II. Double beta decay
nuclear matrix elements (in QRPA)

Fedor Šimkovic
Study of the $0
\nu\beta\beta$-decay is one of the highest priority issues in particle and nuclear physics

APS Joint Study on the Future of Neutrino Physics (physics/0411216)
We recommend, as a high priority, a phased program of sensitive searches for neutrinoless double beta decay (first in the list of recommendations)

ASPERA road map:
- Requirement for construction and operation of two double-beta decay experiments with a European lead role or shared equally with non-European partners (GERDA, COBRA, CUORE, SuperNEMO)
- We finally reiterate the importance of assessing and reducing the uncertainty in our knowledge of the corresponding nuclear matrix elements, experimentally and theoretically.
The double beta decay process can be observed due to nuclear pairing interaction that favors energetically the even-even nuclei over the odd-odd nuclei.

\[
\frac{1}{T^{0\nu}_{1/2}} = \left| \frac{m_{\beta\beta}}{m_e} \right|^2 \quad G^{01}(E_0, Z) \left| M^{0\nu} \right|^2
\]

| Transition | \( G^{01}(E_0, Z) \times 10^{14} \) y | \( Q_{\beta\beta} \) [MeV] | Abund. (%) | \( |M^{0\nu}|^2 \) |
|------------|--------------------------------------|-----------------|------------|-----------------|
| \(^{150}\text{Nd} \rightarrow ^{150}\text{Sm}\) | 26.9 | 3.667 | 6 | ? |
| \(^{48}\text{Ca} \rightarrow ^{48}\text{Ti}\) | 8.04 | 4.271 | 0.2 | ? |
| \(^{96}\text{Zr} \rightarrow ^{96}\text{Mo}\) | 7.37 | 3.350 | 3 | ? |
| \(^{116}\text{Cd} \rightarrow ^{116}\text{Sn}\) | 6.24 | 2.802 | 7 | ? |
| \(^{136}\text{Xe} \rightarrow ^{136}\text{Ba}\) | 5.92 | 2.479 | 9 | ? |
| \(^{100}\text{Mo} \rightarrow ^{100}\text{Ru}\) | 5.74 | 3.034 | 10 | ? |
| \(^{130}\text{Te} \rightarrow ^{130}\text{Xe}\) | 5.55 | 2.533 | 34 | ? |
| \(^{82}\text{Se} \rightarrow ^{82}\text{Kr}\) | 3.53 | 2.995 | 9 | ? |
| \(^{76}\text{Ge} \rightarrow ^{76}\text{Se}\) | 0.79 | 2.040 | 8 | ? |

The NMEs for \(0\nu\beta\beta\)-decay must be evaluated using tools of nuclear theory.

Fedor Simkovic
Neutrinoless double beta decay of $^{110}\text{Pd}$

With its high natural abundance, the new results reveal $^{110}\text{Pd}$ to be an excellent candidate for double-$\beta$ decay studies.

$Q$-Value and Half-Lives for the Double-Beta-Decay Nuclide $^{110}\text{Pd}$

D. Fink, et al.


<table>
<thead>
<tr>
<th></th>
<th>$^{82}\text{Se}$</th>
<th>$^{110}\text{Pd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>34</td>
<td>46</td>
</tr>
<tr>
<td>Abund. (%)</td>
<td>8.73</td>
<td>11.72</td>
</tr>
<tr>
<td>$Q$ [keV]</td>
<td>2 995</td>
<td>2 017.8</td>
</tr>
<tr>
<td>$G^{0\nu}$ [10$^{-15}$ yr$^{-1}$]</td>
<td>10.16</td>
<td>4.815</td>
</tr>
<tr>
<td>$0\nu\beta\beta$ NME</td>
<td>4.64</td>
<td>5.76</td>
</tr>
<tr>
<td>$T^{2\nu}_{1/2}$ [yr]</td>
<td>0.92 $10^{20}$</td>
<td>1.5(6) $10^{20}$(SSD)</td>
</tr>
</tbody>
</table>
If (or when) the $0\nu\beta\beta$ decay is observed two theoretical problems must be resolved

1) What is the mechanism of the decay, i.e., what kind of virtual particle is exchanged between the affected nucleons (quarks).

2) How to relate the observed decay rate to the fundamental parameters, i.e., what is the value of the corresponding nuclear matrix elements.

In double beta decay two neutrons bound in the ground state of an initial even-even nucleus are simultaneously transformed into two protons that are bound in the ground state or excited ($0^+$, $2^+$) states of the final nucleus.

It is necessary to evaluate, with a sufficient accuracy, wave functions of both nuclei, and evaluate the matrix element of the $0\nu\beta\beta$-decay operator connecting them.

This can not be done exactly, some approximation and/or truncation is always needed. Moreover, there is no other analogues observable that can be used to judge directly the quality of the result.
• Exact methods exist up to $A=4$
• Computationally exact methods for $A$ up to 16
• Approximate many-body methods for $A$ up to 60
• Mostly mean-field pictures for $A$ greater than 60 or so

With newer methods and powerful computers, the future of nuclear structure theory is bright!
Many-body Hamiltonian

- Start with the many-body Hamiltonian
  \[ H = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} V_{NN}(r_i - r_j) \]

- Introduce a mean-field \( U \) to yield basis
  \[ H = \sum_i \left( \frac{p_i^2}{2m} + U(r_i) \right) + \sum_{i<j} V_{NN}(r_i - r_j) - \sum_i U(r_i) \]

The success of any nuclear structure calculation depends on the choice of the mean-field basis and the residual interaction!

- The mean field determines the shell structure
- In effect, nuclear-structure calculations rely on perturbation theory
0νββ-decay matrix element (two ways of calculation)
0νββ-decay matrix elements

\[ M^{0\nu} = \frac{4\pi R}{g_A^2} \int \left( \frac{1}{(2\pi)^3} \int \frac{e^{-iq\cdot(\vec{x}_1-\vec{x}_2)}}{|q|} \right) \times \]

\[ \sum_m \frac{<0^+_f|J_{\alpha}^\dagger(\vec{x}_1)|m><m|J_{\alpha}^\dagger(\vec{x}_2)|0^+_i>}{E_m - (E_i + E_f)/2 + |q|} \ d\vec{q} d\vec{x}_1 d\vec{x}_2 \]

Weak hadron current

\[ j^{\rho \dagger} = \bar{\Psi} \gamma^5 \left[ g_V(q^2) \gamma^\rho + ig_M(q^2) \frac{\sigma^{\rho\nu}}{2m_p} q_\nu ight. \]

\[ \left. -g_A(q^2) \gamma^\rho \gamma_5 - g_P(q^2) q^\rho \gamma_5 \right] \Psi , \]

Formfactor

\[ g_V(\vec{q}^2) = \frac{g_V}{(1 + \vec{q}^2/M_V^2)^2} \]

\[ g_A(\vec{q}^2) = \frac{g_A}{(1 + \vec{q}^2/M_A^2)^2} \]

Weak hadron current in a Breit frame

\[ J^{\rho \dagger}(\vec{x}) = \sum_{n=1}^{A} \tau_n^+ \left[ g^{\rho 0} J^0(\vec{q}^2) + \sum_k g^{\rho k} J^n_k(\vec{q}^2) \right] \delta(\vec{x} - \vec{r}_n) \]

\[ J^0(\vec{q}^2) = g_V(q^2) \]

\[ \vec{J}_n(\vec{q}^2) = g_M(q^2) i \frac{\vec{\sigma}_n \times \vec{q}}{2m_p} + g_A(q^2) \vec{\sigma} - g_P(q^2) \frac{\vec{q} \cdot \vec{\sigma}_n}{2m_p} \]
Two one-body operators

$$e^{i\vec{q}\cdot\vec{r}} = 4\pi \sum_l i^l j_l(qr)(Y_{lm}(\Omega_r \cdot Y_{lm}(\Omega_q)))$$

One-body operator

$$\hat{O}_{JM} = \sum_{pn} \frac{\langle p \parallel O_J \parallel n \rangle}{\sqrt{2J+1}} [c_p^+ \tilde{c}_n]_{JM}$$

Decomposition of plane waves

$$\int e^{i\vec{q}_1\cdot\vec{r}_1} e^{-i\vec{q}_2\cdot\vec{r}_2} d\Omega_q = (4\pi)^2 \sum_l (-1)^l \sqrt{2l+1} j_l(qr_1) j_l(qr_2) \{Y_{lm}(\Omega_{r_1}) \otimes Y_{lm}(\Omega_{r_2})\}_{00}$$

$$M_K = \sum_{J,\pi, k_i, k_f} \sum_{pnp'} (-)^J \frac{R}{g_A^2} \int_0^\infty \frac{\mathcal{P}^K_{pnp'p', J}(q)}{|q|(|q| + (\Omega_{j_i}^{k_i} + \Omega_{j_f}^{k_f})/2)} h_{K}(q^2) q^2 dq \times$$

$$\langle 0^+_f || [c_{p'}^+ \tilde{c}_{n'}]_J || J^\pi k_f \rangle \langle J^\pi k_f || \tilde{c}_{n}^+ \rangle \langle \tilde{c}_{n}^+ || [c_p^n]_J || 0^+_i \rangle$$

$$\mathcal{P}^{VV}_{pnp'n', J}(q) = \langle p \parallel O^{(1)}_J(q) \parallel n \rangle \langle p' \parallel O^{(1)}_J(q) \parallel n' \rangle,$$

$$\mathcal{P}^{AA}_{pnp'n', J}(q) = \sum_{L=J, J\pm1} (-)^{J+L+1} \times$$

$$\langle p \parallel O^{(2)}_{LJ}(q) \parallel n \rangle \langle p' \parallel O^{(2)}_{LJ}(q) \parallel n' \rangle,$$

$$\mathcal{P}^{PP}_{pnp'n', J}(q) = \mathcal{P}^{PP}_{pnp'n', J}(q),$$

$$\mathcal{P}^{AP}_{pnp'n', J}(q) = \mathcal{P}^{AA}_{pnp'n', J}(q) - \mathcal{P}^{PP}_{pnp'n', J}(q).$$

Product of one-body matrix elements

7/28/2014
One two-body operators

\[ \langle p|O(1)|n\rangle \langle p'|O(2)|n'\rangle = \langle p, p'|O'(1, 2)|n, n'\rangle \]

Integration over angular part of ν momentum

\[ \int e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)} d\Omega_q = \int e^{i\vec{q} \cdot \vec{r}} d\Omega_q = \sqrt{4\pi} \cdot 4\pi \sum_{lm} i^l j_l(qr) Y_{lm}(\Omega_r) \int Y_{lm}^*(\Omega_q) Y_{00}(\Omega_q) d\Omega_q = 4\pi j_0(qr) \]

Neutrino potential

\[ H_K(r_{12}, E_{J\pi}^k) = \frac{2}{\pi g_A^2} R \int_0^\infty f_K(qr_{12}) \frac{h_K(q^2)qdq}{q + E_{J\pi}^k - (E_i + E_f)/2} \]

with \( f_{F,GT}(qr_{12}) = j_0(qr_{12}), \quad f_T(qr_{12}) = -j_2(qr_{12}) \)

\[ \sigma_{12} = \hat{\sigma}_1 \cdot \hat{\sigma}_2, \]

\[ S_{12} = 3(\hat{\sigma}_1 \cdot \hat{r}_{12})(\hat{\sigma}_2 \cdot \hat{r}_{12}) - \sigma_{12} \]

Nuclear matrix element

\[ M^{0\nu} = -\frac{M_F}{g_A^2} + M_{GT} - M_T \]

\[ M_K = \sum_{J_{\pi},k_i,k_f,J} \sum_{pn,p'n'} (-1)^{j_n+j_p'+J+J} \times \]

\[ \sqrt{2J+1} \left\{ \begin{array}{ccc} j_p & j_n & J \\ j_{n'} & j_{p'} & J' \end{array} \right\} \]

\[ \langle p(1), p'(2); J \mid \bar{f}(r_{12})O_K \bar{f}(r_{12}) \mid n(1), n'(2); J' \rangle \times \]

\[ \langle 0_f^{+} \mid [c_p^{+} \bar{c}_{n'}^{+}]_{J} \mid J^{\pi} k_f \rangle \langle J^{\pi} k_f \mid J^{\pi} k_i \rangle \langle J^{\pi} k_i | 0_i^{+} \rangle \]
Calculation of two-body matrix elements

From j-j to LS coupling

\[ \mathcal{M}^{2\text{body}} = \langle a(1), b(2); J' | O(1, 2) | c(1) d(2); J' \rangle \]

\[ |n_c l_c j_c, n_d l_d j_d; J' M'\rangle = \sum_{SL} \hat{S}^2 \hat{L}^2 \hat{j_c} \hat{j_d} \begin{pmatrix} 1/2 & l_c & j_c \\ 1/2 & l_d & j_d \\ S & L & J' \end{pmatrix} |n_c l_c, n_d l_d, SL; J' M'\rangle \]

Moshinsky transformation to relative coordinates

\[ |n_c l_c n_d l_d; LM_L\rangle = \sum_{n_l} \langle nl, N L, L | n_c l_c, n_d l_d, L \rangle |nl, N L; LM_L\rangle \]

Two-body m.e.

\[ \mathcal{M}_{F,G,T}^{2\text{body}} = \hat{J}' \sum_{SL} \hat{S} \hat{L} \hat{j_a} \hat{j_b} \hat{j_c} \hat{j_d} \begin{pmatrix} 1/2 & l_c & j_c \\ 1/2 & l_d & j_d \\ S & L & J' \end{pmatrix} \begin{pmatrix} 1/2 & l_a & j_a \\ 1/2 & l_b & j_b \\ S & L & J' \end{pmatrix} \]

\[ \times \sum_{n_l} \sum_{n'_l} \langle n_l, N L, L | n_c l_c, n_d l_d, L \rangle \langle n'_l, N' L', L | n_a l_a, n_b l_b, L \rangle \]

\[ \times \langle n'_l, N' L'; L | j_0 (q | \vec{r}_{i,j} |) | nl, N L; L \rangle \langle s_a s_b; S | \hat{\sigma}_1 \cdot \hat{\sigma}_2 \rangle | s_c s_d; S \rangle \]

\[ \langle n'_l, N' L'; L | j_0 (q | \vec{r}_{i,j} |) | nl, N L; L \rangle = \delta_{l'_l} \delta_{N'N} \delta_{LL'} \langle n'_l | j_0 (q | \vec{r}_{i,j} |) | nl \rangle \]

\[ \langle s_a s_b; S | \hat{\sigma}_1 \cdot \hat{\sigma}_2 | s_c s_d; S \rangle = \hat{S} (\delta_{S1} - 3 \delta_{S0}), \]

\[ \langle s_a s_b; S | 1 | s_c s_d; S \rangle = \hat{S} (\delta_{S1} + \delta_{S0}) \]
Realistic NN-interactions used in the QRPA calculations

Modern (phase-shift equivalent) NN potentials

- Nijmegen I - \( P_D = 5.66\% \) - 41 parameters - \( \chi^2/N_{data} = 1.03 \)
- Nijmegen II - \( P_D = 5.64\% \) - 47 parameters - \( \chi^2/N_{data} = 1.03 \)
- Argonne \( V_{18} \) - \( P_D = 5.76\% \) - 40 parameters - \( \chi^2/N_{data} = 1.09 \)
- CD Bonn - \( P_D = 4.85\% \) - 43 parameters - \( \chi^2/N_{data} = 1.02 \)

Brueckner G-matrices from Tuebingen (H. Muether group)

Based upon the OBE model

(1999 NN Database: 5990 pp and np scattering data)

Renormalization of the NN interaction

Difficulty in the derivation of \( V_{eff} \) from any modern NN potential: existence of a strong repulsive core which prevents its direct use in nuclear structure calculations.

Traditional approach to this problem: Brueckner G-matrix method. The G matrix is model-space dependent as well as energy dependent.
$0\nu\beta\beta$-decay NMEs:
QRPA and other approaches
The $0\nu\beta\beta$-decay NME (light $\nu$ exchange mech.)

The $0\nu\beta\beta$-decay half-life

$$\frac{1}{T_{1/2}} = G^{0\nu}(E_0, Z)|M^{0\nu}|^2 |<m_{\beta\beta}|^2,$$

NME = sum of Fermi, Gamow-Teller and tensor contributions

$$M^{0\nu} = \left(\frac{g_A}{1.25}\right)^2 |f| - \frac{M_F^{0\nu}}{g_A} + M_{GT}^{0\nu} + M_{T}^{0\nu} |i\rangle$$

Neutrino potential (about $1/r_{12}$)

$$H_K(r_{12}) = \frac{2}{\pi g_A^2} R \int_0^{\infty} f_K(qr_{12}) \frac{h_K(q^2)qdq}{q + E^m - (E_i + E_f)/2}$$

$$f_{F,GT}(qr_{12}) = j_0(qr_{12}), \quad f_T(qr_{12}) = -j_2(qr_{12})$$

Form-factors: finite nucleon size

$$h_F = g_\nu^2(q^2)$$

$$h_{GT} = g_A^2 \left[1 - \frac{2}{3} \frac{q^2}{q^2 + m_\pi^2} + \frac{1}{3} \left(\frac{q^2}{q^2 + m_\pi^2}\right)^2 \right]$$

$$h_T = g_A^2 \left[\frac{2}{3} \frac{q^2}{q^2 + m_\pi^2} - \frac{1}{3} \left(\frac{q^2}{q^2 + m_\pi^2}\right)^2 \right]$$

Induced pseudoscalar coupling (pion exchange)

$$M_{K=F,GT,T} = \sum_{J^\pi, k_i, k_f, J} \sum_{pnp', n'} (-1)^{j_n + j_p' + J + J} \sqrt{2J + 1} \begin{pmatrix} j_p & j_n & J \\ j_n' & j_p' & J \end{pmatrix}$$

$$\langle p(1), p'(2); J \parallel f(r_{12}) O_{K,f}(r_{12}) \parallel n(1), n'(2); J \rangle$$

Jastrow f. s.r.c.

$$J^\pi = 0^+, 1^+, 2^+ \ldots$$

$$0^-, 1^-, 2^- \ldots$$
In **NSM** a limited valence space is used but all configurations of valence nucleons are included. Describes well properties of low-lying nuclear states. **Technically difficult, thus only few 0νββ-decay calculations**

- Define a valence space
- Derive an effective interaction $H \Psi = E \Psi \rightarrow H_{\text{eff}} \Psi_{\text{eff}} = E \Psi_{\text{eff}}$
- Build and diagonalize Hamiltonian matrix $(10^{10})$
- Transition operator $< \Psi_{\text{eff}} | O_{\text{eff}} | \Psi_{\text{eff}}>$
- Phenomenological input: Energies of states, systematics of B(E2) and GT trans.

$$H = \sum_a \varepsilon_a a_a^+ a_a - \sum_{abcd} \frac{\langle j_a j_b ; JT | V | j_c j_d ; JT \rangle_A}{\sqrt{(1 + \delta_{ab})(1 + \delta_{cd})}} \left[ a_a^+ \otimes a_b^+ \right]^{JT} \otimes \left[ \tilde{a}_c \otimes \tilde{a}_d \right]^{JT} \right|_0^{00}$$

In NSM a limited valence space is used but all configurations of valence nucleons are included. Describes well properties of low-lying nuclear states. Technically difficult, thus only few 0νββ-decay calculations.
Quasiparticle Random Phase Approximation (QRPA)

In QRPA a large valence space is used, but only a class of configurations is included. Describe collective states, but not details of dominantly few particle states. Relative simple, thus more 0nbb-decay calculations.

- Large model space (up 23 s.p.l, $^{150}$Nd – 60 active prot. and 90 neut.)
- Spin-orbit partners included
- Possibility to describe all multipolarities of the intermed. nucl. $J^\pi (\pi=\pm 1, J=0...9)$

$$H = H_0 + g_{ph} V_{ph} + g_{pp} V_{pp}$$

The NSM (Normal State Model) helps to describe quasiparticle states and mean field interaction between them.
The Interacting Boson Model

- The low-lying states of the nucleus, composed by $n$ and $z$ valence nucleons, are modeled in terms of $(n+z)/2$ bosons.
- The bosons have either $L = 0$ (s boson) or $L = 2$ (d boson).
- The bosons can interact through one-body and two-body forces giving rise to bosonic wave functions.
- Any observable can be calculated using these wave functions provided that the relevant operator is employed.

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Projected Hartree-Fock-Bogoliubov Model

**PHFB Model**

States of good angular momentum $J$

\[ |\Psi^J_M\rangle = \frac{2J+1}{8\pi^2} a_j \int d\Omega D^J_{MK}(\Omega) \hat{R}(\Omega) |\Phi_K\rangle \]

Axially symmetric HFB intrinsic state

\[ |\Phi_0\rangle = \prod_{im} \left( u_{im} + v_{im} b^+_im b^+_im \right) \]

where

\[ b^+_im = \sum_m C_{i\alpha m} a^+_im \quad b^+_im = \sum_m (-1)^{l+j-m} C_{i\alpha m} a^+_i-m \]

Hamiltonian:

\[ H = H_{sp} + V(P) + \zeta_{qq} V(QQ) \]

Only quadrupole interaction, GT interaction is missing
Separation of $g_{pp}$ into $g_{pp}^{T=0}$ and $g_{pp}^{T=1}$ adjusted $M^{2\nu}_{F}=0$

$g_{pp}^{T=1}$ adjusted to exp. paring gaps

$g_{pp}^{T=0}$ adjusted $M^{2\nu}_{-exp_{GT}}$

Close values $\square$ and $\blacksquare$ $\Rightarrow$ no new parameter

QRPA and isospin symmetry restoration

F. Š., V. Rodin, A. Faessler, and P. Vogel
PRC 87, 045501 (2013)
\( M^{2\nu}_F \) depends strongly on \( g^{T=1}_{pp} \).

\( M^{2\nu}_{GT} \) does not depend on \( g^{T=1}_{pp} \).
Multipole decomposition

\[ M_{F}^{0} (J; \pi) \]

old par.  
new par.
\[ M^{0\nu} = M_{GT}^{0\nu} \left( 1 + \frac{1}{g_A^2} \frac{M_F^{0\nu}}{M_{GT}^{0\nu}} + \frac{M_T^{0\nu}}{M_{GT}^{0\nu}} \right) \]
Differences: mean field; residual int.; size of the m.s.; many-body appr.

ISM: Menendez et al. NPA 818 (2009) 139

IBM: Barea, Kotila, Iachello, PRC (2013) 014315

EDF: Rodriguez, Martinez-Pinedo, PRL (2010) 105

PHFB: K. Rath et al., PRC 85 (2012) 014308
The $0\nu\beta\beta$-decay NMEs (Status: 2014)

Differences:

i) mean field;
ii) residual int.;
iii) size of the m.s.
iv) many-body appr.

LSSM (small m.s., negative parity states)
PHFB (GT force neglected)
IBM (Hamiltonian truncated)
(R)QRPA (g.s. correlations not accurate enough)

Nobody is perfect:
$g_A=1.25(7)$, CCm or UCOM s.r.c., $r_0=1.20$ fm
\[ \chi_F = \frac{M^{0\nu}_F}{M^{0\nu}_{\text{GT}}} \approx -\frac{1}{3} \]

Fermi:
\[ 1 = \Omega(S=0) + \Omega(S=1) \]

Gamow-Teller:
\[ \sigma \cdot \sigma = -3 \Omega(S=0) + \Omega(S=1) \]
Tensor part of the $0\nu\beta\beta$ NME
(some disagreement)

**ISM:** effect is small, **QRPA(J):** negligible; **PHFB, EDF:** not calculated; **QRPA(TBC), IBM:** up to 10%
Deformed QRPA

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
 & Ref. [24] & Exp. \\
 & this work & Sk3 & SG2 & Ref. [25] & Ref. [26] \\
\hline
$^{76}$Ge & 0.184$^a$ & 0.161 & 0.157 & 0.095(30) & 0.2623(9) \\
$^{76}$Se & -0.018 & -0.181 & -0.191 & 0.163(33) & 0.3090(37) \\
$^{130}$Te & 0.01 & -0.076 & -0.039 & 0.035(23) & 0.1184(14) \\
$^{130}$Xe & 0.13 & 0.108 & 0.161 & - & 0.1837(49) \\
$^{136}$Xe & 0.004 & 0.001 & 0.016 & - & 0.122(10) \\
$^{136}$Ba & -0.021 & 0.009 & 0.070 & - & 0.1258(12) \\
$^{150}$Nd & 0.27 & 0.266 & 0.271 & 0.367(86) & 0.2853(21) \\
$^{150}$Sm & 0.22 & 0.207 & 0.203 & 0.230(30) & 0.1931(21) \\
\hline
\end{tabular}
\end{table}

Skyrme int: Mustonen, Engel, PRC 87 (2013) 064302
Argonne int: Fang, Faessler, Rodin, F.Š., PRC 83 (2011) 034320
Quenching of $g_A$ and two-body currents
Menendez, Gazit, Schwenk, PRL 107 (2011) 062501; MEDEX13 contribution

\[ g_A = 1.269 \]
\[ g_{\text{eff}}^A = 0.75 \, g_A \]

\[(1.269)^4 = 2.6\]

Strength of GT trans. has to be quenched to reproduce experiment

The $0\nu\beta\beta$ operator calculated within effective field theory. Corrections appear as 2-body current predicted by EFT. The 2-body current contributions are related to the quenching of Gamow-Teller transitions found in nuclear structure calc.
Quenching of $g_A$, two-body currents and QRPA
(Suppression of about 20%)
$\langle p \rangle \approx 230 \text{ MeV}, \quad \sqrt{\langle p^2 \rangle} \approx 250 \text{ MeV}$
Anatomy of the 0νββ-decay NMEs
List of reasons, why QRPA-like $0\nu\beta\beta$–decay NME are different

Quasiparticle mean field
fixing of pp,nn (pn) pairing

two-nucleon s.r.c. (~10-20%)

Many-body approximations
QRPA, RQRPA, SRQRPA

finite size of nucleon (~10%)
form factors

Choice of NN interaction
Schem., realistic (Bonn, Paris …)
h.o.t. of nucleon curr. (~30%)
Induced PS, weak magnetism

the closure approximation
the overlap factor
the BCS overlap

p-h interaction ($g_{ph} \approx 1$)
fixed to GT resonance

the axial-vector coupling
$g_A = 1.0$ or 1.25

The size of model space

p-p interaction ($g_{pp}$)
fixed to $2\nu\beta\beta$–decay

Nuclear shape
Spherical - deformed
2νββ-decay in the QRPA

Shell model

Small model space, effective w.f. and operators

Isotope  $T_{1/2}(\text{th.})[y]$  $T_{1/2}(\text{exp.})[y]$

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$T_{1/2}(\text{th.})[y]$</th>
<th>$T_{1/2}(\text{exp.})[y]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{48}$Ca</td>
<td>$3.7 \times 10^{19}$</td>
<td>$4.2 \times 10^{19}$</td>
</tr>
<tr>
<td>$^{76}$Ge</td>
<td>$1.2 \times 10^{21}$</td>
<td>$1.4 \times 10^{21}$</td>
</tr>
<tr>
<td>$^{82}$Se</td>
<td>$3.4 \times 10^{19}$</td>
<td>$9.0 \times 10^{19}$</td>
</tr>
<tr>
<td>$^{130}$Te</td>
<td>$1.3 \times 10^{20}$</td>
<td>$6.1 \times 10^{20}$</td>
</tr>
<tr>
<td>$^{136}$Xe</td>
<td>$1.9 \times 10^{20}$</td>
<td>$8.1 \times 10^{20}$</td>
</tr>
</tbody>
</table>

Strasbourg group, SuperNEMO meeting, Dec. 2003

QRPA

\[
\sum_{m} \frac{< 0_f^+ | \tau^+ \sigma | 1_m^+ > < 1_m^+ | \tau^+ \sigma | 0_i^+ >}{E_m - E_i + \Delta} = M_{GT}^{2\nu} = \sum_{m} \frac{< 0_f^+ | \tau^+ \sigma | 1_m^+ > < 1_m^+ | \tau^+ \sigma | 0_i^+ >}{E_m - E_i + \Delta}
\]

$^{76}$Ge $\rightarrow ^{82}$Se

Strasbourg group, SuperNEMO meeting, Dec. 2003

7/28/2014  Fedor Simkovic  35
The $0\nu\beta\beta$-decay NME: $g_{pp}$ fixed to $2\nu\beta\beta$-decay

Each point: (3 basis sets) x (3 forces) = 9 values

By adjusting of $g_{pp}$ to $2\nu\beta\beta$-decay half-life the dependence of the $0\nu\beta\beta$-decay NME on other things that are not a priori fixed is essentially removed

Rodin, Faessler, Šimkovic, Vogel,
The importance of transition through higher-lying states of $(A,Z+1)$ nucleus

Rodin, Faessler, Šimkovic, Vogel, nucl-th/0503063
Dominance of Pairing mode (J=0)

Two types of decompositions (Particle-particle) and (particle-hole)

Sensitivity to g_{pp} of 1^+ state

A comparison with NSM for the same model space
Decomposition in pp and nn channels

\[ \langle p(1), p'(2); \mathcal{J} \parallel f(r_{12})O_K f(r_{12}) \parallel n(1), n'(2); \mathcal{J} \rangle \]

- $^{76}$Ge: $M^{0v} = 3.98$
- $^{100}$Mo: $M^{0v} = 2.74$
- $^{130}$Te: $M^{0v} = 2.67$

**J=0**
Pairing mode

**J≠0**
Non-pairing mode

The radial dependence of $M^{0n}$ for the three indicated nuclei. The contributions summed over all components shown in the upper panel. The `pairing' $J = 0$ and `broken pairs' $J \neq 0$ parts are shown separately below. Note that these two parts essentially cancel each other for $r > 2-3$ fm. This is a generic behavior. Hence the treatment of small values of $r$ and large values of $q$ are quite important.
Large Scale Shell Model
Menendez, Poves, Caurier, Nowacki,
Arxive:0901.3760 [nucl-th]

PHFB
P.Rath, R. Chandra, K. Chaturverdi,
P.Raina, J.G. Hirsch,
to be published in PRC

Nuclear physics

Nucleon physics

7/28/2014  Fedor Simkovic
A consistent approach for the $0\nu\beta\beta$-decay (pairing, s.r.c, g.s.c. calculated with the same NN potential- BonnCD, Argon)

Neutrino potential: $I(r)/r$

$$I(r) = \frac{2}{\pi} \int_0^\infty \frac{\sin(qr)}{(q + E_{\text{aver}})(1 + q^2/E_{\text{cut}}^2)} dq$$

$$|\Psi>_{\text{corr.}} = f(r_{12}) |\Psi>$$

$$O_{\text{corr.}}(r_{12}) = f(r_{12})O(r_{12})f(r_{12})$$

Two-nucleon short range correlations

Nucleon–Nucleon Potential
Neutrinoless double beta decay matrix elements

F.Š., Faessler, Muether, Rodin, Stauf, PRC 79, 055501 (2009)
It is of interest to see the contribution of individual orbits to the $0\nu\beta\beta$ matrix element. Within QRPA and its generalization this can be done by using the basic formula:

$$M_K = \sum_{J^\pi,k_z,k_f,J\,pnp'/n'} \sum_{J_n, J_{n'}} (-1)^{J_n + J_{n'}} + J + J' \times \sqrt{2J + 1} \left\{ \begin{array}{ccc} J & J & J \\ J' & J' & J' \end{array} \right\} \times$$

$$\langle p(1), p'(2); J || \bar{f}(r_{12})O_K f(r_{12}) || n(1), n'(2); J \rangle \times \langle 0^+_f || [\tilde{c}_{p'}^{\dagger} \tilde{c}_{n'}] J || J^\pi k_f \rangle \langle J^\pi k_f | J^\pi k_i \rangle \langle J^\pi k_f i || [c_p^{\dagger} \tilde{c}_n] J || 0^+_i \rangle$$

Summing over all indeces except $n,n'$ (or $p,p'$) will tell give us the required contribution. Note that it can be positive or negative.
Contribution of individual neutron orbits to $M^{0\nu}$ for $^{76}\text{Ge}$ $0\nu\beta\beta$ decay

Note the large positive contributions along the diagonal (pairing) and the negative off-diagonal contributions (higher seniority). The valence orbits dominate, but $f_7/2$ and $g_9/2$ contribute noticeably.
Contribution of different configurations to $M^{0\nu}$ for $^{76}$Ge $0\nu\beta\beta$ decay
Closure approximation

How good is the closure approximation?

Comparison between the QRPA $M^{0\nu}$ with the proper energies of the virtual intermediate states (symbols with arrows) and the closure approximation (lines) with different $\langle E_n - E_i \rangle$.

Note the mild dependence on $\langle E_n - E_i \rangle$ and the fact that the exact results are reasonably close to the closure approximation results for $\langle E_n - E_i \rangle < 20$ MeV.
Constraining the $0\nu\beta\beta$-decay NMEs

Nucleons that change from neutrons to protons are valence neutrons
Proton, neutron removing transfer reaction

$^\text{76}\text{Ge} \rightarrow ^\text{76}\text{Se}$

J. Schiffer, B. Kay, P. Grabmayr et al.

$\eta_j^{exp} = \langle 0^+_{init} | \sum_m c_{j,m}^+ c_{j,m} | 0^+_{init} \rangle$

QRPA(A) $\equiv$ BCS (WS)
QRPA(B) $\equiv$ BCS (AWS)

Suhonen, Civitarese, PLB 668, 277 (2008)

How can we take into account theoretically the constraint represented by the experimentally determined occupancies?

The experiment fixes for the final nucleus

\[ n_j^{\text{exp}} = \langle 0^+_{\text{init}} | \sum c_{j,m}^+ c_{j,m} | 0^+_{\text{init}} \rangle \text{ and the same} \]

In BCS \( n_j^{\text{BCS}} = v_j^2 \times (2j+1) \) depends only on \( v_j \) which in turn depends on the mean field eigenenergies.

In QRPA the ground state includes correlations and thus

\[ n_j^{\text{QRPA}} = (2j+1) \times [v_j^2 + (u_j^2 - v_j^2) \xi_j] \]

\[ \xi_j = (2j+1)^{-1/2} \langle 0^+_{\text{qrpa}} | [\alpha_j^+ \alpha_j]^0 | 0^+_{\text{qrpa}} \rangle \]

depends on the quasiparticle content of the correlated ground state.
Initial and adjusted mean field levels

While $n_j^{\text{exp}}$ and $n_j^{\text{BCS}}$ are constrained by $\Sigma n_j = N$ (or Z) the $n_j^{\text{QRPA}}$ are not constrained by that requirement. The particle number is not conserved, even on average. Thus the QRPA must be modified to remedy this $\Rightarrow$ Selfconsistent Renormalized QRPA

<table>
<thead>
<tr>
<th>$^{76}\text{Ge} \rightarrow ^{76}\text{Se}$</th>
<th>prev.</th>
<th>new</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jastrow s.r.c.</td>
<td>4.24(0.44)</td>
<td>3.49(0.23)</td>
</tr>
<tr>
<td>UCOM s.r.c.</td>
<td>5.19(0.54)</td>
<td>4.60(0.39)</td>
</tr>
</tbody>
</table>

F.Š., A. Faessler, P. Vogel, PRC 79, 015502 (2009)
Staircase plot (running sum) of the contributions to the $2\nu\beta\beta$ decay ($^{76}\text{Ge} \rightarrow ^{76}\text{Se}$)
Shell structure of the mean field changed

Anisotropic harmonic oscillator
Nuclear deformation

\[ \beta = \sqrt{\frac{\pi}{5}} \frac{Q_p}{Z r_c^2} \]

Exp. I (nuclear reorientation method)
Exp. II (based on measured E2 trans.)
Theor. I (Rel. mean field theory)
Theor. II (Microsc.-Macrosc. Model of Moeller and Nix)

Till now, in the QRPA-like calculations of the $0\nu\beta\beta$-decay NME spherical symmetry was assumed

The effect of deformation on NME has to be considered

<table>
<thead>
<tr>
<th>Nucl.</th>
<th>Exp. I</th>
<th>Exp. II</th>
<th>Theor. I</th>
<th>Theor. II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{48}$Ca</td>
<td>0.00</td>
<td>0.101</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{48}$Ti</td>
<td>+0.17</td>
<td>0.269</td>
<td>-0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{76}$Ge</td>
<td>+0.09</td>
<td>0.26</td>
<td>0.16</td>
<td>0.14</td>
</tr>
<tr>
<td>$^{76}$Se</td>
<td>+0.16</td>
<td>0.31</td>
<td>-0.24</td>
<td>-0.24</td>
</tr>
<tr>
<td>$^{82}$Se</td>
<td>+0.10</td>
<td>0.19</td>
<td>0.13</td>
<td>0.15</td>
</tr>
<tr>
<td>$^{82}$Kr</td>
<td></td>
<td>0.20</td>
<td>0.12</td>
<td>0.07</td>
</tr>
<tr>
<td>$^{96}$Zr</td>
<td></td>
<td>0.081</td>
<td>0.22</td>
<td>0.22</td>
</tr>
<tr>
<td>$^{96}$Mo</td>
<td>+0.07</td>
<td>0.17</td>
<td>0.17</td>
<td>0.08</td>
</tr>
<tr>
<td>$^{100}$Mo</td>
<td>+0.14</td>
<td>0.23</td>
<td>0.25</td>
<td>0.24</td>
</tr>
<tr>
<td>$^{100}$Ru</td>
<td>+0.14</td>
<td>0.22</td>
<td>0.19</td>
<td>0.16</td>
</tr>
<tr>
<td>$^{116}$Cd</td>
<td>+0.11</td>
<td>0.19</td>
<td>-0.26</td>
<td>-0.24</td>
</tr>
<tr>
<td>$^{116}$Sn</td>
<td>+0.04</td>
<td>0.11</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{128}$Te</td>
<td>+0.01</td>
<td>0.14</td>
<td>-0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{128}$Xe</td>
<td></td>
<td>0.18</td>
<td>0.16</td>
<td>0.14</td>
</tr>
<tr>
<td>$^{130}$Te</td>
<td>+0.03</td>
<td>0.12</td>
<td>0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{130}$Xe</td>
<td></td>
<td>0.17</td>
<td>0.13</td>
<td>-0.11</td>
</tr>
<tr>
<td>$^{136}$Xe</td>
<td></td>
<td>0.09</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{136}$Ba</td>
<td></td>
<td>0.12</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{150}$Nd</td>
<td>+0.37</td>
<td>0.28</td>
<td>0.22</td>
<td>0.24</td>
</tr>
<tr>
<td>$^{150}$Sm</td>
<td>+0.23</td>
<td>0.19</td>
<td>0.18</td>
<td>0.21</td>
</tr>
</tbody>
</table>
The suppression of the NME depends on relative deformation of initial and final nuclei
F.Š., Pacearescu, Faessler.
NPA 733 (2004) 321

Systematic study of the deformation effect on the 2νββ-decay NME within deformed QRPA
Alvarez,Sarriguren, Moya,Pacearescu, Faessler, F.Š.,
QRPA with realistic forces in deformed nuclei


\[
\langle pp_p n \rho_n | G | p'p'_p n' \rho_{n'} \rangle = \sum_J \sum_{(N_0l_j)_p} \sum_{(N_0l_j)_n} \sum_{(N_0l_j')_p} \sum_{(N_0l_j')_n} B^{(p)}_{(N_0l_j)_p} B^{(n)}_{(N_0l_j)_n} B^{(p')}_{(N_0l_j')_p} B^{(n')}_{(N_0l_j')_n} \times (-1)^{J_n - \Omega_n} (-1)^{J_n' - \Omega_n'} C^{JK}_{j_p \Omega_p j_n \Omega_n} C^{JK}_{j_p' \Omega_p' j_n' \Omega_n'} \times \langle (N_0l_j)_p (N_0l_j)_n, J | G | (N_0l_j')_p (N_0l_j')_n', J \rangle
\]

G-matrix elements in spherical single particle basis
Bonn CD potential
**Effect of nuclear deformation on $0\nu\beta\beta$-decay**

SuperNEMO (SNO+): about 56 kg of $^{150}$Nd => 0.1 eV

$0\nu\beta\beta$-decay of $^{150}$Nd in different models with half-lives for $m_{\beta\beta}=50$ meV

<table>
<thead>
<tr>
<th></th>
<th>QRPA [6] a</th>
<th>this work ($\beta_2 = 0$) b</th>
<th>this work</th>
<th>pseudo-SU(3) [8]</th>
<th>PHFB [9]</th>
<th>IBM-2 [10]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^{0\nu}$</td>
<td>5.17</td>
<td>5.78</td>
<td>3.16</td>
<td>1.57</td>
<td>1.61</td>
<td>2.32</td>
</tr>
<tr>
<td>$T^{0\nu}_{1/2}$, $10^{25}$ y</td>
<td>1.72</td>
<td>1.38</td>
<td>4.60</td>
<td>18.7</td>
<td>17.7</td>
<td>8.54</td>
</tr>
</tbody>
</table>

$\langle m_{\beta\beta} \rangle = 50$ meV

[5] Barea and Iachello, PRC 79 (2009), IBM approach

7/28/2014  Fedor Simkovic  57
On the relation between $0\nu\beta\beta$-decay and $2\nu\beta\beta$-decay (GT) NMEs

F.Š., R. Hodák, A. Faessler, P. Vogel, PRC 83, 015502 (2011)

$$M^{0\nu} = M^{0\nu}_{GT} \left( 1 + \frac{1}{g_A^2 M^{0\nu}_{GT}} + \frac{M^{0\nu}}{M^{0\nu}_{GT}} \right)$$
2νββ-decay NMEs

\[ \frac{1}{T_{2\nu-\text{exp}}^{1/2}} = G_{2\nu}^{2}(E_{0}, Z) \, g_{A}^{4} \, |M_{GT}^{2\nu}|^{2} \]

**Why the spread of the 2νββ NMEs is large and of the 0νββ NMEs is small?**

**Are both type of NMEs related?**

*Differences among 2νββ-decay NMEs: up to factor 10*
The cross sections of \((t,^3\text{He})\) and \((d,^2\text{He})\) reactions give \(B(GT^\pm)\) for \(\beta^+\) and \(\beta^-\), product of the amplitudes \((B(GT)^{1/2})\) entering the numerator of \(M^{2\nu}_{GT}\).

\[
M^{2\nu}_{GT} = \sum_m \frac{M_{GT}^+(m) M_{GT}^-(m)}{Q_{\beta\beta}/2 + m_e + E_x(1^+_m) - E_0}
\]

- \(2\nu\beta\beta\)-matrix element
  - \(0.16 \pm 0.04\) MeV\(^{-1}\)
  - with \(G^{(2\nu)} = 3.4 \times 10^{-20}\) MeV\(^2\) a\(^{-1}\)
  - Closure \(2\nu\beta\beta\)-decay NME

\[
M^{2\nu}_{GT-cl} = \sum_m M_{GT}^+(m) M_{GT}^-(m)
\]

- \(2\nu\beta\beta\) - half-life
  - \((1.1 \pm 0.2) \times 10^{21}\) a
  - recommended. exp. value:
    - \((1.5 \pm 0.1) \times 10^{21}\) a

\[
g_A^2 M^{2\nu}_{GT-cl} = \frac{3D}{\sqrt{ft_{EC}ft_{\beta^-}}}
\]

Grewe, …Frekers at al, PRC 78, 044301 (2008)
Going to relative coordinates:

\[ M_{GT-cl}^{2\nu} = \int_0^\infty C_{GT-cl}^{2\nu}(r)dr \]

**r- relative distance of two nucleons**

A connection between closure 2νββ and 0νββ GT NMEs

F.Š., R. Hodák, A. Faessler, P. Vogel, PRC 83, 015502 (2011)

\[ M_{GT}^{0\nu} = \int_0^\infty H_{GT}^{0\nu}(r)C_{GT-cl}^{2\nu}(r)dr \]

**Neutrino potential**

\[ H(r) = R^2 \frac{2}{\pi} \int_0^\infty j_0(qr) \frac{q}{q + E} f_{FNS}^2(q^2)g_{HOT}(q^2)dr \]

**Neutrino potential prefer short distances**
Closure 2νββ GT NME

The only non-zero contribution from $J^{\pi}=1^+$

$$M_{\text{GT-cl}}^{2\nu} = \sum_{J^{\pi},m} \langle 0_f^+ |\tau^+ \bar{\sigma} | J^{\pi}, m > \cdot \langle J^{\pi}, m |\tau^+ \bar{\sigma} | 0_i^+ >$$

$$= \sum_{m} \langle 0_f^+ |\tau^+ \bar{\sigma} | 1^+, m > \cdot \langle 1^+, m |\tau^+ \bar{\sigma} | 0_i^+ >$$

$$M_{\text{GT-cl}}^{2\nu} = \sum_{J^{\pi}} \int_0^\infty C_{\text{GT-}J^{\pi}}^{2\nu}(r) dr$$

7/28/2014
\( M^{0\nu}_{GT} \) depends weakly on \( g_A/g_{pp} \) and QRPA approach unlike \( M^{2\nu}_{GT} \)

\[
M^{0\nu}_{GT}(r_0) = \int_0^{r_0} H^{0\nu}_{GT}(r) C^{2\nu}_{GT-cl}(r) dr
\]

Different QRPA-like approaches

Dependence on axial-vector coupling

Nucleon Nuclear physics
### Phenomenological estimation of $M^{0\nu}_{GT}$

| Nucleus | $T_{1/2}^{2\nu-exp}$ [y] | $|M_{GT}^{2\nu-exp}|$ [MeV$^{-1}$] | $|M_{GT-cl}^{2\nu}|$ | $|M_{0\nu-ph}^{\nu}|$ | $|M_{GT}^{2\nu}|$ [MeV$^{-1}$] | $|M_{GT-cl}^{2\nu}|$ | $|M_{0\nu-ph}^{\nu}|$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $^{48}Ca$ | $4.4 \times 10^{19}$ | 0.046 | - | - | 0.083 | 0.220 | 1.98 |
| $^{76}Ge$ | $1.5 \times 10^{21}$ | 0.0.136 | - | - | 0.159 | 0.522 | 5.46 |
| $^{96}Zr$ | $2.3 \times 10^{19}$ | 0.090 | - | - | - | 0.222 | 3.45 |
| $^{100}Mo$ | $7.1 \times 10^{18}$ | 0.231 | 0.350 | 4.02 | - | - | - |
| $^{116}Cd$ | $2.8 \times 10^{19}$ | 0.126 | 0.349 | 4.21 | 0.064 | 0.305 | 3.67 |
| $^{128}Te$ | $1.9 \times 10^{24}$ | 0.126 | 0.033 | 0.41 | - | - | - |

### Neutrino potential

\[
H(r) = R^2 \frac{2}{\pi} \int_0^\infty j_0(qr) \frac{q}{q + E} f_{\text{FNS}}(q^2) g_{\text{HOT}}(q^2) dq
\]

with Taylor expansion

\[
j_0(qr) = 1 - \frac{1}{6} (qr)^2 + \frac{1}{120} (qr)^4 - \cdots = 1 - \mathcal{F}(r)
\]

\[
M_{0\nu}^{\nu} = H_{GT}(r = 0) M_{GT-cl}^{2\nu} - \int_0^\infty \mathcal{F}(r) C_{GT-cl}^{2\nu}(r) dr = M_{GT-cl}^{0\nu-ph} - M_{GT}^{0\nu-rest}
\]

**A: Phenomen. prediction:** Too large ($\sim$ factor 2)

**B: Need to be calculated** Not negligible

---

**There is no proportionality between $M^{0\nu}_{GT}$ and $M^{2\nu}_{GT}$**
There is no proportionality between $0\nu\beta\beta$-decay and $2\nu\beta\beta$-decay NM!!

Frekers et al.  
Charge exchange reactions
Both 2νββ and 0νββ operators connect the same states. Both change two neutrons into two protons.

Explaining 2νββ-decay is necessary but not sufficient
2νββ-decay

Gamow-Teller transitions

Continuum states

1+

GT resonance

QRPA

RQRPA

shell model

low lying states

virtual states

SSD hypothesis

(A,Z)

(A,Z+1)

(A,Z+2)

\[
M^{2ν}_{GT} = \sum_{m} \frac{<0^+_f || τ^+ σ || 1^+_m> <1^+_m || τ^+ σ || 0^+_i>}{E_m - E_i + \Delta}
\]

\[
(T_{1/2}^{2ν})^{-1} = G^{2ν} |M^{2ν}_{GT}|^2
\]

deduced from measured \(T_{1/2}^{2ν}\)

Differences in NME: by factor ~ 10
2νββ-decay within the field theory


Weak interaction Hamiltonian

\[ \mathcal{H}^\beta(x) = \frac{G_F}{\sqrt{2}} \left[ \bar{e}_L(x) \gamma_\alpha \nu_e L(x) \right] j_\alpha(x) + h.c. \]

2nbb-decay amplitude

\[ \langle f | S^{(2)} | i \rangle = \frac{(-i)^2}{2} \left( \frac{G_F}{\sqrt{2}} \right)^2 L_{\mu\nu}(p_1, p_2, k_1, k_2) J_{\mu\nu}(p_1, p_2, k_1, k_2) \]

\[ - (p_1 \leftrightarrow p_2) - (k_1 \leftrightarrow k_2) + (p_1 \leftrightarrow p_2)(k_1 \leftrightarrow k_2) \]

Hadron part of amplitude

\[ J_{\mu\nu}(p_1, p_2, k_1, k_2) = \int e^{-i(p_1+k_1)x_1} e^{-i(p_2+k_2)x_2} \]

\[ \text{out} \langle p_f | T(J_\mu(x_1)J_\nu(x_2)) | p_i \rangle > \text{in} \ dx_1 dx_2 \]
A sum over intermediate nuclear states represents a sum over all meson and gamma exchange correlations of two beta decaying nucleons.

\[ T(J_\mu(x_1)J_\nu(x_2)) = J_\mu(x_1)J_\nu(x_2) \quad \text{(two } \beta \text{ decays)} \]
\[ + \Theta(x_{20} - x_{10})[J_\nu(x_2), J_\mu(x_1)] \quad \text{(2}\nu\beta\beta \text{ decay)} \]

\[ J_\alpha(0, \vec{x}) = \sum_n \tau_n^+ (\delta_{\alpha A} + ig_A(\vec{\sigma})_k \delta_{\alpha k}) \delta(\vec{x} - \vec{x}_n) \]

\[ J^{2\beta2\nu}_{\mu\nu}(p_1, p_2, k_1, k_2) = -i2M_{GT} \delta_{\mu k} \delta_{\nu k} \]
\[ \times 2\pi \delta(E_f - E_i + p_{10} + k_{10} + p_{20} + k_{20}), \quad k = 1, 2, 3, \]
Integral representation of $M_{GT}$

$$M_{GT} = \frac{i}{2} \int_0^\infty (e^{i(p_{10}+k_{10}-\Delta)t} + e^{i(p_{20}+k_{20}-\Delta)t}) M_{AA}(t) \, dt$$

with

$$M_{AA}(t) = \langle 0_+^f | \frac{1}{2} [A_k(t/2), A_k(-t/2)] | 0_+^i \rangle$$

$$A_k(t) = e^{iHt} A_k(0) e^{-iHt}, \quad A_k = \sum_i \tau_i^+ (\vec{q}_i)_k, \quad k = 1, 2, 3.$$  

Completeness:  
$$\sum_n |n><n|=1$$

$$\langle A'| J_\alpha(x_1) J_\beta(x_2) | A \rangle = \sum_n \langle A'| J_\alpha(0, \vec{x}_1) | n \rangle \langle n | J_\beta(0, \vec{x}_2) | A \rangle \times$$

$$e^{-i(E'-E_n)x_{10}} e^{-i(E_n-E)x_{20}}$$

$$\int_0^\infty e^{-iat} \, dt \Rightarrow \lim_{\epsilon \to 0} \int_0^\infty e^{-i(a-i\epsilon)t} \, dt = \lim_{\epsilon \to 0} \frac{-i}{a-i\epsilon}$$

$$M_{GT} = \sum_n \frac{\langle 0_+^f | A(0)^n_k | 1_+^n \rangle \langle 1_+^n | A(0)^n_k | 0_+^i \rangle}{E_n - E_i + \Delta}$$
Double beta decay is a two-body process

\[ H = \text{one-body} + \text{two-body}, \quad A_k(0) = \text{one-body} \]

\[ A_k(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H[H...[H, A_k(0)]...]] \]

\[
\begin{array}{ccc}
n=0 & [H^{(0)} A] & [1] \\
\end{array}
\]

\[ If \quad H \approx \text{one-body op.} \quad \implies A_k(t) \text{ is one-body op.} \]
$r_{12}$ -dependence of the $2\nu\beta\beta$-decay NME

$M_{GT}^{2\nu} = \int C^{2\nu}(r) \, dr$

$$M_{GT}^{2\nu} = \sum_{J^\pi, k_i, k_f, J} \sum_{\text{pn}p'n'} (-1)^{j_n + j_{p'} + J + J} \times \sqrt{2J + 1} \left\{ \begin{array}{ccc} j_p & j_n & J \\ j_{n'} & j_{p'} & J \end{array} \right\} \times \langle p(1), p'(2); J \parallel \sigma(1) \cdot \sigma(2) \parallel n(1), n'(2); J \rangle \times \langle 0^+_f \parallel [c_p^+ \tilde{c}_{n'}]_J \parallel J^\pi k_f \rangle \langle J^\pi k_f \parallel J^\pi k_i \rangle \langle J^\pi k_f i \parallel [c_p^+ \tilde{c}_n]_J \parallel 0^+_i \rangle$$
Decomposition of $C^{2\nu}$ on multipole contributions

\[ M_{GT}^{2\nu} = <0^+_J | \tau^+ | \sigma \sum_{J^+,m} \frac{|J^\pi_m > < J^\pi_m |}{E_m - (E_i + E_f)/2} \tau^+ | \sigma | 0^+_i \rangle > \]

\[ = \sum_m <0^+_J | \tau^+ | \sigma | 1^+_{m'} > < 1^+_{m'} | \tau^+ | \sigma | 0^+_i \rangle > \]

\[ \int C_J(r) \, dr = M_{GT}^{2\nu} \text{ for } J^\pi = 1^+ \]

\[ = 0 \text{ for } J^\pi \neq 1^+ \]
BCS limit \((g_{ph}=g_{pp}=0)\) and decomposition of \(M_{GT}\) on pairing and broken pairs contributions

\[
\langle p(1), p'(2); J \parallel \sigma(1) \cdot \sigma(2) \parallel n(1), n'(2); J \rangle
\]
Single State Dominance (\(^{100}\text{Mo}, ^{106}\text{Cd}, ^{116}\text{Cd}, ^{128}\text{Te} \ldots\))

HSD, higher levels contribute to the decay

SSD, \(1^+\) level dominates in the decay

(Abad et al., 1984, Ann. Fis. A 80, 9)

\[^{100}\text{Mo} \quad \begin{array}{c} 0^+ \\ 1^+ \end{array} \quad ^{100}\text{Tc} \quad 0^+ \quad ^{100}\text{Ru} \quad ^{106}\text{Tc} \quad ^{106}\text{Ru} \quad ^{106}\text{Ag} \quad ^{106}\text{Cd} \quad ^{106}\text{Pd} \]

\[E_i-E_f = -0.041\text{ MeV}\]

\[E_i-E_f = 0.705\text{ MeV}\]

\[E_i-E_f = -0.343\text{ MeV}\]
SSD – theoretical studies

\[
M_{GT}^K = \sum_m \left( \frac{M_m^i (1^+) M_m^f (1^+)}{E_m - E_i + e_{10} + \nu_{10}} + \frac{M_m^i (1^+) M_m^f (1^+)}{E_m - E_i + e_{20} + \nu_{20}} \right)
\]

\[
M_{GT}^K = M_{GT}^L (\nu_{10} \leftrightarrow \nu_{20})
\]

SSD

\[
\Rightarrow \frac{M_1^i (1^+) M_1^f (1^+)}{E_1 - E_i + e_{10} + \nu_{10}} + \frac{M_1^i (1^+) M_1^f (1^+)}{E_1 - E_i + e_{20} + \nu_{20}} \Rightarrow 2 \frac{M_1^i (1^+) M_1^f (1^+)}{E_1 - E_i + \Delta}
\]

Common approx.

\[
e_{10} + \nu_{10} \approx e_{20} + \nu_{20} \approx (E_i - E_f) / 2 \equiv \Delta
\]

\[
E_1 - E_i \approx 0 \text{ or neg.} \Rightarrow \text{sensitivity to lepton energies in energy denominators}
\]

\[
\Rightarrow \text{SSD and HSD offer different differential characteristics}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Isotope} & \text{f.s.} & T_{1/2}^{\text{(SSD)][y]}} & T_{1/2}^{\text{(exp.)}[y]} \\
\hline
{^{100}}\text{Mo} & 0_{\text{g.s.}} & 6.8 \times 10^{18} & 6.8 \times 10^{18} \\
& 0_{1} & 4.2 \times 10^{20} & 6.1 \times 10^{18} \\
{^{116}}\text{Cd} & 0_{\text{g.s.}} & 1.1 \times 10^{19} & 2.6 \times 10^{19} \\
{^{128}}\text{Te} & 0_{\text{g.s.}} & 1.1 \times 10^{25} & 2.2 \times 10^{24} \\
& \text{EC/EC} & & & \\
{^{106}}\text{Cd} & 0_{\text{g.s.}} & >4.4 \times 10^{21} & >5.8 \times 10^{17} \\
{^{130}}\text{Ba} & 0_{\text{g.s.}} & 5.0 \times 10^{22} & 4.0 \times 10^{21} \\
\hline
\end{array}
\]

Domin, Kovalenko, Šimkovic, Semenov, NPA 753, 337 (2005)
SSD differential characteristics

2νβ−β−–decay

$^{100}\text{Mo} \rightarrow ^{100}\text{Ru}$

Do not depend on $M^iM^f$

2νEC/β+–decay

$^\nu \nu \beta$ decay

$^\nu \nu \beta$ decay

7/28/2014 Fedor Simkovic
$^{100}$Mo $2\beta 2\nu$: Experimental Study of SSD Hypothesis

NEMO 3 exp.

Single electron spectrum different between SSD and HSD

\[ \chi^2/\text{ndf} = 139. / 36 \]

$^{100}$Mo $2\beta 2\nu$ single energy distribution in favour of Single State Dominant (SSD) decay

\[
\begin{align*}
\text{HSD: } T_{1/2} & = 8.61 \pm 0.02 \text{ (stat)} \pm 0.60 \text{ (syst)} \times 10^{18} \text{ y} \\
\text{SSD: } T_{1/2} & = 7.72 \pm 0.02 \text{ (stat)} \pm 0.54 \text{ (syst)} \times 10^{18} \text{ y}
\end{align*}
\]
2νββ-decay

The half life for 2νββ decay

\[ \left[ T^{2\nu\beta\beta} \right]^{-1} = \frac{1}{4} \frac{\int d\Gamma}{\ln 2} = \frac{m_e}{8\pi^2} \frac{1}{\ln 2} (G_\beta m_e^2)^4 I^{2\nu} (0^+) \]

with

\[ I^{2\nu} (0^+) = \frac{1}{m_e} \int_{m_e}^{Q-m_e} F_0 (Z_f, p_{10}) p_1 p_{10} dp_{10} \]
\[ \times \int_{m_e}^{Q-p_{10}} F_0 (Z_f, p_{20}) p_2 p_{20} dp_{20} \int_0^{Q-p_{10}-p_{20}} k_{10}^2 k_{20}^2 |M^{2\nu}|^2 dk_{10}, \]

The nuclear matrix element

\[ |M^{2\nu}|^2 = g_V^4 \left[ |M^K_F|^2 - Re \left\{ M^K_F M^K_L^* \right\} + |M_L^F|^2 \right] - \\
g_V^2 g_A^2 Re \left\{ M^K_F M^K_L^{GT*} + M^K_{GT*} M^K_L^F \right\} + g_A^4 \left[ \frac{|M^K_{GT} + M_L^{GT}|^2}{4} + \frac{1}{12} |M^K_{GT} - M_L^{GT}|^2 \right] \]
Fermi and Gamow-Teller matrix elements

\[ M^F_\Omega = \frac{1}{2} \sum_n \left\langle 0^+_f \left| \sum_m \tau^+_m \right| 0^+_n \right\rangle \left\langle 0^+_n \left| \sum_m \tau^+_m \right| 0^+_i \right\rangle \Omega_n , \]
\[ M^{GT}_\Omega = \frac{1}{2} \sum_n \left\langle 0^+_f \left| \sum_m \tau^+_m (\sigma_m)_k \right| 1^+_n \right\rangle \left\langle 1^+_n \left| \sum_m \tau^+_m (\sigma_m)_k \right| 0^+_i \right\rangle \Omega_n , \]

with \( \Omega_n = K_n, L_n \)

\[ K_n = \frac{1}{(2E_n - E_i - E_f)/2 + \epsilon_K} + \frac{1}{(2E_n - E_i - E_f)/2 - \epsilon_K} \]
\[ L_n = \frac{1}{(2E_n - E_i - E_f)/2 + \epsilon_L} + \frac{1}{(2E_n - E_i - E_f)/2 - \epsilon_L} \]

\[ \epsilon_K = (p_{20} + k_{20} - p_{10} - k_{10})/2 \]
\[ \epsilon_L = (p_{10} + k_{20} - p_{20} - k_{10})/2 \]

From above expressions we can see that in the limit, \( 2E_n - E_i - E_f = 0 \), the Fermi and Gamow-Teller matrix elements vanish.
3. Study of the $2\nu\beta\beta$ Gamow-Teller nuclear matrix elements within SO(8) model

The Hamiltonian will contain $T=0, S=1, L=0$ and $T=1, S=0, L=0$ two body particle-particle interactions and $T=1, S=1, L=0$ particle-hole interactions and will be expressed in terms of the generators of an SO(8) algebra. We denote it as the SO(8) model.

$$H = -g_{pair} \sum_{M_T} A_{0,1}^\dagger(0, M_T) A_{0,1}(0, M_T) - g_{pp} \sum_{M_S} A_{1,0}^\dagger(M_S, 0) A_{1,0}(M_S, 0) + g_{ph} \sum_{\mu, \nu} F_{\nu}^\mu \dagger F_{\nu}^\mu$$

$$A_{S,T}^\dagger(M_S, M_T) = \sum_{m_S, m_T, m} \sqrt{\frac{\Omega}{2}} C_{lmvm}^{m_S} C_{m_S m_T}^{S,M_S} C_{m_S m_T}^{T,M_T} a_{lmm'sm}^\dagger a_{lmm'm'}^\dagger$$

$$F_{\alpha}^b = \sum_{m, m_S, m_t} \langle (m_s + a)(m_t + b) | \sigma_a \tau_b | m_s m_t \rangle a_{nlm(m_s+a)(m_t+b)}^\dagger a_{nlmm'sm'}$$

The 12 operators of type $A$ and 9 operators of type $F$, together with 7 operators $S$, $T$ and $N$, represents 28 generators of group SO(8).

In the case of zero seniority, the wave functions for given \( l \) shell are labeled as

\[
|N, S, S_z, T, T_z, n >
\]

Matrix elements of the SO(8) generators are known in this basis.

When the \( g_{pp}^{T=0} = g_{\text{pair}} \), the Hamiltonian is the diagonal in this basis, on the other hand it mixes the quantum number \( n\pm2 \).

For given \((S,T)\), the values of SU(4) quantum numbers are

\[ n = S + T, S + T + 2, \ldots, N/2 \]

We divide this schematic Hamiltonian in the SU(4) spin-isospin symmetry conservation part and in the SU(4) breaking part.

\[ H = -g_{\text{pair}} \left( \sum_{M_T} A^\dagger_{0,1}(0, M_T)A_{0,1}(0, M_T) + \sum_{M_S} A^\dagger_{1,0}(M_S, 0)A_{1,0}(M_S, 0) \right) + g_{ph} \sum_{\mu, \nu} F^\mu_{\nu} F^\nu_{\mu} + (g_{\text{pair}} - g_{pp}^{T=0}) \sum_{M_S} A^\dagger_{1,0}(M_S, 0)A_{1,0}(M_S, 0) \]

Because the Gamow-Teller operators connect only states with the same SU(4) quantum numbers, the Gamow-Teller nuclear matrix element vanishes in the SU(4) conservation limit.
Using the perturbation theory up to the second order of the parameter \((g_{\text{pair}} - g_{p p}^{T=0})\), we can find that only the ground state of the intermediate nucleus is dominant in the G-T matrix element.

\[
M_{G T}^{2 \nu} \approx \langle 0 | \hat{\sigma}_{\tau^-} | 1_{n=1} \rangle \langle 1_{n=1} | \hat{\sigma}_{\tau^-} | 0 \rangle / (2E_{n=1} - E_f - E_i)/2
\]

where the nominator has a form

\[
\frac{144 \sqrt{\frac{231}{35}}}{10g_{\text{pair}} + 20g_{ph}} \left( \frac{g_{\text{pair}} - g_{pp}^{T=0}}{35(-10g_{\text{pair}} - 20g_{ph})^2} - \frac{267(g_{\text{pair}} - g_{p p}^{T=0})^2}{35(-10g_{\text{pair}} - 20g_{ph})^2} \right)
\]

\( \Omega = 12, N = 20, \)

and the energy denominator

\[
(2E_{n=1} - E_i - E_f)/2 = 5g_{\text{pair}} + 9g_{ph} + (g_{\text{pair}} - g_{pp}^{T=0}) \frac{39}{5} + \frac{(g_{\text{pair}} - g_{p p}^{T=0})^2}{g_{\text{pair}} + 2g_{ph}} \frac{1249263}{171500}
\]
$g_{\text{pair}} = 1.5 \text{ MeV}, \ g_{\text{ph}} = 1.5 g_{\text{pair}}$

- **Exact solution**
- **1st order of perturbation theory**
- **2nd order of perturbation theory**

![Graph showing the relationship between $g_{\text{pp}}/g_{\text{pair}}$ and $M_{GT}[\text{MeV}^{-1}]$. The graph includes lines for the exact solution and the first and second order of perturbation theory.]
3. **Energy-weighted sum rule involving $\Delta Z=2$ nuclei**

We propose an energy-weighted sum rule with connection of nuclei participating in the double beta decay. We have

$$\sum_n (2E_n - E_i - E_f) \left< f \left| O_{F,GT} \right| n \right> \left< n \left| O_{F,GT} \right| i \right> = \left< f \left| [O_{F,GT}, [H, O_{F,GT}]] \right| i \right>$$

**For SO(8) model**

$$\sum_n (2E_n - E_i - E_f) \left< f \left| \vec{\sigma} \tau^- \right| n \right> \left< n \left| \vec{\sigma} \tau^- \right| i \right> =$$

$$12 (g_{pp}^{T=0} - g_{pair}) \left< f \left| A^\dagger_{0,1}(0,-1) A_{0,1}(0,1) \right| i \right> - 2g_{ph} \left< f \left| \vec{\sigma} \tau^- \cdot \vec{\sigma} \tau^- \right| i \right> - 6g_{ph} \left< f \left| T^- T^- \right| i \right>$$
What is the meaning of quantity \((2E_{n=1} - E_i - E_f)\)?
### Instead of Conclusion:
*There is a need for supporting experiments*

#### Nuclear matrix elements:

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Frank Avignone:

Nuclear Matrix Elements are as important as DATA