$, $a = \exp[2\pi(V_B - E)/\hbar\omega_B]$ and $b = \pi\hbar/\mu R_B^2\omega_B$ we obtain

$$\sigma_F(E) = \frac{\pi}{k^2}\int_0^\infty \frac{dy}{1 + a\exp(by)} = \frac{\pi}{k^2}\frac{1}{b}\ln\left(1 + \frac{1}{a}\right).$$

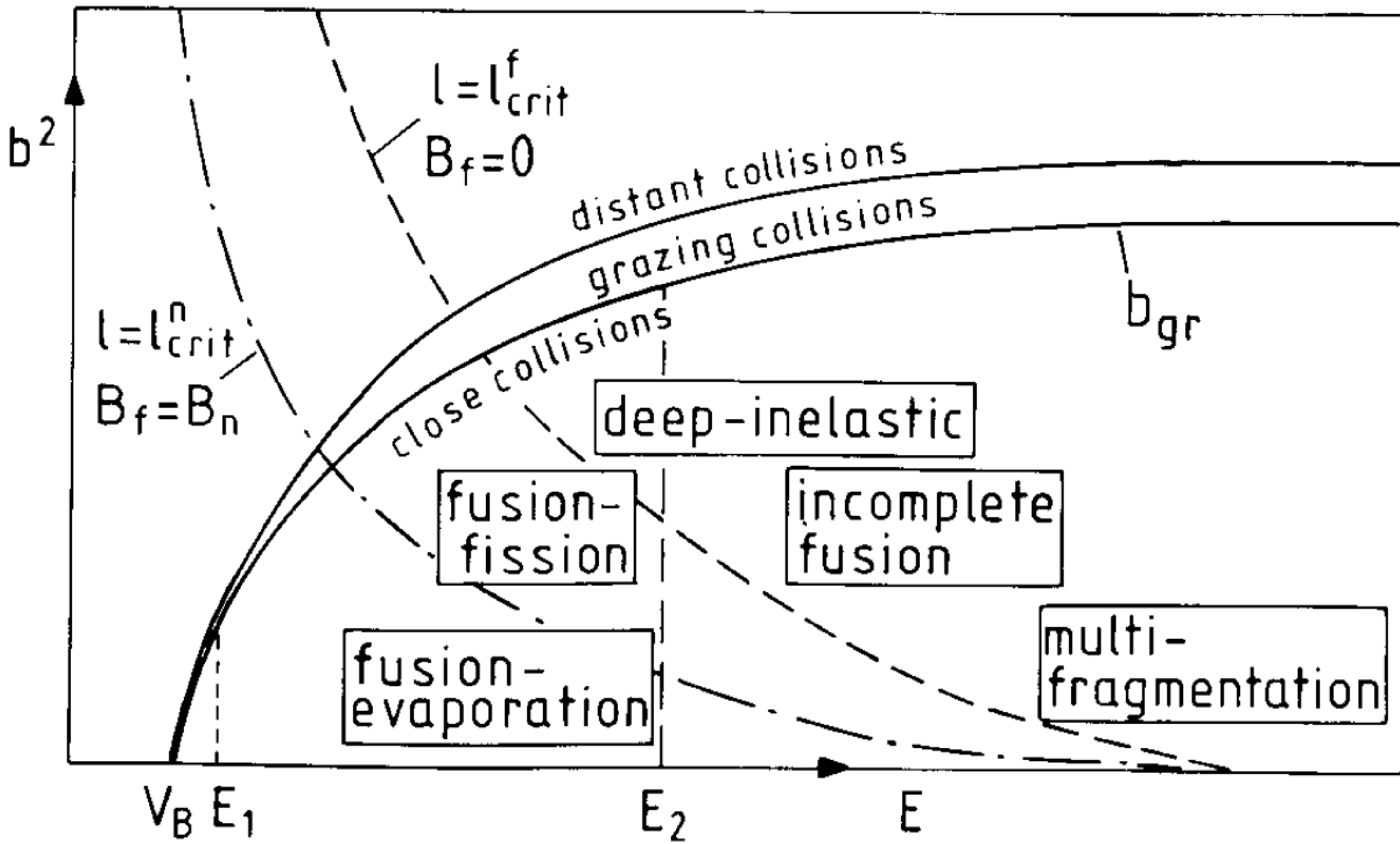
Going back to the original parameters, we arrive at the *Wong formula* for the fusion cross section

$$\sigma_F(E) = \frac{\hbar\omega_B R_B^2}{2E}\ln\{1 + \exp[2\pi(E - V_B)/\hbar\omega_B]\}.$$

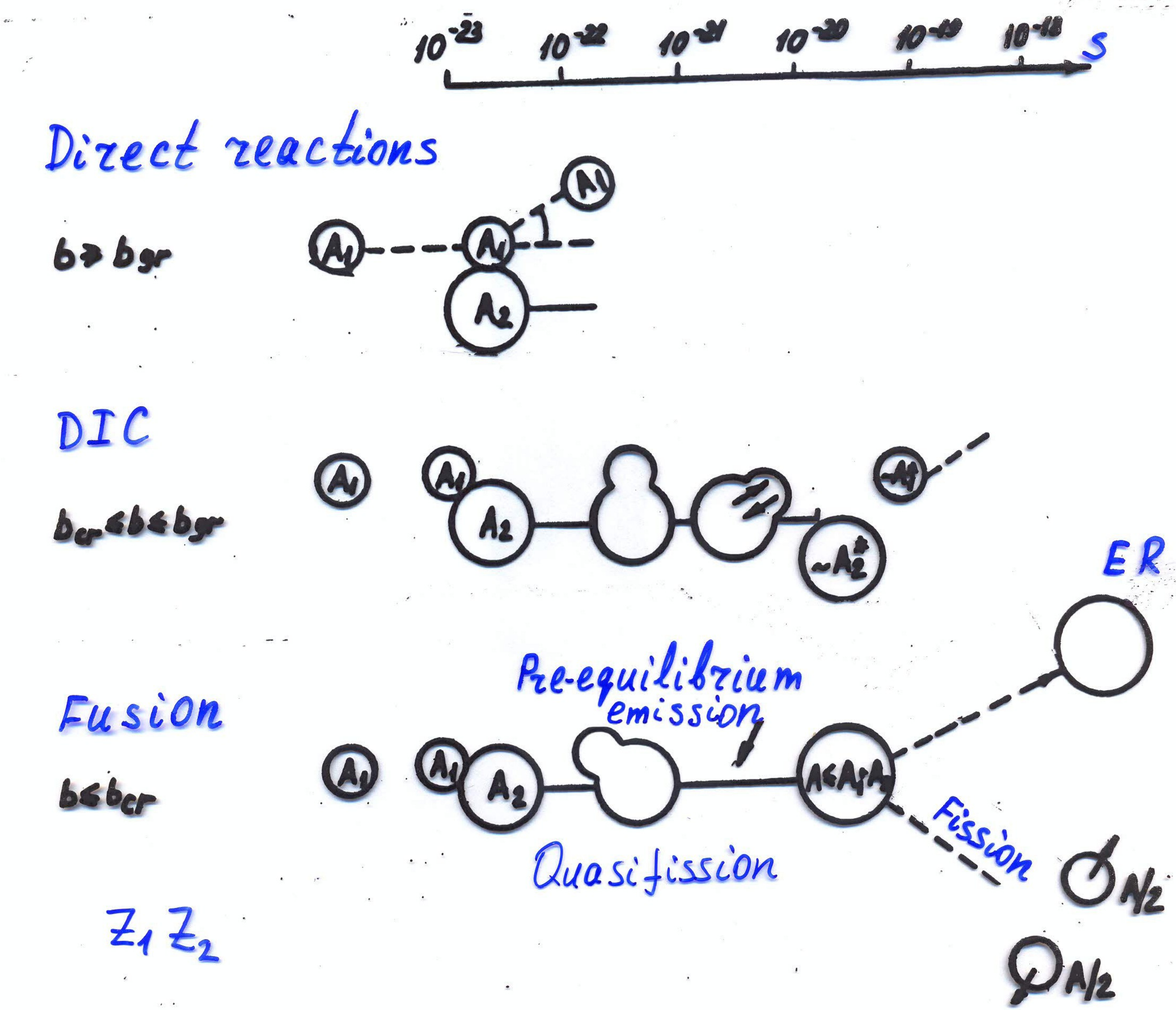
$$\sigma_F(E) = \begin{cases} \pi R_B^2[1 - (V_B/E)] & \text{for } E > V_B, \\ \left(\hbar\omega_B R_B^2/2E\right)\exp[-2\pi(V_B - E)/\hbar\omega_B] & \text{for } E < V_B. \end{cases}$$



$+^{238}U$



b²
l = l^f_crit
B_f = 0
distant collisions
grazing collisions
close collisions
b_gr
l = l^n_crit
B_f = B_n
deep-inelastic
fusion-fission
incomplete fusion
fusion-evaporation
multi-fragmentation
V_B
E_1
E_2
E



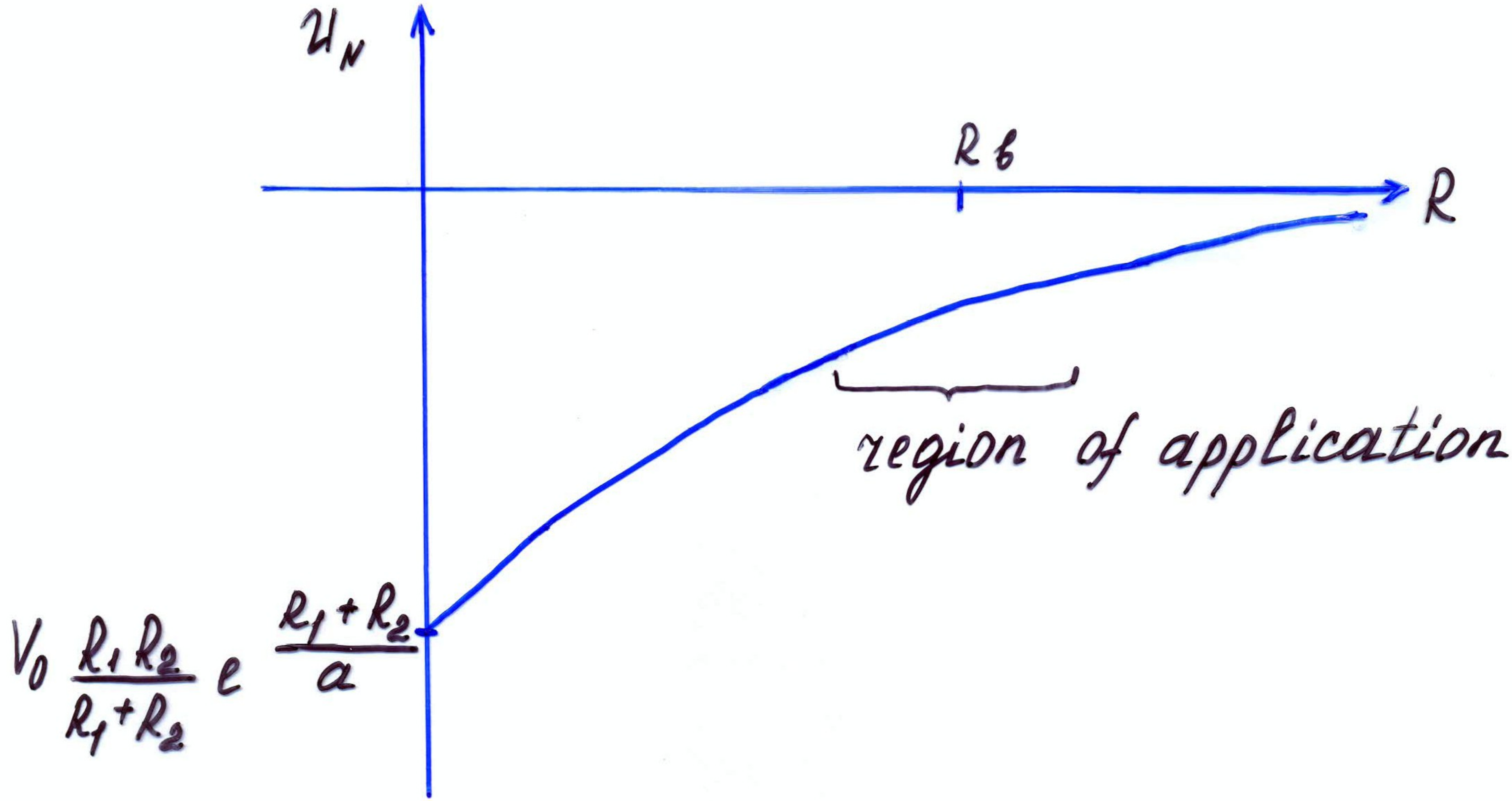
10^{-23} 10^{-22} 10^{-21} 10^{-20} 10^{-19} 10^{-18} s
Direct reactions
$b > b_{gr}$
A_1
A_1
A_1
A_2
DIC
$b_{cr} < b < b_{gr}$
A_1
A_1
A_2
$\sim A_1$
$\sim A_2^*$
ER
Pre-equilibrium emission
Fusion
$b < b_{cr}$
A_1
A_1
A_2
$A \leq A_1 + A_2$
Fission
Quasifission
A/2
A/2
$Z_1 Z_2$

Nucleus-nucleus potential

$$U(R, J) = U_N(R) + U_{Coul}(R) + U_{rot}(R, J)$$

Phenomenological potentials

$$U_N(R) = V_0 \frac{R_1 R_2}{R_1 + R_2} \exp\left(-\frac{R - R_1 - R_2}{a}\right)$$

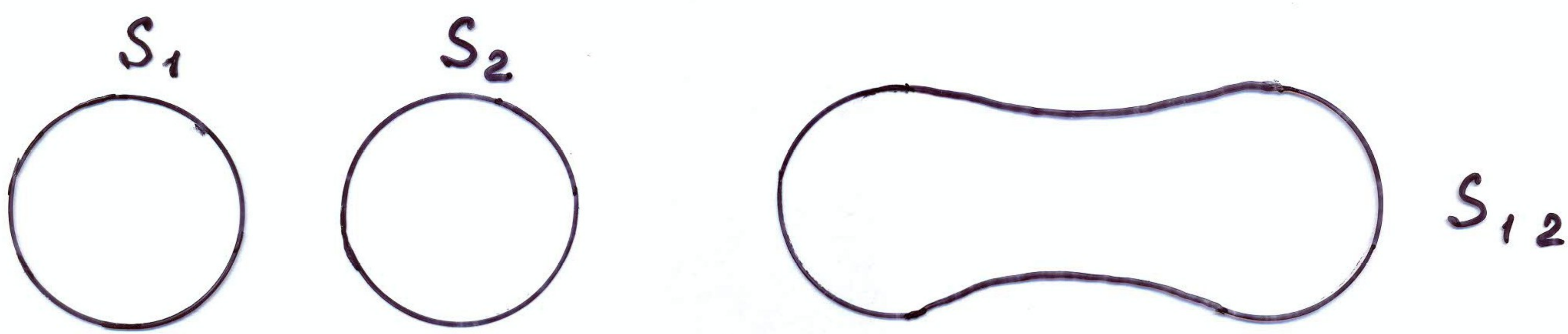


$V_0 < 0 \quad \approx -50\ MeV, \quad a \approx 0{,}5\ fm$

$R_i = r_0 A_i^{1/3}, \quad r_0 \approx 1{,}3\ fm$

Adiabatic approach: the smooth change of internal structure of approaching nuclei, equilibrium $\rho(\vec{r})$ at each R

$$U_N(R) = \sigma \underbrace{(S_{12} - S_1 - S_2)}_{\text{the change of surface}}$$



$$\sigma \approx 0{,}95 \ MeV \cdot fm^{-2}$$

Sudden approximation remains the structures of interacting nuclei

$$\rho(\vec{r}) = \rho_1(\vec{r}) + \rho_2(\vec{r})$$

small compressibility of nuclear matter → repulsive core

Energy density approach

$$\langle \Psi(R) | \hat{H} | \Psi(R) \rangle = \int d\vec{r}\, \varepsilon(\rho)$$

$$U_N(R) = \int d\vec{r} \left\{ \varepsilon(\rho_1 + \rho_2) - \varepsilon(\rho_1) - \varepsilon(\rho_2) \right\}$$

parametrization of $\varepsilon(\rho)$

V. N. Bragin, M.V Zhukov, Part. Nucl. 15(1984)725

$$U_N(R) = \bar{C} \begin{cases} -34\, e^{-0{,}27 s^2} & , \ s > -1{,}6 \text{ fm} \\ -34 + 5{,}4\,(s + 1{,}6)^2 & , \ s < -1{,}6 \text{ fm} \end{cases}$$

$$\bar{C} = C_1 C_2 / (C_1 + C_2)$$

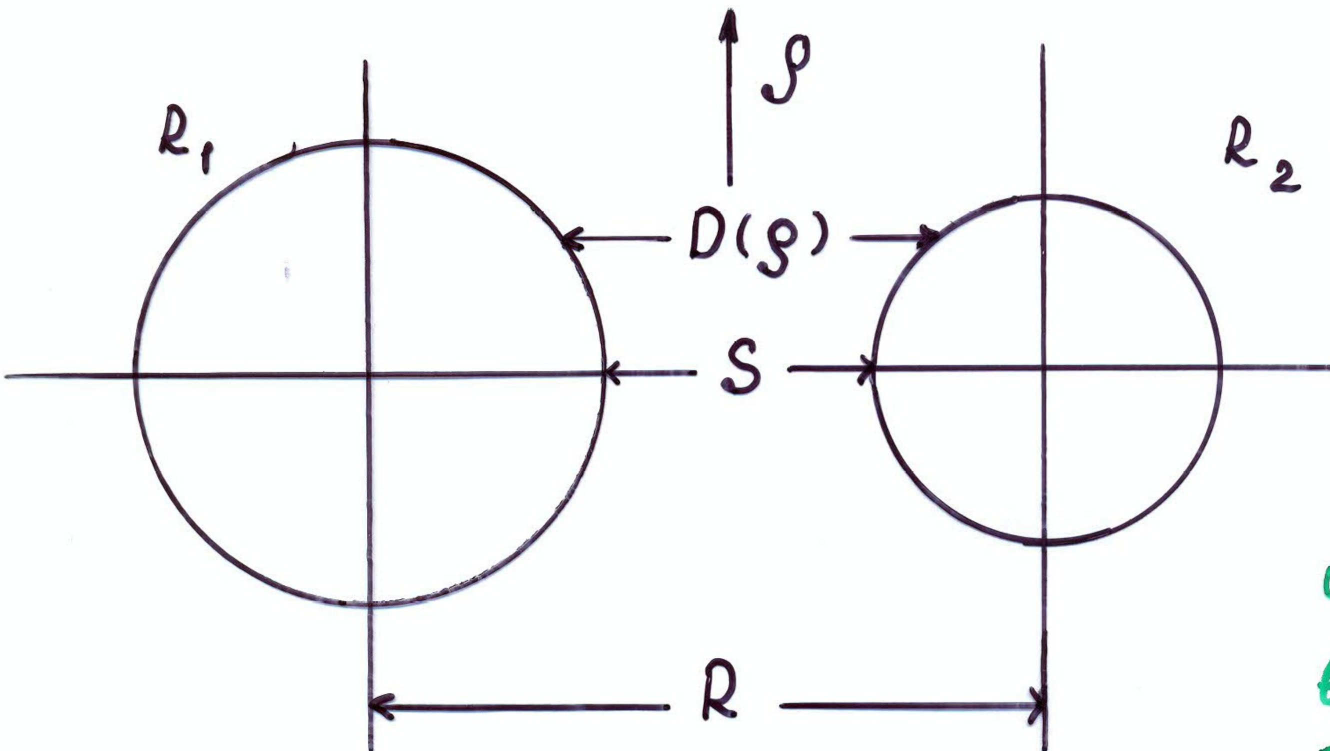
$$s = R - C_1 - C_2 \ , \quad R_i = 1{,}16\, A_i^{1/3} \text{ fm} \ , \quad C_i = R_i - 1/R_i$$

repulsive core due to the condition of saturation of nuclear forces in $\varepsilon(\rho)$

$$\varepsilon(\rho) = \tau + \rho\, V(\rho, \alpha) + \frac{\hbar^2}{8m}\, \eta\, (\nabla \rho)^2$$

$$\alpha = \frac{\rho^N - \rho^Z}{\rho^N + \rho^Z} \ , \quad \tau \sim \rho^{5/3}$$

Proximity potential



surface density of interaction energy of two plane layer

$$U_N(R) = \int dS\, e(D) = 2\pi \bar{R}_{12} \int_{D=S}^{\infty} dD\, e(D) =$$

$$= 4\pi \sigma b \bar{R}_{12}\, \Phi(\zeta)$$

$\zeta = s/b$, $b \approx 1 fm$, $\bar{R}_{12} = R_1 R_2 / (R_1 + R_2)$ - the reduced curvature radius,

$$\Phi(\zeta) = \int_{\zeta}^{\infty} d\zeta' \, \varphi(\zeta)$$

$$\varphi(\zeta) = e(b\zeta')/(2\sigma)$$

$$
\Phi(\zeta) = \begin{cases}
-1{,}7817 + \zeta , & \zeta < 0 \text{ (adiabatic limit)} \\
-1{,}7817 + 0{,}927\zeta + 0{,}143\zeta^2 - 0{,}09\zeta^3 , & \zeta < 0 \text{ (sudden limit)} \\
-1{,}7817 + 0{,}927\zeta + 0{,}1696\zeta^2 - 0{,}05148\zeta^3 , & 0 < \zeta < 1{,}9475 \\
-4{,}41\, e^{-\zeta/0{,}7176} , & \zeta \geq 1{,}9475
\end{cases}
$$

DOUBLE FOLDING POTENTIAL

$$U_N(R) = \int \rho_1(\mathbf{r_1})\rho_2(\mathbf{R} - \mathbf{r_2})F(\mathbf{r_1} - \mathbf{r_2})d\mathbf{r_1}d\mathbf{r_2}$$

The method allows us to take into account the finite size of interacting nuclei by their densities. However, there is a question of the choice of the nucleon-nucleon interaction. The microscopic theories were developed together with the phenomenological approaches.

With the density-independent nucleon-nucleon interaction U_N is deep and does not take into account the exchange effects connected with antisymmetrization. These effects are separately treated excluding the forbidden states of the deep potential well from consideration.

The density dependence of the nucleon-nucleon interaction allows one take into account the exchange and saturation effects phenomenologically. Among that kind of interactions, the Skyrme-type interactions are often used due to their simple structure. Without momentum dependence the expression for the Skyrme interaction reduces to the expression for local interaction

$$F(\mathbf{r}_1 - \mathbf{r}_2) = C_0\left(F_{in}\frac{\rho(\mathbf{r}_1)}{\rho_{00}} + F_{ex}\left(1 - \frac{\rho(\mathbf{r}_1)}{\rho_{00}}\right)\right)\delta(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r})$$

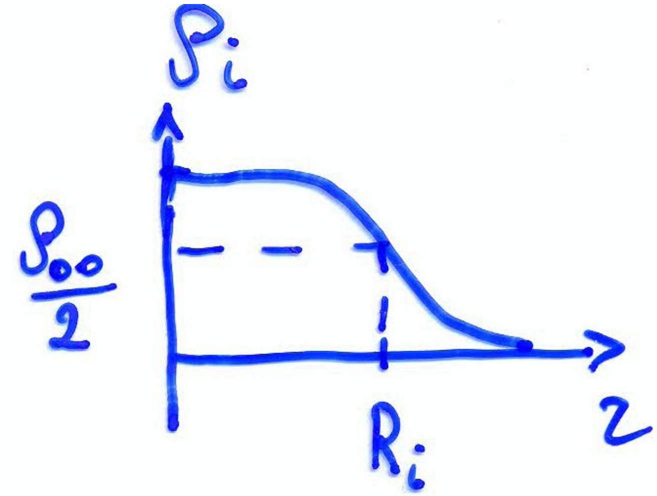
$$U_N(R) = C_0\{\frac{F_{in} - F_{ex}}{\rho_{00}}\left(\int \rho_1^2(\mathbf{r})\rho_2(\mathbf{R}-\mathbf{r})d\mathbf{r} + \int \rho_1(\mathbf{r})\rho_2^2(\mathbf{R}-\mathbf{r})d\mathbf{r}\right)$$

$$+F_{ex}\int \rho_1(\mathbf{r})\rho_2(\mathbf{R}-\mathbf{r})d\mathbf{r}\}$$

$C_0 \simeq 300$ MeV fm^3, $\rho_{00} \simeq 0.17$ fm^{-3}

$F_{in} \simeq 0.1$ $F_{ex} \simeq -2.6$

Two-parameter Woods-Saxon function



$$\rho_i(\mathbf{r}) = \frac{\rho_{00}}{1 + \exp[(r - R_i(\theta_i, \varphi_i))/a_i]}$$

or symmetrized Woods-Saxon function

$$\rho_i(\mathbf{r}) = \frac{\rho_{00} \sinh[R_i(\theta_i, \varphi_i)/a_i]}{\cosh[R_i(\theta_i, \varphi_i)/a_i] + \cosh[r/a_i]}$$

For light spherical nuclei,

$$\rho_i(\mathbf{r}) = A_i(\gamma^2 / \pi)^{3/2} \exp[-\gamma^2 r^2]$$

$$\rho_i^2(r) = -\rho_{00} a_i \sinh\frac{R_{0i}}{a_i} \frac{d}{dR_{0i}} \frac{\rho_i(r)}{\sinh\frac{R_{0i}}{a_i}}$$

$$\int \rho_1^2(\mathbf{r}) \rho_2(\mathbf{R} - \mathbf{r}) d\mathbf{r}$$

$$= -4\pi\rho_{00} a_i \sinh\frac{R_{0i}}{a_i} \frac{d}{dR_{0i}} \frac{1}{\sinh\frac{R_{0i}}{a_i}} \int_0^\infty \rho_1(p)\rho_2(p) j_0(pR) p^2 dp$$

$$f(\vec{x}) = \frac{1}{(2\pi)^3} \int d\vec{p}\; \tilde{f}(\vec{p})\, e^{-i\vec{p}\vec{x}}$$

$$\tilde{f}(\vec{p}) = \int d\vec{x}\; e^{i\vec{p}\vec{x}} f(\vec{x})$$ — the Fourier transform of $f(\vec{x})$

$$U(R) = \int d\vec{r}_1\, d\vec{r}_2\; \rho_1(\vec{r}_1)\, \mathcal{F}(\vec{r}_{12} = \vec{R} + \vec{r}_2 - \vec{r}_1)\, \rho_2(\vec{r}_2) =$$

$$= \frac{1}{(2\pi)^3} \int d\vec{p}\, d\vec{r}_1\, d\vec{r}_2\; \tilde{\mathcal{F}}(\vec{p})\, e^{-i\vec{p}(R + \vec{r}_2 - \vec{r}_1)}\, \rho_1(\vec{r}_1)\rho_2(\vec{r}_2) =$$

$$= \frac{1}{(2\pi)^3} \int d\vec{p}\; \tilde{\mathcal{F}}(\vec{p})\, \rho_1(\vec{p})\, \rho_2(-\vec{p})\, e^{-i\vec{p}\vec{R}}$$

$$U_{coul}(R) = \frac{2e^2 Z_1 Z_2}{(2\pi)^2} \int d\vec{p}\, e^{-i\vec{p}\vec{R}} \rho_1^z(\vec{p}) \rho_2^z(-\vec{p}) \frac{1}{p^2}$$

$$\tilde{f}(\vec{p}) = \int d\vec{z}\, \frac{1}{z} e^{i\vec{p}\vec{z}} = 2\pi \int z\,dz \int_{-1}^{1} dx\, e^{ipzx} =$$

$$= \frac{2\pi}{ip} \int_0^\infty dz \left(e^{ipz} - e^{-ipz} \right)$$

$$= \frac{2\pi}{ip} \lim_{\eta \to 0} \int_0^\infty dz \left(e^{ipz - \eta z} - e^{-ipz - \eta z} \right)$$

$$= \frac{2\pi}{ip} \times \left[-\frac{1}{ip} + \frac{1}{-ip} \right] = \frac{4\pi}{p^2}$$

$$\rho_i(p) = \frac{\sqrt{2\pi} a_i R_{0i} \rho_{00}}{p \sinh(\pi a_i p)} \left(\frac{\pi a_i}{R_{0i}} \sin(pR_{0i}) \coth(\pi a_i p) - \cos(pR_{0i}) \right)$$

$a_1 = a_2 = a$, poles at $p = in/a$, $n = 1, 2 \ldots$

$$\int \rho_1^2(\mathbf{r}) \rho_2(\mathbf{R} - \mathbf{r}) d\mathbf{r}$$

$$= -\frac{4\pi}{3} \rho_{00}^3 \frac{a^2}{R} \sinh\frac{R_{01}}{a} \frac{d}{dR_{01}} \frac{1}{\sinh\frac{R_{01}}{a}} \sum_{n=1}^{\infty} \frac{1}{n} \exp[-\frac{nR}{a}]$$

$$\times \left\{ \left[R^3 + \frac{3a}{n} \left(R^2 + \frac{2Ra}{n} + \frac{2a^2}{n^2} \right) - 3a^2 (R + \frac{n}{a}) \left(\frac{2\pi^2}{3} + \frac{R_{01}^2 + R_{02}^2}{a^2} \right) \right] \right.$$

$$\times \sinh\frac{nR_{01}}{a} \sinh\frac{nR_{02}}{a} + 2R_{01}(\pi^2 a^2 + R_{01}^2) \cosh\frac{nR_{01}}{a} \sinh\frac{nR_{02}}{a}$$

$$\left. + 2R_{02}(\pi^2 a^2 + R_{02}^2) \cosh\frac{nR_{02}}{a} \sinh\frac{nR_{01}}{a} \right\}$$

$R > R_{01} + R_{02}$

two light nuclei

$$\int \rho_1^2(\mathbf{r})\rho_2(\mathbf{R}-\mathbf{r})d\mathbf{r}$$

$$= \pi A_1^2 A_2 \left(\frac{\gamma_1^2}{\pi}\right)^3 \left(\frac{\gamma_2^2}{\pi}\right)^{3/2} \frac{\sqrt{\pi}}{(2\gamma_1^2+\gamma_2^2)^{3/2}} \exp\left[-\frac{2\gamma_1^2\gamma_2^2}{2\gamma_1^2+\gamma_2^2} R^2\right]$$

spherical light-spherical heavy nuclei

$$U_N(R) = 2C_0 A_1 \left(\frac{\gamma_1^2}{\pi}\right)^{1/2} \exp[-\gamma_1^2 R^2] \frac{1}{R}$$

$$\times \int_0^\infty \exp[-\gamma_1^2 r^2] \frac{\rho_2(r)}{\rho_{00}} [(F_{in} - F_{ex})(\rho_2(r) \sinh(2\gamma_1^2 Rr)$$

$$+ \frac{A_1}{4} \left(\frac{\gamma_1^2}{\pi}\right)^{3/2} \exp[-\gamma_1^2 (r^2 + R^2)] \sinh(4\gamma_1^2 Rr))$$

$$+ \rho_{00} F_{ex} \sinh(2\gamma_1^2 Rr)] r dr$$

Relationship of Double Folding Potential and Proximity Potential

$$U_N(R) \approx C_0 \left\{ (F_{in} - F_{ex}) \left(2 - a_1 \frac{\partial}{\partial R_{01}} - a_2 \frac{\partial}{\partial R_{02}} \right) + F_{ex} \right\}$$

$$\times \int \rho_1(\vec{r}) \rho_2(\vec{r} - \vec{R}) \, d\vec{r}$$

$$\left(1 - a_i \frac{\partial}{\partial R_i} \right) \rho_i = \left(1 - \frac{e^{\frac{r - R_i}{a_i}}}{1 + e^{\frac{r - R_i}{a_i}}} \right) \rho_i = \frac{\rho_i^2}{\rho_{00}}$$

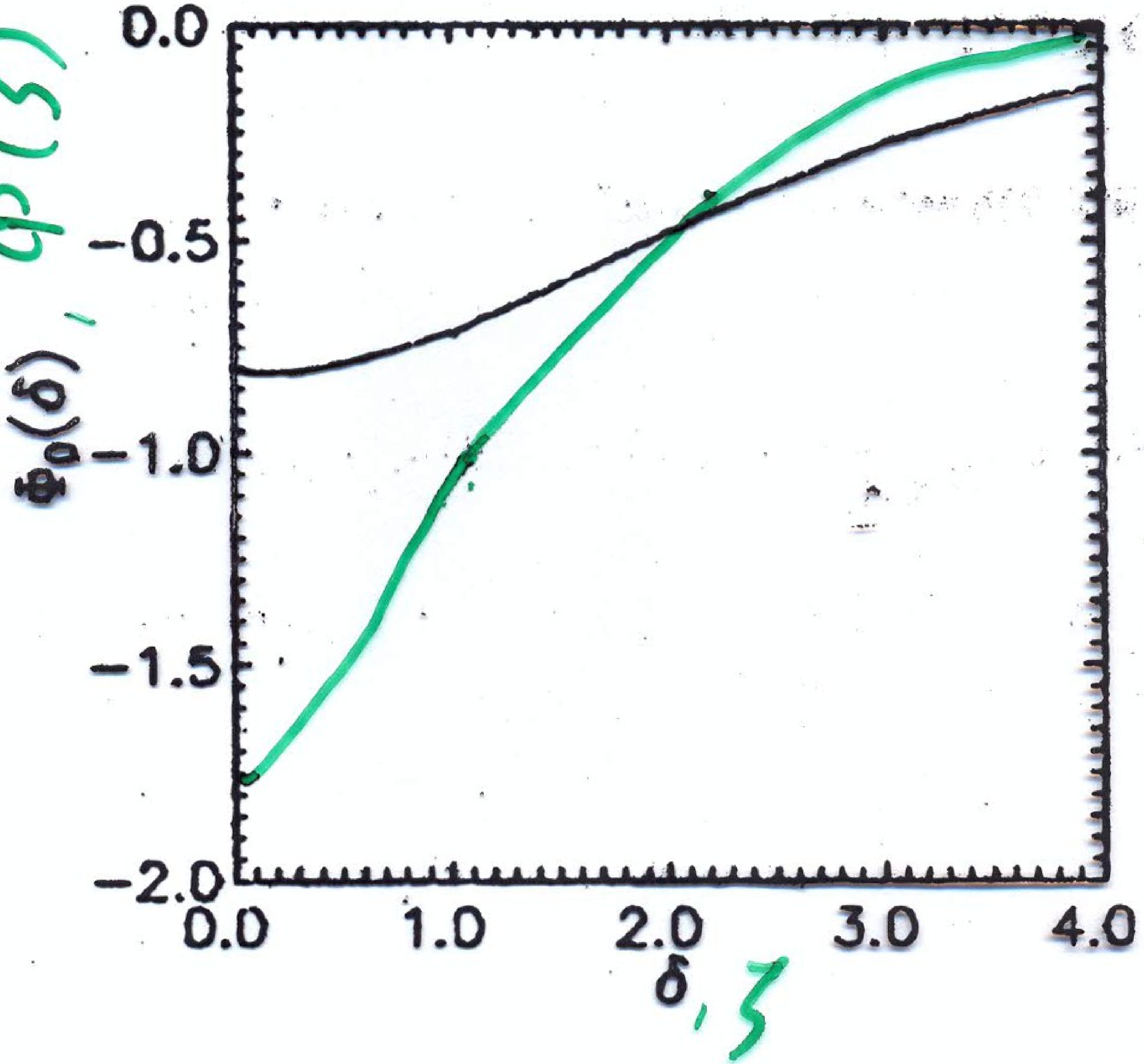
$$a_1 = a_2 = a$$

$$U_N(R) \approx 2\pi \rho_{00}^2 c_0 a^2 \frac{R_{01} R_{02}}{R_0}$$

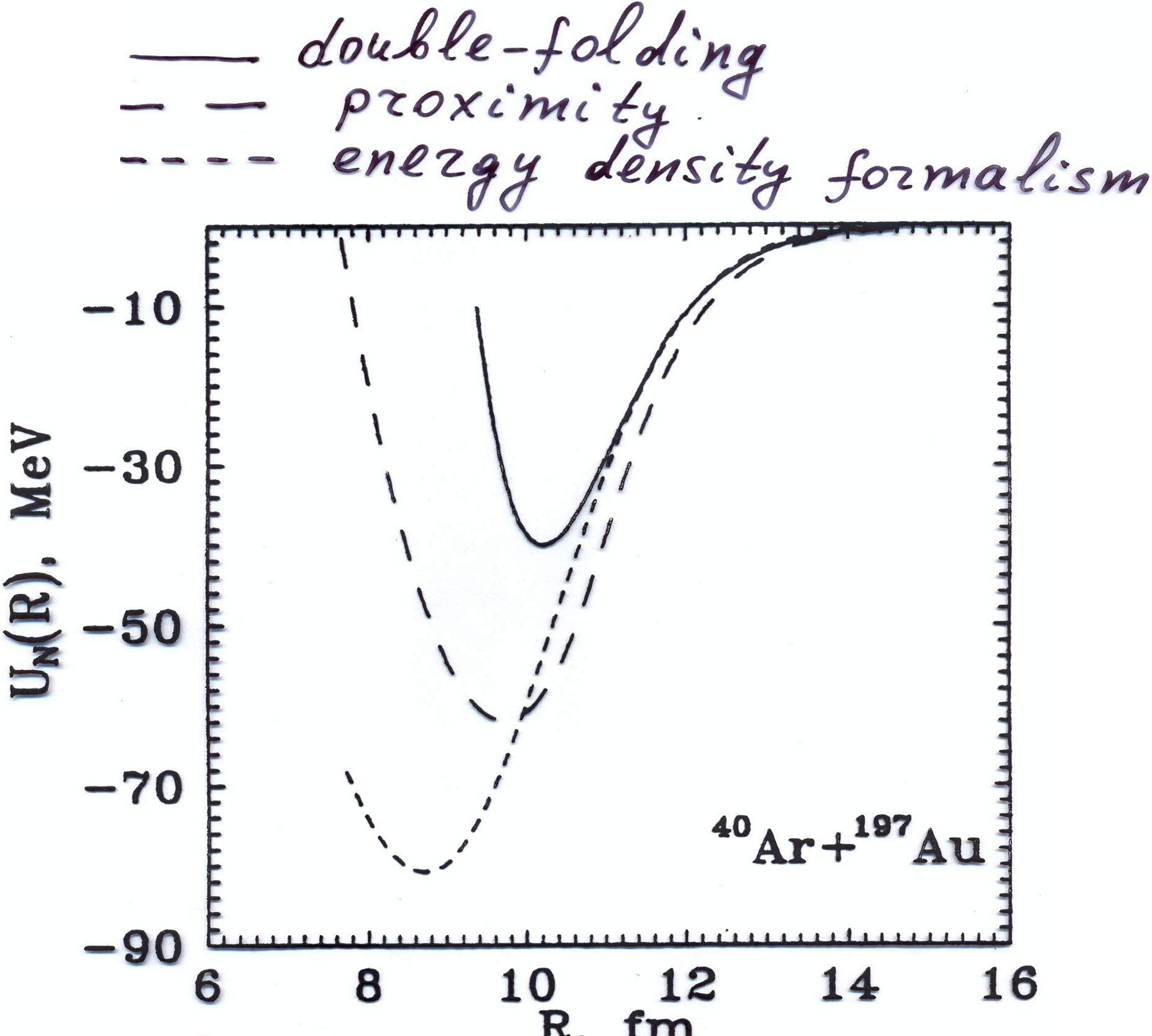
$$\times \Bigg\{ \underbrace{\sum_{n=1}^{\infty} e^{-n\delta} \left[\frac{2F_{in} - F_{ex}}{n^2} (1 + n\delta) - 2(F_{in} - F_{ex})\delta \right]}_{\Phi_0(\delta)}$$

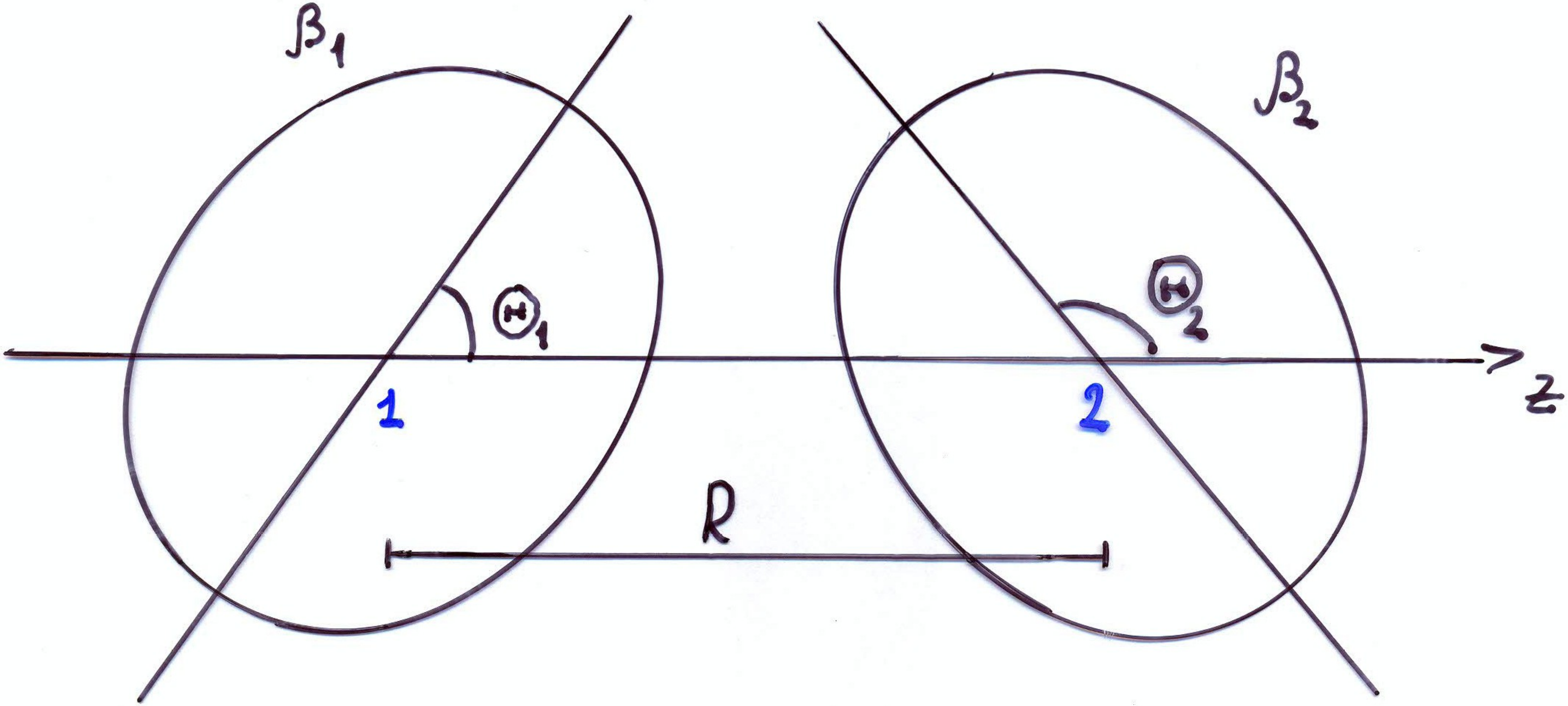
$$+ \frac{R_0^2}{2R_{01}R_{02}} \frac{a}{R_0} \Phi_1(\delta) + \frac{R_0^2}{6R_{01}R_{02}} \left(\frac{a}{R_0}\right)^2 \Phi_2(\delta)$$

Where $\delta = (R - R_{01} - R_{02})/a$ and $R_0 = R_{01} + R_{02}$



0.0
-0.5
-1.0
-1.5
-2.0
0.0
1.0
2.0
3.0
4.0
δ

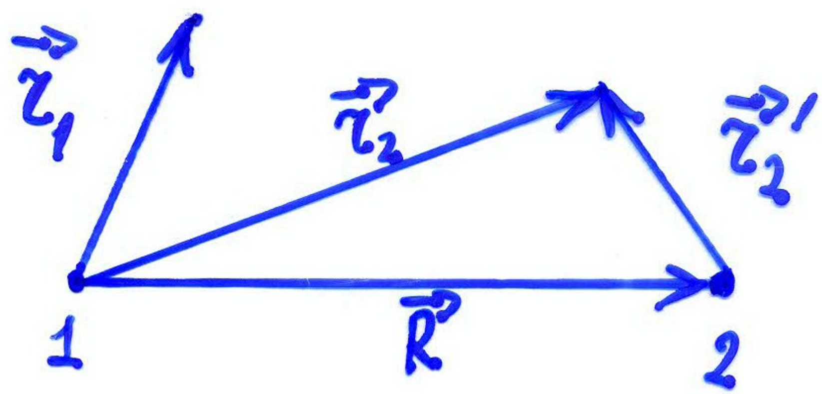

double-folding
proximity
energy density formalism
$U_N(R)$, MeV
−10
−30
−50
−70
−90
6
8
10
12
14
16
R, fm
$^{40}Ar+^{197}Au$



β_1
β_2
Θ_1
Θ_2
1
2
z
R

COULOMB POTENTIAL

$$U_{Coul}(R) = e^2 Z_1 Z_2 \int \frac{\rho_1^z(\mathbf{r}_1)\rho_2^z(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\mathbf{r}_1 d\mathbf{r}_2$$



$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_1^l}{r_2^{l+1}} Y_{lm}(\theta_1, \varphi_1) Y_{lm}^*(\theta_2, \varphi_2)$$

at $\quad r_1 < r_2$

$$Y_{lm}(\theta,\varphi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{2(l+m)!}}\, P_l^m(\cos\theta)\frac{1}{\sqrt{2\pi}}\, e^{im\varphi}$$

$$\frac{1}{r_2^{l+1}} Y_{lm}^*(\theta_2,\varphi_2) = \frac{1}{|\mathbf{R}+\mathbf{r}_2'|^{l+1}} Y_{lm}^*(\theta_2,\varphi_2)$$

$$= \sqrt{\frac{1}{(2l)!}} \sum_{\substack{l_1,l_2=0 \\ l_2-l_1=l}} (-1)^{l_1+l_2} \sqrt{\frac{(2l_2+1)!}{(2l_1+1)!}} C_{l_1 m, l_2 0}^{lm} \frac{r_2'^{l_1}}{R^{l_2+1}} Y_{lm}^*(\theta_2',\varphi_2')$$

$$r_2' < R$$

$$U_{Coul}(R)=e^2 Z_1 Z_2 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\int r_1^l Y_{lm}(\theta_1,\varphi_1)\rho_1^z(\mathbf{r}_1)d\mathbf{r}_1$$

$$\times\sqrt{\frac{1}{(2l)!}}\sum_{\substack{l_1,l_2=0\\ l_2-l_1=l}}(-1)^{l_1+l_2}\sqrt{\frac{(2l_2+1)!}{(2l_1+1)!}}C^{lm}_{l_1 m, l_2 0}\frac{1}{R^{l_2+1}}\int r_2'^{l_1}\rho_2^z(\mathbf{r}_2')Y^*_{l_1 m}(\theta_2',\varphi_2')d\mathbf{r}_2'$$

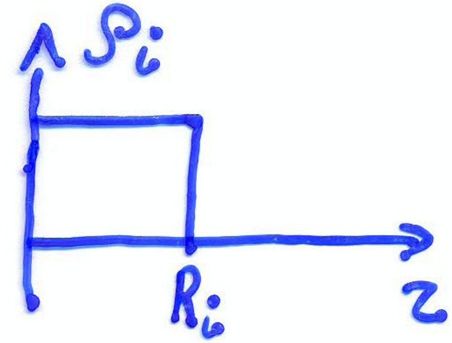
$l=l_1=l_2=m=0$; $l=0,\ l_1=l_2=2,\ m=0$; $l=l_2=2,\ l_1=0,\ m=0$

shape parametrization

$$R_i(\theta_{i0}) = R_{0i}(1 + \beta_i Y_{20}(\theta_{i0}))$$

$$Y_{20}(\theta_{i0}) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta_{i0} - 1)$$

$$Y_{20}(\theta_{i0}) = \sqrt{\frac{4\pi}{5}} \sum_m (-1)^m Y_{2m}(\theta_i, \varphi_i) Y_{2m}(\Theta_i, \Phi_i)$$



$$\rho_i^z(\mathbf{r_i}) = \rho_0^z S(r - R_i(\theta_{i0}))$$

$$= \rho_0^z [S(r - R_{0i}) + R_{0i}\beta_i Y_{20}(\theta_{i0})\delta(r - R_{0i})$$

$$-\frac{1}{2}(R_{0i}\beta_i Y_{20}(\theta_{i0}))^2 \delta'(r - R_{0i})]$$

$$(Y_{20}(\theta_{i0}))^2 = \frac{4\pi}{5}\sum_{m_1,m_2}(-1)^{m_1+m_2}Y_{2m_1}(\Theta_i,\Phi_i)Y_{2m_2}(\Theta_i,\Phi_i)Y_{2m_1}(\theta_i,\varphi_i)Y_{2m_2}(\theta_i,\varphi_i)$$

$$= \frac{4\pi}{5}\sum_{m_1,m_2}(-1)^{m_1+m_2}Y_{2m_1}(\Theta_i,\Phi_i)Y_{2m_2}(\Theta_i,\Phi_i)\sum_L\sqrt{\frac{25}{4\pi(2L+1)}}C^{L0}_{2020}C^{LM}_{2m_1 2m_2}Y_{LM}(\theta_i,\varphi_i)$$

$$\times Y_{20}(\theta_i)\cdots\int d\Omega_i \quad , \qquad \int Y_{LM}(\theta_i,\varphi_i)Y_{20}(\theta_i)\,d\Omega_i = \delta_{L,2}\,\delta_{M,0}$$

$$\frac{4\pi}{5}\sqrt{\frac{5}{4\pi}}C^{20}_{2020}\sum_{m_1,m_2}C^{20}_{2m_1 2m_2}Y_{2m_1}(\Theta_i,\Phi_i)Y_{2m_2}(\Theta_i,\Phi_i) = [C^{20}_{2020}]^2\,Y_{20}(\Theta_i)$$

$$\int r_i^l Y_{lm}(\theta_i,\varphi_i)\rho_i^z(\mathbf{r}_i)d\mathbf{r}_i =$$

$$|\, l = 0, m = 0 \,|$$

$$= Z_i / \sqrt{4\pi}$$

$$|\, l = 2, m = 0 \,|$$

$$= Z_i \sqrt{\frac{4\pi}{5}}\left(\frac{3}{4\pi} R_{0i}^2 \beta_i Y_{20}(\Theta_i) + \frac{3}{7\pi}\sqrt{\frac{5}{4\pi}}[R_{0i}\beta_i]^2 Y_{20}(\Theta_i)\right)$$

$$C_{2020}^{20} = -\sqrt{\frac{2}{7}}, \quad C_{2020}^{00} = \frac{1}{\sqrt{5}}, \quad C_{0020}^{20} = 1$$

$$U_{Coul}(R) = \frac{e^2 Z_1 Z_2}{R} + \frac{3}{5}\frac{e^2 Z_1 Z_2}{R^3}\sum_{i=1,2} R_{0i}^2 \beta_i Y_{20}(\Theta_i)$$

$$+\frac{12}{35}\sqrt{\frac{5}{4\pi}}\frac{e^2 Z_1 Z_2}{R^3}\sum_{i=1,2}[R_{0i}\beta_i]^2 Y_{20}(\Theta_i)$$

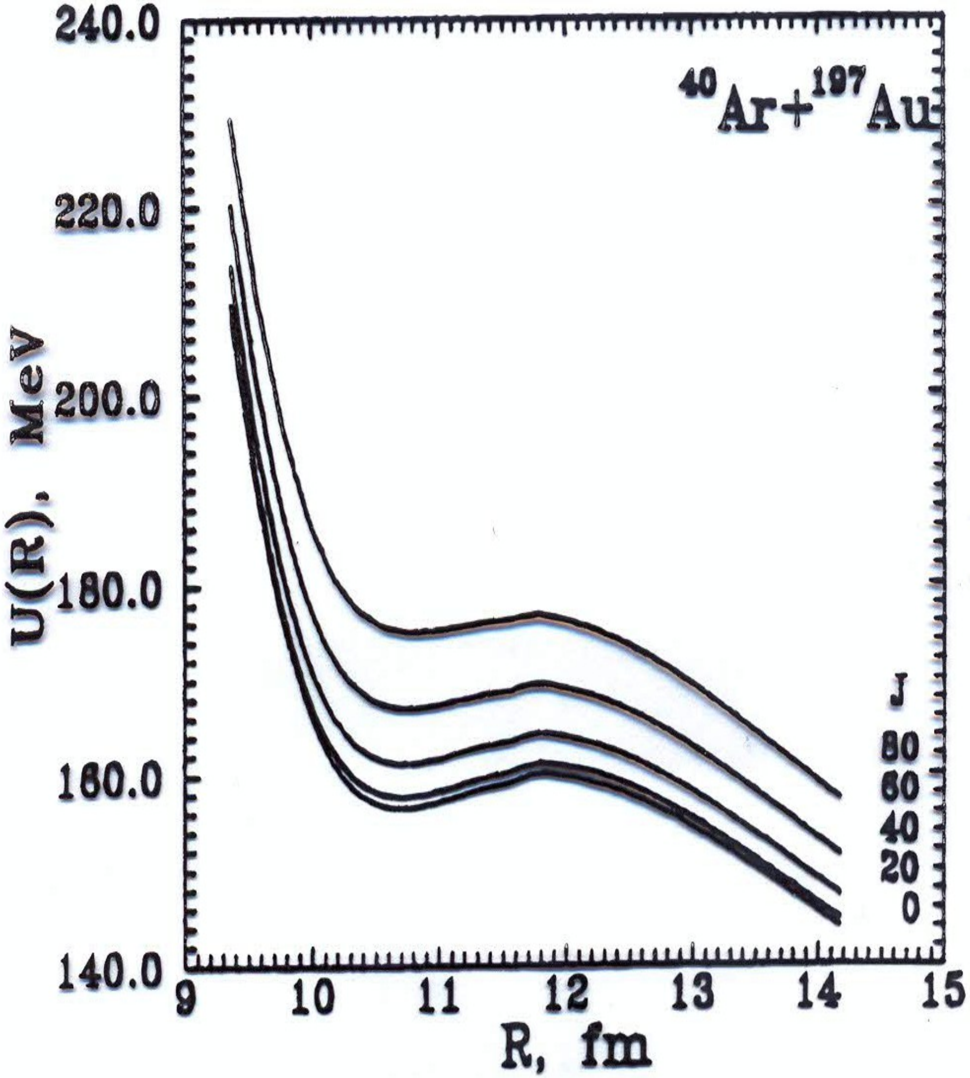
$$U_{rot}(R) = \frac{\hbar^2 \varphi J(\varphi J+1)}{2(J_1+J_2+\mu R^2)} + \frac{\hbar^2 (1-\varphi) J((1-\varphi)J+1)}{2\mu R^2}$$

the parameter φ characterizes the contribution of the rolling

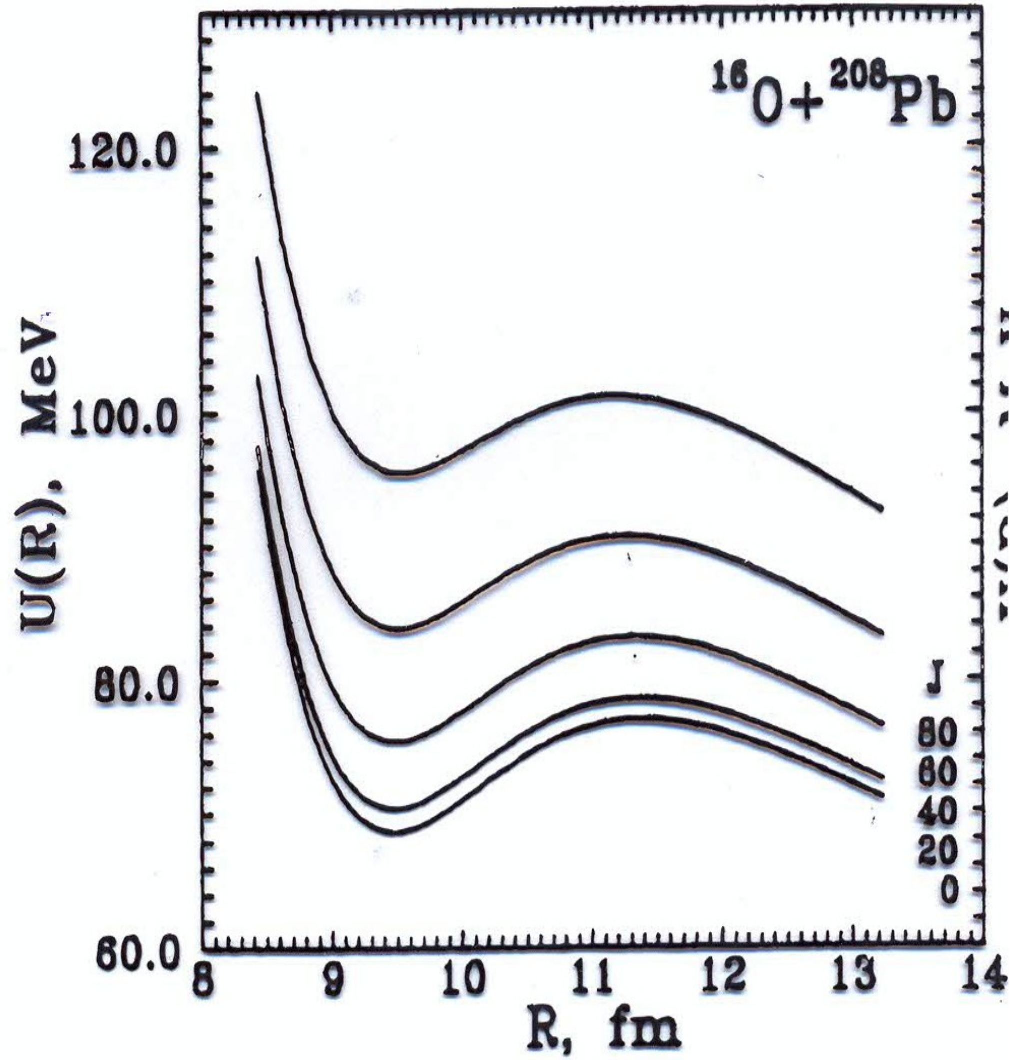
$$\varphi = 0 : \quad \frac{\hbar^2 J(J+1)}{2\mu R^2}$$

$$\varphi = 1 : \quad \frac{\hbar^2 J(J+1)}{2(J_1+J_2+\mu R^2)}$$

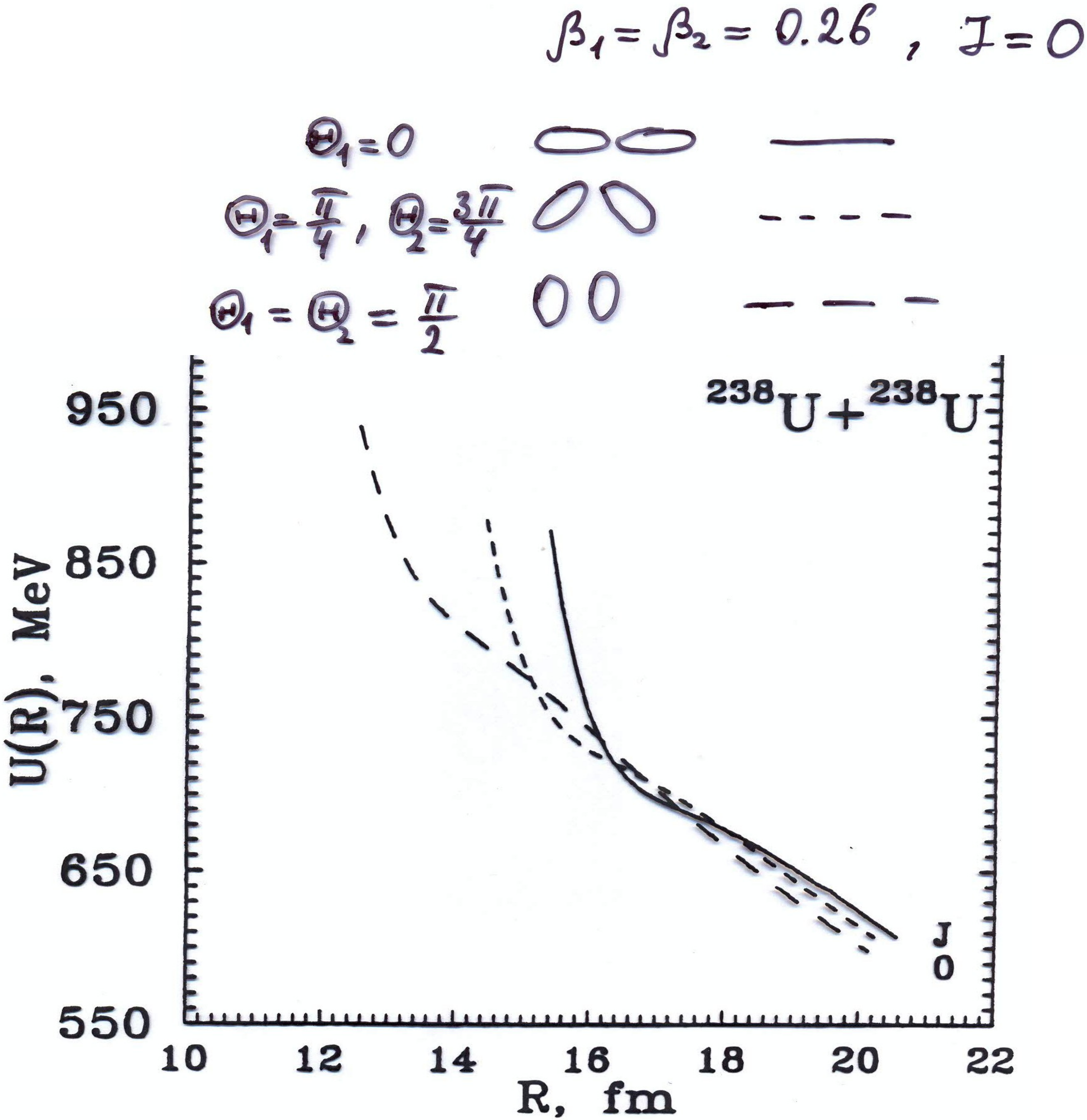
sticking condition



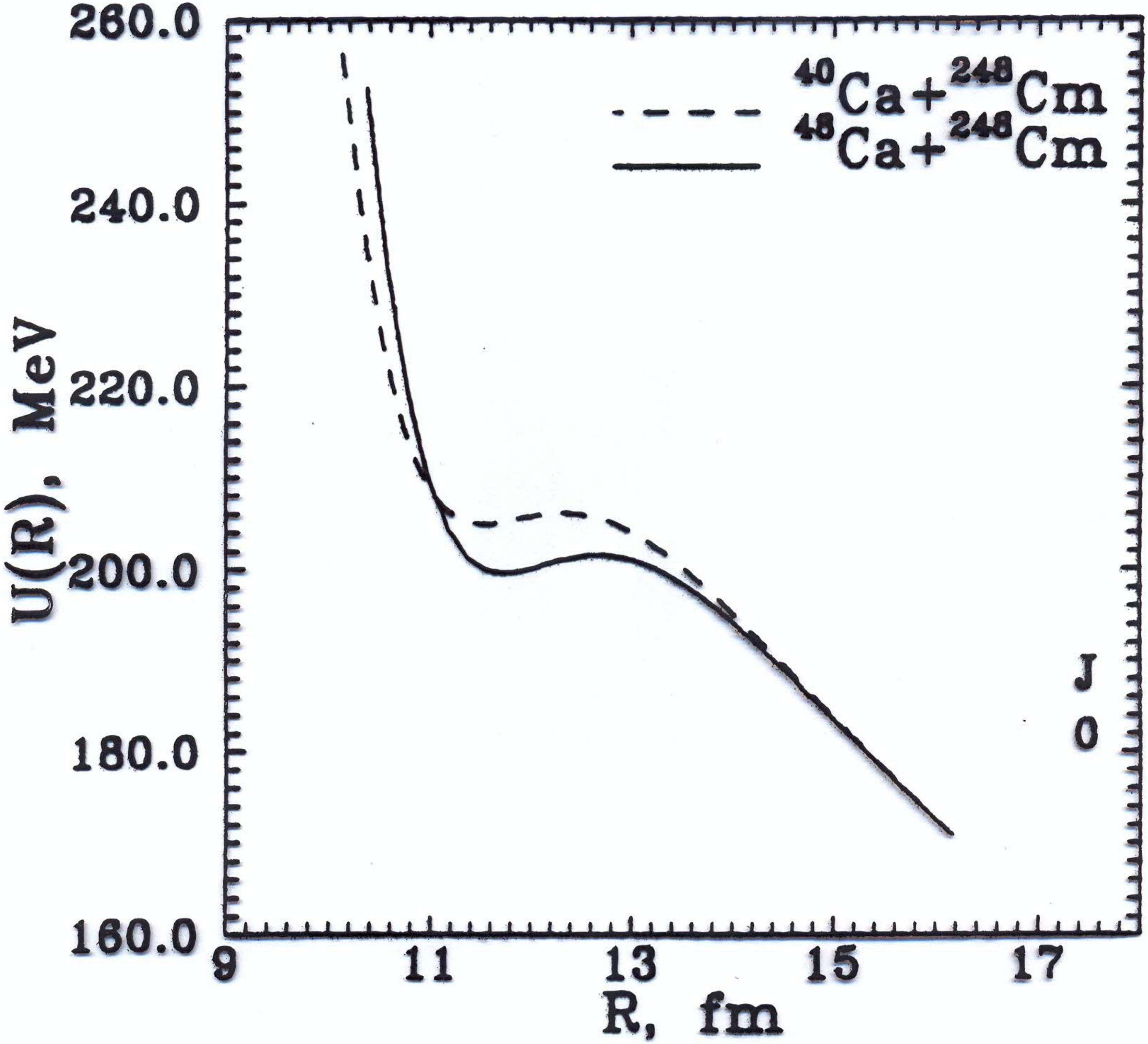
240.0
220.0
200.0
180.0
160.0
140.0
U(R), MeV
$^{40}Ar+^{197}Au$
J
80
60
40
20
0
9
10
11
12
13
14
15
R, fm



120.0
100.0
80.0
60.0
U(R), MeV
$^{16}O+^{208}Pb$
J
80
60
40
20
0
8
9
10
11
12
13
14
R, fm



$\beta_1 = \beta_2 = 0.26$, $J = 0$
$\Theta_1 = 0$
$\Theta_1 = \frac{\pi}{4}$, $\Theta_2 = \frac{3\pi}{4}$
$\Theta_1 = \Theta_2 = \frac{\pi}{2}$
$^{238}U + ^{238}U$
J
0
950
850
750
650
550
10
12
14
16
18
20
22
U(R), MeV
R, fm



260.0
240.0
220.0
200.0
180.0
160.0
U(R), MeV
9
11
13
15
17
R, fm
^{40}Ca+^{248}Cm
^{48}Ca+^{248}Cm
J
0

Classical desription

The Rayleigh dissipation function

$$\mathcal{R} = -\tfrac{1}{2}K_r(r)\dot{r}^2 - \tfrac{1}{2}K_\phi(r)r^2\dot{\phi}^2,$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} = \frac{\partial\mathcal{R}}{\partial\dot{q}_i} \quad \text{for } i = 1, 2.$$

$$q_1 = r \text{ and } q_2 = \phi.$$

$$K_r(r) = K_r^0\left(\frac{dV_N(r)}{dr}\right)^2, \quad K_\phi(r) = K_\phi^0\left(\frac{dV_N(r)}{dr}\right)^2,$$

$$K_r^0 = 4.0 \times 10^{-23}\ \mathrm{s\,MeV^{-1}}, \quad K_\phi^0 = 0.01 \times 10^{-23}\ \mathrm{s\,MeV^{-1}}.$$

$$\dot{r} = \frac{p_r}{\mu},$$

$$\dot{p}_r = -\frac{dV(r)}{dr} + \frac{L^2}{\mu r^3} - \frac{K_r(r)}{\mu} p_r,$$

$$\dot{\phi} = \frac{L}{\mu r^2},$$

$$\dot{L} = -\frac{K_\phi(r)}{\mu} L.$$

$$L = \mu r^2 \dot{\phi} \qquad\qquad p_r = \mu \dot{r}$$

$$\dot{r} = \frac{p_r}{\mu},$$

$$\dot{p}_r = -\frac{\partial V}{\partial r} + \frac{L^2}{\mu r^3} - \frac{K_r}{\mu} p_r - \sum_{i=\mathrm{P,T}} \tfrac{1}{2} K_{ra_i} \frac{\pi_i}{B_i},$$

$$\dot{\phi} = \frac{L}{\mu r^2},$$

$$\dot{L} = -\frac{K_\phi}{\mu} L,$$

$$\dot{a}_i = \frac{\pi_i}{B_i},$$

$$\dot{\pi}_i = -\frac{\partial V}{\partial a_i} - \sum_{j=\mathrm{P,T}} K_{a_i a_j} \frac{\pi_j}{B_j} - \tfrac{1}{2} K_{ra_i} \frac{p_r}{\mu} - C_i a_i,$$

$i = \mathrm{P, T}$. vibrational momentum $\pi_i = B_i \dot{a}_i$

$$p_{r(n+1)} = p_{r(n)} - \left(\frac{\partial V}{\partial r} - \frac{L^2}{\mu r^3} + K_r \frac{p_r}{\mu} + \sum_i K_{ra_i} \frac{\pi_i}{B_i} \right)_n \tau + \sqrt{D_{r(n)} \tau}\, w_r(t_n),$$

$$r_{n+1} = r_n + \frac{p_{r(n)} + p_{r(n+1)}}{2\mu} \tau,$$

$$L_{n+1} = L_n - K_{\phi(n)} L_n \tau + \sqrt{D_{\phi(n)} \tau}\, w_\phi(t_n),$$

$$\phi_{n+1} = \phi_n + \frac{L_n + L_{n+1}}{2\mu r_n^2} \tau,$$

$$\pi_{i(n+1)} = \pi_{i(n)} - \left(\frac{\partial V}{\partial a_i} + \sum_j K_{a_i a_j} \frac{\pi_j}{B_j} + K_{ra_i} \frac{p_r}{\mu} + C_i a_i \right)_n \tau + \sqrt{D_{a_i} \tau}\, w_{a_i}(t_n),$$

$$a_{i(n+1)} = a_{i(n)} + \frac{\pi_{i(n)} + \pi_{i(n+1)}}{2B_i} \tau;$$

Models of complete fusion with adiabatic and diabatic potentials

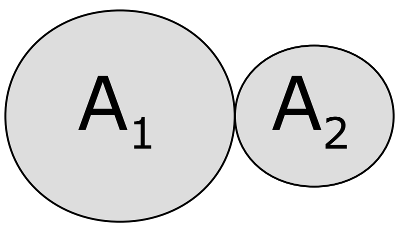
Two main collective coordinates are used for the description of the fusion process:

1. Relative internuclear distance R
2. Mass asymmetry coordinate η for transfer

Idea of Volkov (Dubna) to describe fusion reactions with the dinuclear system concept:

Fusion is assumed as a transfer of nucleons (or clusters) from the lighter nucleus to the heavier one in a dinuclear configuration.

This process is describable with the mass asymmetry coordinate $\eta=(A_1-A_2)/(A_1+A_2)$.



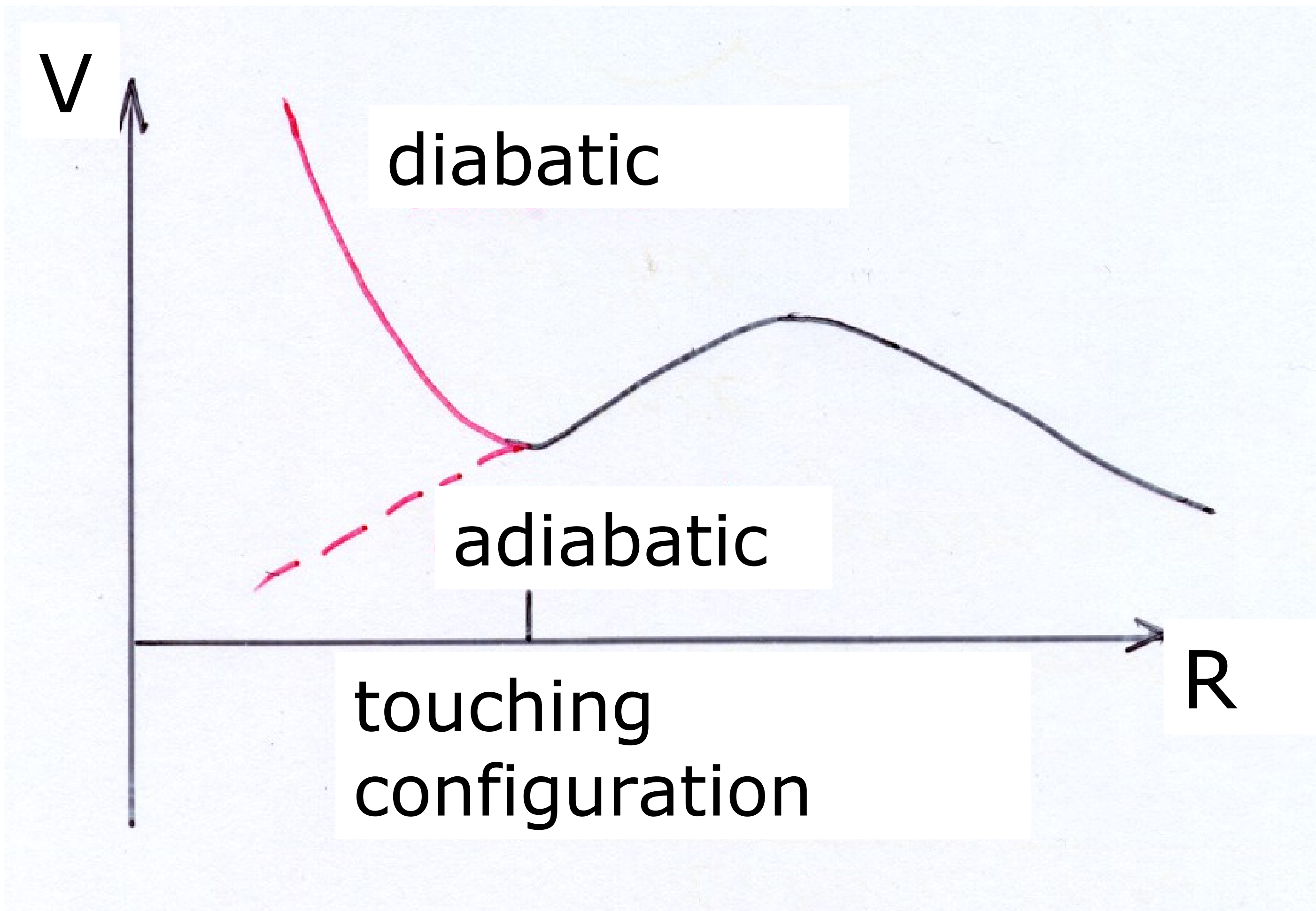
If A_1 or A_2 get small, then $|\eta|\rightarrow 1$ and the system fuses.

The dinuclear system model uses two main degrees of freedom to describe the fusion and quasifission processes:

1. Relative motion of nuclei, capture of target and projectile into dinuclear system, decay of the dinuclear system: quasifission

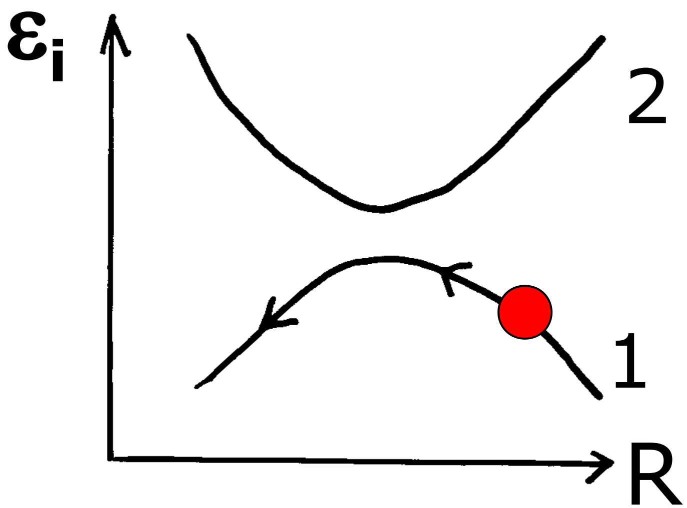
2. Transfer of nucleons between nuclei, change of mass and charge asymmetries leading to fusion and quasifission

Description of fusion dynamics depends strongly whether adiabatic or diabatic potential energy surfaces are assumed.

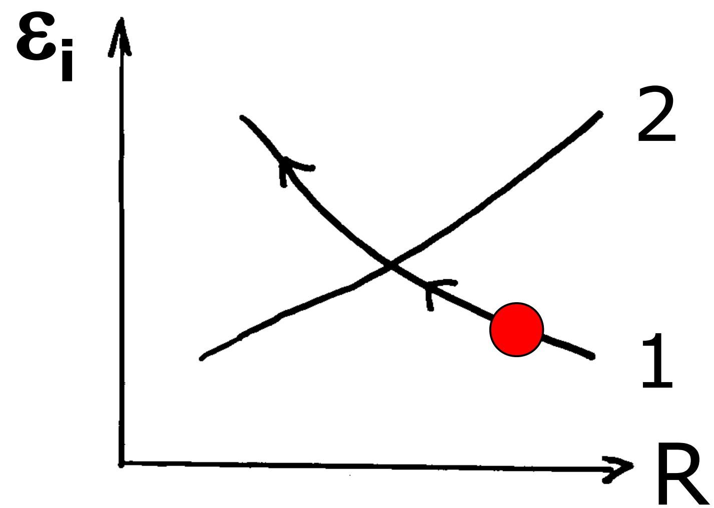


Diabatic potentials are repulsive at smaller internuclear distances $R<R_t$.

Explanation with two-center shell model:



adiabatic model

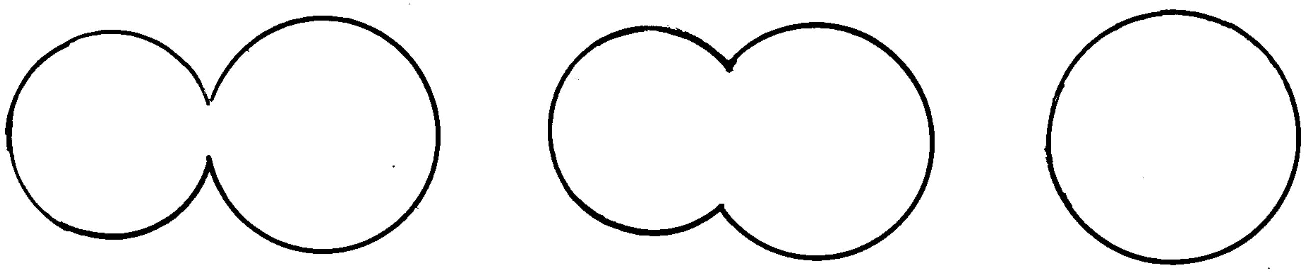


diabatic model

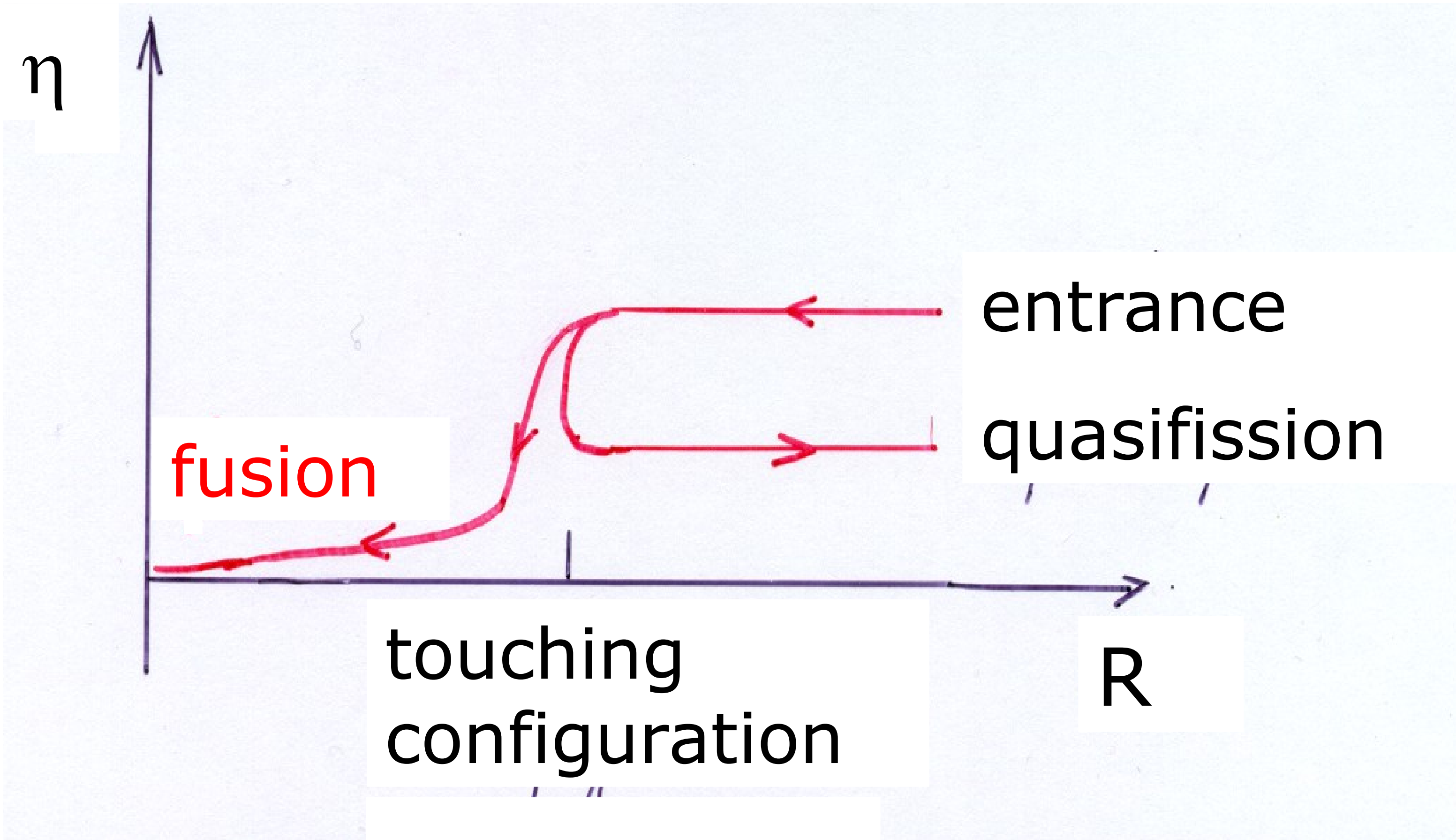
Velocity between nuclei leads to diabatic occupation of single-particle levels, Pauli principle between nuclei

a)Models using adiabatic potentials

Minimization of potential energy, essentially adiabatic dynamics in the internuclear distance, nuclei melt together.

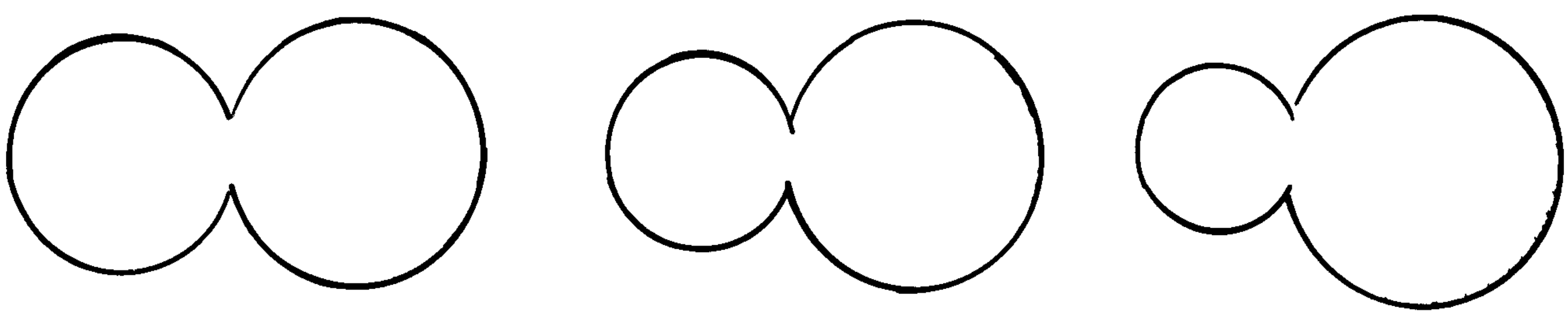


Large probabilities of fusion for producing nuclei with similar projectile and target nuclei.

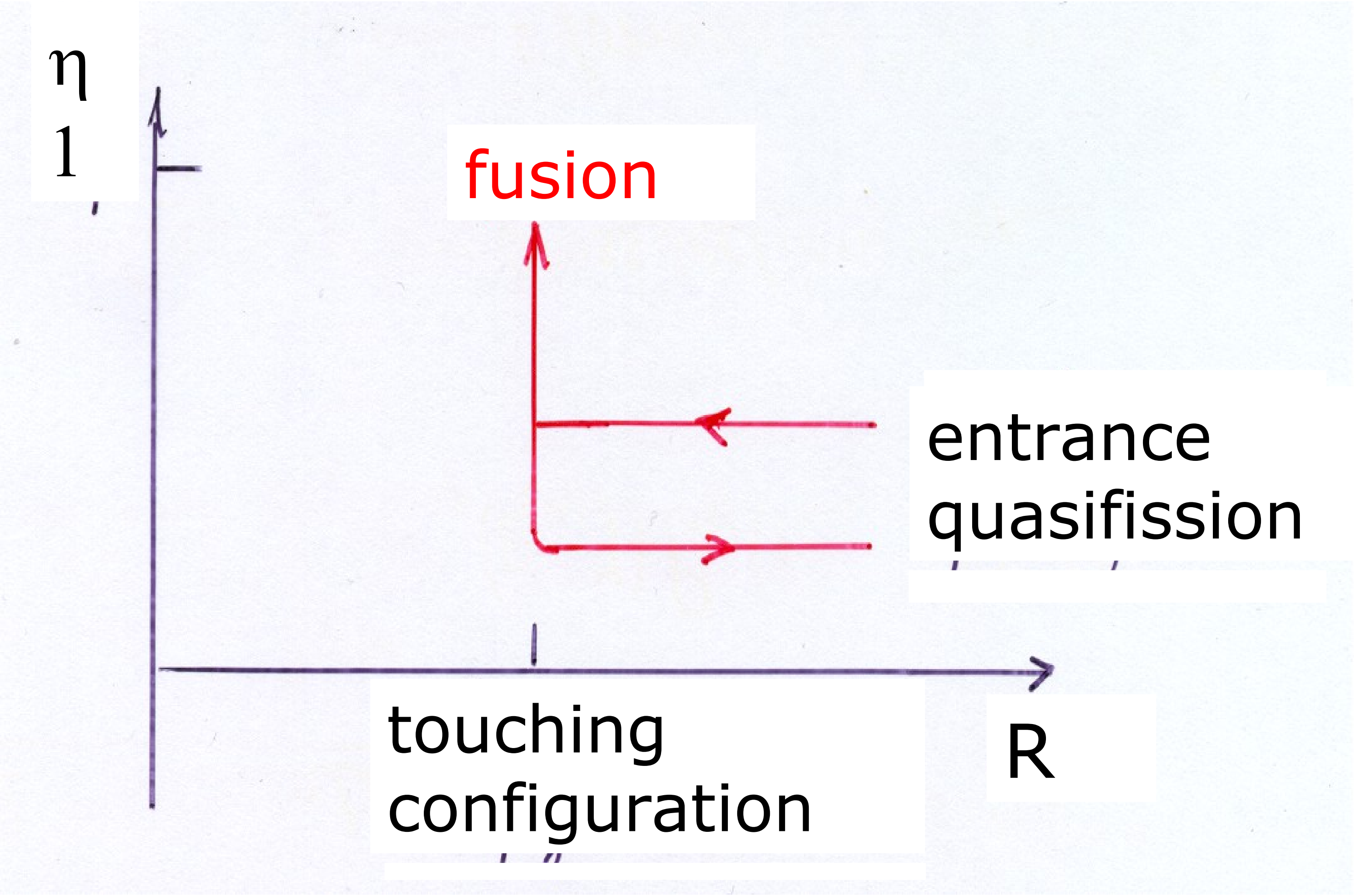

η
entrance
quasifission
fusion
touching
configuration
R

b) Dinuclear system (DNS) concept

Fusion by transfer of nucleons between the nuclei (idea of V. Volkov, also von Oertzen), mainly dynamics in mass asymmetry degree of freedom, use of diabatic potentials, e.g. calculated with the diabatic two-center shell model.



$\eta \to 1$


η
1
fusion
entrance
quasifission
touching
configuration
R

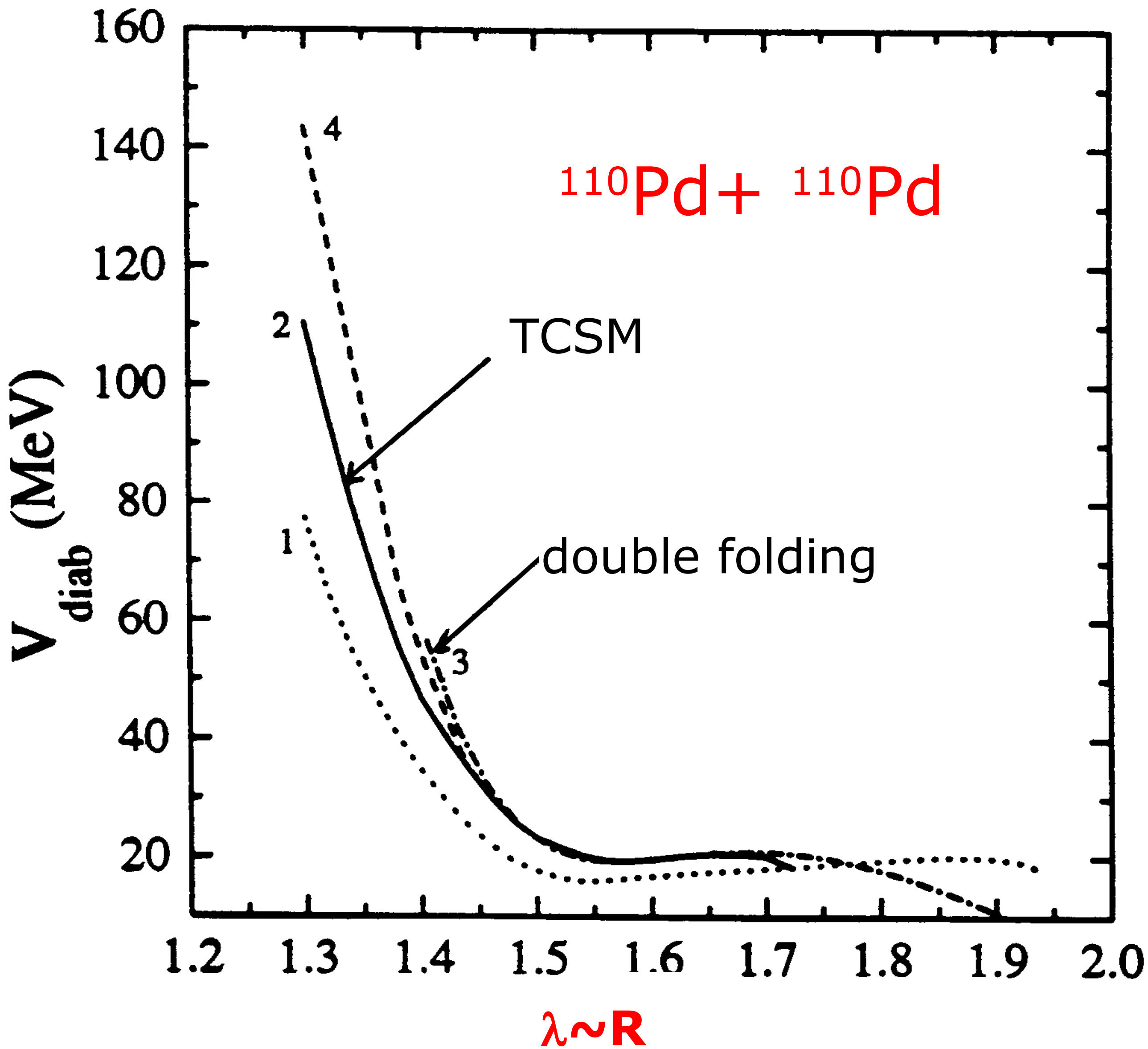


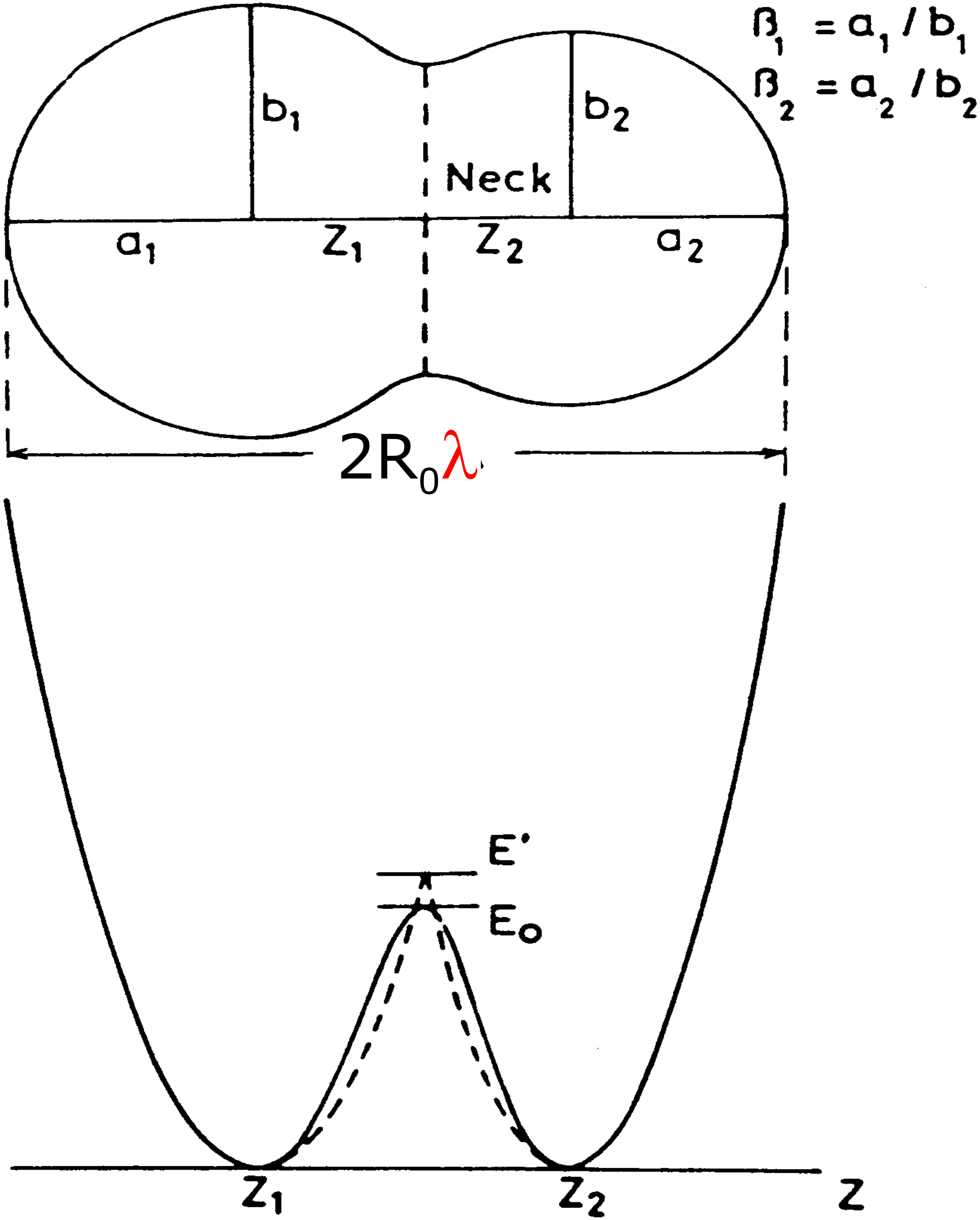
Fig.2. Diabatic potentials for the systems:

(1) $^{100}Mo + ^{100}Mo$ (TCSM)

(2) $^{110}Pd + ^{110}Pd$ (TCSM)

(3) $^{110}Pd + ^{110}Pd$ (double folding)

(4) Neck parameter ε is diminished with decreasing λ


$\beta_1 = a_1 / b_1$
$\beta_2 = a_2 / b_2$
b_1
b_2
Neck
a_1
Z_1
Z_2
a_2
$2R_0\lambda$
E'
E_o
Z_1
Z_2
Z

Calculation of diabatic potential:

$$U_{diab}(\lambda) = U_{adiab}(\lambda) + \sum_{\alpha} \epsilon_{\alpha}^{diab}(\lambda) n_{\alpha}^{diab}(\lambda) - \sum_{\alpha} \epsilon_{\alpha}^{adiab}(\lambda) n_{\alpha}^{adiab}(\lambda)$$

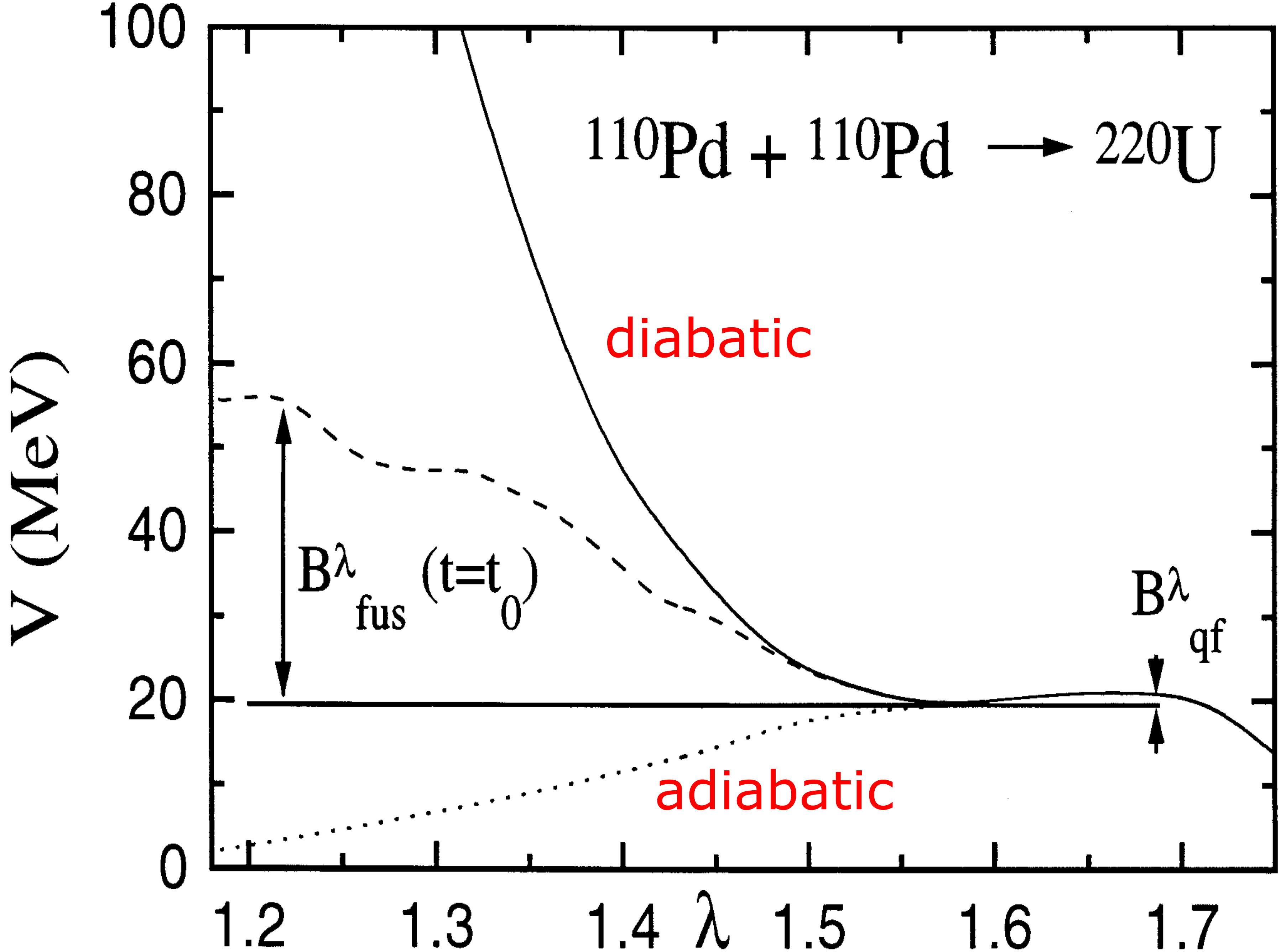
$\epsilon_{\alpha}^{diab}(\lambda), \epsilon_{\alpha}^{adiab}(\lambda)$ = single particle energies

$n_{\alpha}^{diab}(\lambda), n_{\alpha}^{adiab}(\lambda)$ = occupation numbers

Diabatic occupation numbers depend on time:

$$\boxed{n_{\alpha}^{diab}(\lambda, t)}$$

De-excitation of diab. levels with relaxation time, depending on single particle width.



100
80
60
40
20
0
V (MeV)
1.2
1.3
1.4
λ
1.5
1.6
1.7
110Pd + 110Pd → 220U
diabatic
adiabatic
Bλfus (t=t0)
Bλqf

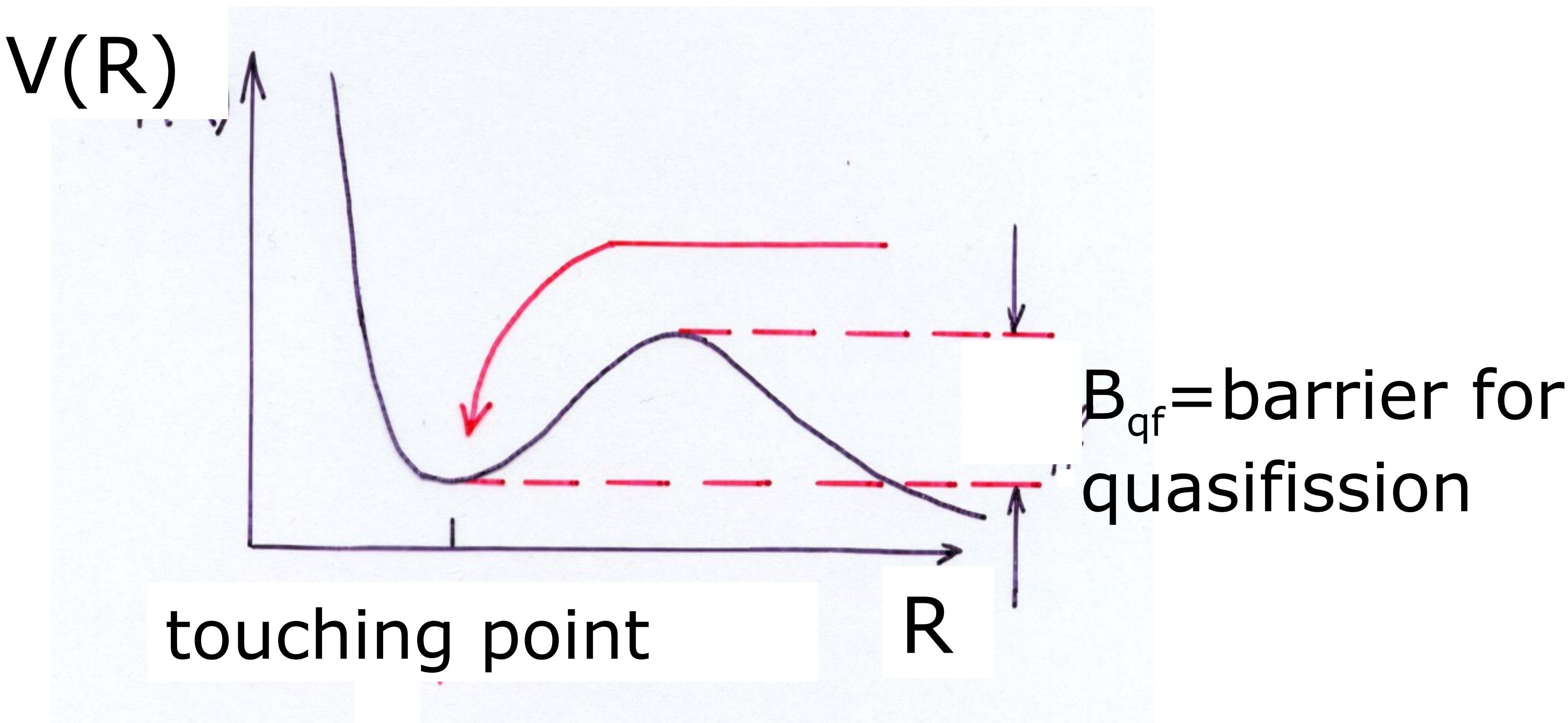
Dynamics of fusion in the dinuclear system model

Evaporation residue cross section for the production of superheavy nuclei:

$$\sigma_{ER}(E_{cm}, J) = \sum_{J=0}^{J_{max}} \sigma_{cap}(E_{cm}, J) P_{CN}(E_{cm}, J) W_{sur}(E_{cm}, J)$$

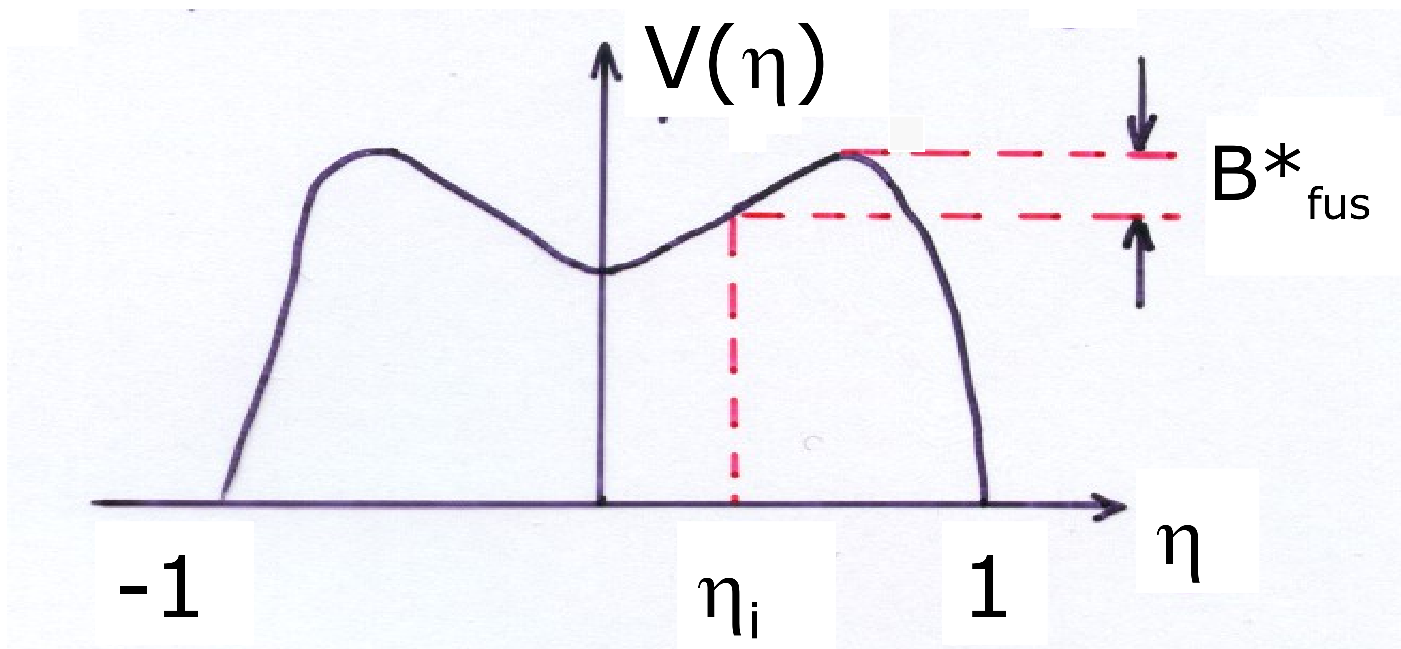
a) Partial capture cross section σ_{cap}

Dinuclear system is formed at the initial stage of the reaction, kinetic energy is transferred into potential and excitation energy.

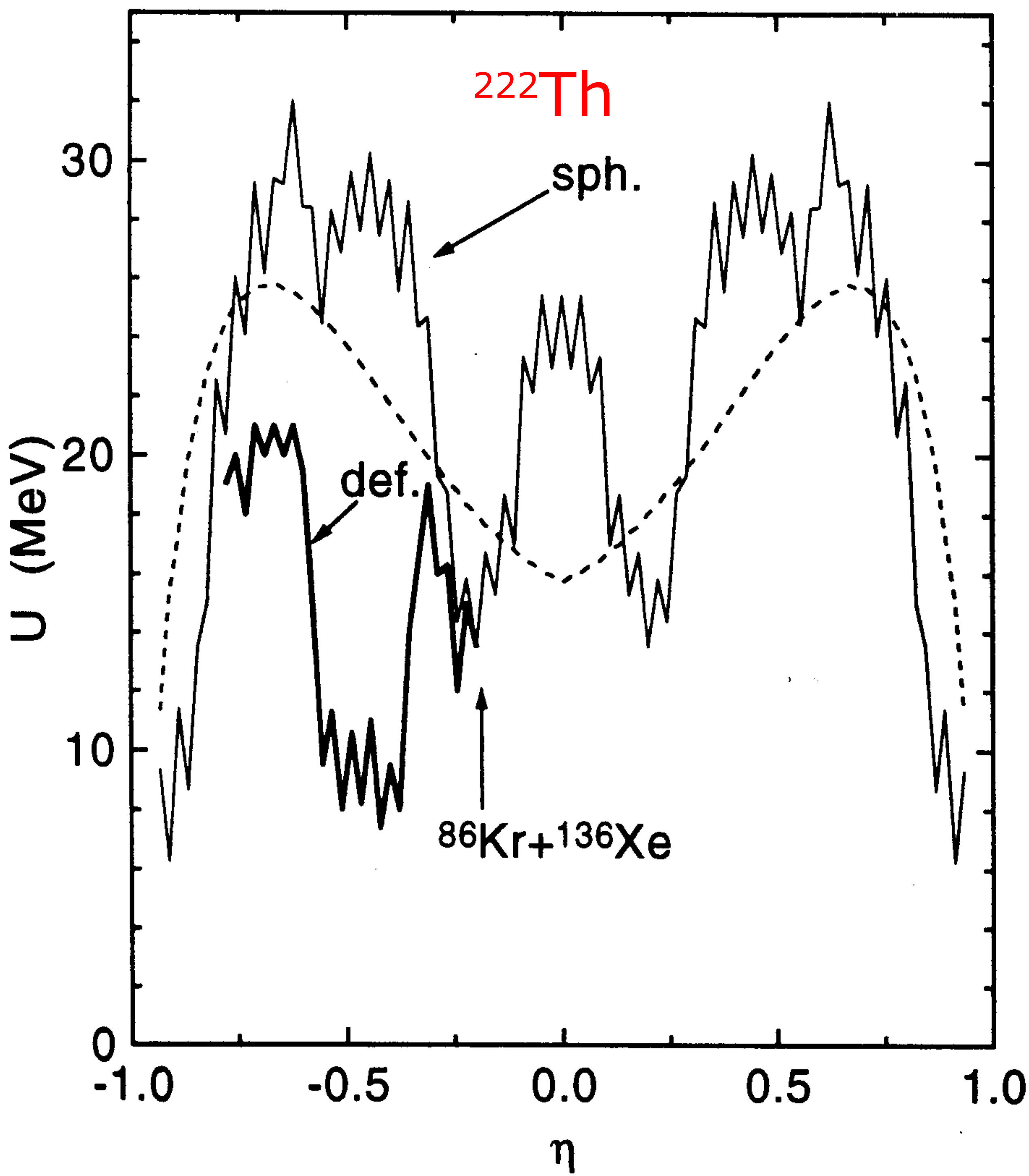


b) Probability for complete fusion P_{CN}

DNS evolves in mass asymmetry coordinate by diffusion processes toward fusion and in the relative coordinate toward the decay of the dinuclear system which is quasifission.



B^*_{fus}=inner fusion barrier



^{222}Th
sph.
def.
^{86}Kr+^{136}Xe
U (MeV)
η
30
20
10
0
-1.0
-0.5
0.0
0.5
1.0

Competition between fusion and quasifission, both processes are treated simultaneously.

Calculation of P_{CN} and mass and charge distributions in η and R:

Fokker-Planck equation, master equations, Kramers approximation

Kramers formula for P_{CN}:

Rate for fusion: $\Lambda_{\eta\ fus}$

Rate for quasifission: $\Lambda_{qf} = \Lambda_R + \Lambda_{\eta\ sym}$, i.e. decay in R and diffusion in η to more symmetric DNS.

$$P_{CN} = \frac{\Lambda_{\eta fus}}{\Lambda_{\eta fus} + \Lambda_{qf}}$$

$$P_{CN} \sim \exp(-(B^*_{fus} - B_{qf})/kT)$$

Cold fusion (Pb-based reactions): $\Lambda_R \gg \Lambda_{\eta sym}$

Hot fusion (^{48}Ca projectiles): $\Lambda_R \ll \Lambda_{\eta sym}$

$$\Lambda_k^{\mathrm{Kr}} = \frac{1}{2\pi}\,\frac{\omega_k \omega_{\bar{k}}}{\omega_R^{B_k}\omega_\eta^{B_k}}\left(\sqrt{\left[\frac{\Gamma}{2\hbar}\right]^2 + (\omega_k^{B_k})^2} - \frac{\Gamma}{2\hbar}\right)\exp\left(-B_k/\theta\right).$$

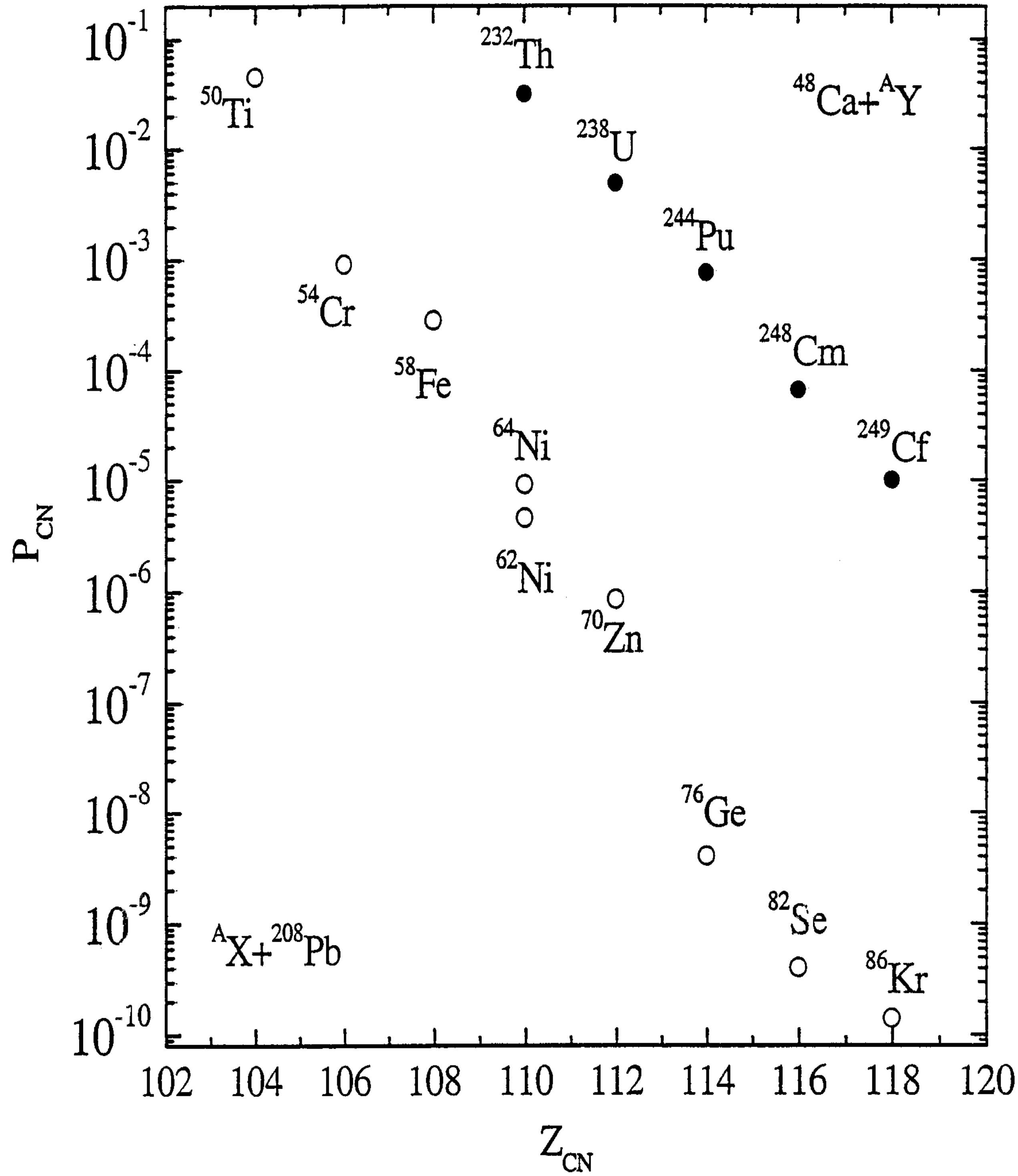
Competition between fusion and quasifission, both processes are treated simultaneously.

Calculation of P_{CN} and mass and charge distributions in $\boldsymbol{\eta}$ and R:

Fokker-Planck equation, master equations, Kramers approximation.

$$P_{CN} \approx \frac{1.25 \exp[-(B_{\eta}^{fus} - B_{qf})/T]}{1 + 1.25 \exp[-(B_{\eta}^{fus} - B_{qf})/T]}$$

$$B_{qf} = \min(B_{qf}^{R}, B_{qf}^{\eta})$$



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
P_{CN}
102
104
106
108
110
112
114
116
118
120
Z_{CN}
$^{48}Ca + ^{A}Y$
$^{A}X + ^{208}Pb$
^{50}Ti
^{54}Cr
^{58}Fe
^{64}Ni
^{62}Ni
^{70}Zn
^{76}Ge
^{82}Se
^{86}Kr
^{232}Th
^{238}U
^{244}Pu
^{248}Cm
^{249}Cf

$$\frac{d}{dt}P_{Z,N}(t) = \Delta^{(-,0)}_{Z+1,N}P_{Z+1,N}(t) + \Delta^{(+,0)}_{Z-1,N}P_{Z-1,N}(t)$$
$$+ \Delta^{(0,-)}_{Z,N+1}P_{Z,N+1}(t) + \Delta^{(0,+)}_{Z,N-1}P_{Z,N-1}(t)$$
$$- \left(\Delta^{(-,0)}_{Z,N} + \Delta^{(+,0)}_{Z,N} + \Delta^{(0,-)}_{Z,N} + \Delta^{(0,+)}_{Z,N}\right)P_{Z,N}(t)$$
$$- (\Lambda^{qf}_{Z,N} + \Lambda^{fis}_{Z,N})P_{Z,N}(t)$$

Rates Δ depend on single-particle energies and temperature related to excitation energy.

Only one-nucleon transitions are assumed.

$\Lambda^{qf}_{Z,N}$: rate for quasifission

$\Lambda^{fis}_{Z,N}$: rate for fission of heavy nucleus

The charge and mass yields for quasifission can be expressed

$$Y_{Z,N}(t_0) = \Lambda^{qf}_{Z,N} \int_0^{t_0} P_{Z,N}(t)\,dt$$

The time t_0 of reaction is determined by solving the normalization condition

$$\sum_{Z,N} Y_{Z,N}(t_0) + P_{CN} \approx 1$$

$$P_{CN} = \sum_{Z<Z_{BG},\, N<N_{BG}} P_{Z,N}(t_0)$$

Z_{BG}=8-14 in the reactions considered

Survival probability W_{sur}

De-excitation of excited compound nucleus by neutron, alpha, proton and gamma emissions in competition with fission. The survival probability under the evaporation of a certain sequence *s* of *x* particles is calculated as:

$$W^{s}_{sur}(E_{CN}) \approx P_s(E_{CN}) \prod_{i_s=1}^{x} \frac{\Gamma_{i_s}(E_{i_s})}{\Gamma_t(E_{i_s})}$$

$\boldsymbol{P_s}$ =probability of realisation of s channel The total width for the compound nucleus decay is the sum of partial widths.

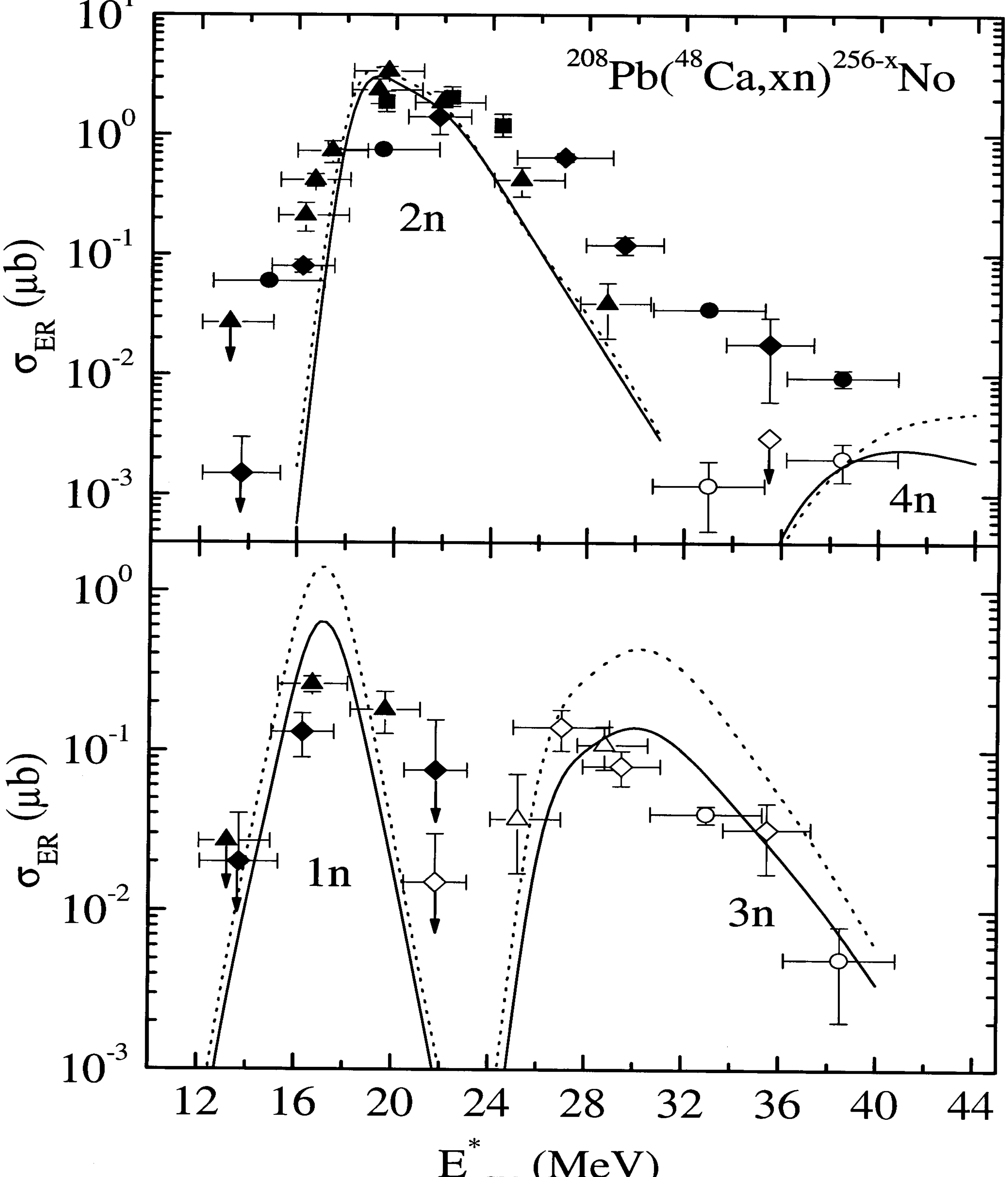
The fission barrier $\boldsymbol{B_f}$ has a liquid drop part $\boldsymbol{B_f}^{LD}$ and a microscopical part $\boldsymbol{B_f}^{M}$.

$\boldsymbol{B_f}^{\mathrm{LD}}$=1.9-3.2 MeV for Pu and Cm isotopes

$$B_f^M \approx -\Delta W_{gr}^A$$

$$B_f(E_{CN}) = B_f^{LD} + B_f^M \exp[-E_{CN}/E_D]$$

$$E_D = 0.4\mathrm{A}^{4/3}/a$$



$^{208}Pb(^{48}Ca,xn)^{256-x}No$
2n
4n
1n
3n
σ_{ER} (μb)
σ_{ER} (μb)
E^*_{CN} (MeV)

References

V. V. Volkov, Phys. Rep. **44**, 93 (1978).

W. U. Schröder and J. R. Huizenga, in *Treatise on Heavy-Ion Science*, edited by D. A. Bromley (Plenum, New York, 1984), Vol. 2, p. 115.

P.Fröbrich, R.Liepperhiede, Theory of nuclear reactions (Clarendon Press, Oxford, 1996)

G. G. Adamian, N. V. Antonenko, and W. Scheid, in *Lecture Notes in Physics, Clusters in Nuclei*, Vol. 2, edited by C. Beck (Springer, Berlin, 2012).