University of Southampton
School of Physics and Astronomy

elusives invisiblesPlus
neutrinos, dark matter & dark energy physics

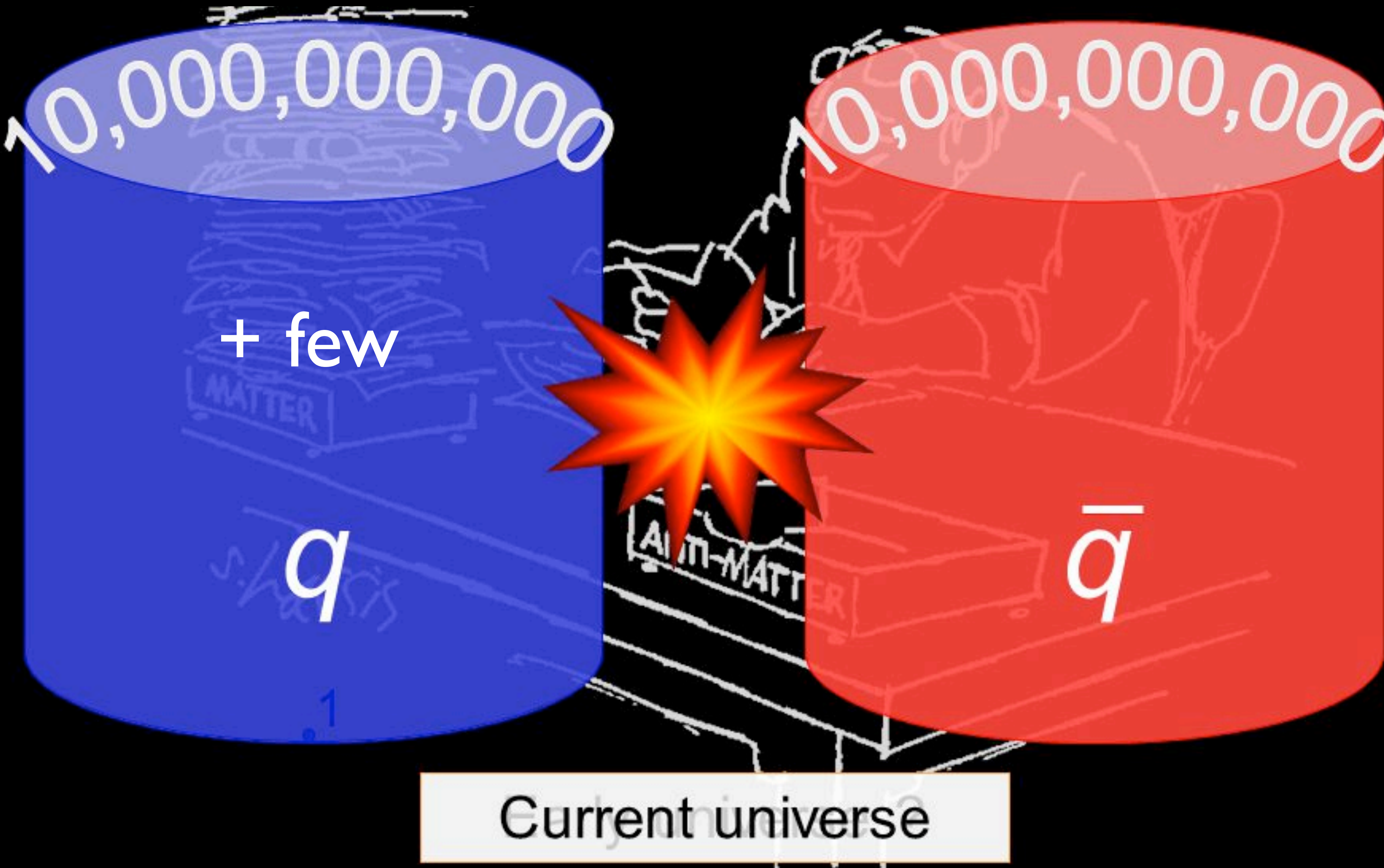
Theory of Neutrinos, Masses and Mixings

Steve King

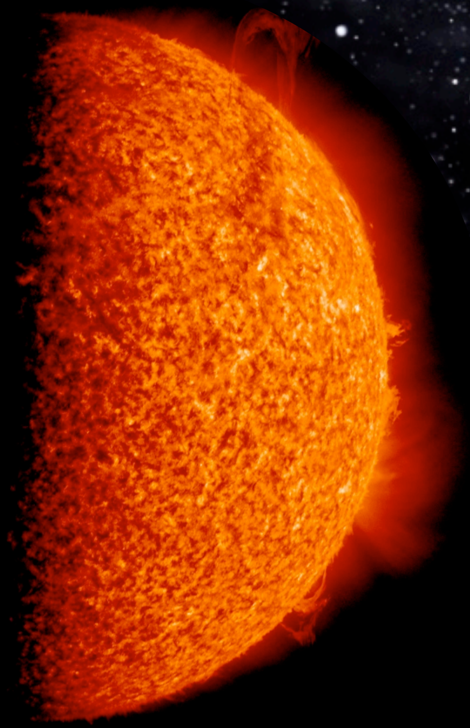
Introduction

At the end of Inflation the Universe was empty, cold and bare…

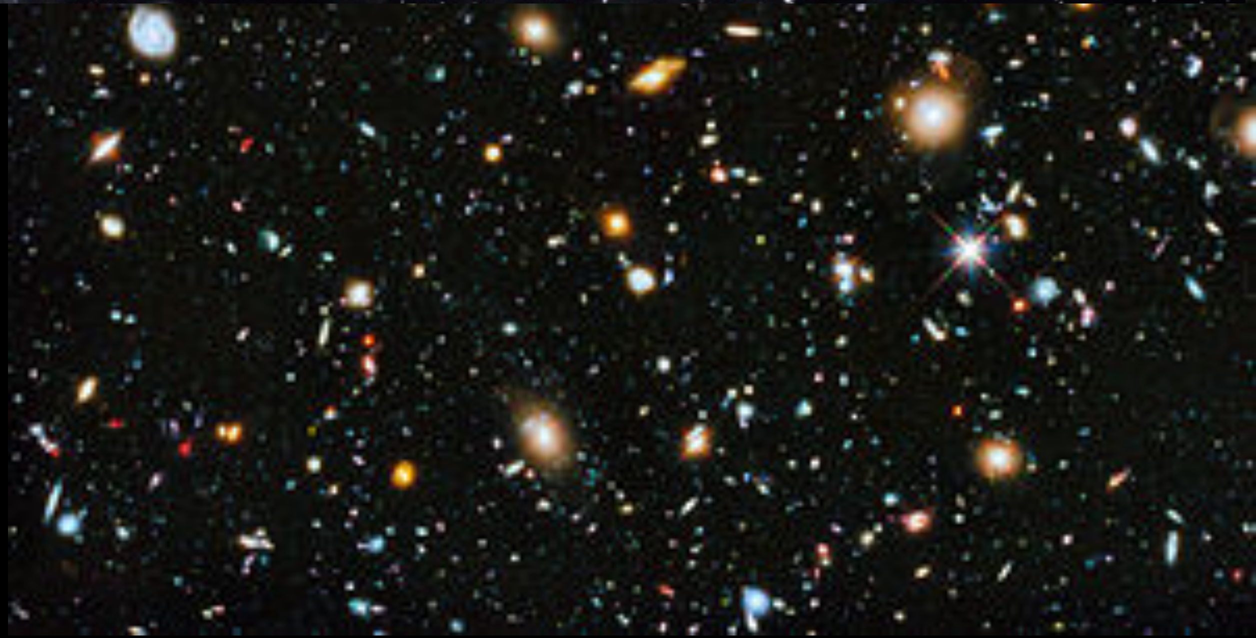
After reheating a very slight excess of matter was somehow generated



Giving the observed baryon asymmetry of the Universe

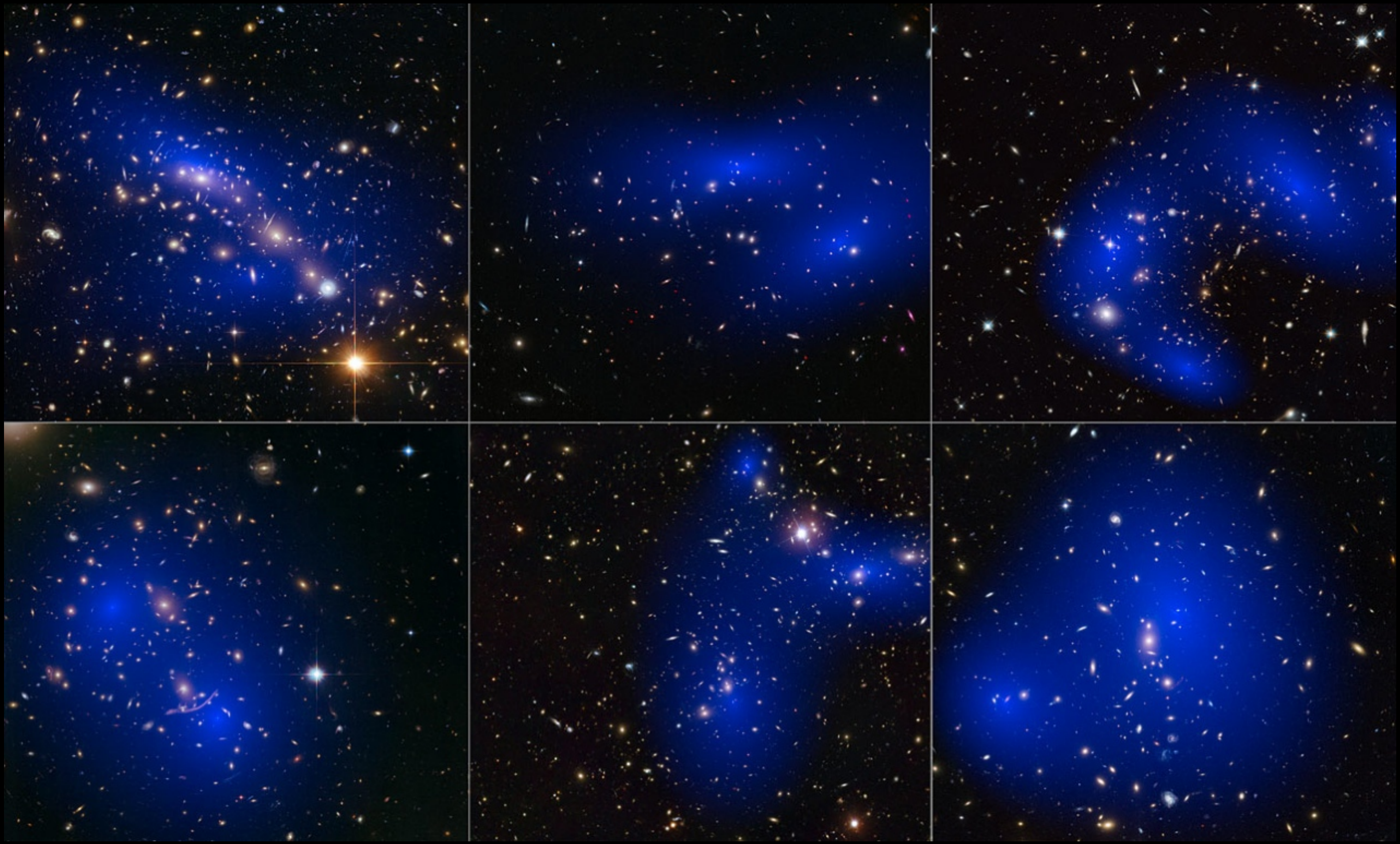


$$\eta_B = \frac{n_B - n_{\bar{B}}}{n_\gamma} = \frac{n_B}{n_\gamma} \approx 6 \times 10^{-10}$$



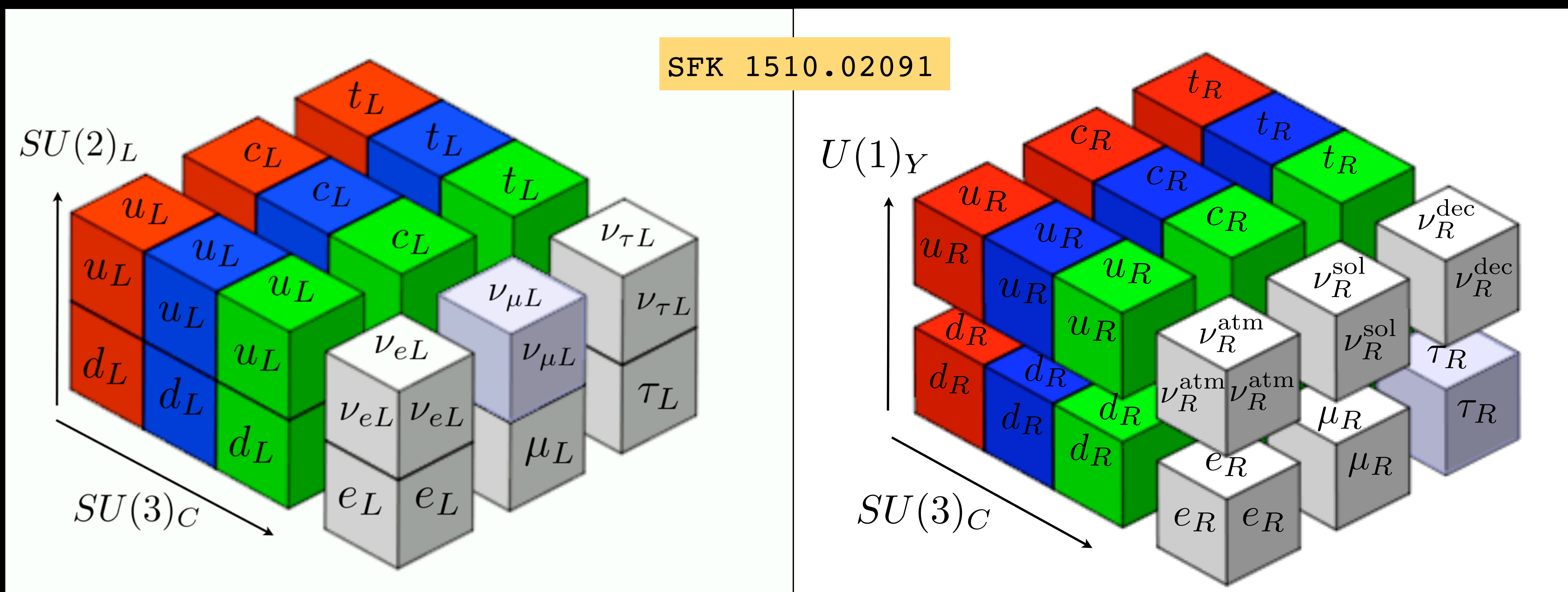
heic1506 — Science Release

Dark Matter?



Dark Energy?

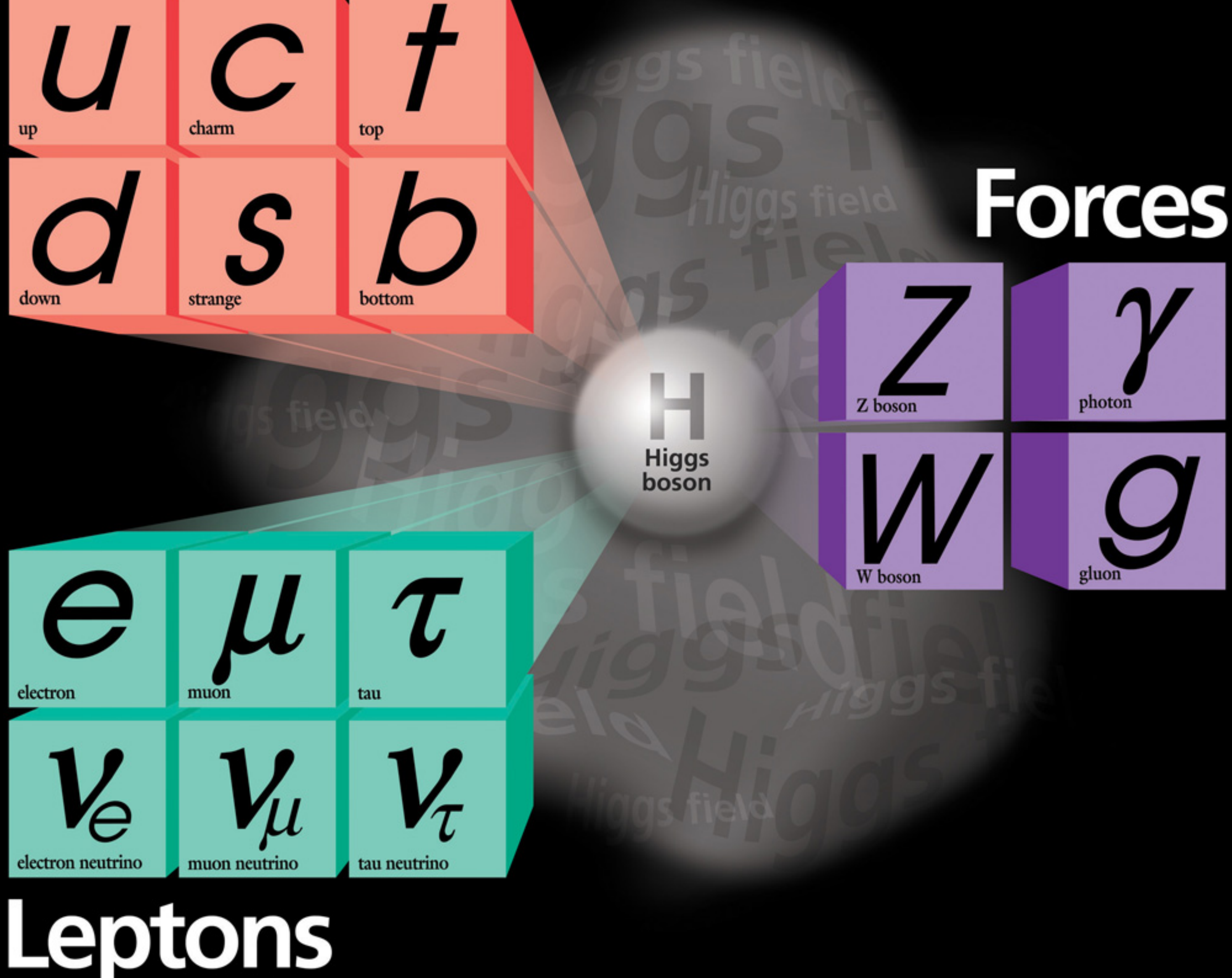
The Standard Model



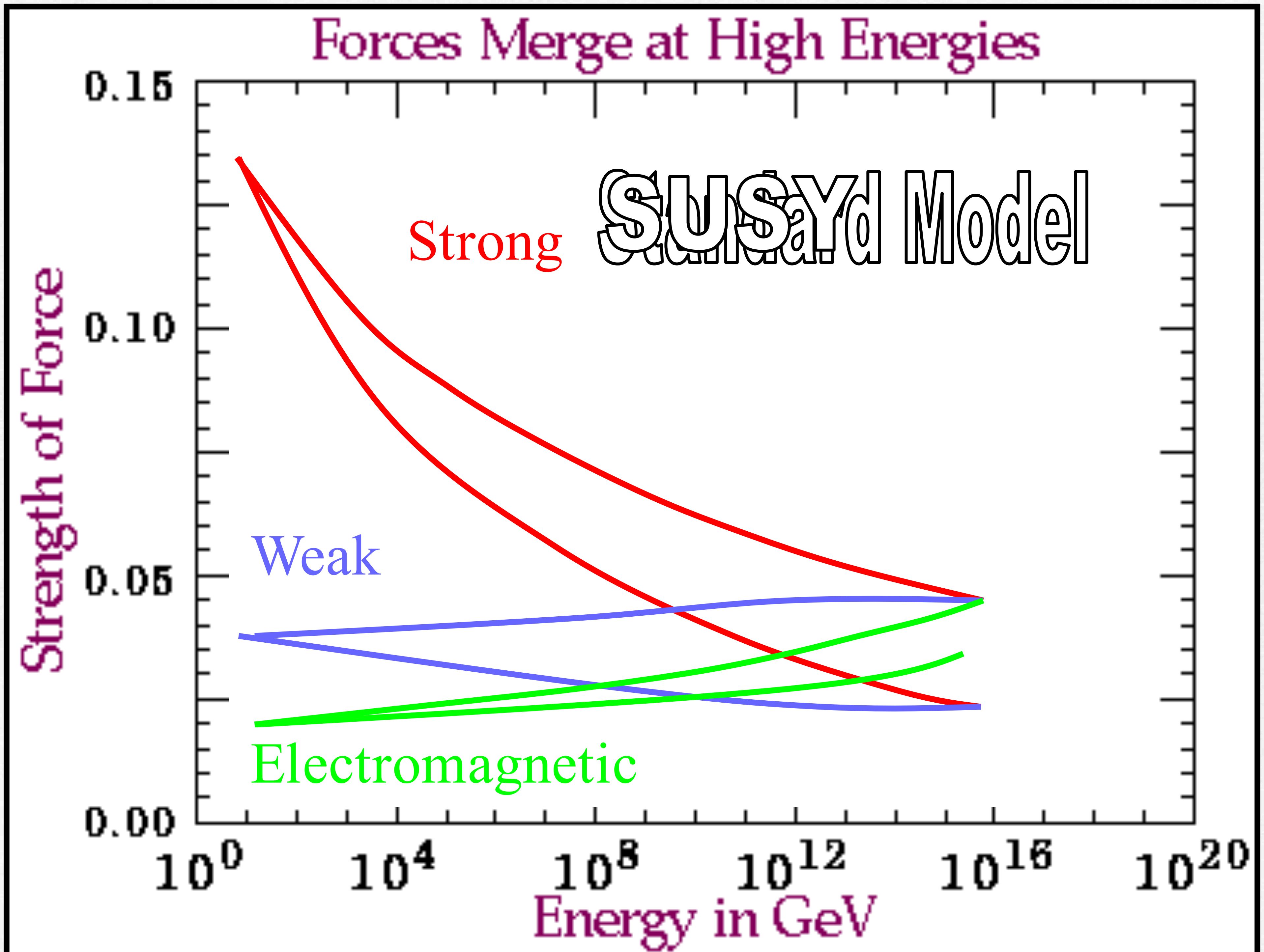
leaves many questions unanswered...

Why is the Higgs mass light?

$$m_H \ll m_{Planck}$$

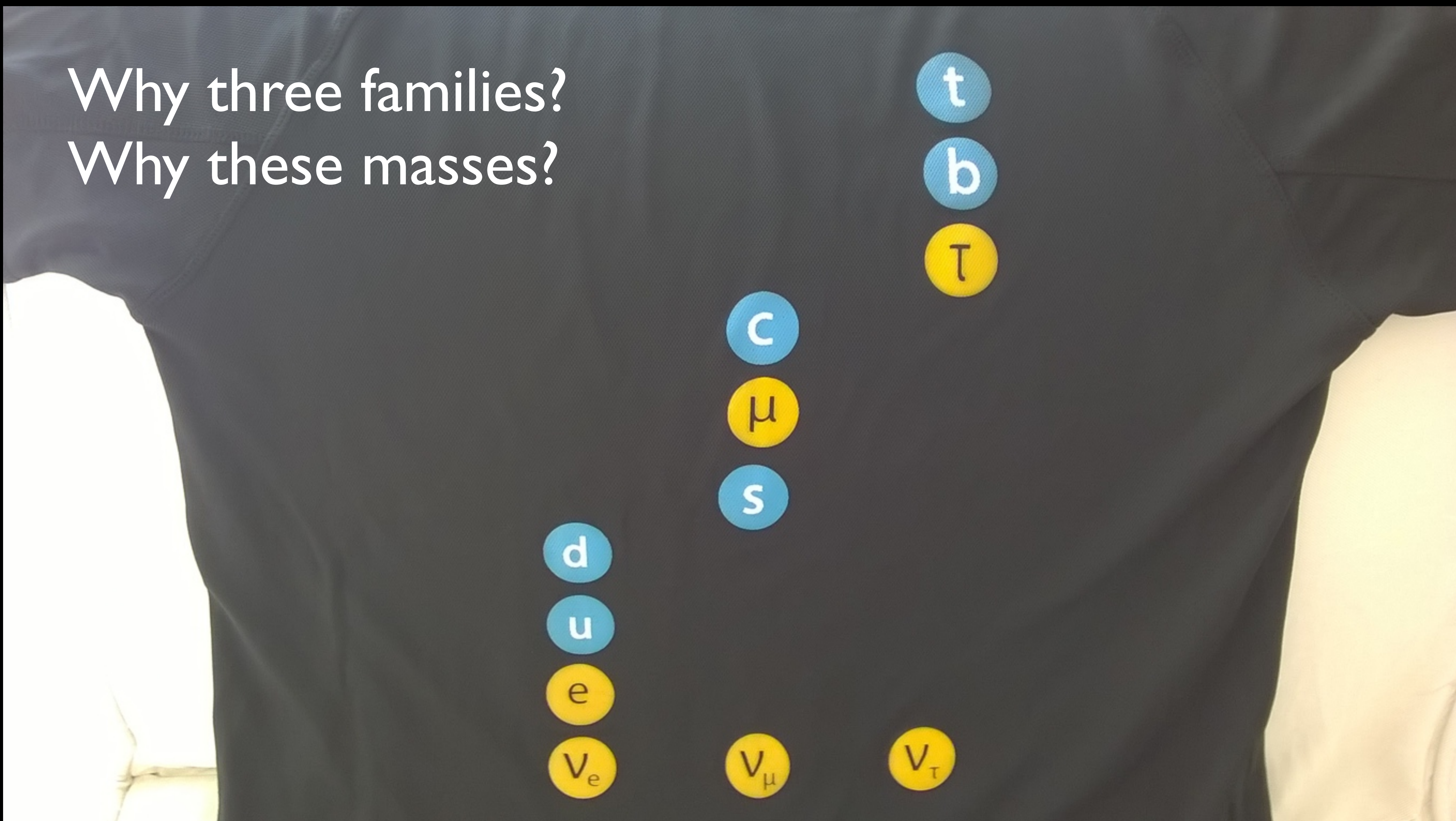


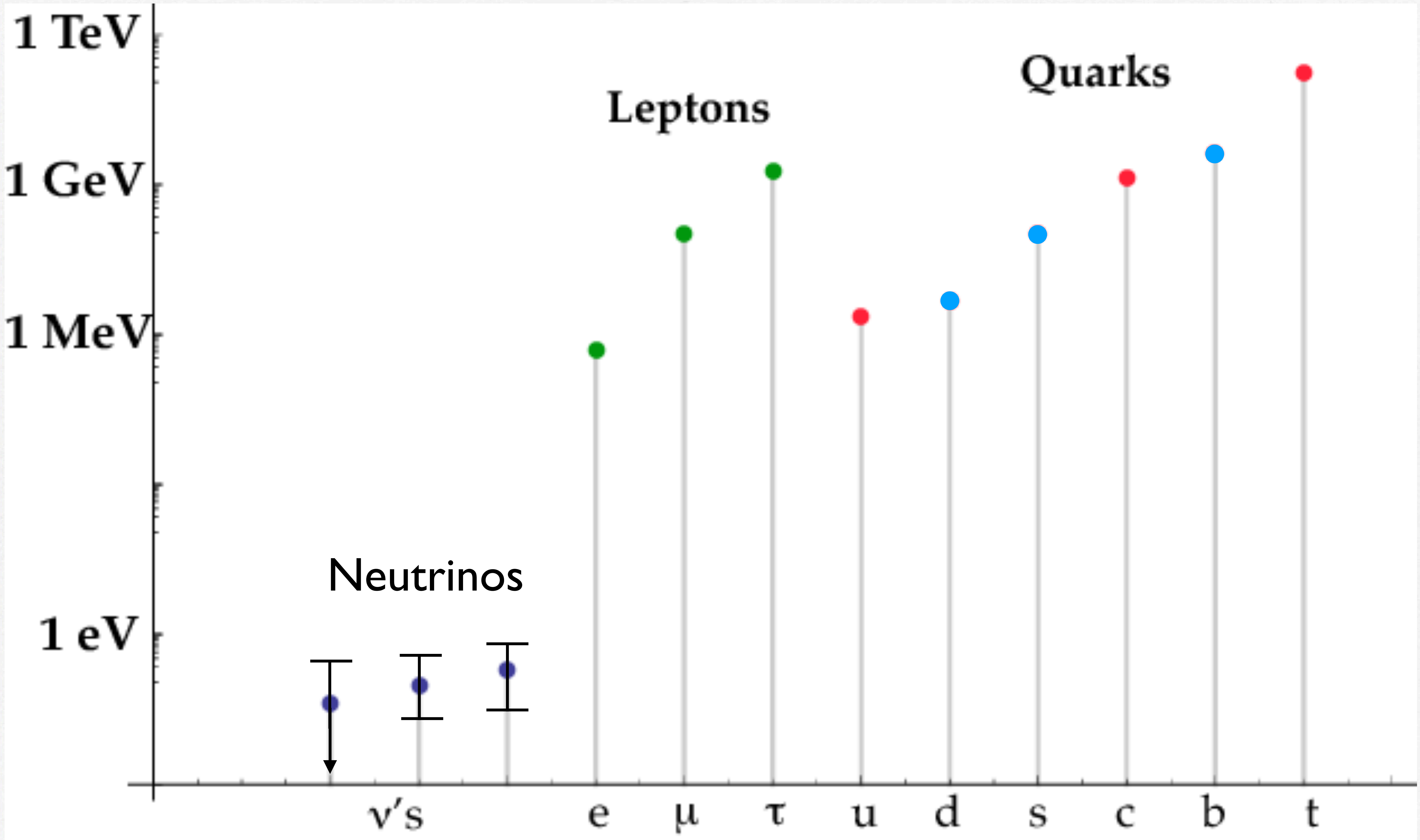
Unification?



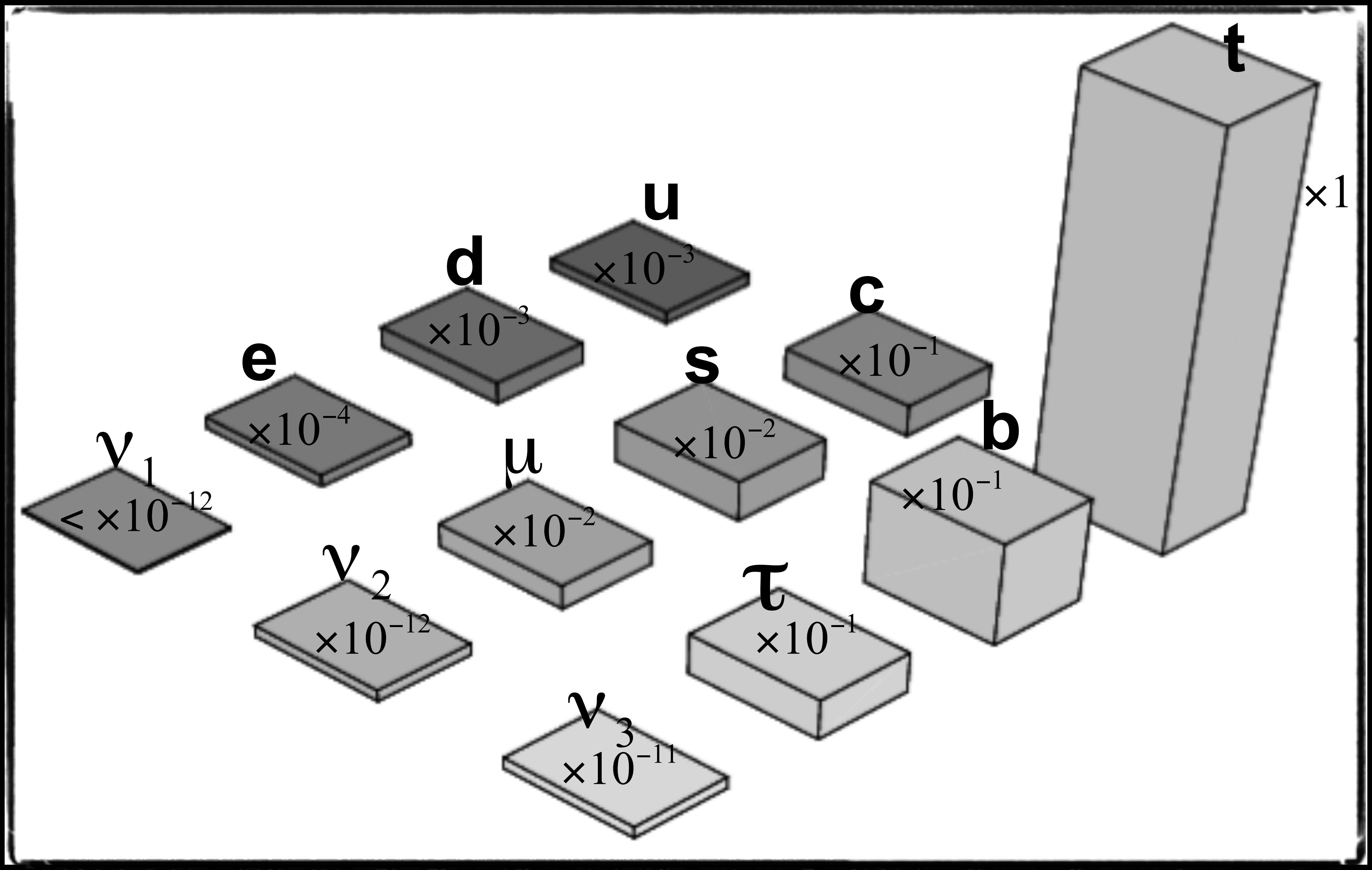
Flavour

Why three families?
Why these masses?

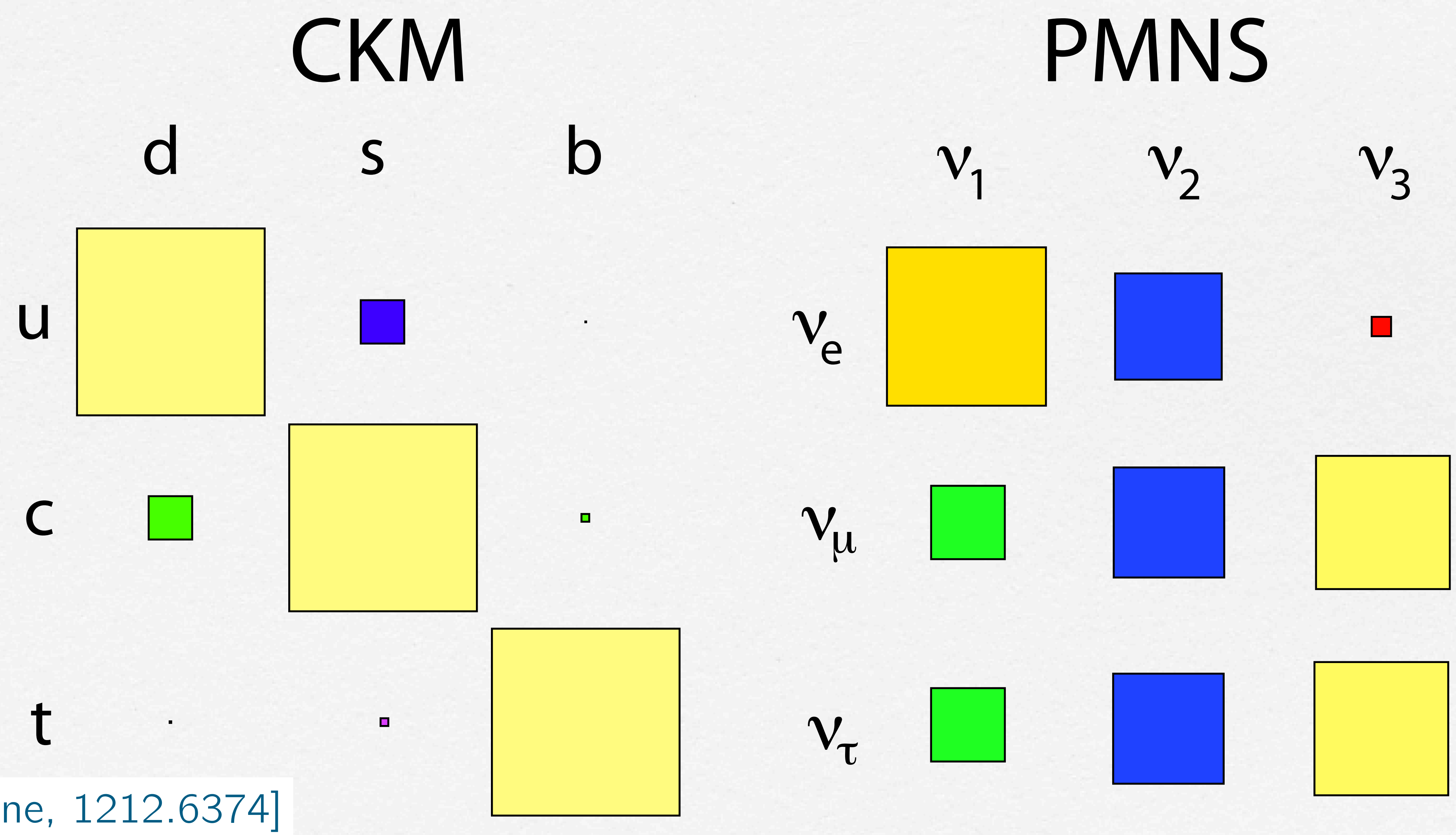



1 TeV
1 GeV
1 MeV
1 eV
Leptons
Quarks
Neutrinos
ν's
e
μ
τ
u
d
s
c
b
t

Origin of quark and lepton masses?



Origin of quark and lepton mixing?



[Stone, 1212.6374]

Neutrino mass and mixing

- ☐ Neutrinos have tiny masses (much less than electron)
- ☐ Neutrinos mix a lot (unlike the quarks)
- ☐ Up to 9 new params: 3 masses, 3 angles, 3 phases
- ☐ Origin of mass and mixing is unknown

Implications for PP and Cosmology

These lectures

- Origin of neutrino mass
  See-saw mechanisms, RPV SUSY, Extra dimensions,...
- Unification of matter, forces and flavour
  SUSY, GUTs, Family Symmetry,...

Particle Physics

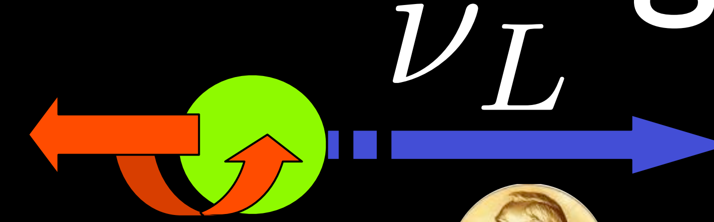
- Baryon asymmetry of the universe?
  Leptogenesis
- Dark Matter?
  Warm dark matter
- Inflation?
  Sneutrino inflation
- Dark energy? $\Lambda \sim m_\nu^4$

Cosmology

From review SFK 1701.04413

Neutrino Masses and Mixings

A Brief (and incomplete) History of Neutrino Mass and Mixing

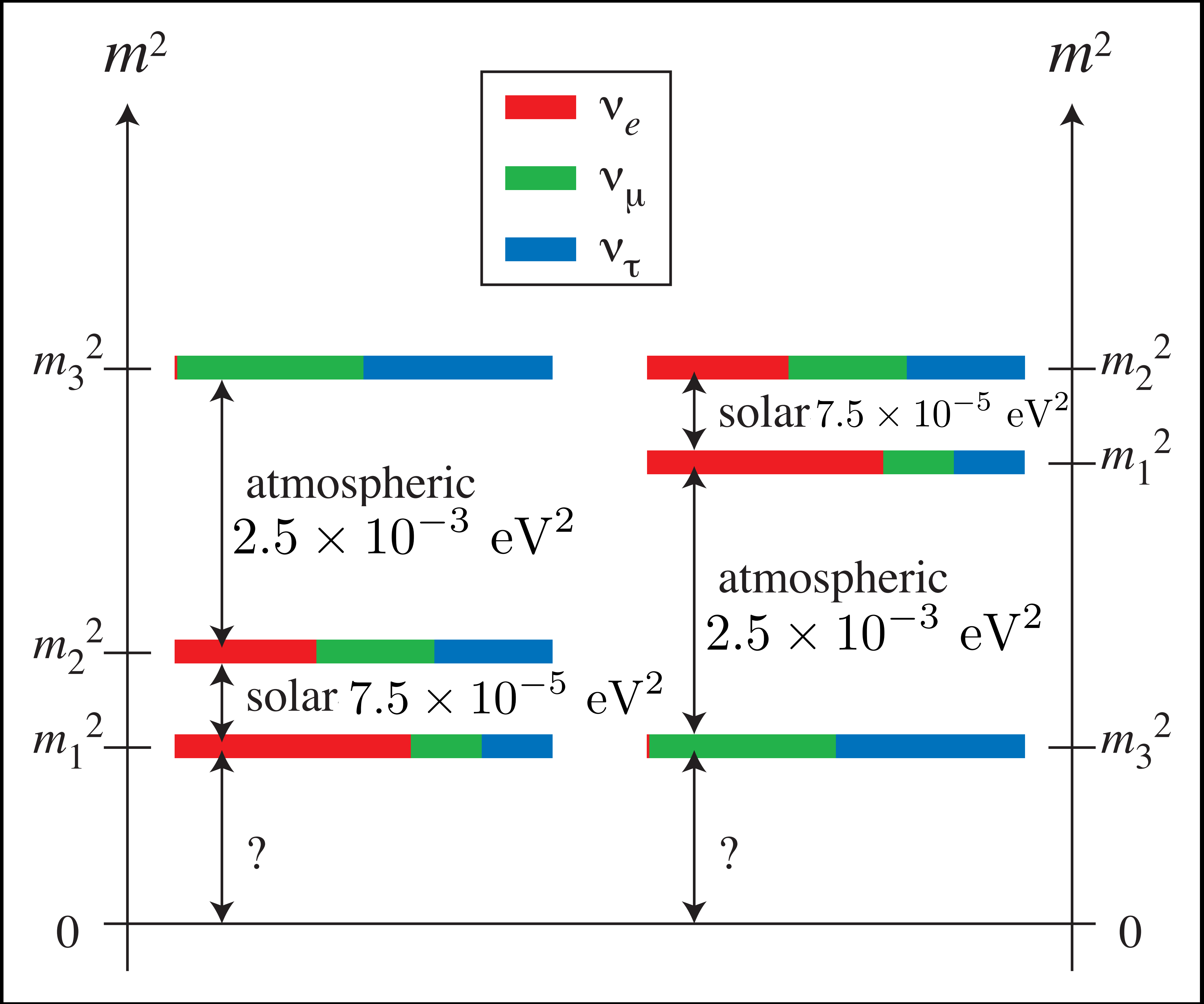


- ☑ Atmospheric ν_μ disappear, large θ_{23} (1998)  SK
- ☑ Solar ν_e disappear, large θ_{12} (2002)  SK, SNO
- ☑ Solar ν_e are converted to ν_μ+ν_τ (2002) SNO
- ☑ Reactor anti-ν_e disappear/reappear (2004) Kamland
- ☑ Accelerator ν_μ disappear (2006) MINOS
- ☑ Accelerator ν_μ converted to ν_τ (2010) OPERA
- ☑ Accelerator ν_μ converted to ν_e , θ_{13} hint (2011) T2K
- ☑ Reactor anti-ν_e disappear, θ_{13} meas. (2012) DB, Reno

The 6 observables in neutrino oscillations

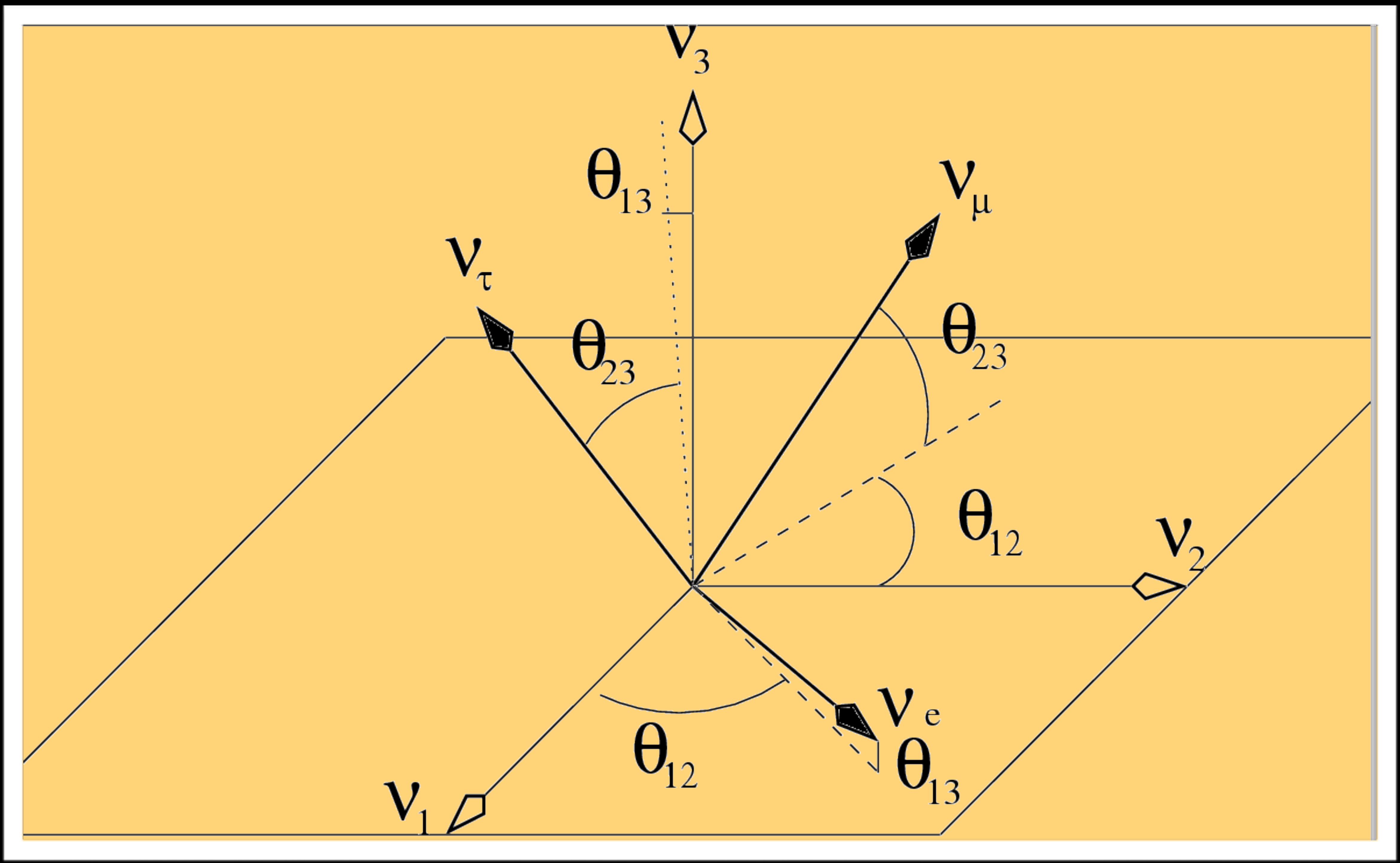
- ✳The atmospheric mass squared difference Δm^2_{31}
- ✳The solar mass squared difference Δm^2_{21}
- ✳The atmospheric angle θ_{23}
- ✳The solar angle θ_{12}
- ✳The reactor angle θ_{13}
- ✳The CP violating phase δ

2 Mass Squared Differences

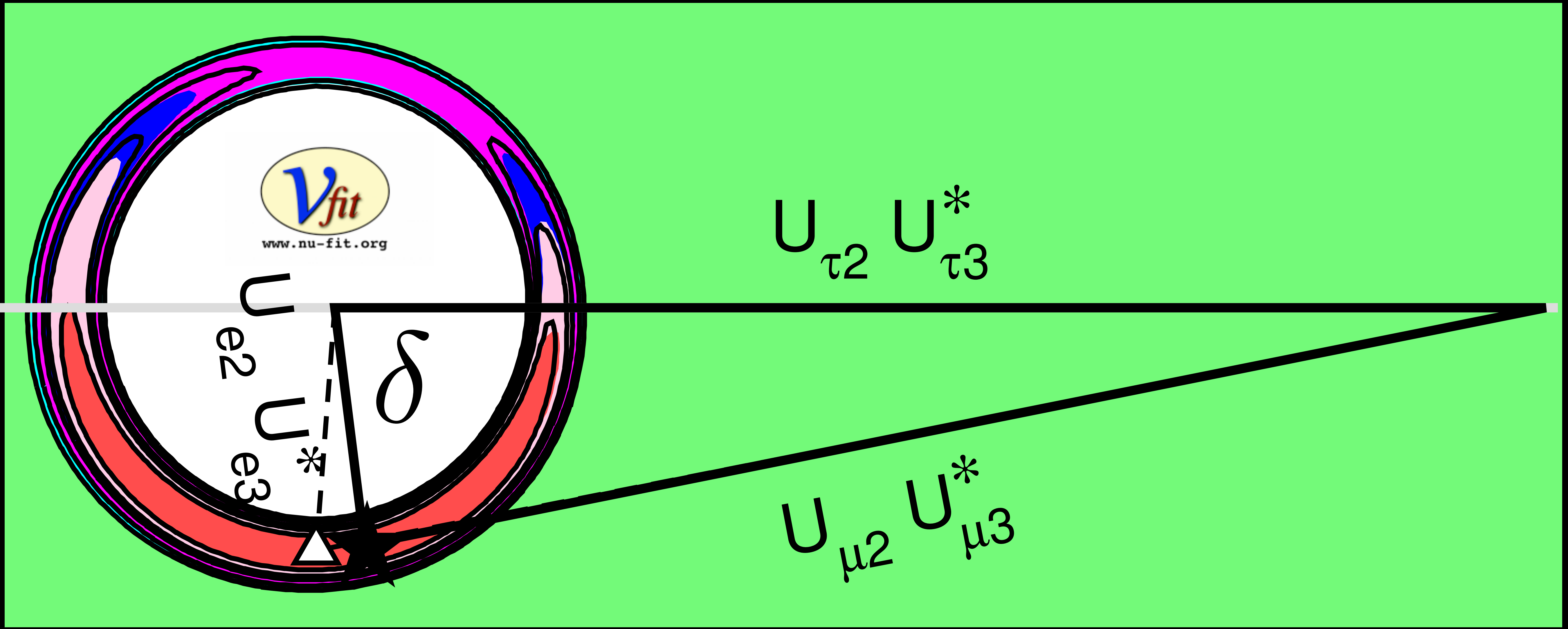


From review SFK 1701.04413

The 3 Lepton Mixing Angles



The oscillation observable CP Violating Phase



PMNS Lepton mixing matrix

Standard Model states

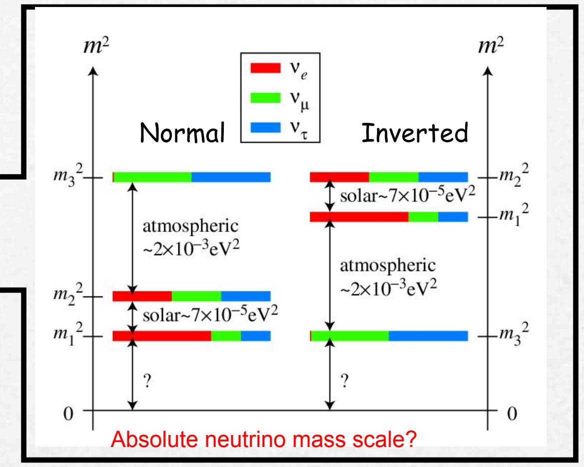
$$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L$$

PMNS matrix

Neutrino mass states

Pontecorvo
Maki
Nakagawa
Sakata

$$\begin{pmatrix}\nu_e \\ \nu_\mu \\ \nu_\tau\end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix} \begin{pmatrix}\nu_1 \\ \nu_2 \\ \nu_3\end{pmatrix}$$



$$U_{PMNS} = \begin{pmatrix}1 & 0 & 0 \\ 0 & c^l_{23} & s^l_{23} \\ 0 & -s^l_{23} & c^l_{23}\end{pmatrix} \begin{pmatrix} c^l_{13} & 0 & s^l_{13}e^{-i\delta^l} \\ 0 & 1 & 0 \\ -s^l_{13}e^{i\delta^l} & 0 & c^l_{13}\end{pmatrix} \begin{pmatrix} c^l_{12} & s^l_{12} & 0 \\ -s^l_{12} & c^l_{12} & 0 \\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 \\ 0 & \frac{\alpha_{21}}{2} & 0 \\ 0 & 0 & \frac{\alpha_{31}}{2}\end{pmatrix}$$

$s^l_{ij} = \sin(\theta^l_{ij})$

$c^l_{ij} = \cos(\theta^l_{ij})$

Atmospheric | Reactor | Solar | Majorana

Oscillation phase δ^l

Majorana phases α_{21}, α_{31}

3 masses + 3 angles + 3 phases =
9 new parameters for SM

PMNS and CKM mixing

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

For Majorana neutrinos → $\times \,\mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$

Same form for quarks and leptons
(but very different angles)

Quark vs Lepton mixings

	θ_{12}	θ_{23}	θ_{13}	δ
Quarks	$13°$ $\pm 0.1°$	$2.4°$ $\pm 0.1°$	$0.2°$ $\pm 0.05°$	$70°$ $\pm 5°$
Leptons	$34°$ $\pm 1°$	$45°$ $41° \pm 1°$ $50° \pm 1°$	$8.5°$ $\pm 0.15°$	$-90°$ $\pm 50°$

Latest NuFIT Fit 3.0

NuFIT 3.0 (2016)

	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 0.83$)		Any Ordering
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range	3σ range
$\sin^2\theta_{12}$	$0.306^{+0.012}_{-0.012}$	$0.271 \to 0.345$	$0.306^{+0.012}_{-0.012}$	$0.271 \to 0.345$	$0.271 \to 0.345$
$\theta_{12}/^\circ$	$33.56^{+0.77}_{-0.75}$	$31.38 \to 35.99$	$33.56^{+0.77}_{-0.75}$	$31.38 \to 35.99$	$31.38 \to 35.99$
$\sin^2\theta_{23}$	$0.441^{+0.027}_{-0.021}$	$0.385 \to 0.635$	$0.587^{+0.020}_{-0.024}$	$0.393 \to 0.640$	$0.385 \to 0.638$
$\theta_{23}/^\circ$	$41.6^{+1.5}_{-1.2}$	$38.4 \to 52.8$	$50.0^{+1.1}_{-1.4}$	$38.8 \to 53.1$	$38.4 \to 53.0$
$\sin^2\theta_{13}$	$0.02166^{+0.00075}_{-0.00075}$	$0.01934 \to 0.02392$	$0.02179^{+0.00076}_{-0.00076}$	$0.01953 \to 0.02408$	$0.01934 \to 0.02397$
$\theta_{13}/^\circ$	$8.46^{+0.15}_{-0.15}$	$7.99 \to 8.90$	$8.49^{+0.15}_{-0.15}$	$8.03 \to 8.93$	$7.99 \to 8.91$
$\delta_{\rm CP}/^\circ$	261^{+51}_{-59}	$0 \to 360$	277^{+40}_{-46}	$145 \to 391$	$0 \to 360$
$\frac{\Delta m^2_{21}}{10^{-5}\ \rm eV^2}$	$7.50^{+0.19}_{-0.17}$	$7.03 \to 8.09$	$7.50^{+0.19}_{-0.17}$	$7.03 \to 8.09$	$7.03 \to 8.09$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \rm eV^2}$	$+2.524^{+0.039}_{-0.040}$	$+2.407 \to +2.643$	$-2.514^{+0.038}_{-0.041}$	$-2.635 \to -2.399$	$\begin{bmatrix} +2.407 \to +2.643 \\ -2.629 \to -2.405 \end{bmatrix}$

Lisi et al 1703.04471

Parameter	Ordering	Best fit	1σ range	2σ range	3σ range
$\delta m^2/10^{-5}\ \mathrm{eV}^2$	NO, IO, Any	7.37	7.21 – 7.54	7.07 – 7.73	6.93 – 7.96
$\sin^2\theta_{12}/10^{-1}$	NO, IO, Any	2.97	2.81 – 3.14	2.65 – 3.34	2.50 – 3.54
$\lvert\Delta m^2\rvert/10^{-3}\ \mathrm{eV}^2$	NO	2.525	2.495 – 2.567	2.454 – 2.606	2.411 – 2.646
	IO	2.505	2.473 – 2.539	2.430 – 2.582	2.390 – 2.624
	Any	2.525	2.495 – 2.567	2.454 – 2.606	2.411 – 2.646
$\sin^2\theta_{13}/10^{-2}$	NO	2.15	2.08 – 2.22	1.99 – 2.31	1.90 – 2.40
	IO	2.16	2.07 – 2.24	1.98 – 2.33	1.90 – 2.42
	Any	2.15	2.08 – 2.22	1.99 – 2.31	1.90 – 2.40
$\sin^2\theta_{23}/10^{-1}$	NO	4.25	4.10 – 4.46	3.95 – 4.70	3.81 – 6.15
	IO	5.89	4.17 – 4.48 $\oplus$ 5.67 – 6.05	3.99 – 4.83 $\oplus$ 5.33 – 6.21	3.84 – 6.36
	Any	4.25	4.10 – 4.46	3.95 – 4.70 $\oplus$ 5.75 – 6.00	3.81 – 6.26
δ/π	NO	1.38	1.18 – 1.61	1.00 – 1.90	0 – 0.17 $\oplus$ 0.76 – 2
	IO	1.31	1.12 – 1.62	0.92 – 1.88	0 – 0.15 $\oplus$ 0.69 – 2
	Any	1.38	1.18 – 1.61	1.00 – 1.90	0 – 0.17 $\oplus$ 0.76 – 2

Valle et al 1708.01186

parameter	best fit $\pm$ 1σ	2σ range	3σ range
Δm^2_{21} $[10^{-5}\text{eV}^2]$	7.56$\pm$0.19	7.20–7.95	7.05–8.14
$\lvert\Delta m^2_{31}\rvert$ $[10^{-3}\text{eV}^2]$ (NO)	2.55$\pm$0.04	2.47–2.63	2.43–2.67
$\lvert\Delta m^2_{31}\rvert$ $[10^{-3}\text{eV}^2]$ (IO)	2.49$\pm$0.04	2.41–2.57	2.37-2.61
$\sin^2\theta_{12}/10^{-1}$	$3.21^{+0.18}_{-0.16}$	2.89–3.59	2.73–3.79
$\theta_{12}/^\circ$	$34.5^{+1.1}_{-1.0}$	32.5–36.8	31.5–38.0
$\sin^2\theta_{23}/10^{-1}$ (NO)	$4.30^{+0.20}_{-0.18}$ [a]	3.98–4.78 & 5.60–6.17	3.84–6.35
$\theta_{23}/^\circ$	41.0$\pm$1.1	39.1–43.7 & 48.4–51.8	38.3–52.8
$\sin^2\theta_{23}/10^{-1}$ (IO)	$5.96^{+0.17}_{-0.18}$ [b]	4.04–4.56 & 5.56–6.25	3.88–6.38
$\theta_{23}/^\circ$	50.5$\pm$1.0	39.5–42.5 & 48.2–52.2	38.5–53.0
$\sin^2\theta_{13}/10^{-2}$ (NO)	$2.155^{+0.090}_{-0.075}$	1.98–2.31	1.89–2.39
$\theta_{13}/^\circ$	$8.44^{+0.18}_{-0.15}$	8.1–8.7	7.9–8.9
$\sin^2\theta_{13}/10^{-2}$ (IO)	$2.140^{+0.082}_{-0.085}$	1.97–2.30	1.89–2.39
$\theta_{13}/^\circ$	$8.41^{+0.16}_{-0.17}$	8.0–8.7	7.9–8.9
δ/π (NO)	$1.40^{+0.31}_{-0.20}$	0.85–1.95	0.00–2.00
$\delta/^\circ$	252^{+56}_{-36}	153–351	0–360
δ/π (IO)	$1.44^{+0.26}_{-0.23}$	1.01–1.93	0.00–0.17 & 0.79–2.00
$\delta/^\circ$	$259\ ^{+47}_{-41}$	182–347	0–31 & 142–360

Open Questions


Is CP violated in the leptonic sector? (Probably)


Is the atmospheric angle in first or second octant?


Are neutrino masses NO or IO ? (NO preferred)

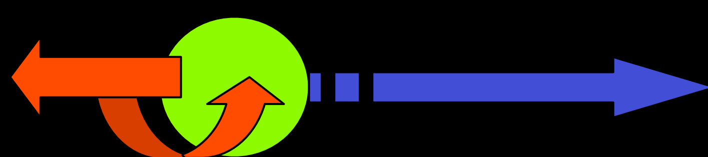

What is the lightest neutrino mass?


Are neutrino masses Dirac or Majorana?

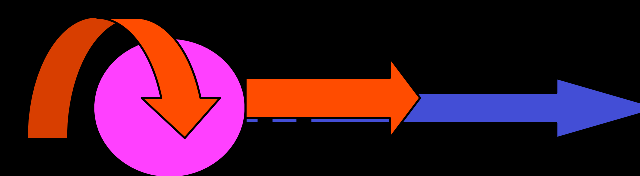
Dirac or Majorana?

SFK 0712.1750

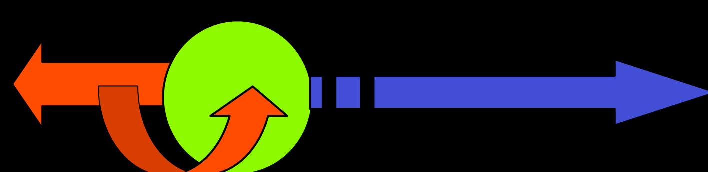
ν_L | **Dirac** | ν_R



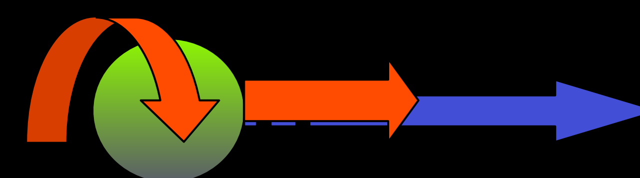
$$m_{\mathrm{D}}\bar{\nu}_L\nu_R$$



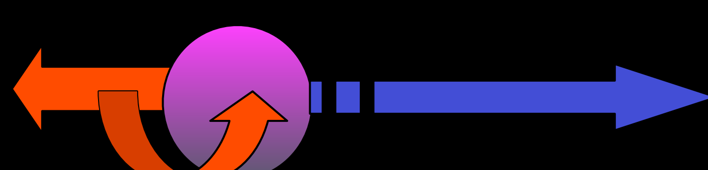
ν_L | **Majorana** | ν_L^c



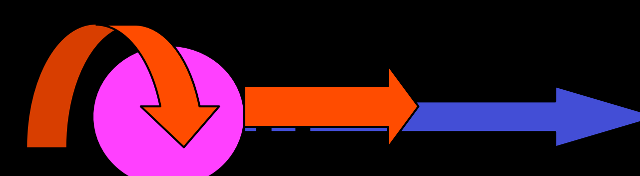
$$m_{\mathrm{L}}\bar{\nu}_L\nu_L^c$$



ν_R^c | **Majorana** | ν_R



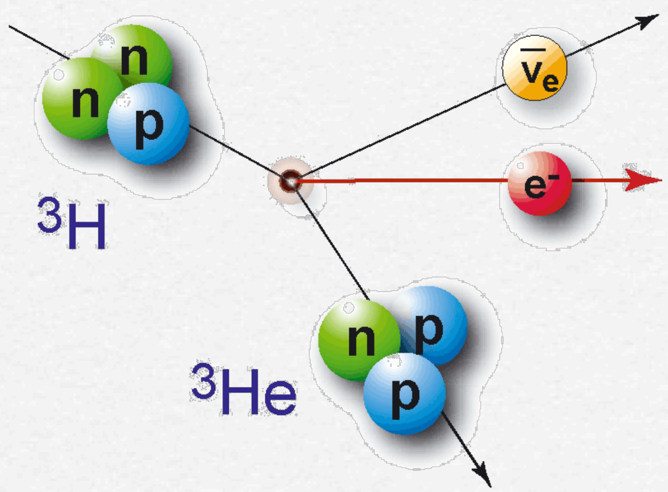
$$M_{\mathrm{R}}\bar{\nu}_R^c\nu_R$$



Experimental determination of neutrino mass

Majorana only
(no signal if Dirac)

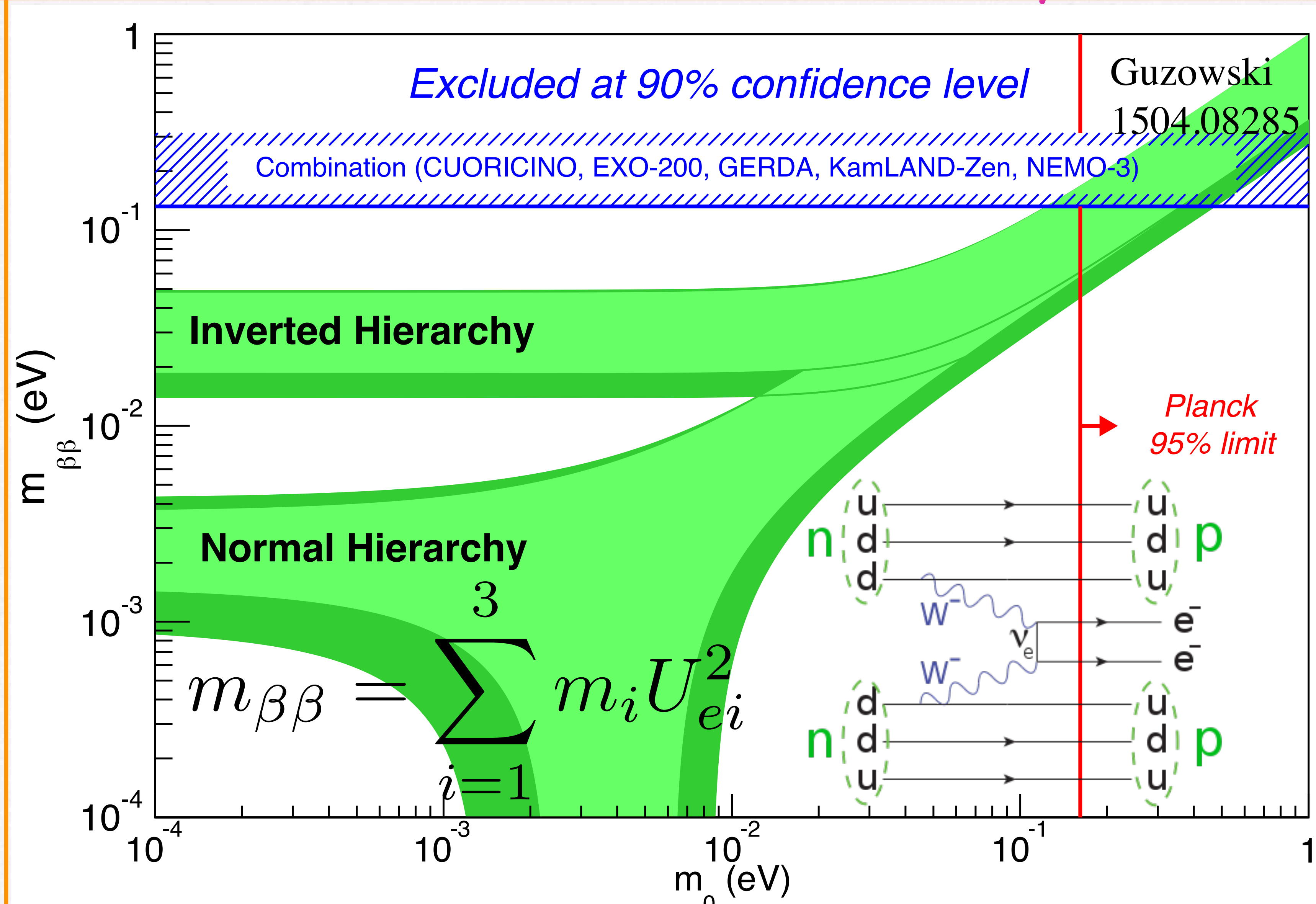
Tritium beta decay



$$m_{\nu_e}^2 = \sum_{i=1}^{3} |U_{ei}|^2 m_i^2$$

Present Mainz	< 2.2 eV
KATRIN	~0.35eV

Neutrinoless double beta decay



Majorana mass sum rules

King, Merle, Stuart 1307.2901

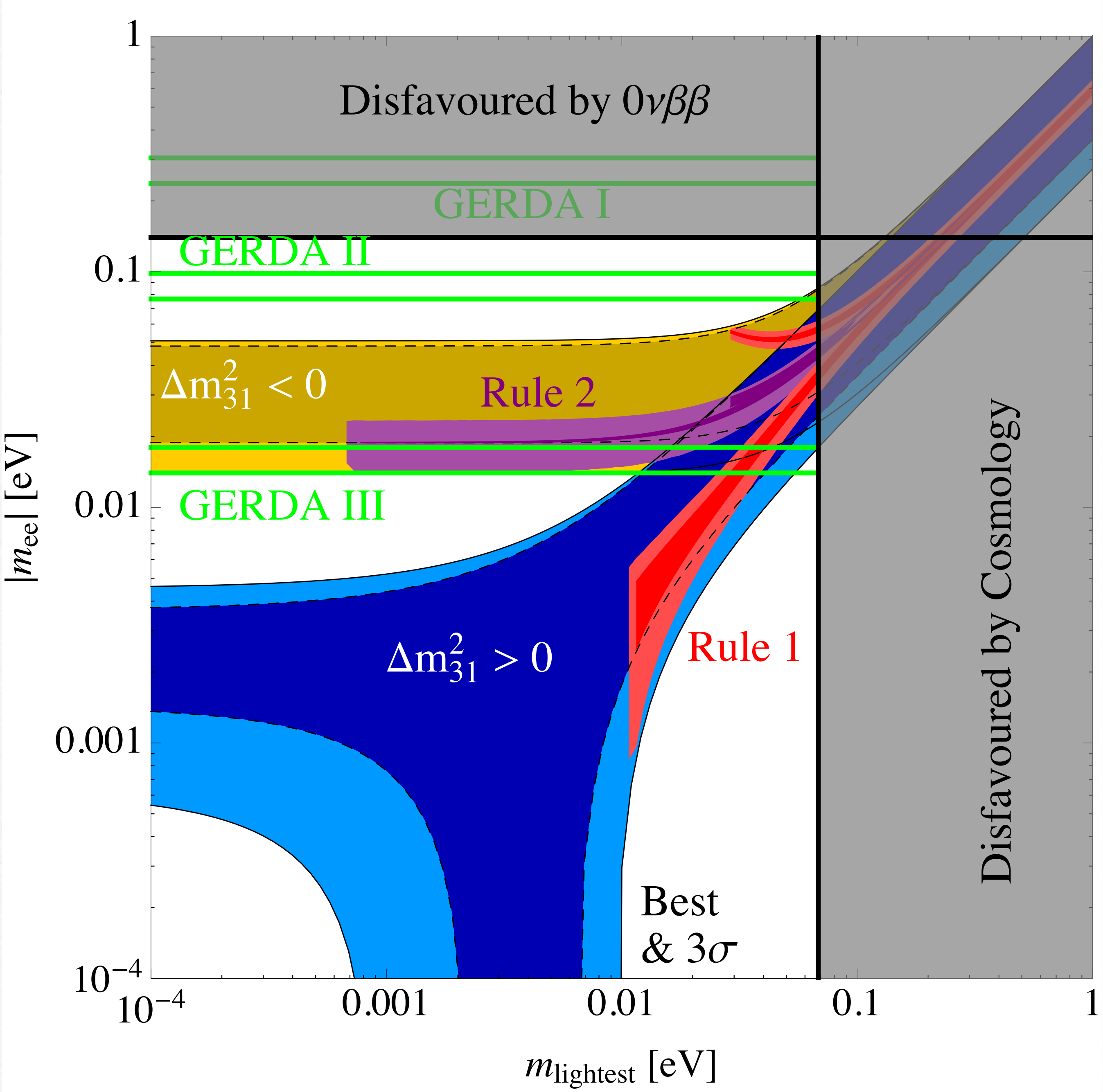
Rule 1

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{m_3}$$

Rule 2

$$m_1 + m_2 = m_3$$

Give restricted regions



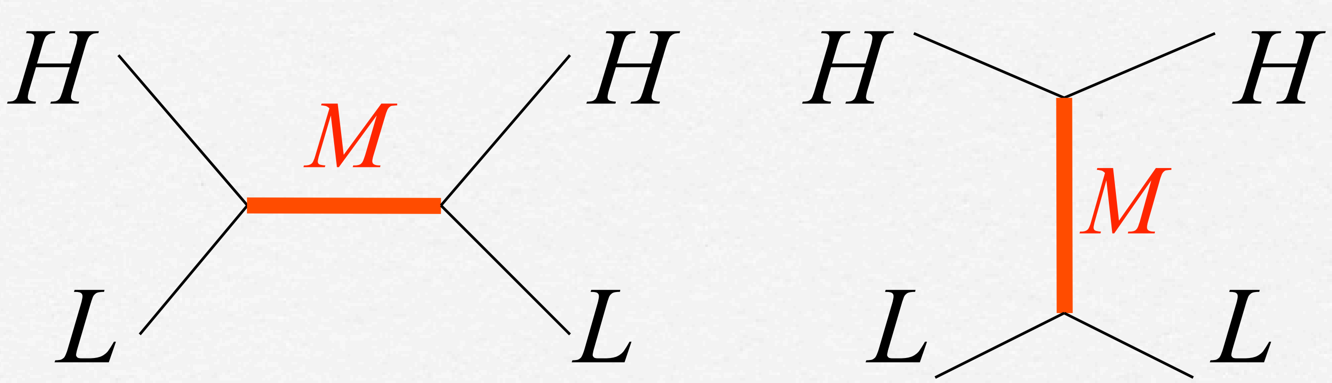
Is Majorana mass renormalisable?

Renormalisable
ΔL =2 operator $\lambda_\nu LL\Delta$ where Δ is light Higgs triplet with VEV < 8GeV from ρ parameter

Non-renormalisable
ΔL =2 operator $$\frac{\lambda_\nu}{M} LLHH = \frac{\lambda_\nu}{M}\left\langle H^0\right\rangle^2 \overline{\nu}_{eL}\nu^c_{eL}$$ Weinberg

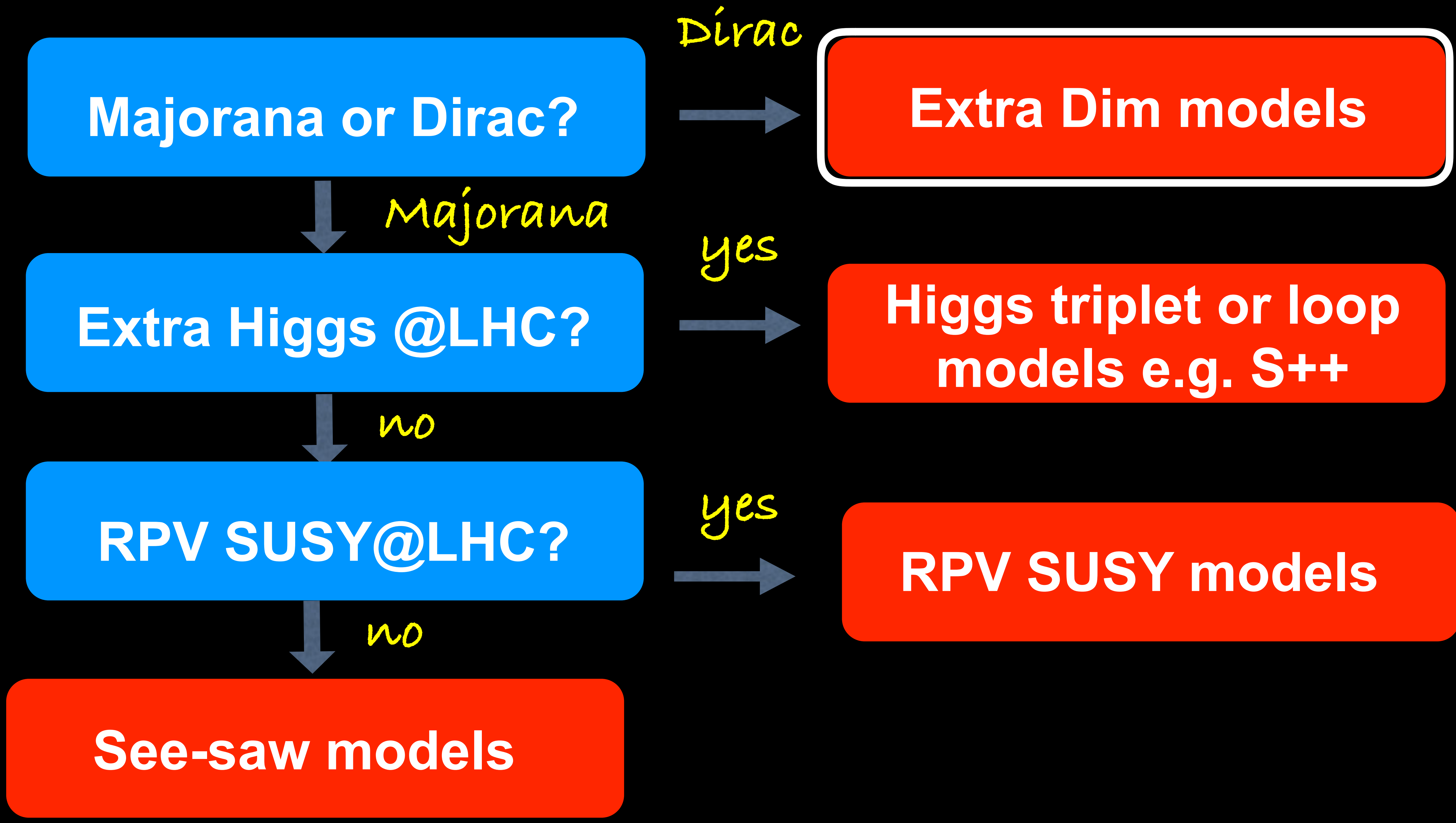
This is nice because it gives naturally small Majorana neutrino masses $m_{LL} \sim \langle H^0\rangle^2/M$ where M is some high energy scale

The high mass scale can be associated with some heavy particle of mass M being exchanged (can be singlet or triplet)



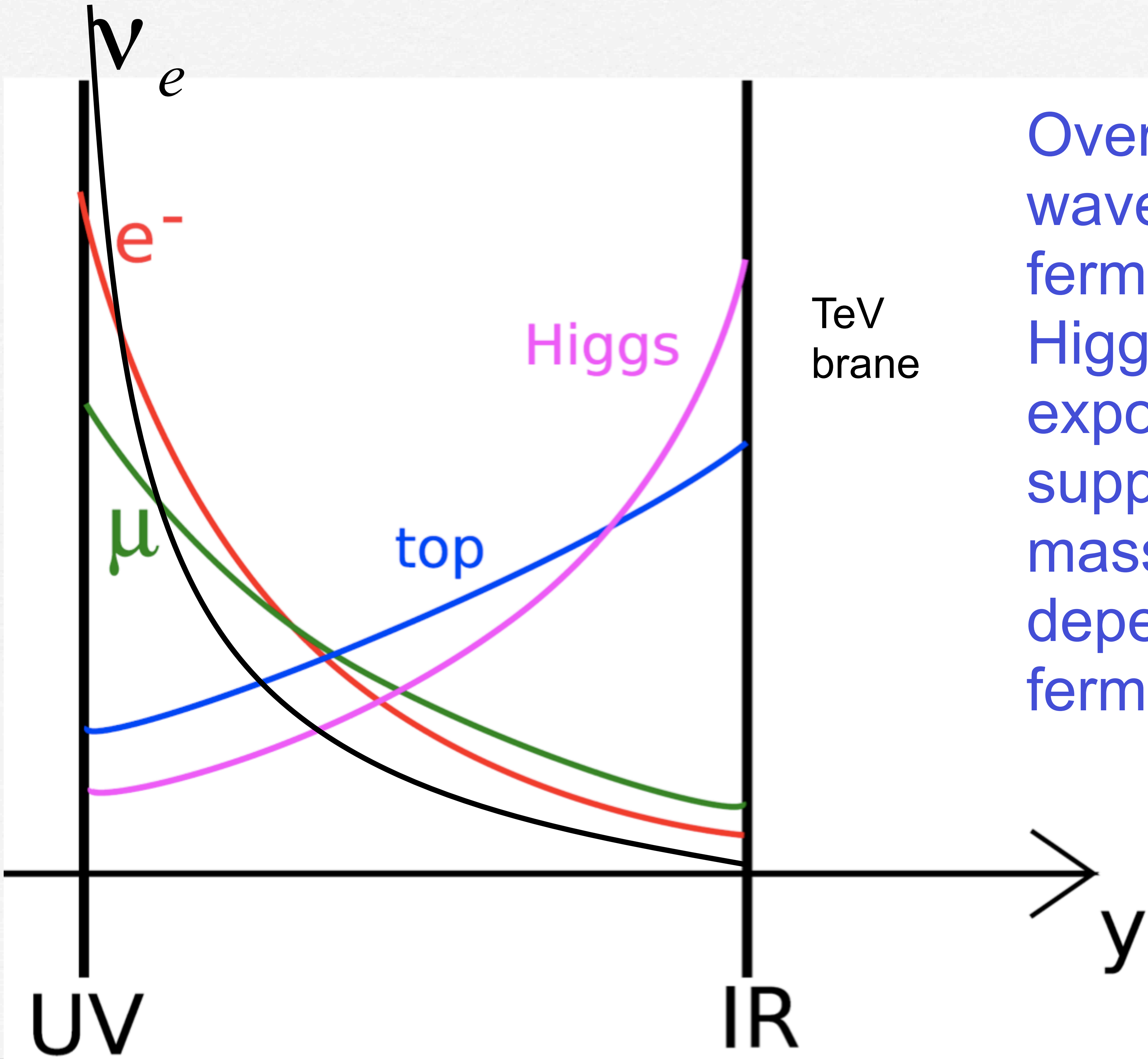
See-saw mechanisms

Roadmap of neutrino mass



Extra dimensions

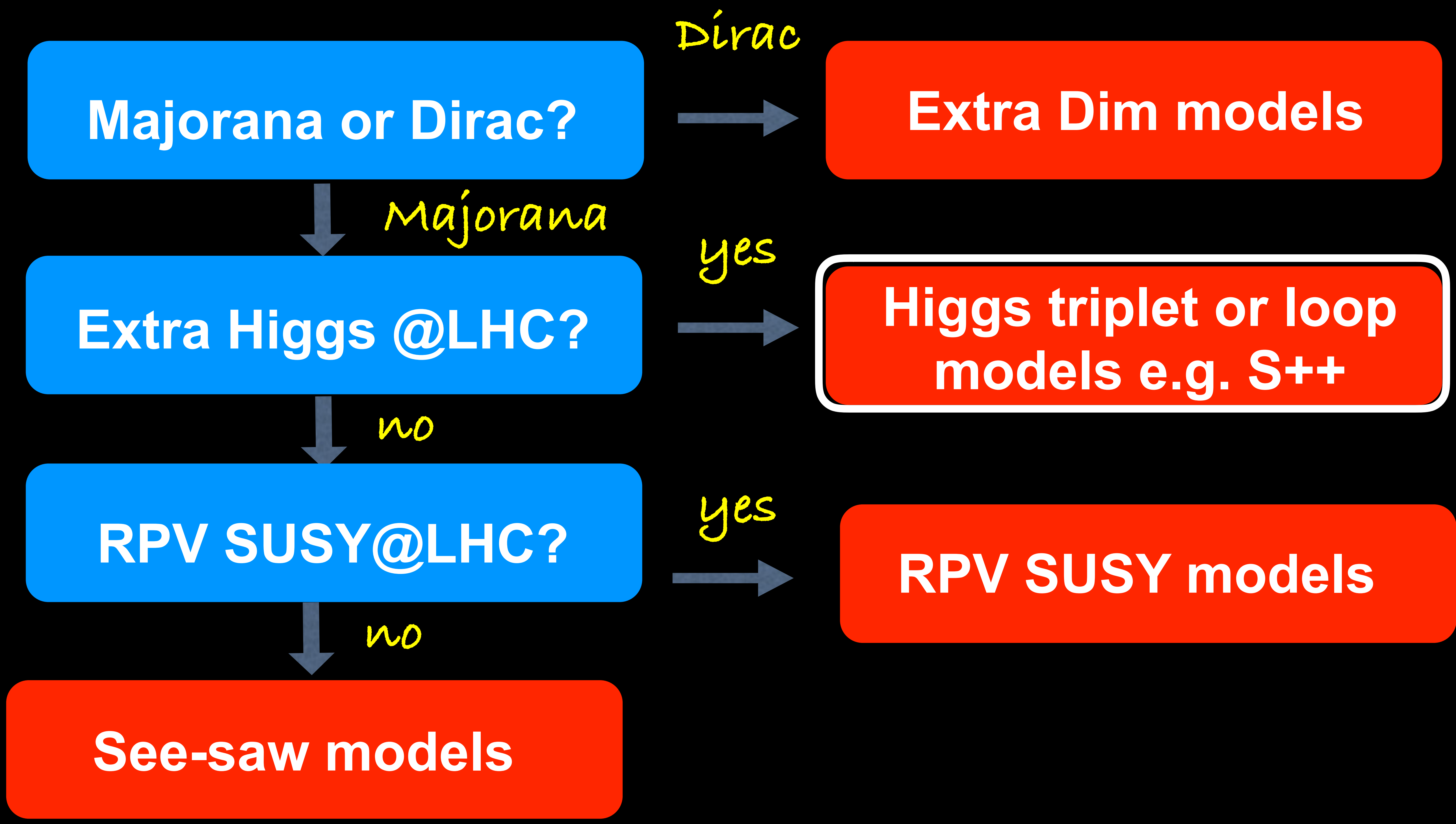
Planck brane



TeV brane

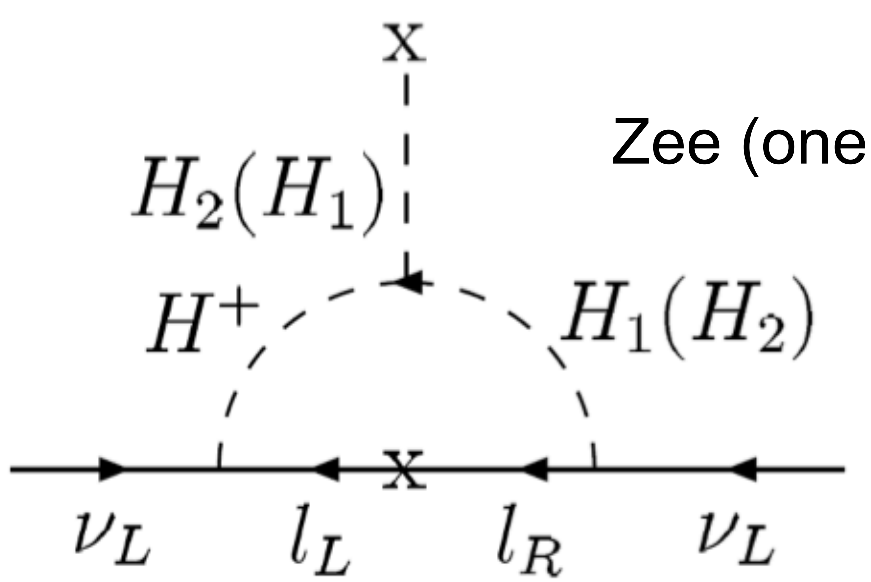
Overlap wavefunction of fermions with Higgs gives exponentially suppressed Dirac masses, depending on the fermion profiles

Roadmap of neutrino mass

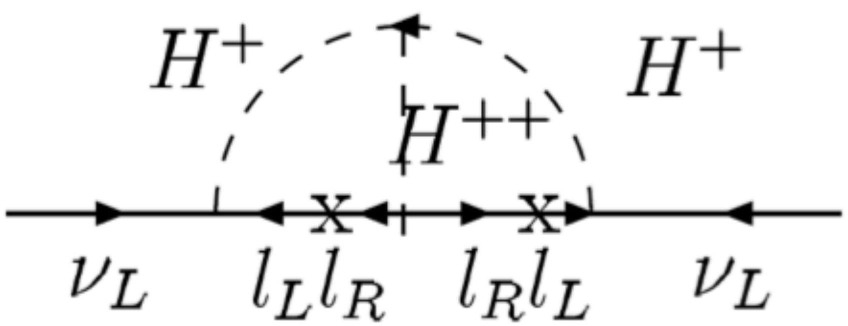


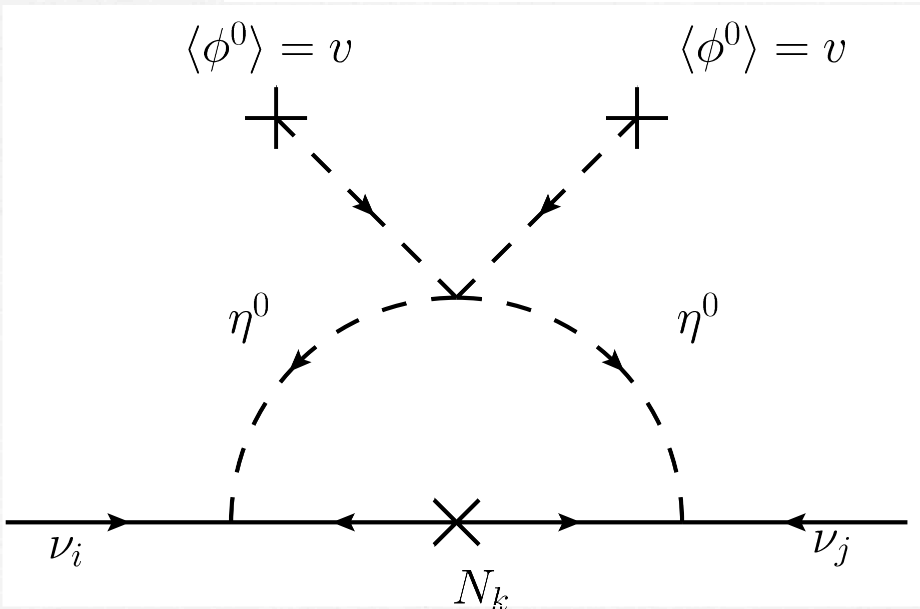
Loop Models of Neutrino Mass

Zee (one loop)

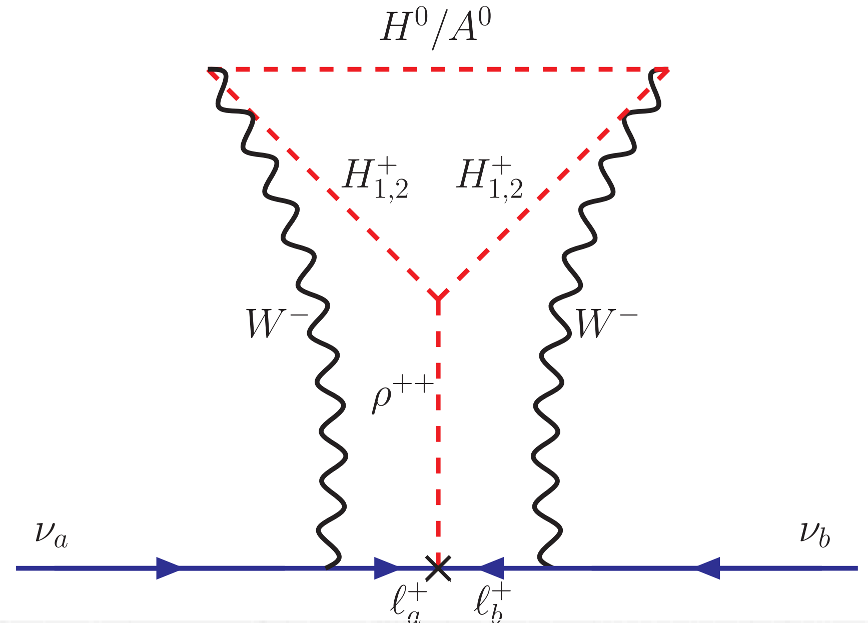


Babu (two loop)

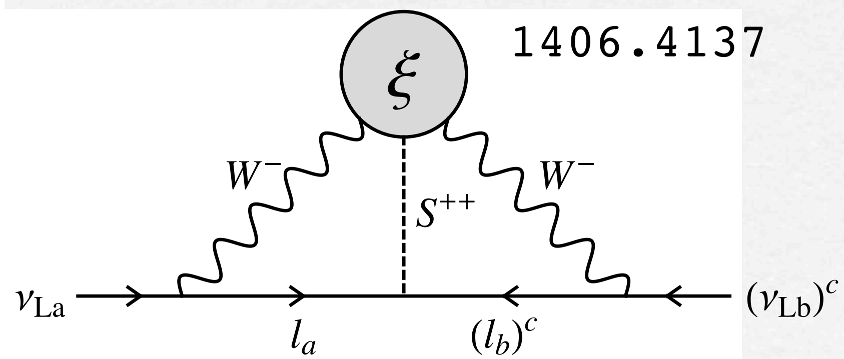




Scotogenic model

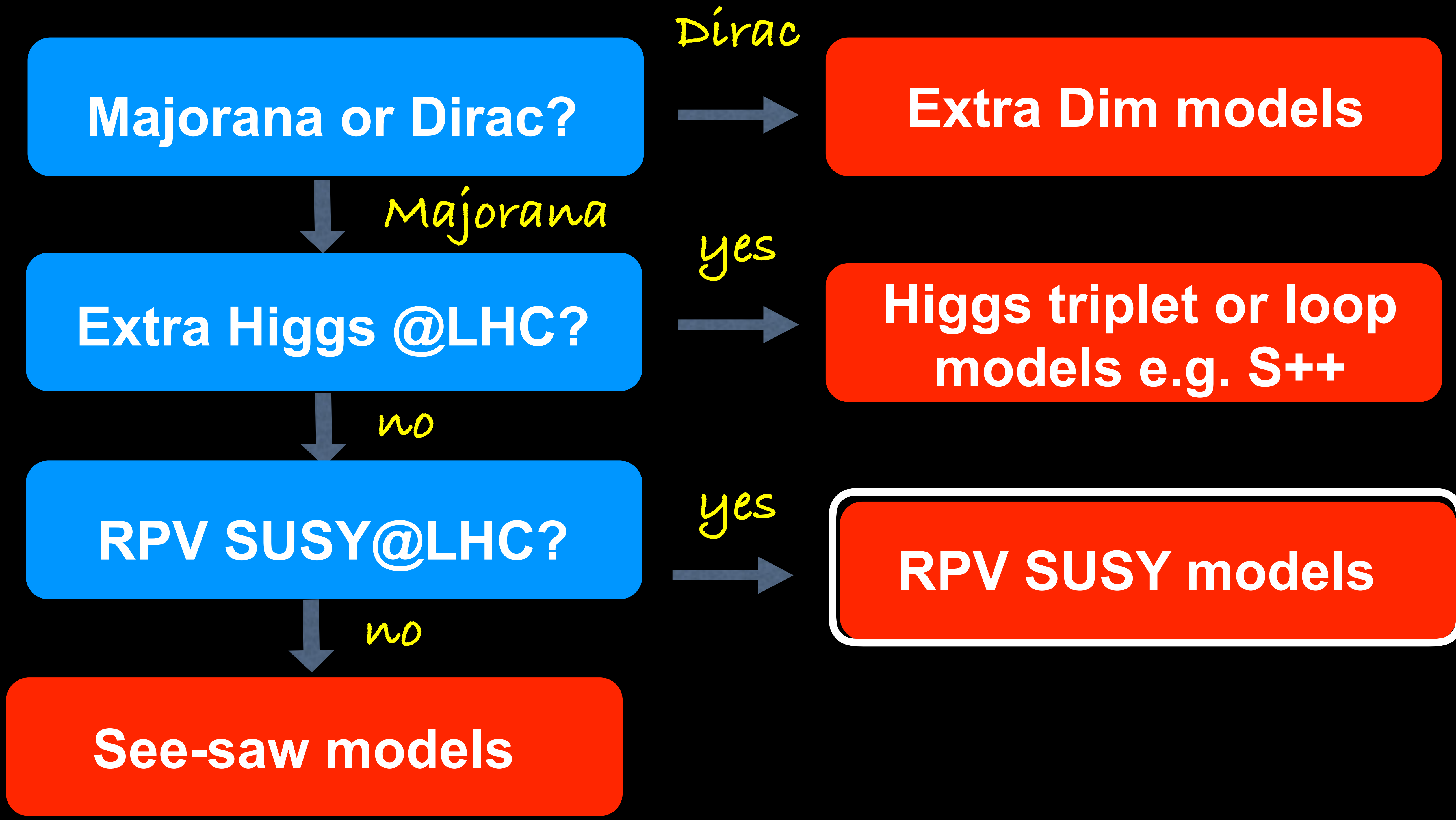


Cocktail model



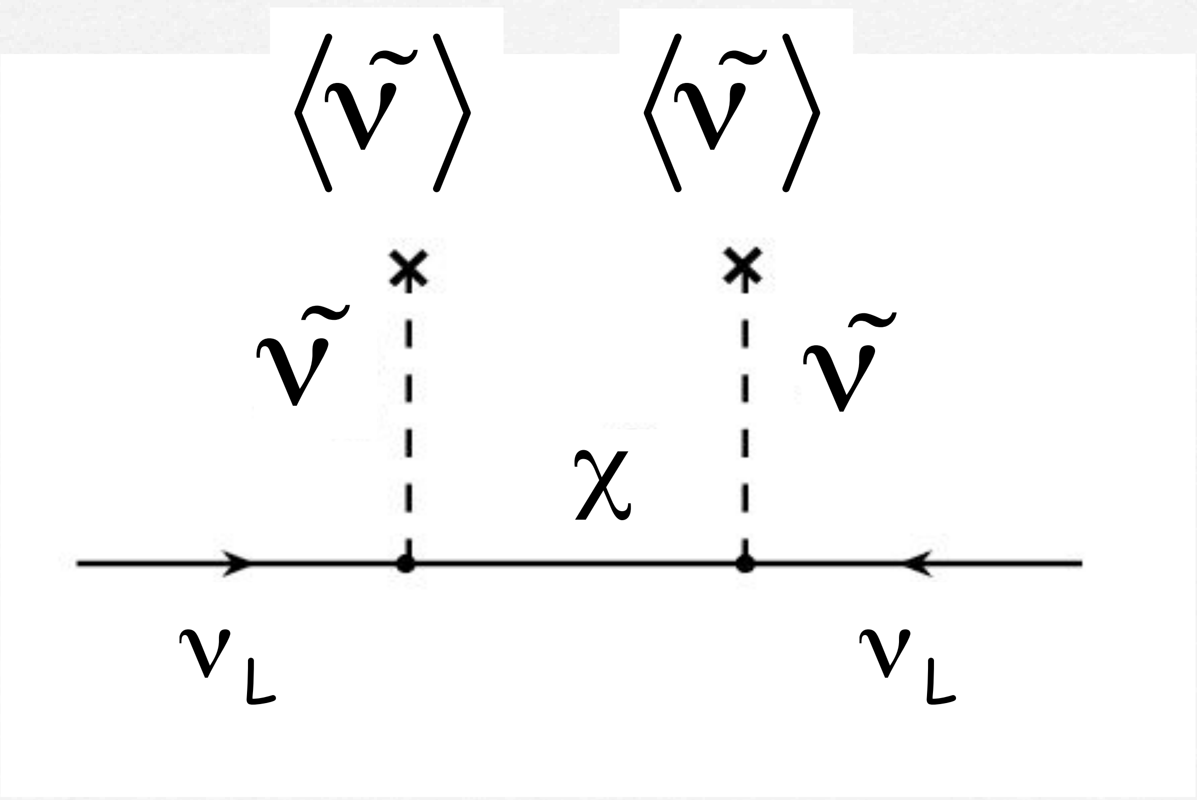
Effective theory

Roadmap of neutrino mass



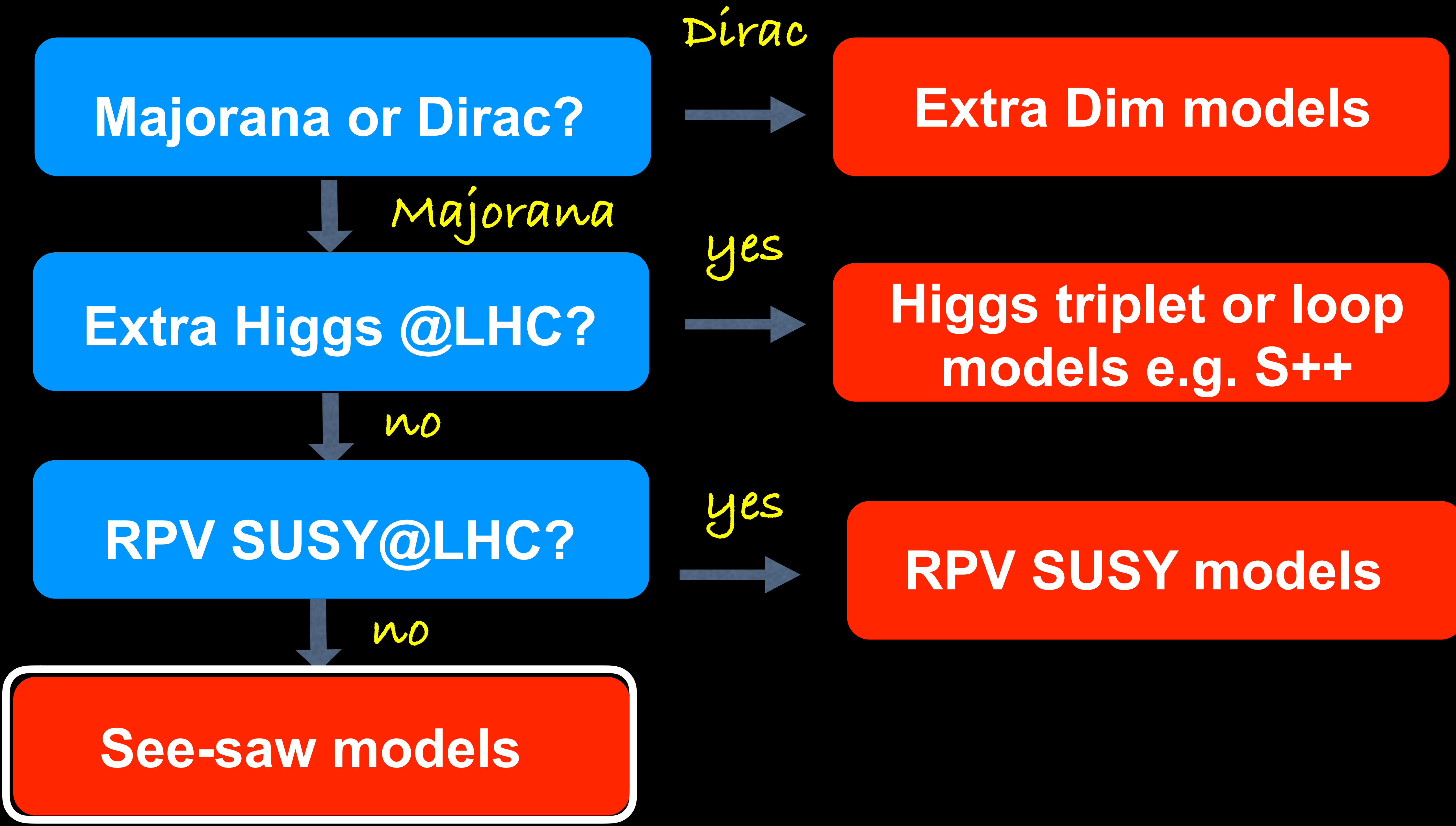
R-Parity Violating SUSY

- Majorana masses can be generated via RPV SUSY
- Scalar partners of lepton doublets (slepton doublets) have same quantum numbers as Higgs doublets
- If R-parity is violated then sneutrinos may get (small) VEVs inducing a mixing between neutrinos and neutralinos χ



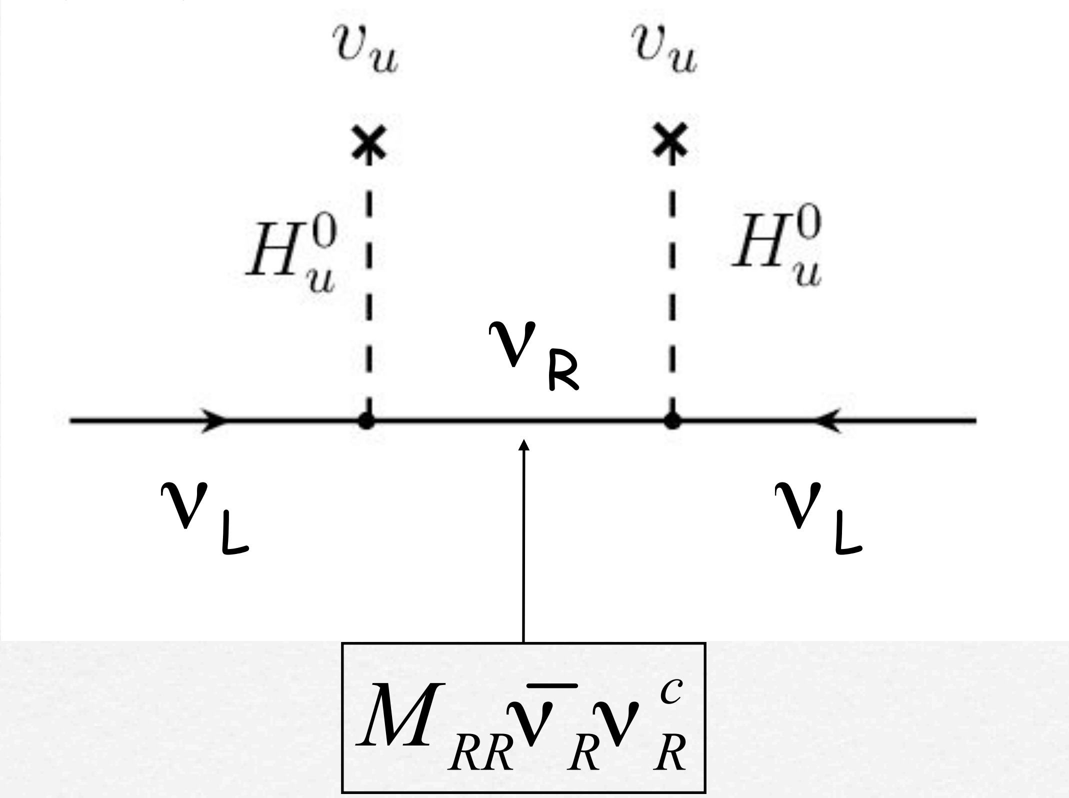
$$m_{LL}^{\nu} \approx \frac{\langle \tilde{\nu} \rangle^2}{M_\chi} \approx \frac{MeV^2}{TeV} \approx eV$$

Roadmap of neutrino mass



Type I see-saw mechanism

P. Minkowski (1977), Gell-Mann, Glashow, Mohapatra, Ramond, Senjanovic, Slanski, Yanagida (1979/1980), Schechter and Valle (1980)…

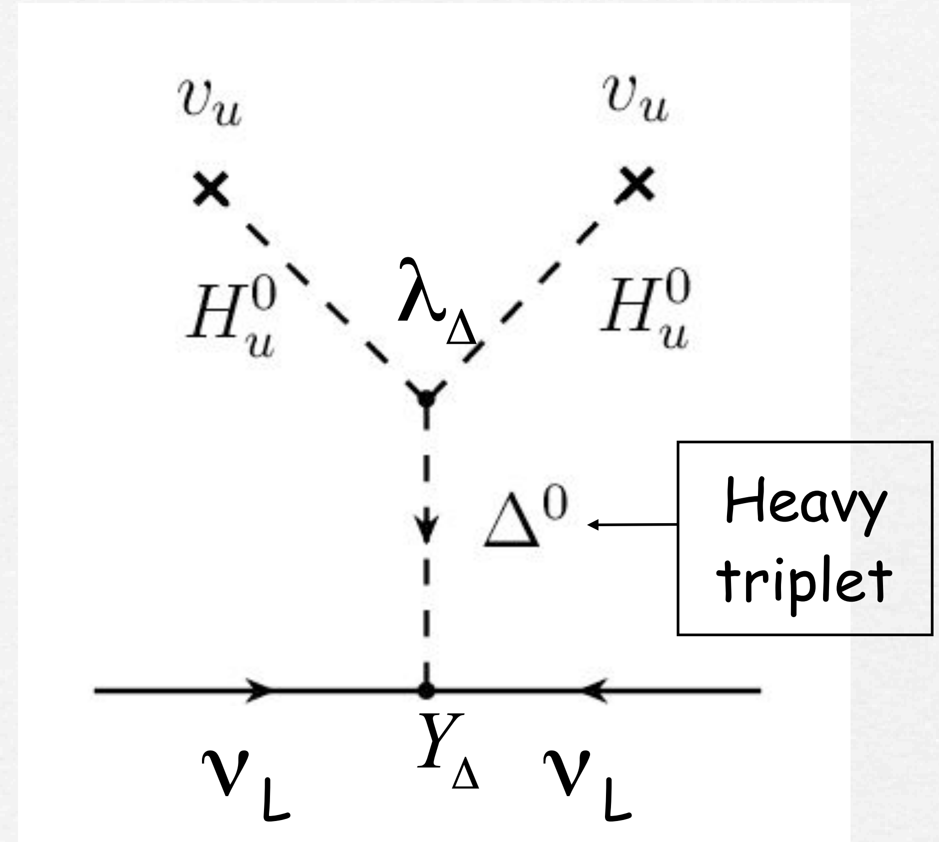


$$m_{LL}^{I} \approx -m_{LR} M_{RR}^{-1} m_{LR}^{T}$$

Type I

Type II see-saw mechanism (SUSY)

Lazarides, Magg, Mohapatra, Senjanovic, Shafi, Wetterich, Schechter and Valle…



$$m_{LL}^{II} \approx \lambda_\Delta Y_\Delta \frac{v_u^2}{M_\Delta}$$

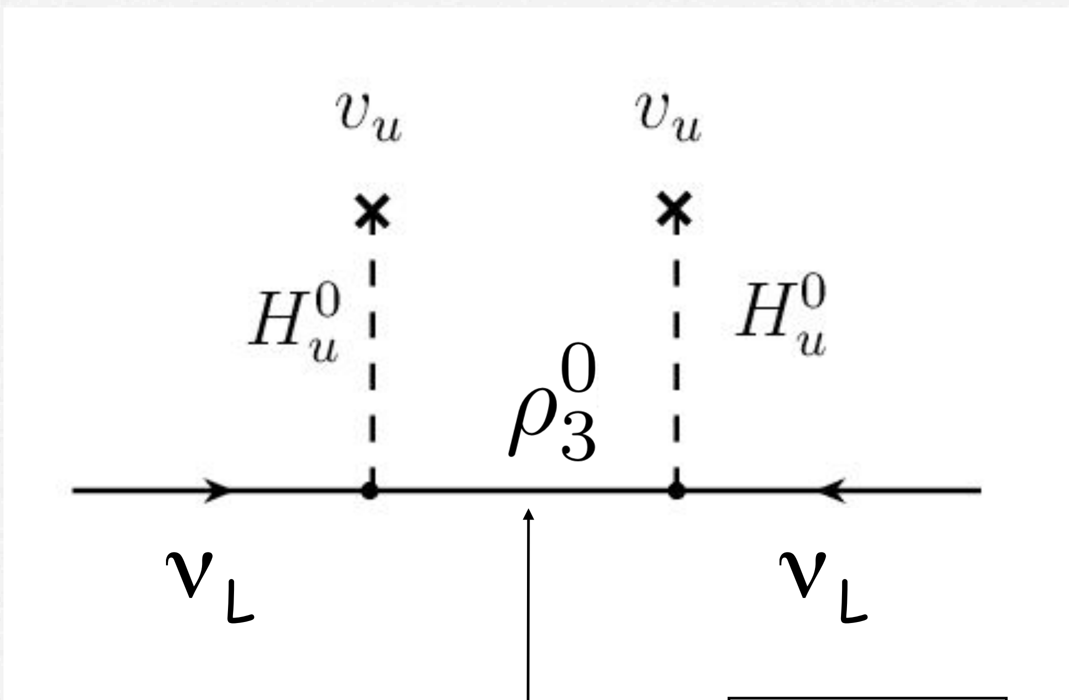
Type II

Type III see-saw mechanism

Foot, Lew, He, Joshi; Ma...

Supersymmetric adjoint SU(5)

Perez et al; Cooper, SFK, Luhn,...



$SU(2)_L$ fermion triplet

$M_\rho \rho\rho$

$$m_{LL}^{III} \approx -m_{LR} M_\rho^{-1} m_{LR}^T$$

Type III

See-saw w/extra singlets S

Inverse see-saw

Wyler, Wolferstein; Mohapatra, Valle

$$\begin{pmatrix} 0 & M_D & 0 \\ M_D^T & 0 & M \\ 0 & M^T & \mu \end{pmatrix}$$

M ≈ TeV➔LHC

$$M_\nu = M_D M^{T^{-1}} \mu M^{-1} M_D^T$$

Linear see-saw

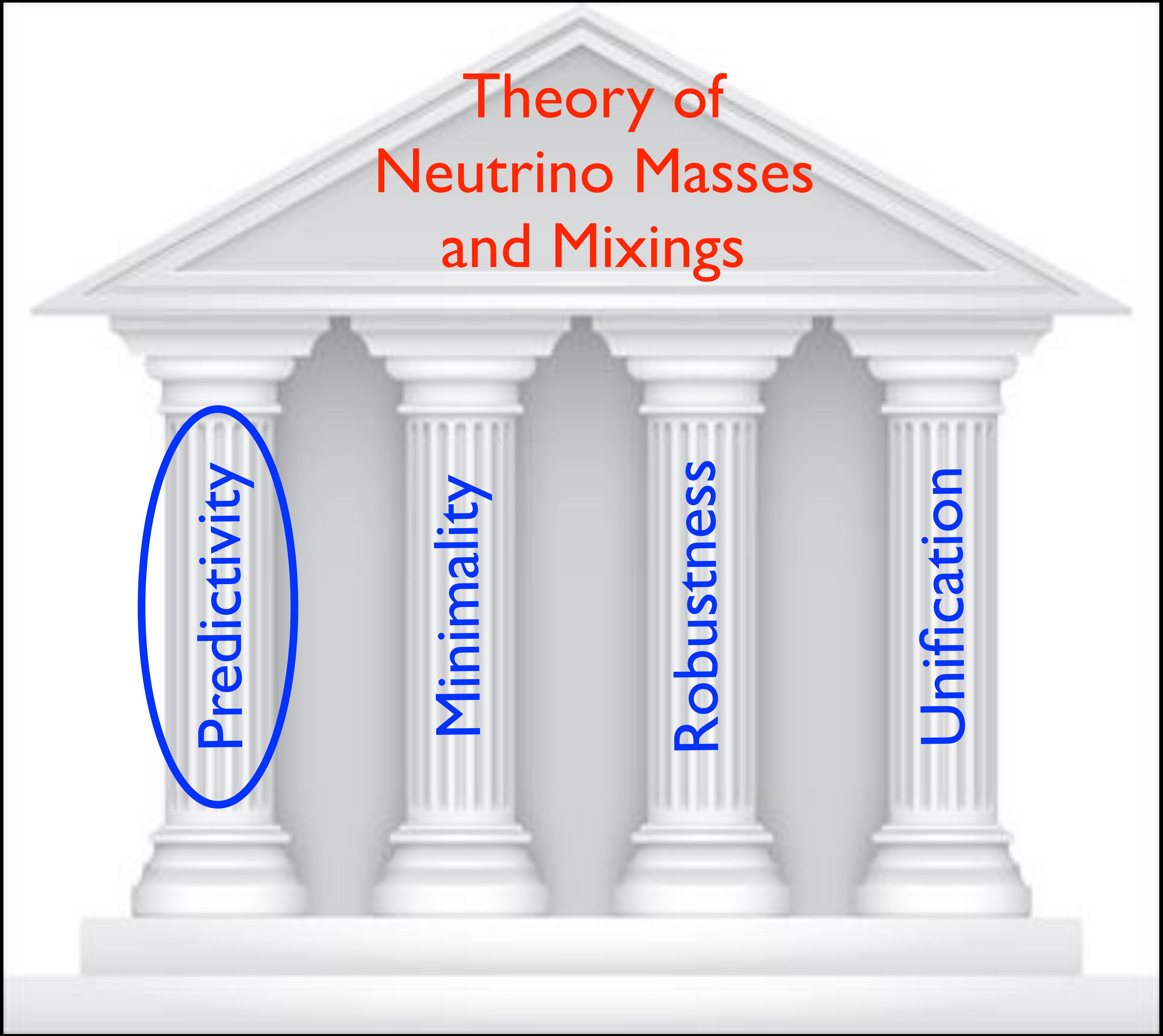
$$\begin{pmatrix} 0 & M_D & M_L \\ M_D^T & 0 & M \\ M_L^T & M^T & 0 \end{pmatrix}$$

Malinsky, Romao, Valle

$$M_\nu = M_D (M_L M^{-1})^T + (M_L M^{-1}) M_D{}^T$$

LFV predictions

From review SFK 1701.04413



Predictivity

Quark mixing matrix V_{CKM}

$$V^{U_L} Y^U_{LR} V^{U_R \dagger} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \quad V^{D_L} Y^D_{LR} V^{D_R \dagger} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$$

Defined as $V_{CKM} = V^{U_L} V^{D_L \dagger}$ 5 phases removed

Lepton mixing matrix U_{PMNS}

Light neutrino Majorana mass matrix ↓

$$V^{E_L} Y^E_{LR} V^{E_R \dagger} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \quad V^{\nu_L} m^\nu_{LL} V^{\nu_L T} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

Defined as $U_{PMNS} = V^{E_L} V^{\nu_L \dagger}$ 3 phases removed

Simple Lepton Mixing Ansatze

$$\theta_{13} = 0^o \quad \theta_{23} = 45^o$$

- Bimaximal

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} P \quad \theta_{12} = 45^o$$

- Tri-bimaximal

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P \quad \theta_{12} = 35.26^o$$

- Golden ratio

$$U_{GR} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \tan\theta_{12} = \frac{1}{\phi} \qquad \theta_{12} = 31.7^o$$

Tri-Bimaximal Deviations

0710.0530

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}$$

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s)\,, \qquad \sin\theta_{23} = \frac{1}{\sqrt{2}}(1+a)\,, \qquad \sin\theta_{13} = \frac{r}{\sqrt{2}}$$

s = solar

a = atmospheric

r = reactor

Tri-Bimaximal-Cabibbo Mixing

1205.0506, 1304.6264

TBC corresponds to

$$s = a = 0, \; r = \theta_C$$

$$\theta_{12} = 35^\circ \qquad \theta_{23} = 45^\circ \qquad \theta_{13} = \frac{\theta_C}{\sqrt{2}} = 9.2^o$$

$$U_{TBC} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\theta_C e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\theta_C \cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}\theta_C \cos\delta) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}}(1-\theta_C \cos\delta) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}\theta_C \cos\delta) & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Tri-maximal Mixing

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}$$

$$U_{\rm TM1} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix} \qquad U_{\rm TM2} \approx \begin{pmatrix} - & \frac{1}{\sqrt{3}} & - \\ - & \frac{1}{\sqrt{3}} & - \\ - & -\frac{1}{\sqrt{3}} & - \end{pmatrix}$$

- ☐ Tri-maximal 1 $s \approx 0,\ a \approx r\cos\delta$
- ☐ Tri-maximal 2 $s \approx 0,\ a \approx -\frac{1}{2}r\cos\delta$

Charged lepton corrections

1410.7573

$U_{PMNS} = V^{E_L} V^{\nu_L \dagger}$

Cabibbo-like

Tri-bimaximal

$$U_{\rm PMNS} = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \cdots & \cdots & \frac{s^e_{12}}{\sqrt{2}}e^{-i\delta^e_{12}} \\ \cdots & \cdots & \frac{c^e_{12}}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Solar sum rule

SFK, Antusch, ...

$$\theta_{12} \approx 35.26^o + \theta_{13}\cos\delta$$

$$\frac{|U_{\tau 1}|}{|U_{\tau 2}|} = \frac{|s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta}|}{|-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}|} = \frac{1}{\sqrt{2}} \rightarrow \cos\delta = \frac{t_{23}s^2_{12} + s^2_{13}c^2_{12}/t_{23} - \frac{1}{3}(t_{23} + s^2_{13}/t_{23})}{\sin 2\theta_{12}s_{13}}$$

Lepton Mixing Sum Rules

- Solar sum rules (from ch lepton corr)

Exact relations

Bimaximal $\theta_{12} \approx 45^o + \theta_{13} \cos\delta$ $\quad |U_{\tau 1}|/|U_{\tau 2}| = 1$

Tri-bimaximal $\theta_{12} \approx 35^o + \theta_{13} \cos\delta$ $\quad |U_{\tau 1}|/|U_{\tau 2}| = 1/\sqrt{2}$

Golden Ratio $\theta_{12} \approx 32^o + \theta_{13} \cos\delta$ $\quad |U_{\tau 1}|/|U_{\tau 2}| = 1/\varphi$

Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$

- Atm. sum rules (TM1 or TM2)

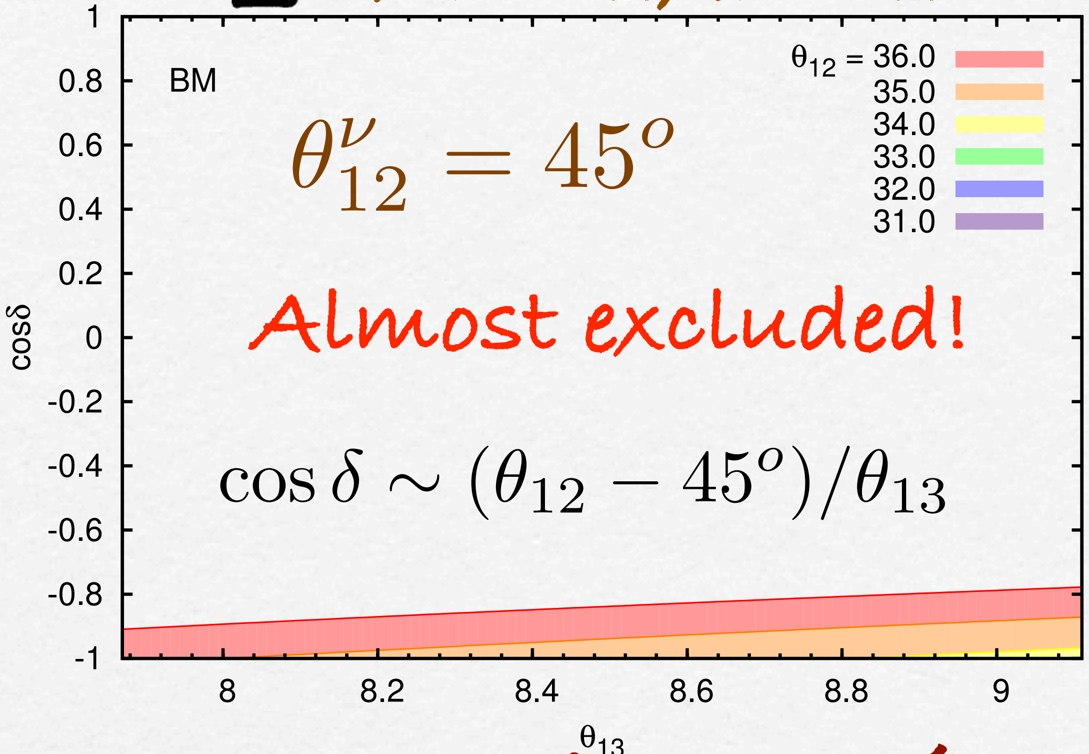
Trimaximal1 $\theta_{23} \approx 45^o + \sqrt{2}\theta_{13} \cos\delta$

$$|U_{e1}| = \sqrt{2/3}$$
$$|U_{\mu 1}| = |U_{\tau 1}| = \frac{1}{\sqrt{6}}$$

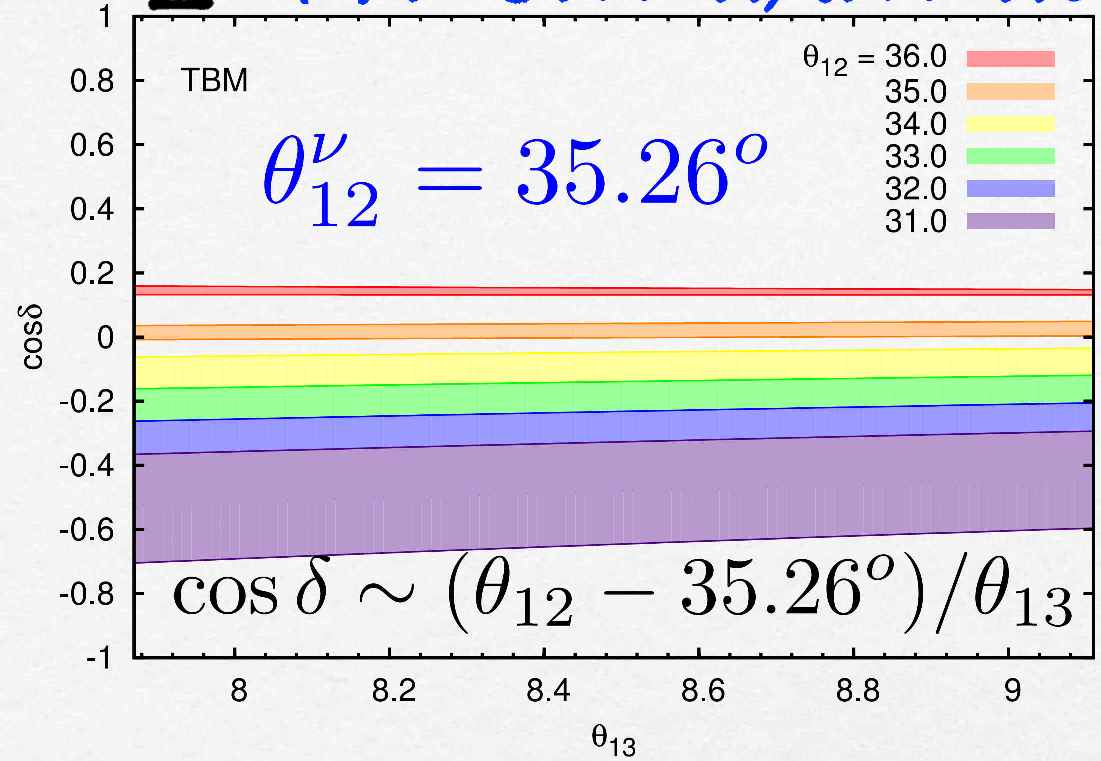
Trimaximal2 $\theta_{23} \approx 45^o - \frac{\theta_{13}}{\sqrt{2}} \cos\delta$

$$|U_{e2}| = |U_{\mu 2}| = |U_{\tau 2}| = \frac{1}{\sqrt{3}}.$$

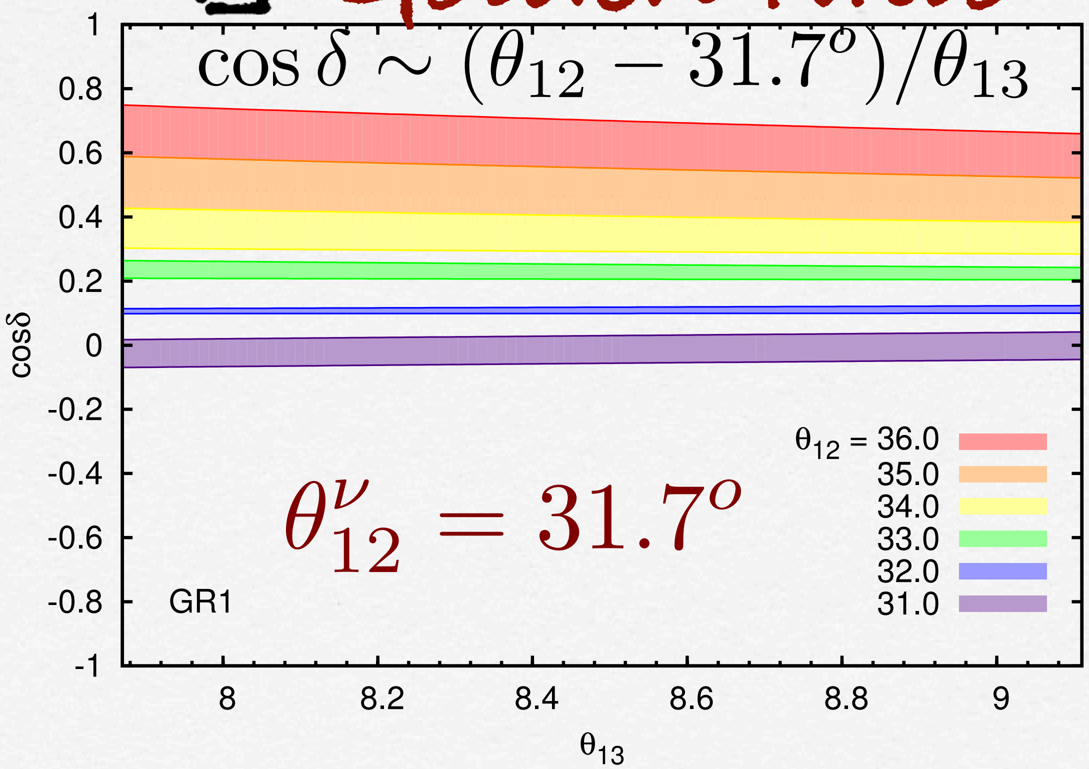
Bimaximal



Tri-bimaximal



Golden ratio



Solar Sum Rules

$$\cos\delta = \frac{t_{23}s_{12}^2 + s_{13}^2 c_{12}^2/t_{23} - s_{12}^{\nu 2}(t_{23} + s_{13}^2/t_{23})}{\sin 2\theta_{12} s_{13}}.$$

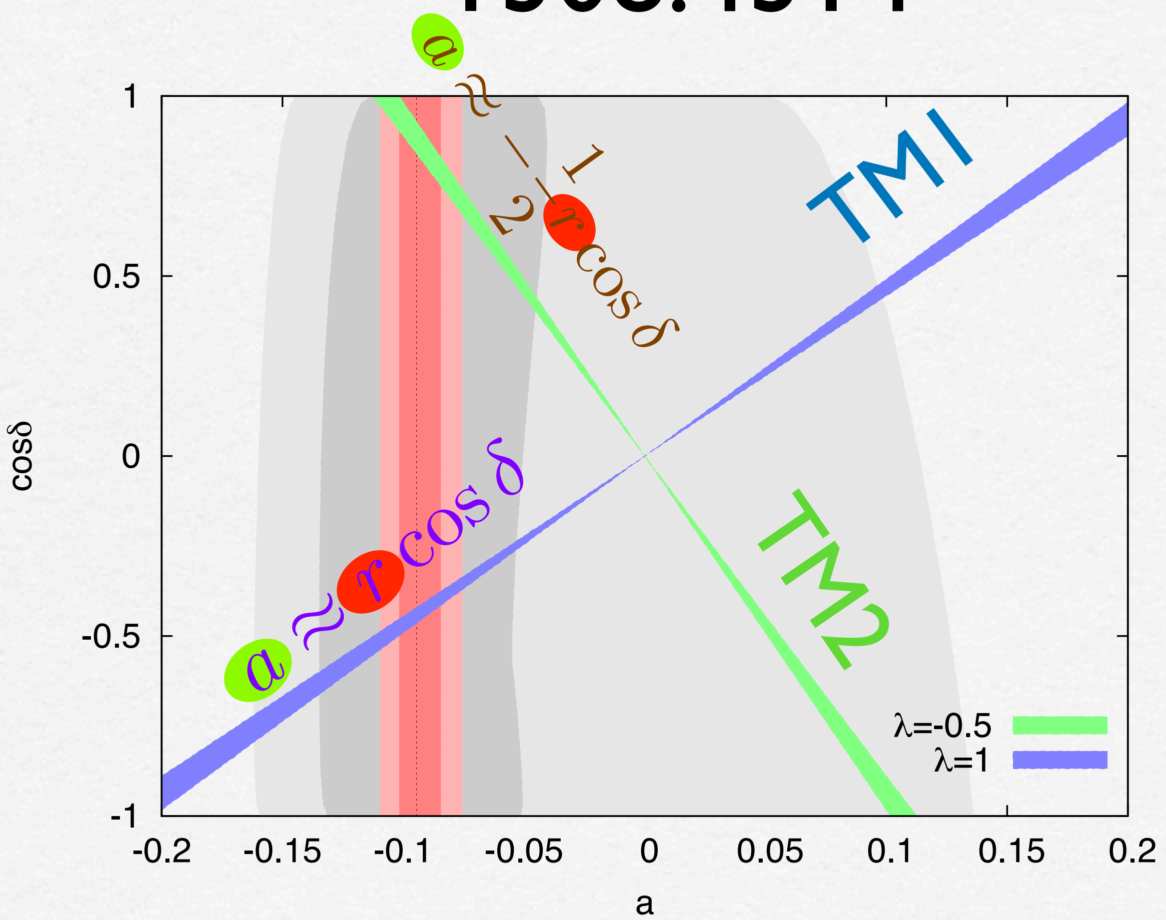
Ballett, SK, Luhn, Pascoli, Schmidt
1410.7573

useful approximation

$$\cos\delta \sim (\theta_{12} - \theta_{12}^{\nu})/\theta_{13}$$

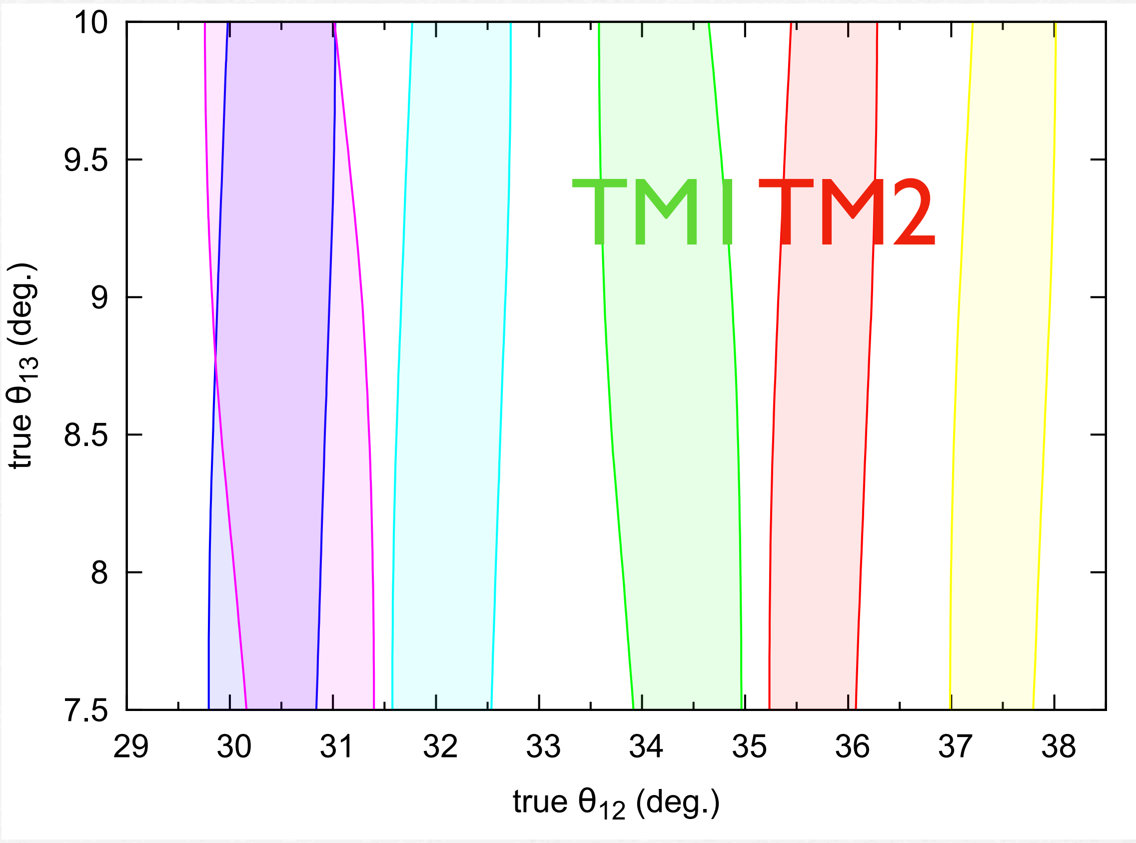
Atmospheric Sum Rules

1308.4314



$$\sin\theta_{23} = \frac{1+a}{\sqrt{2}}$$

1406.0308



5 sigma allowed regions after JUNO

Questions?

Tutorial Questions

1. The PMNS matrix for Dirac neutrinos is [1],

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}, \tag{1}$$

where $s_{13} = \sin\theta_{13}$, etc.

(a) Show that tri-bimaximal mixing defined by

$$s_{13} = 0,\ s_{12} = \frac{1}{\sqrt{3}},\ s_{23} = \frac{1}{\sqrt{2}}, \tag{2}$$

implies the tri-bimaximal (TB) mixing matrix,

$$U_{\mathrm{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3}$$

(b) Consider the reactor, solar and atmospheric parameters r,s,a which parameterise the deviations from tri-bimaximal mixing [2],

$$s_{13} = \frac{r}{\sqrt{2}}, \;\; s_{12} = \frac{(1+s)}{\sqrt{3}}, \;\; s_{23} = \frac{(1+a)}{\sqrt{2}}. \tag{4}$$

By expanding the PMNS mixing matrix to first order in the small parameters r, s, a, it is possible to show (although you do not need to do this) that,

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}. \tag{5}$$

Verify that for TB mixing $r = s = a = 0$, the mixing matrix reduces to U_{TB}.

Show that, for $s \approx 0, \;\; a \approx r\cos\delta$, the first column of the mixing matrix approximately corresponds to that of TB mixing (TM1 mixing).

Similarly show that for $s \approx 0, \;\; a \approx -(r/2)\cos\delta$, the second column of the mixing matrix approximately corresponds to that of TB mixing (TM2 mixing).

(c) Show that the relations $a \approx r\cos\delta$ and $a \approx -(r/2)\cos\delta$ imply the approximate "atmospheric sum rules" of the form,

$$\theta_{23} - 45^\circ \approx C \times \theta_{13}\cos\delta \tag{6}$$

and find the constant C in each case. [**Hint**: take the sine of both sides of the Eq.6, assuming $\sin\theta_{13} \approx \theta_{13}$, then expand $\sin(\theta_{23} - 45^\circ)$ and use definitions of r, a.]

Then discuss how well these so called "atmospheric sum rules" are satisfied by current data on the atmospheric and reactor mixing angles and how future precision measurements of these angles will fix the CP violating phase δ [3].

(d) If the charged lepton mixing matrix has a Cabibbo-like mixing angle [1],

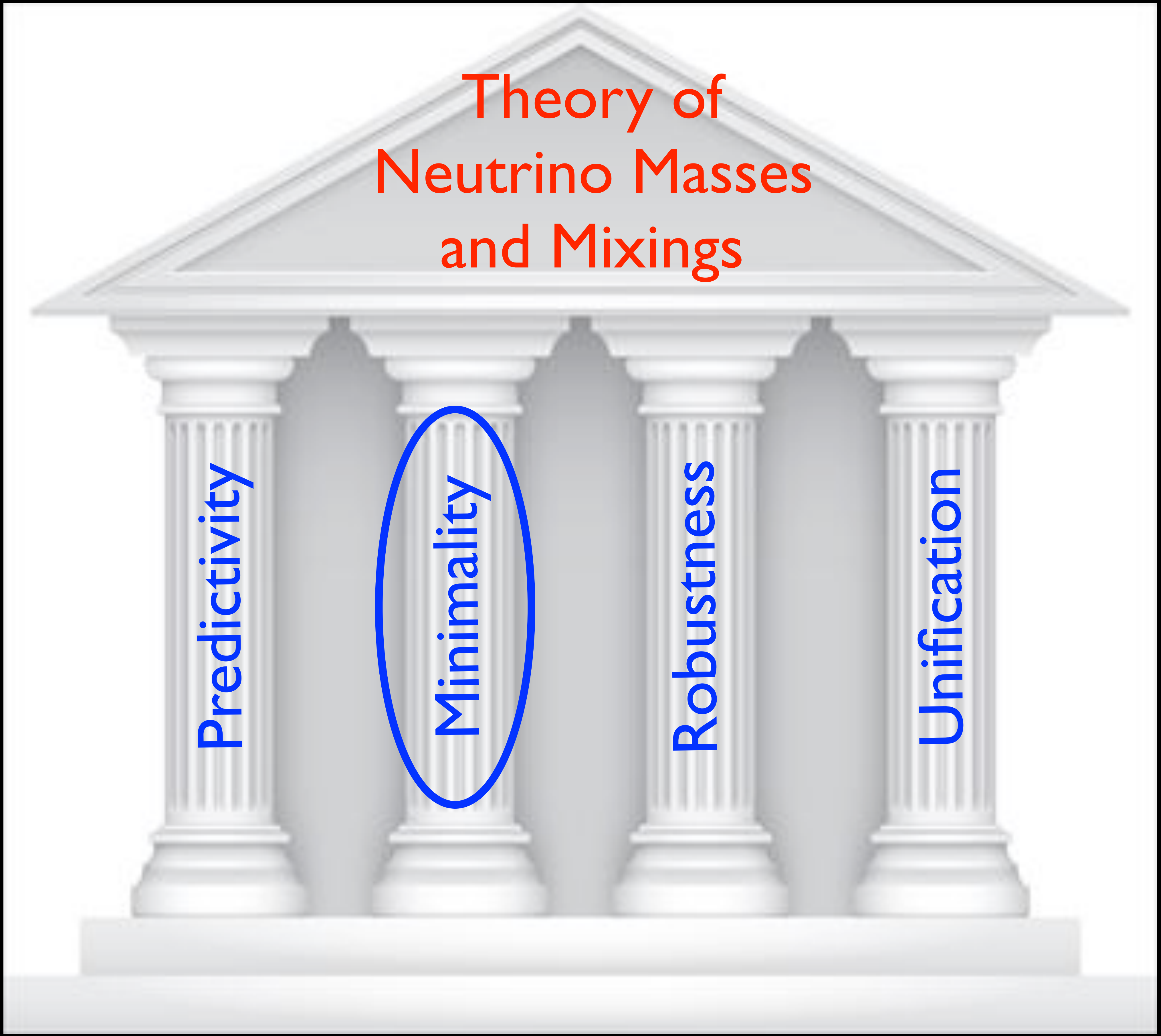
$$U_e = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

calculate the (1,3), (3,1) and (3,3) elements of PMNS matrix $U = U_e U_{\rm TB}$ (you don't need to calculate the whole matrix). Comparing the absolute value of the (1,3) element to that of the standard parameterisation of the PMS matrix, find s_{13} in terms of s^e_{12} and show that choosing $\theta^e_{12} = \theta_C \approx 13^\circ$ (the Cabibbo angle) gives a reasonable value for the reactor angle [7]. Comparing the absolute value of the (3,1) and (3,3) elements to that of the standard parameterisation of the PMS matrix, find relations between PMNS parameters. By combining and expanding these relations show that they lead to the approximate "solar sum rule",

$$\theta_{12} - 35^\circ \approx \theta_{13}\cos\delta, \tag{12}$$

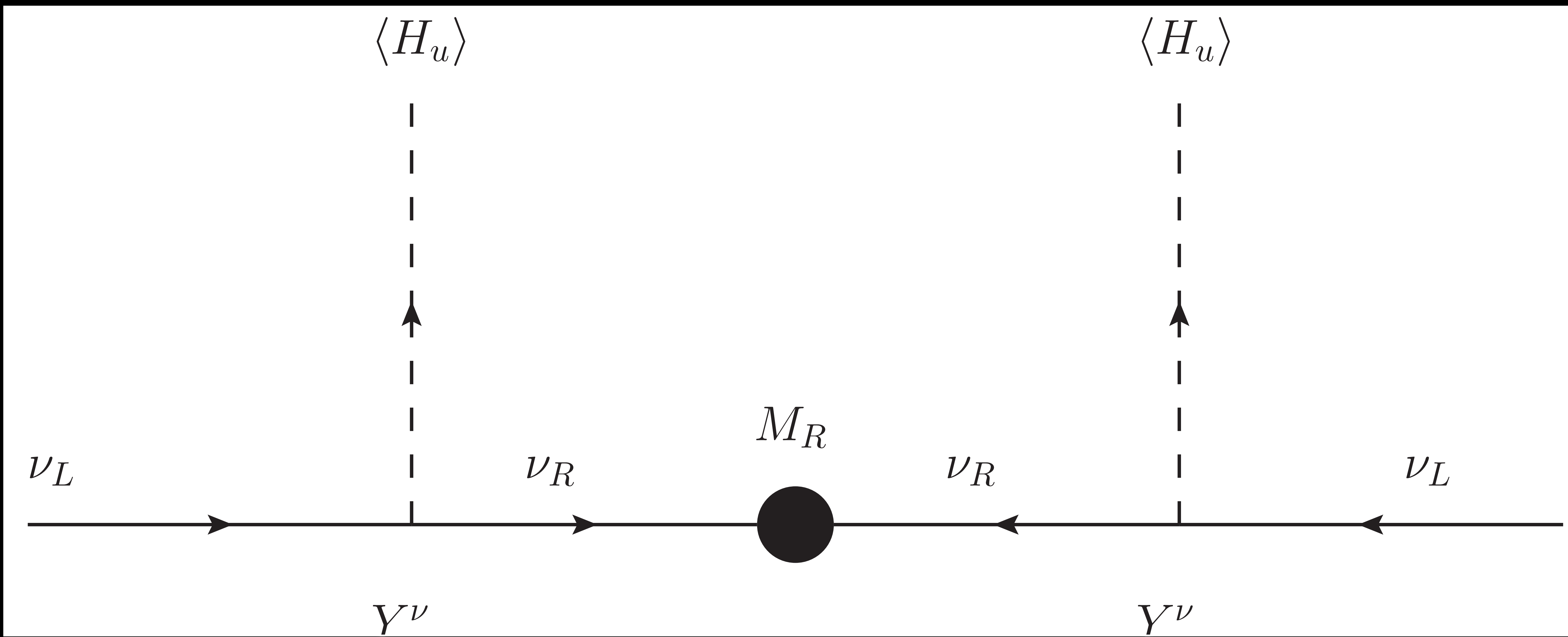
[**Hint**: take the sine of both sides of the Eq.12, assuming $\sin\theta_{13} \approx \theta_{13}$ as well as $\sin 35^\circ \approx 1/\sqrt{3}$.] Discuss the resulting prediction for the CP phase δ [7].

From review SFK 1701.04413



Minimality

Minimal Type I seesaw



Seesaw mechanism

Minkowski; Yanagida;
Gell-Mann, Ramond,
Slansky; Glashow;
Mohapatra, Senjanivic;
Schechter, Valle;...

$$m_{\mathrm{L}} = 0$$

$$\begin{pmatrix} \bar{\nu}_L & \bar{\nu}_R^c \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix} \begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix}$$

One family

Eigenvalue equation for m_ν (up to phases)

$$m_D^2 = m_\nu M_R + m_\nu^2$$

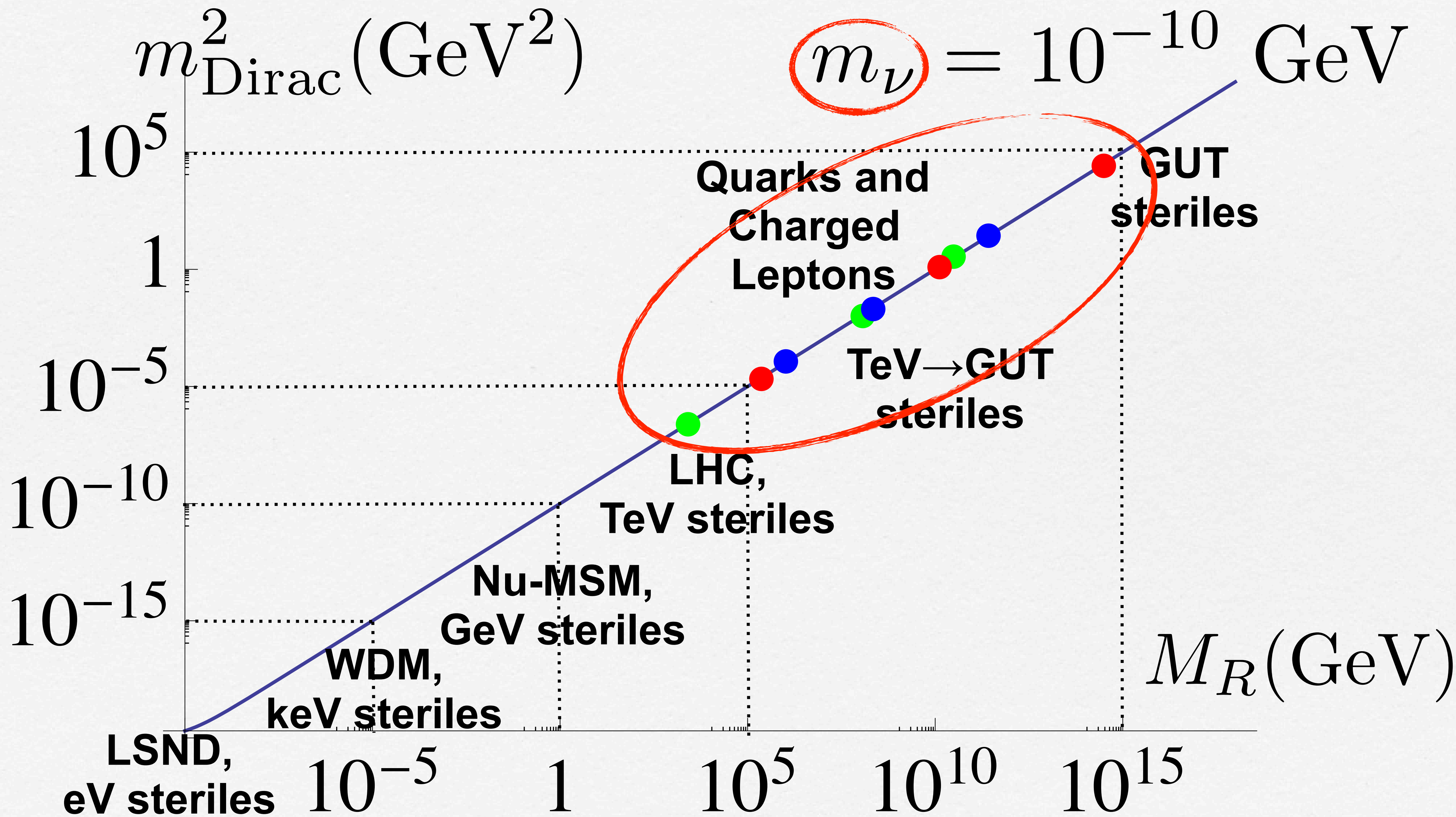
$$m_\nu \ll M_R \qquad m_\nu \approx \frac{m_D^2}{M_R} \sim 0.1\mathrm{eV}$$

Classic seesaw

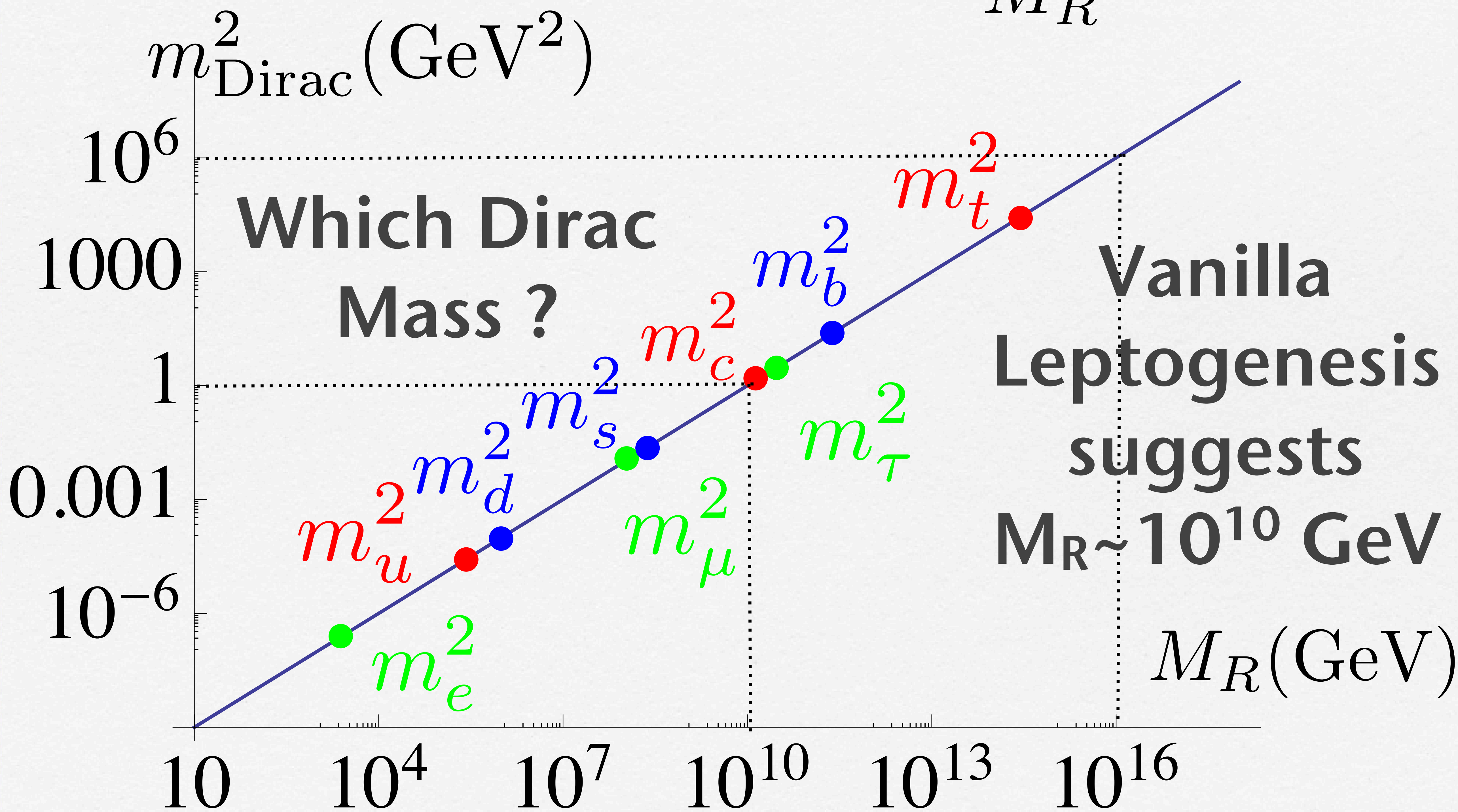
The seesaw line

$$m^2_{\rm Dirac} = m_\nu M_R + m_\nu^2$$

$$m_\nu = 10^{-10}\ {\rm GeV}$$



Classic seesaw: $m_\nu \approx \frac{m^2_{\text{Dirac}}}{M_R} = 0.1\text{eV}$



Seesaw mechanism

Minkowski; Yanagida;
Gell-Mann, Ramond,
Slansky; Glashow;
Mohapatra, Senjanivic;
Schechter, Valle;...

$$m_{\mathrm{L}} = 0$$

$$\begin{pmatrix} \bar{\nu}_L & \bar{\nu}^c_R \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix} \begin{pmatrix} \nu^c_L \\ \nu_R \end{pmatrix}$$

Three families

$$M_R = \begin{pmatrix} M_{\mathrm{atm}} & 0 & 0 \\ 0 & M_{\mathrm{sol}} & 0 \\ 0 & 0 & [M_{\mathrm{dec}}] \end{pmatrix}$$

$$m^D = \begin{pmatrix} m^D_{e,\mathrm{atm}} & m^D_{e,\mathrm{sol}} & [m^D_{e,\mathrm{dec}}] \\ m^D_{\mu,\mathrm{atm}} & m^D_{\mu,\mathrm{sol}} & [m^D_{\mu,\mathrm{dec}}] \\ m^D_{\tau,\mathrm{atm}} & m^D_{\tau,\mathrm{sol}} & [m^D_{\tau,\mathrm{dec}}] \end{pmatrix}$$

$$m^\nu = m^D M_R^{-1} m^{D^T}$$

$$\frac{m^D_{\mathrm{atm}} {m^D_{\mathrm{atm}}}^T}{M_{\mathrm{atm}}} \gg \frac{m^D_{\mathrm{sol}} {m^D_{\mathrm{sol}}}^T}{M_{\mathrm{sol}}} \left[\gg \frac{m^D_{\mathrm{dec}} {m^D_{\mathrm{dec}}}^T}{M_{\mathrm{dec}}} \right]$$

$$\Rightarrow m_3 \gg m_2 \gg m_1$$

SFK'98

Sequential dominance

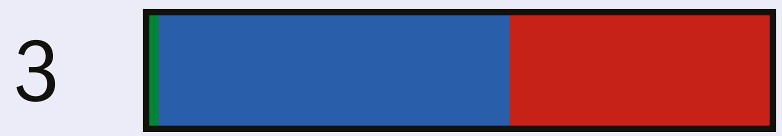
Sequential dominance

SFK'98

Simple way to understand MH

normal ordering

$$m_3 \sim 50\ \text{meV}$$



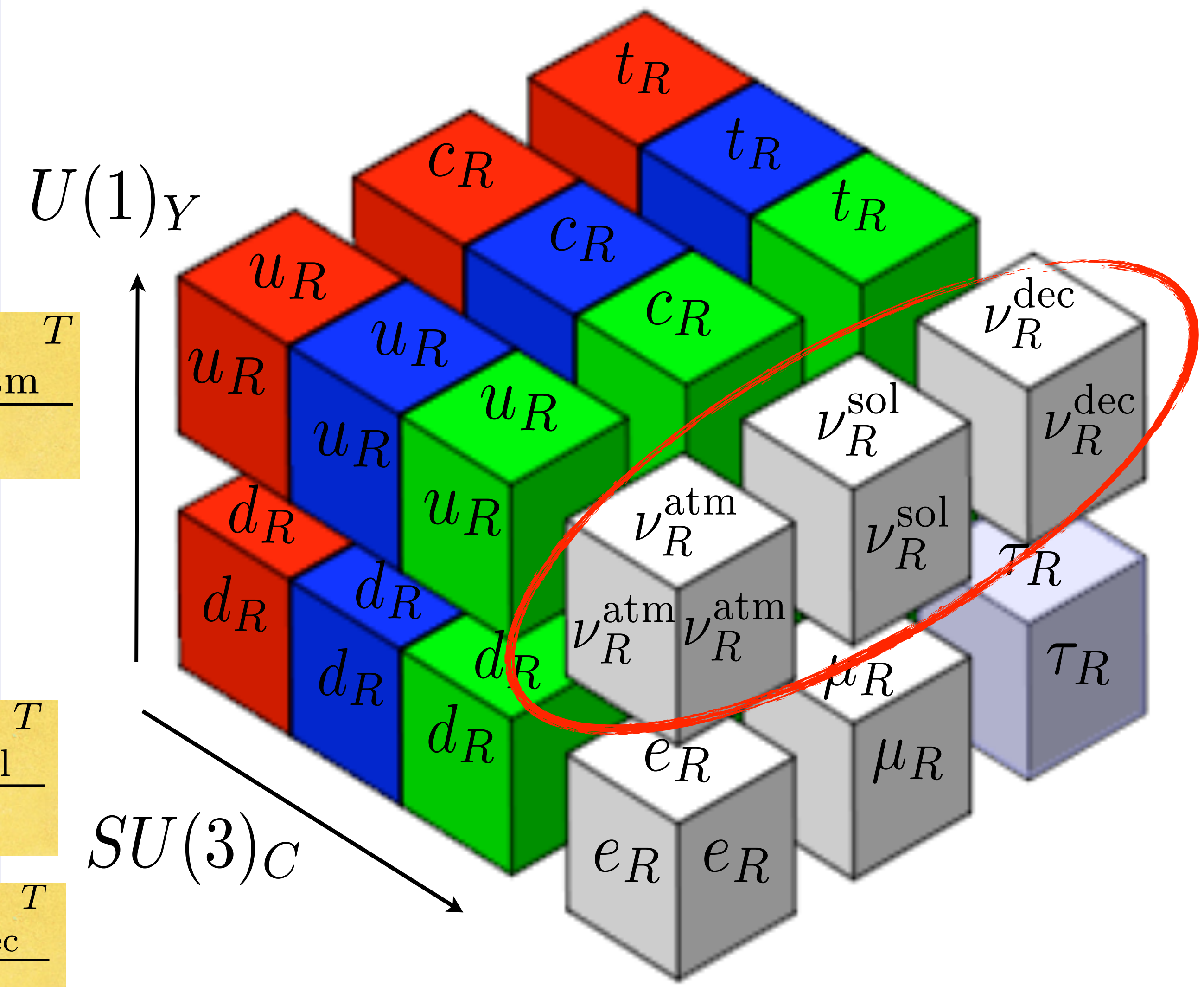
$$\frac{m^D_{\text{atm}} m^{D\,T}_{\text{atm}}}{M_{\text{atm}}}$$

$$m_2 \sim 9\ \text{meV}$$

$$\frac{m^D_{\text{sol}} m^{D\,T}_{\text{sol}}}{M_{\text{sol}}}$$

$$\frac{m^D_{\text{dec}} m^{D\,T}_{\text{dec}}}{M_{\text{dec}}}$$

$$m_1 \ll m_2$$



Littlest Seesaw

SFK
1304.6264,
1512.07531

- Two right-handed neutrinos (RHN) $\nu_R^{\rm atm}$ $\nu_R^{\rm sol}$
- Diagonal $M_R = \begin{pmatrix} M_{\rm atm} & 0 \\ 0 & M_{\rm sol} \end{pmatrix}$ completely decoupled $\nu_R^{\rm dec}$
- Diagonal (m_e, m_μ, m_τ)
- Constrained Sequential Dominance (CSD3):

$$m_D = \begin{pmatrix} 0 & b \\ a & 3b \\ a & b \end{pmatrix} \quad \text{or} \quad m_D = \begin{pmatrix} 0 & b \\ a & b \\ a & 3b \end{pmatrix}$$

Enforced by symmetry (see later)

SFK,Luhn
1607.05276

The Littlest Seesaw

Low energy neutrino mass matrices after seesaw:

$$m^{\nu}_{\mathrm{LSA}} = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix},$$

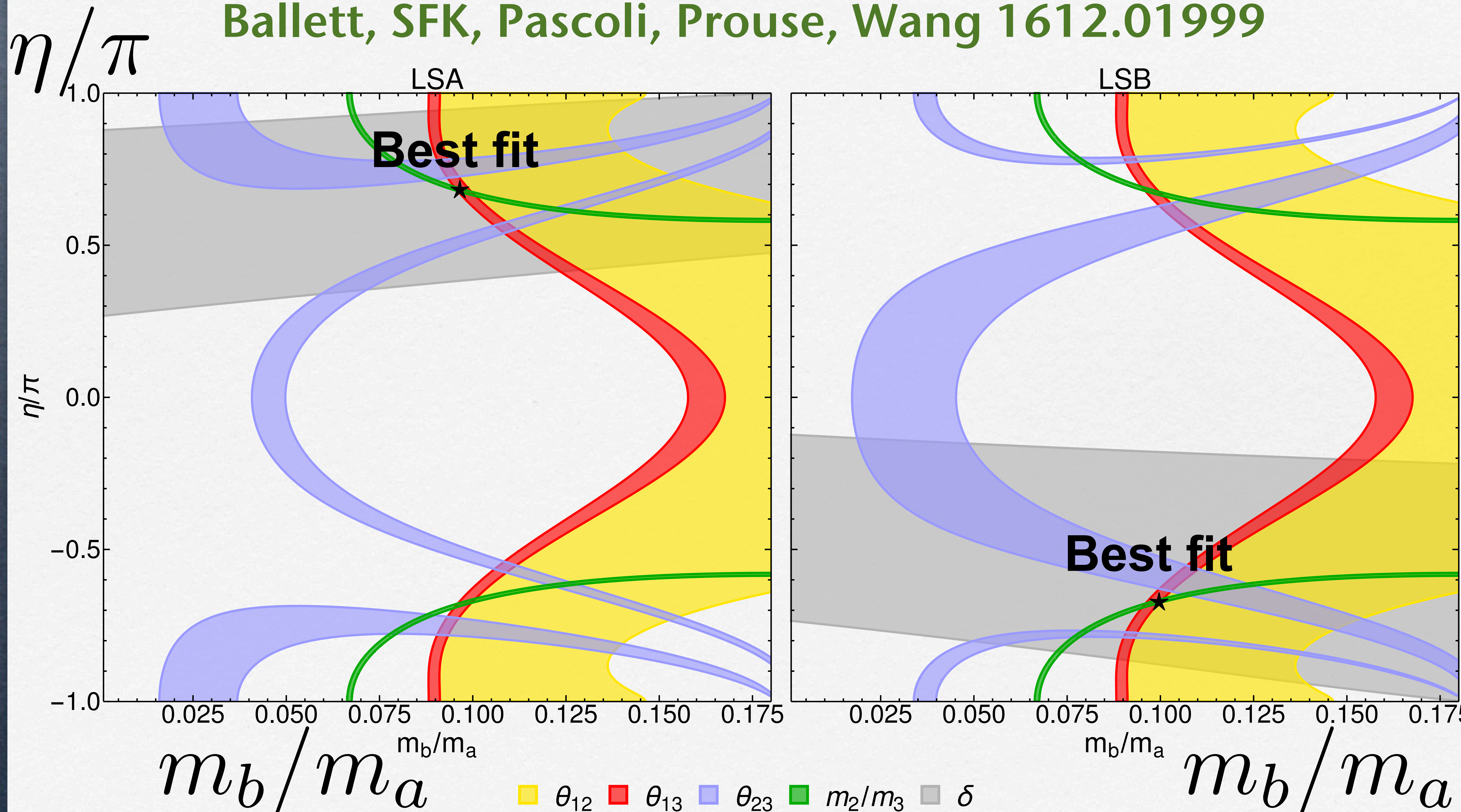
$$m^{\nu}_{\mathrm{LSB}} = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{pmatrix}.$$

SD $m_a \gg m_b$ predicts NO with $m_1=0$

Depends on 3 parameters: ma, mb, eta

Littlest Seesaw vs current data

Ballett, SFK, Pascoli, Prouse, Wang 1612.01999



Best Fit LS Predictions

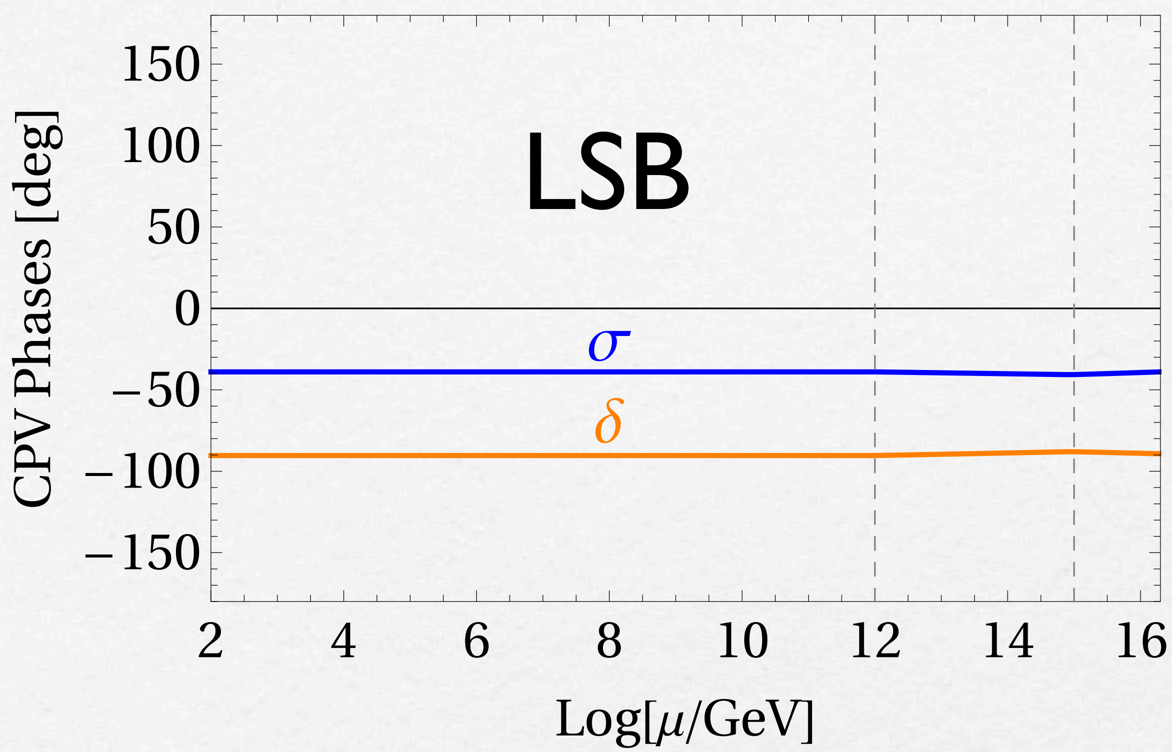
NO with $m_1=0$

	LSA		LSB		NuFIT 3.0
	η free	η fixed	η free	η fixed	global fit
m_a [meV]	27.19	26.74	26.95	26.75	
m_b [meV]	2.654	2.682	2.668	2.684	—
η [rad]	0.680π	$2\pi/3$	-0.673π	$-2\pi/3$	
θ_{12} [°]	34.36	34.33	34.35	34.33	$33.72^{+0.79}_{-0.76}$
θ_{13} [°]	8.46	8.60	8.54	8.60	$8.46^{+0.14}_{-0.15}$
θ_{23} [°]	45.03	45.71	44.64	44.28	$41.5^{+1.3}_{-1.1}$
δ [°]	-89.9	-86.9	-91.6	-93.1	-71^{+38}_{-51}
Δm^2_{21} [10^{-5}eV^2]	7.499	7.379	7.447	7.390	$7.49^{+0.19}_{-0.17}$
Δm^2_{31} [10^{-3}eV^2]	2.500	2.510	2.500	2.512	$2.526^{+0.039}_{-0.037}$
$\Delta\chi^2$ / d.o.f	4.1 / 3	5.6 / 4	3.9 / 3	4.5 / 4	—

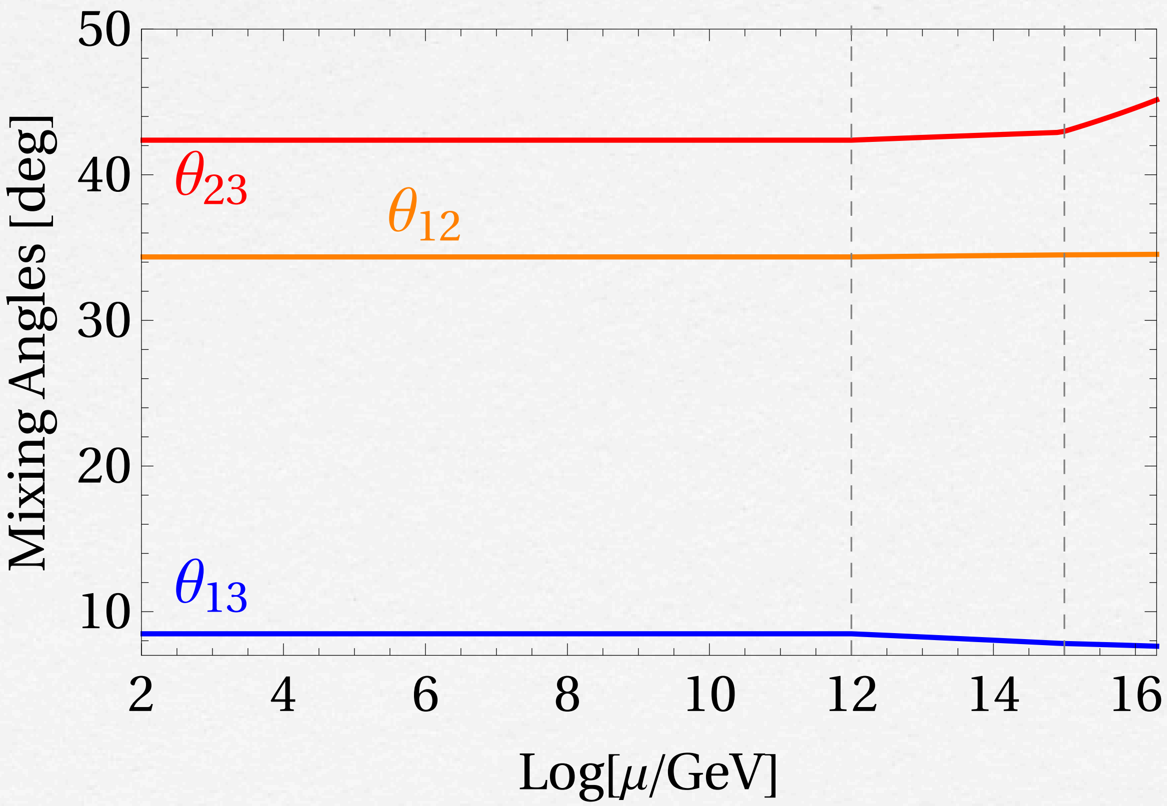
Renormalisation Group Corrections

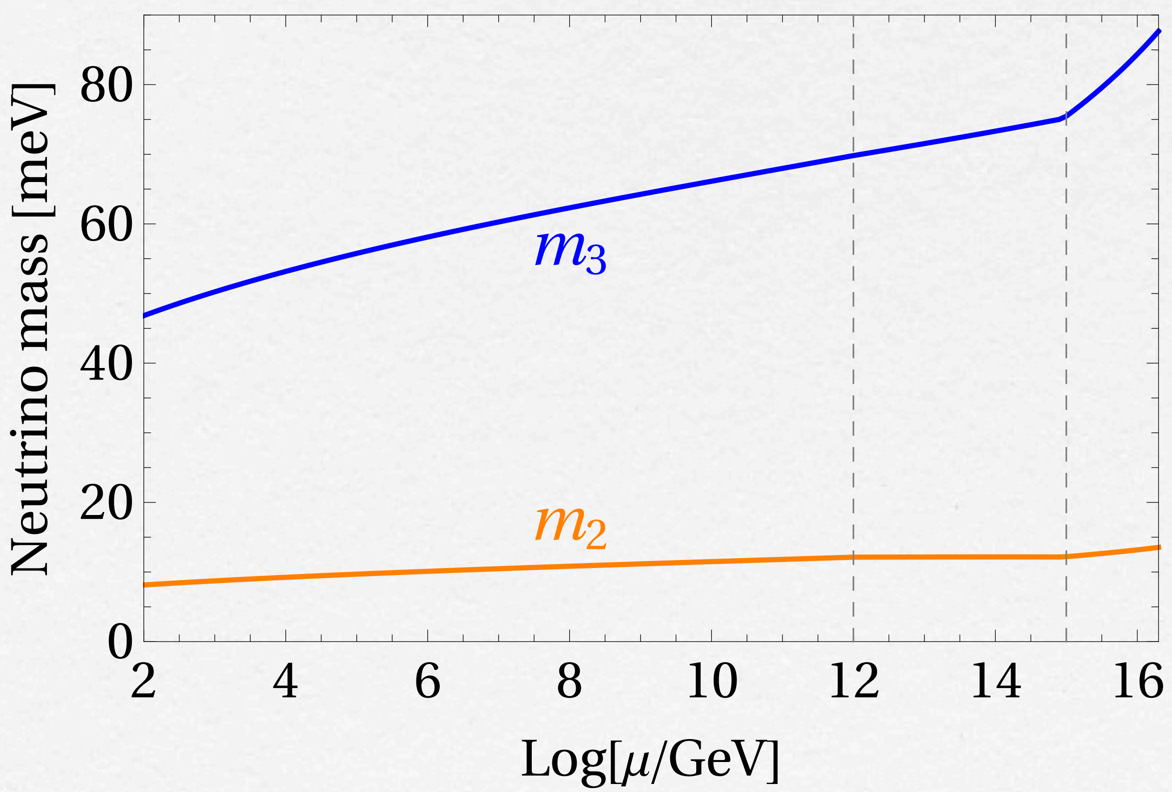
SM with $M_{atm} = 10^{15}$ GeV and $M_{sol} = 10^{12}$ GeV

	Λ_{GUT}	M_{atm}	M_{sol}	Λ_{EW}
θ_{13}(deg)	7.62574	7.81215	8.47979	8.4798
θ_{12}(deg)	34.5348	34.4977	34.3575	34.3572
θ_{23}(deg)	45.1425	42.9816	42.3751	42.3744
m_2(meV)	13.537	12.2035	12.1317	8.73113
m_3(meV)	87.6802	75.4657	69.8112	50.2431
δ_{CP}(deg)	–89.2885	–88.0086	–90.3508	–90.3507
σ_{CP}(deg)	–38.9558	–40.649	–38.9917	–38.9917

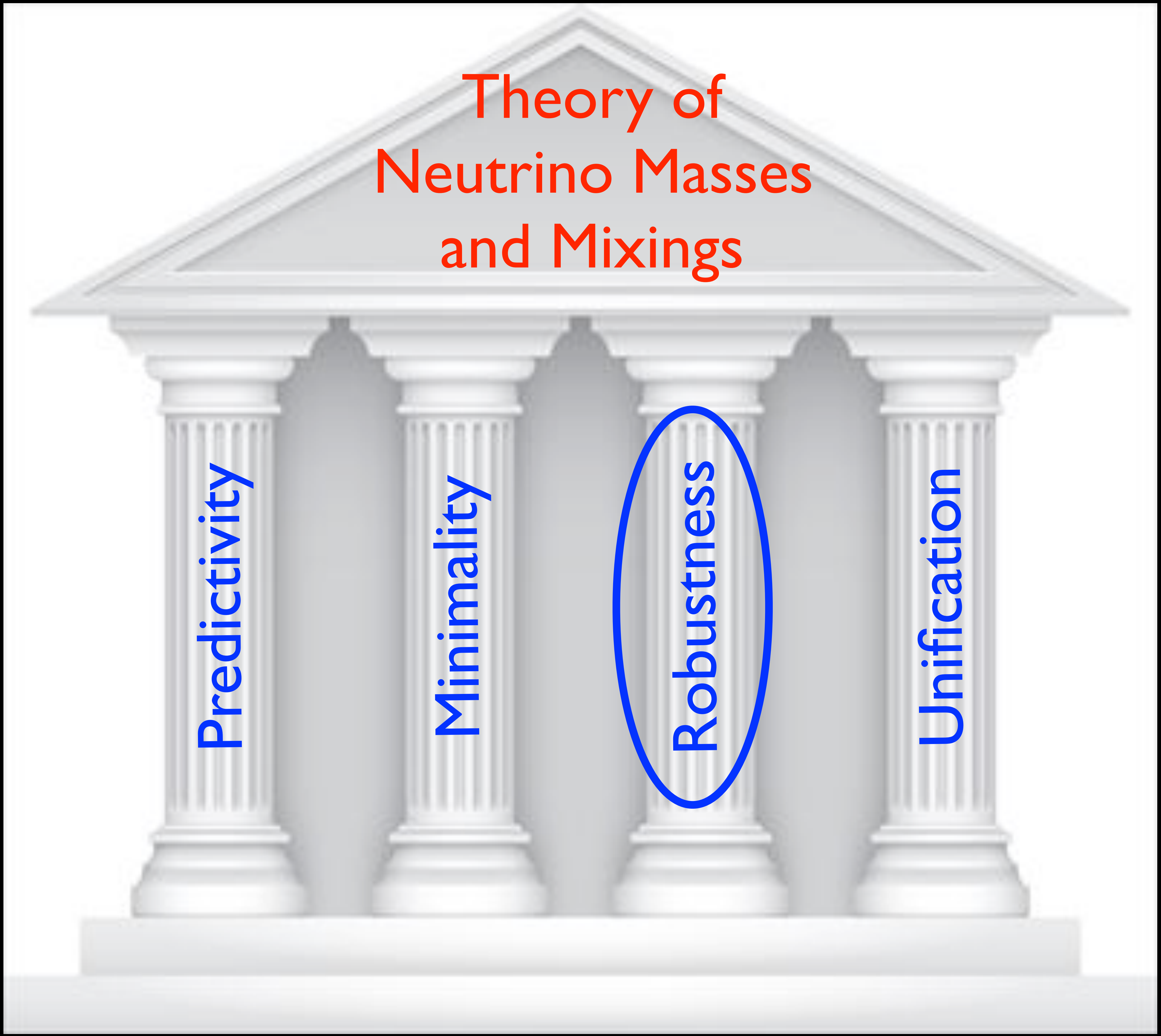


Geib, SFK (to appear)



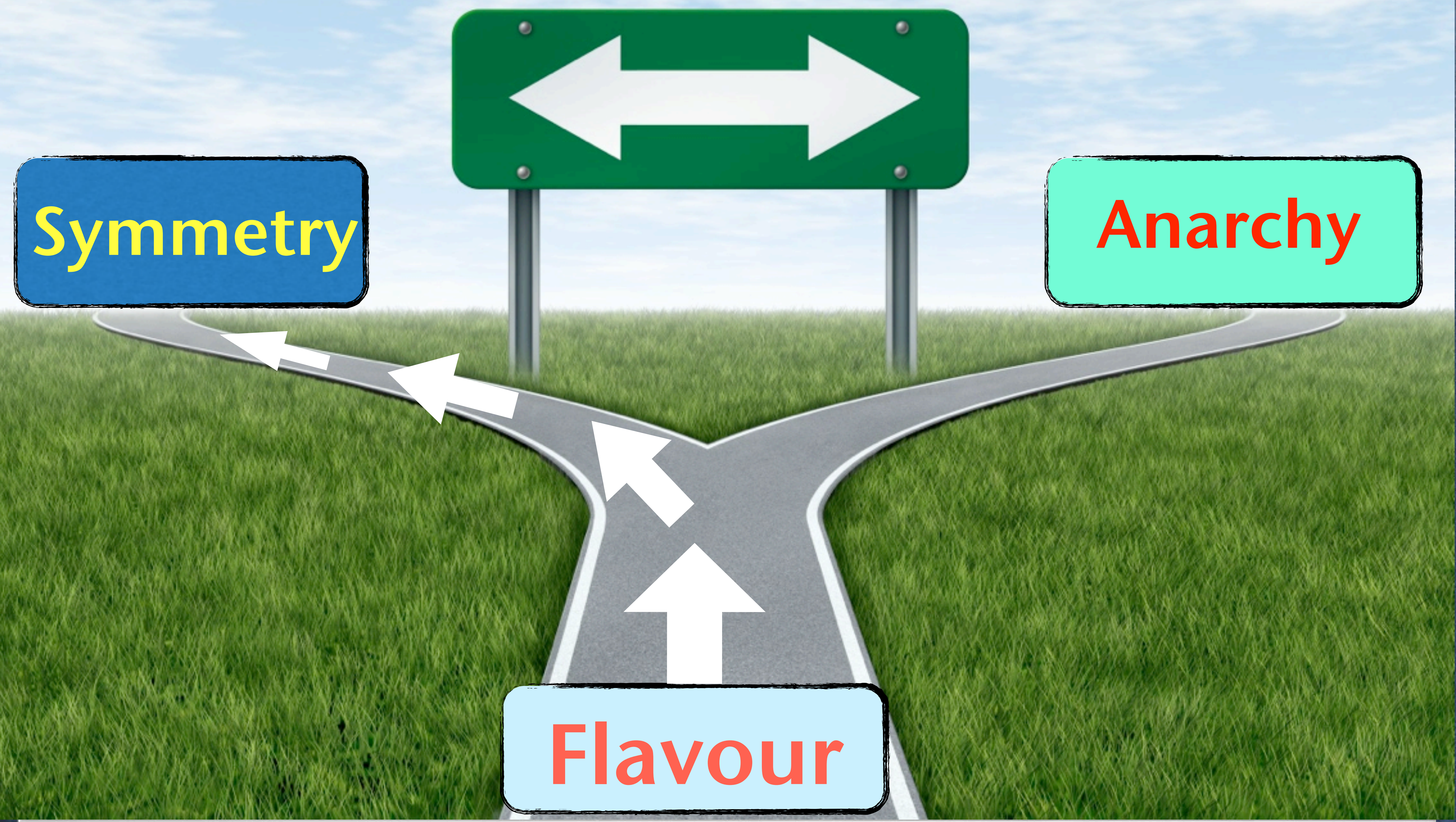


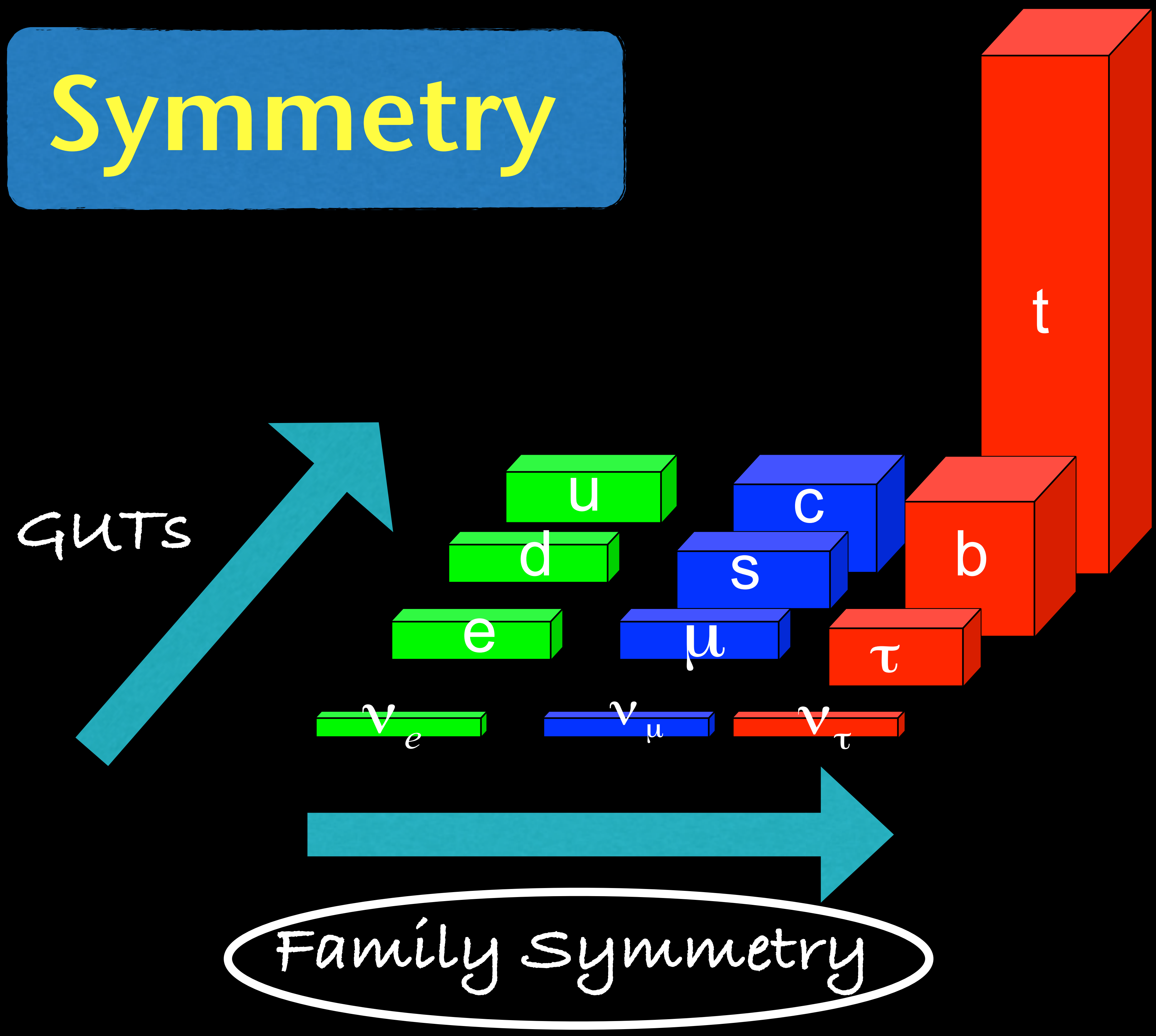
From review SFK 1701.04413



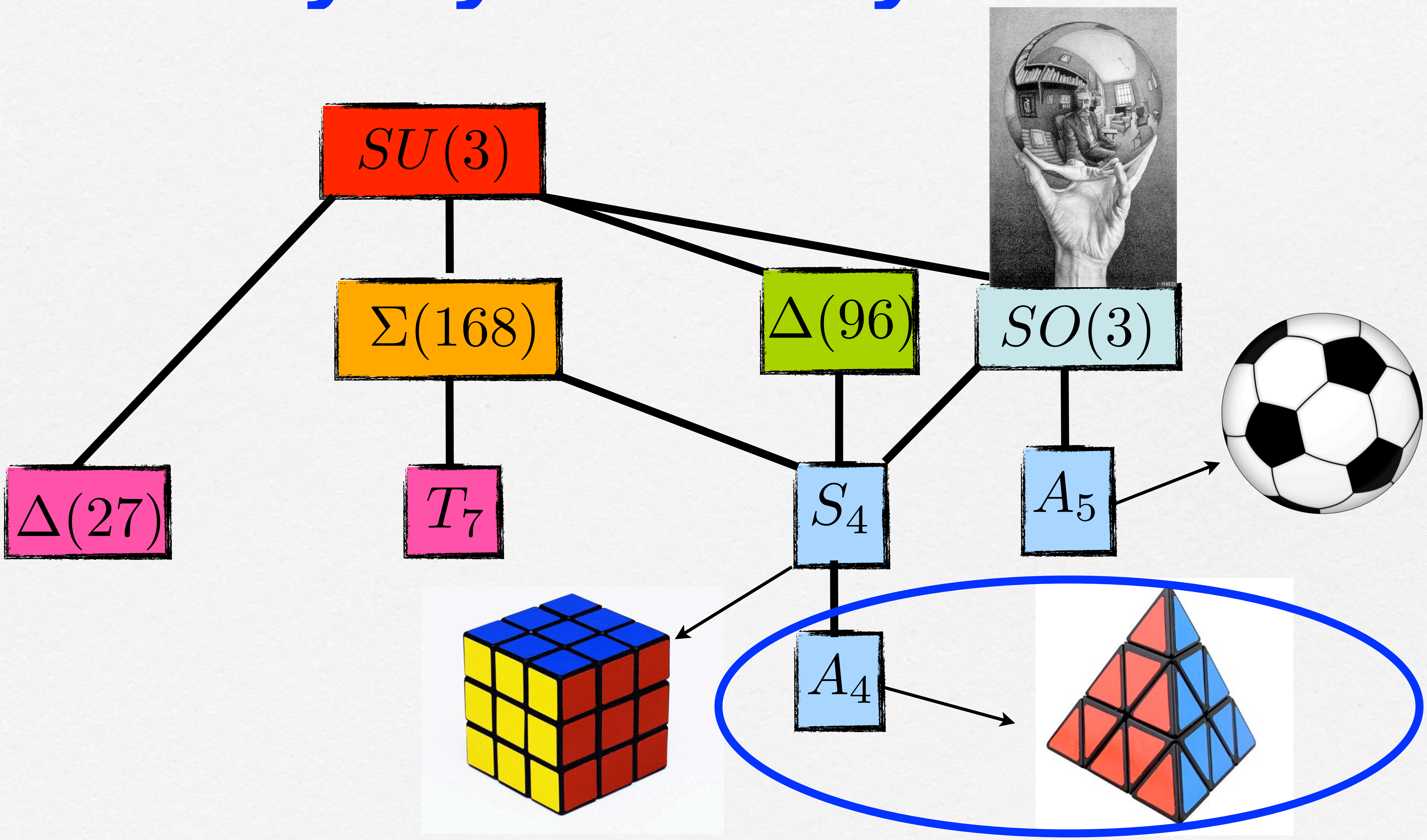
Robustness

Towards a Theory of Flavour




Symmetry
GUTs
u
d
e
ν_e
c
s
μ
ν_μ
t
b
τ
ν_τ
Family Symmetry

Family Symmetry



A_4

Symmetry of the tetrahedron

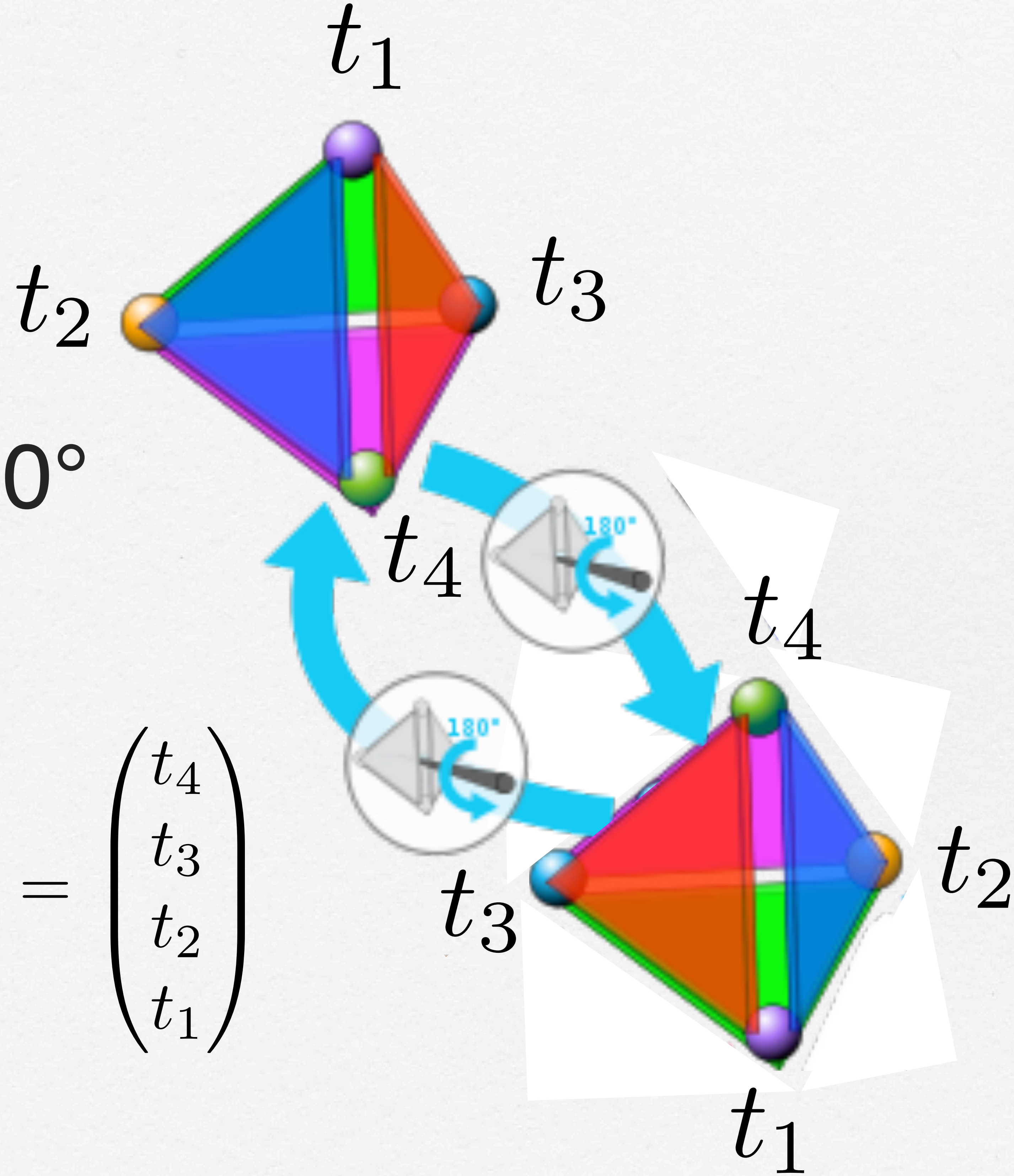
Vertices labelled by t_i

A_4

- rotation by 180°

$$S$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_4 \\ t_3 \\ t_2 \\ t_1 \end{pmatrix}$$

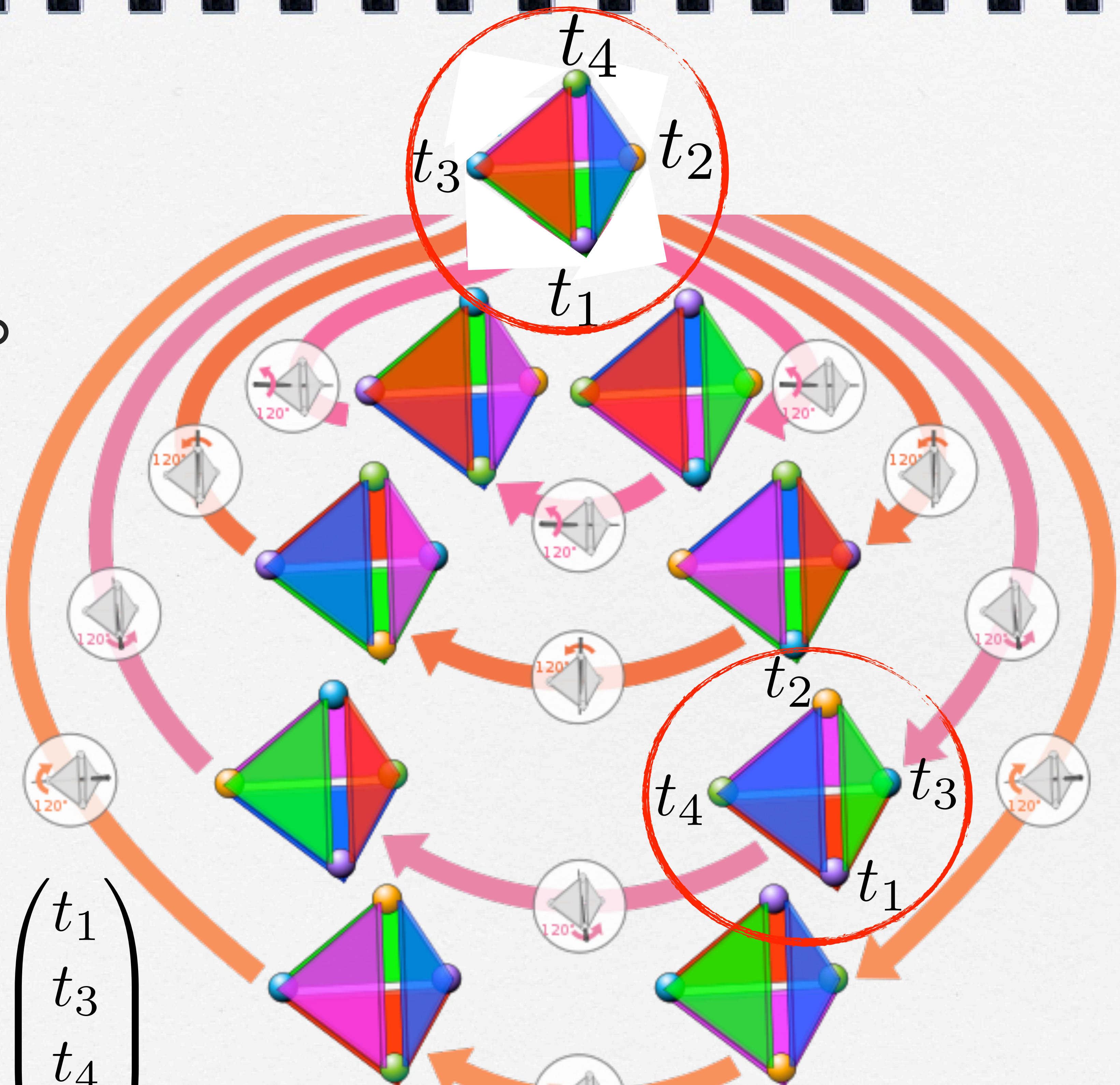


A_4

- rotation by 120° anti-clockwise (seen from a vertex)

$$T$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_3 \\ t_4 \\ t_2 \end{pmatrix}$$



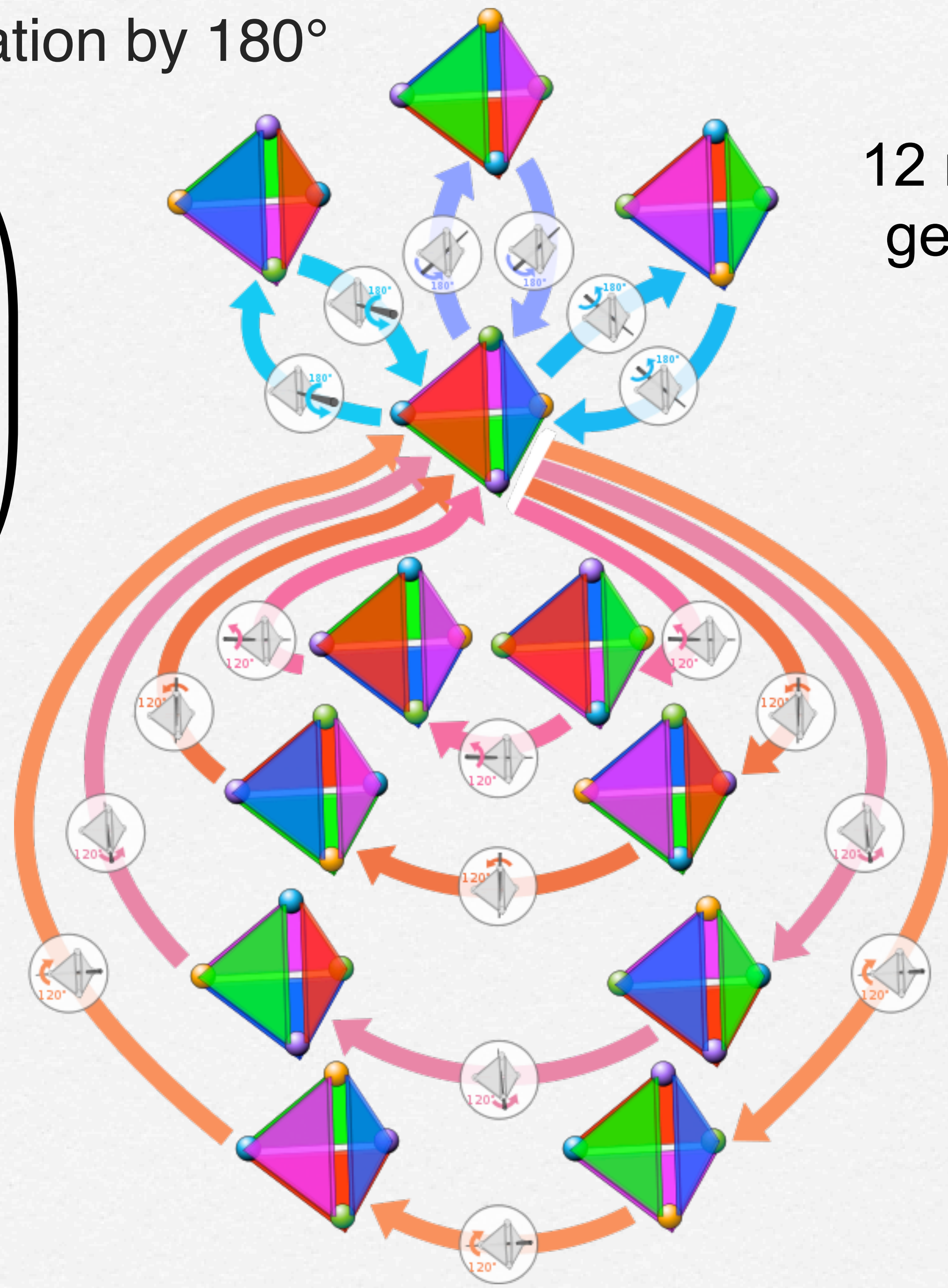
A_4

- 4 × rotation by 120° clockwise (seen from a vertex) T-type rotations
- 4 × rotation by 120° anti-clockwise (ditto) T-type rotations
- 3 × rotation by 180° S-type rotations

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Diagonal

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



12 rotations ("group elements") generated by products of S,T ("generators")

$$S^2 = T^3 = I$$
$$(ST)^3 = I$$

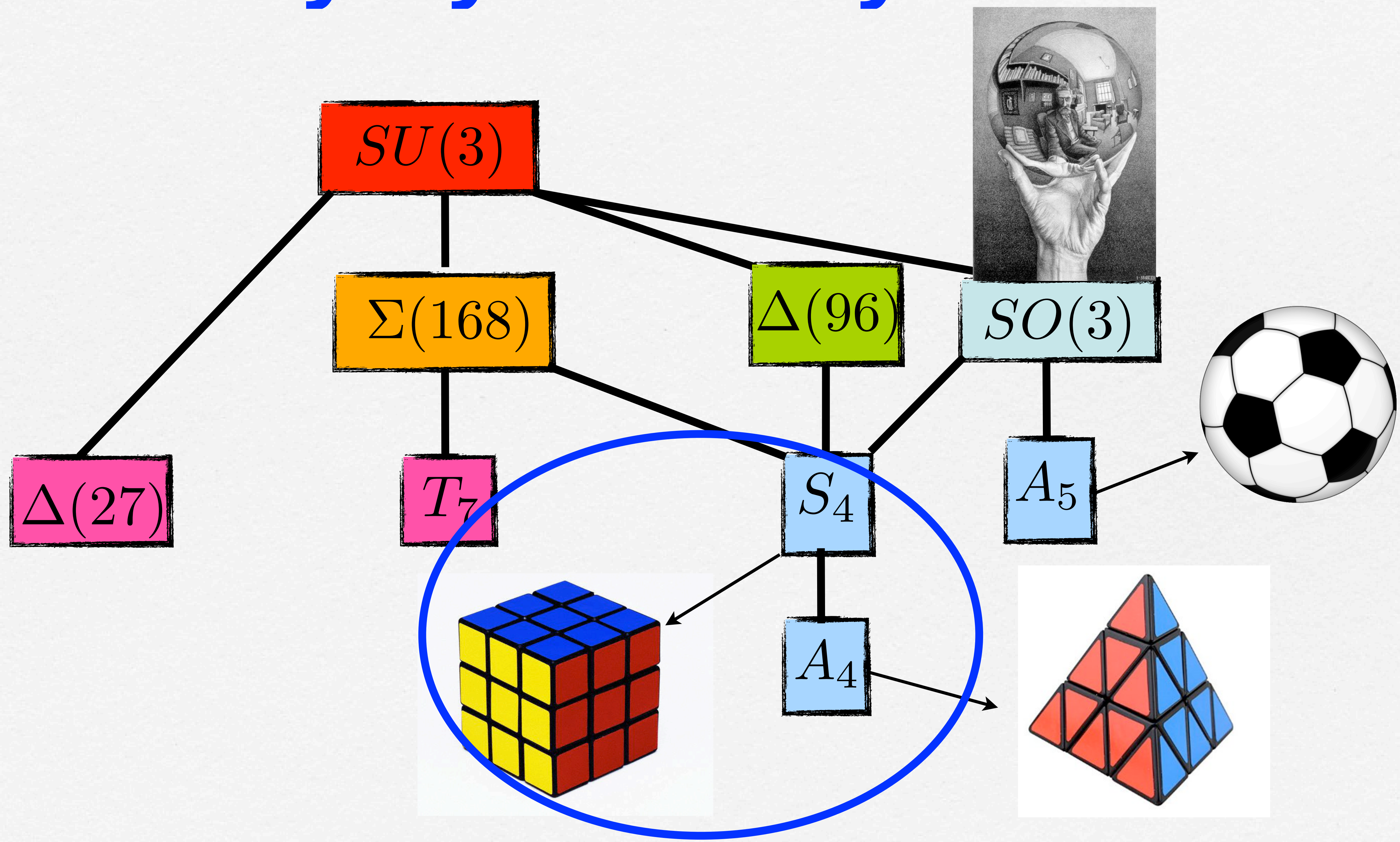
Block diagonal (rotate about first vertex)

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

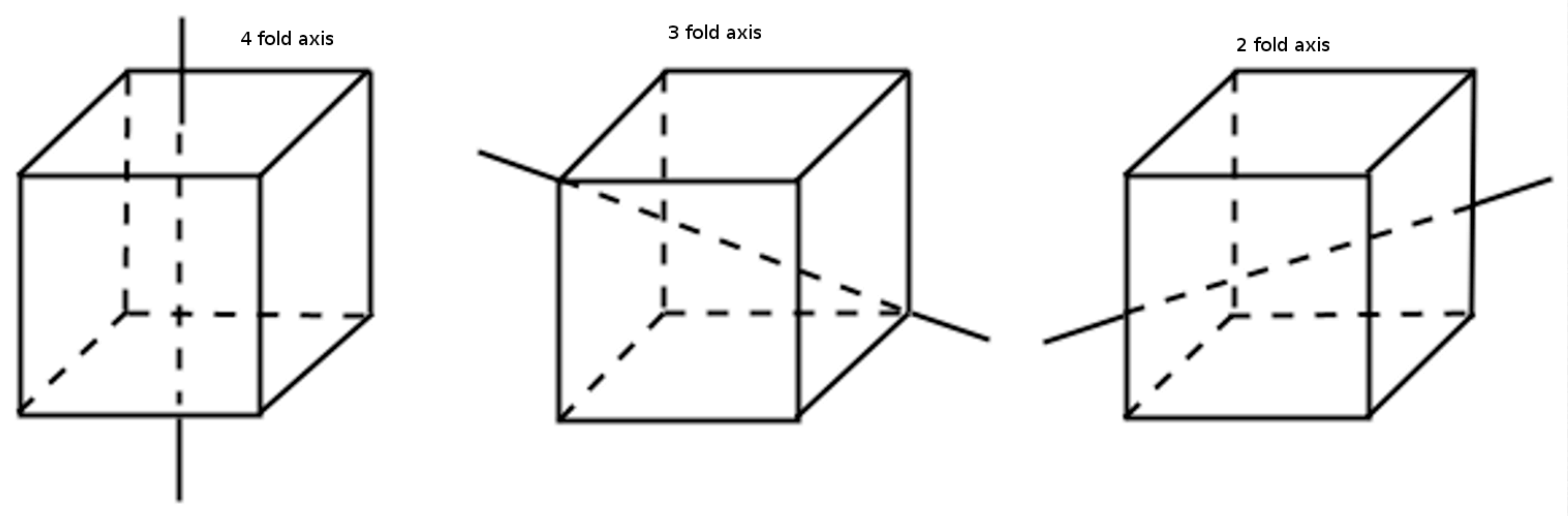
Since S,T are block diagonal, the 4 dimensional matrix of vertex transformations is equivalent to a triplet plus singlet

$$4 \to 3 \oplus 1$$

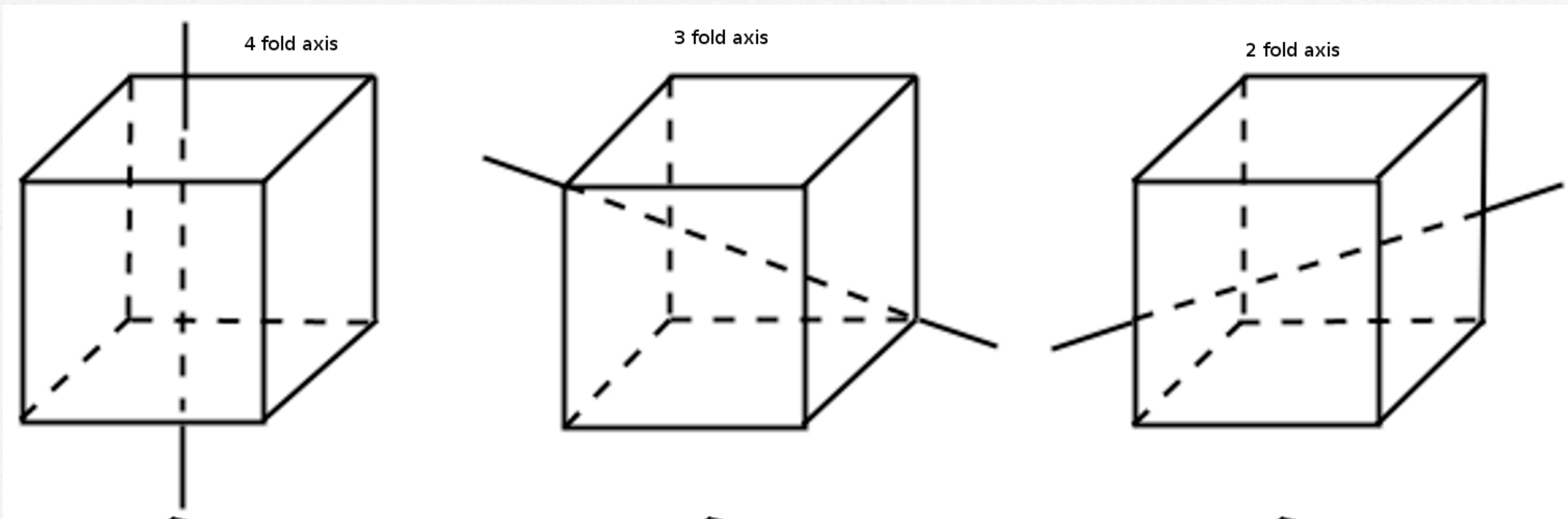
Family Symmetry

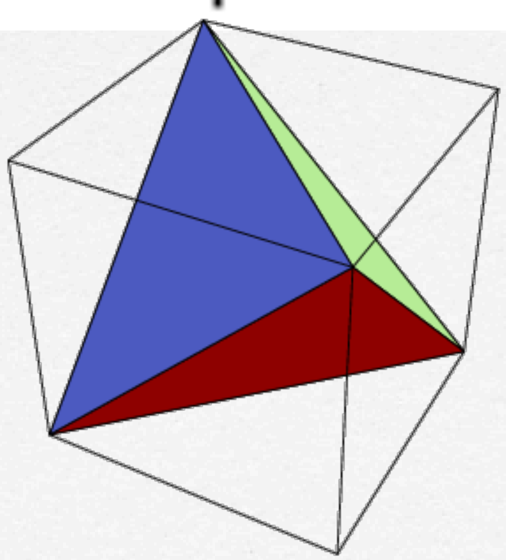


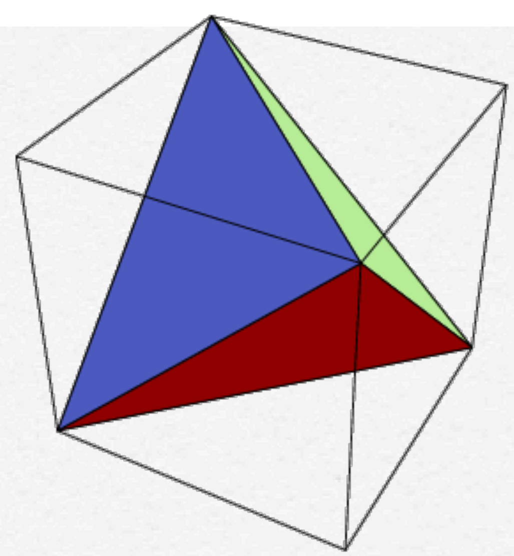
- S_4 rotation symmetry of a cube

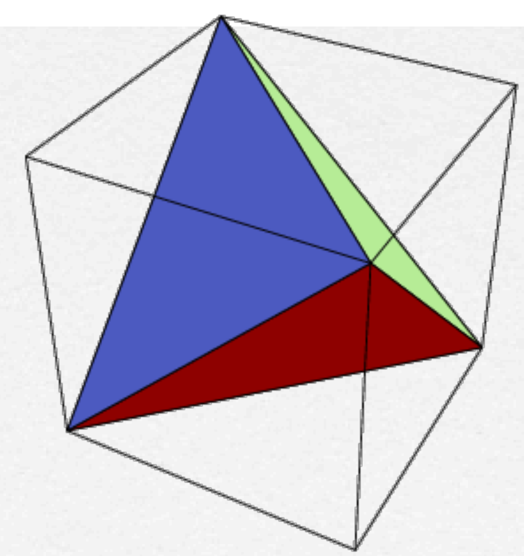


S_4 rotation symmetry of a cube









- 2 fold symmetry of the tetrahedron S
- 3 fold symmetry of the tetrahedron T
- Not a symmetry of the tetrahedron U

Group theory

S_4	A_4	S	T	U
$\mathbf{1}, \mathbf{1}'$	$\mathbf{1}$	1	1	± 1
$\mathbf{2}$	$\begin{pmatrix} \mathbf{1}'' \\ \mathbf{1}' \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{3}, \mathbf{3}'$	$\mathbf{3}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\mathbf{3} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ preserves } T, \text{ breaks } S, U,$$

$$\mathbf{3}' \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ preserves } T, U \text{ breaks } S,$$

$$\mathbf{3} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ preserves } S \text{ breaks } T, U,$$

$$\mathbf{3}' \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ preserves } S, U \text{ breaks } T,$$

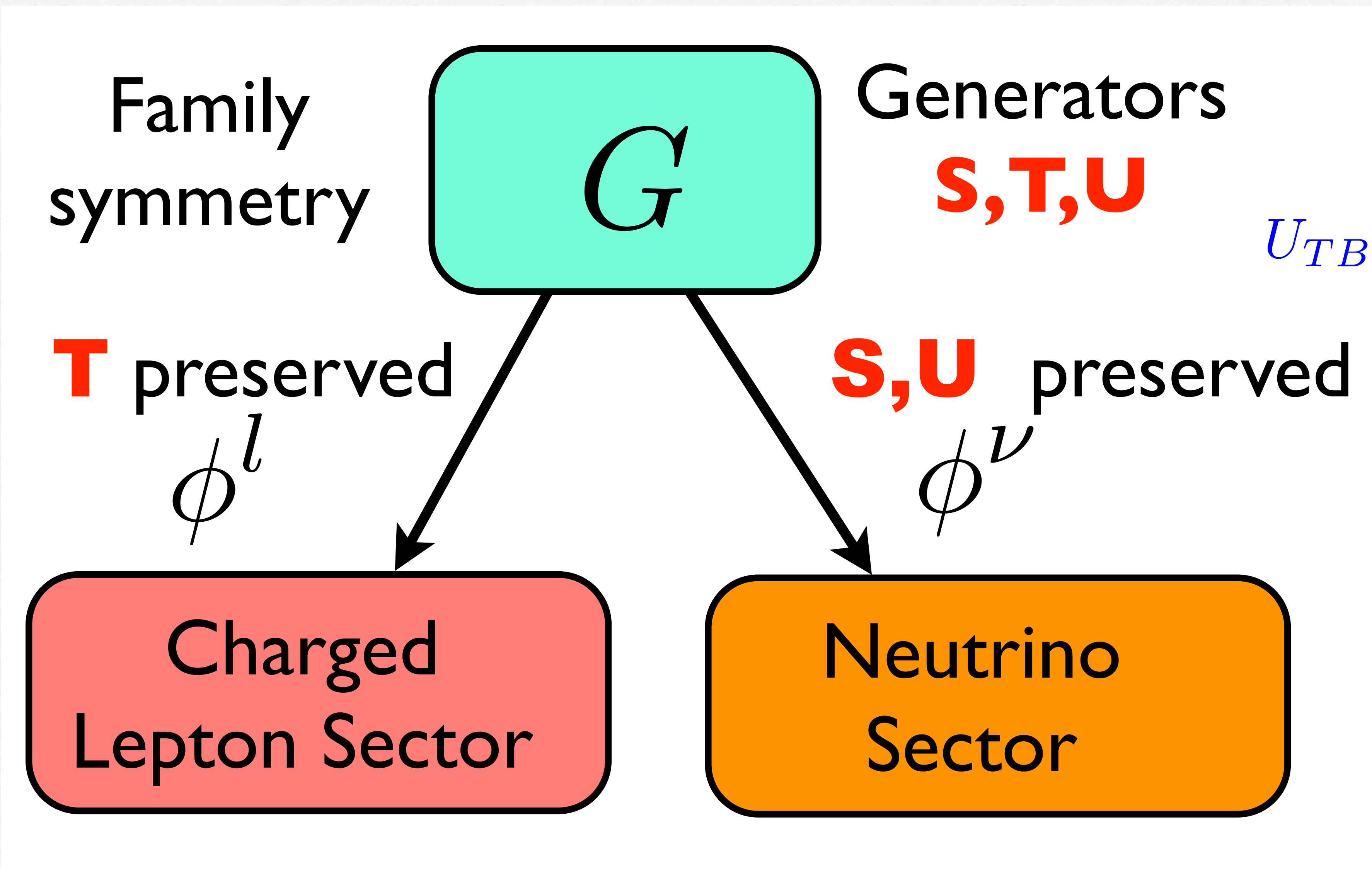
$$\mathbf{3}' \sim \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \text{ preserves } SU \text{ breaks } T, U,$$

$$\mathbf{3}' \sim \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \text{ preserves } SU \text{ breaks } T, U.$$

Family symmetry is spontaneously broken by Higgs fields called "flavons" but some symmetry may be preserved by particular vacuum alignments

} Littlest Seesaw

Direct Models



Historically predicted TB mixing from S_4

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now $\Delta(6n^2)$

Is the only viable symmetry class - predicts zero Dirac CPV

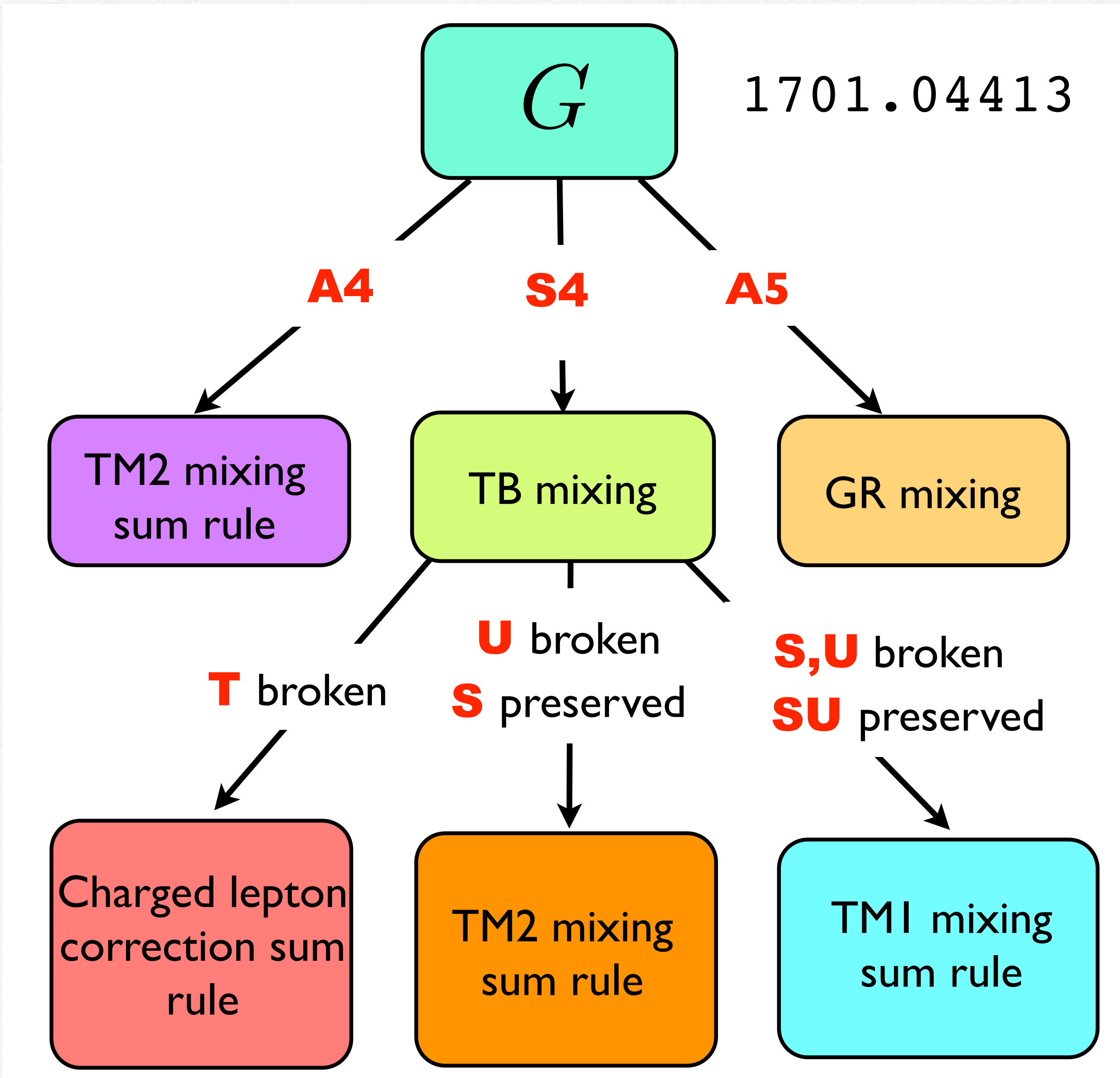
Holthausen,Lim, Lindner; SFK,Neder,Stuart; Lavoura,Ludl; Fonseca,Grimus

$$T^\dagger(M_e^\dagger M_e)T = M_e^\dagger M_e$$

$$m_\nu = S^T m_\nu S$$

$$m_\nu = U^T m_\nu U$$

Semi-Direct Models



Here S,U and T are partly preserved as subgroups of some family symmetry

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

TM1 TM2

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s)$$

$$\sin\theta_{23} = \frac{1}{\sqrt{3}}(1+a)$$

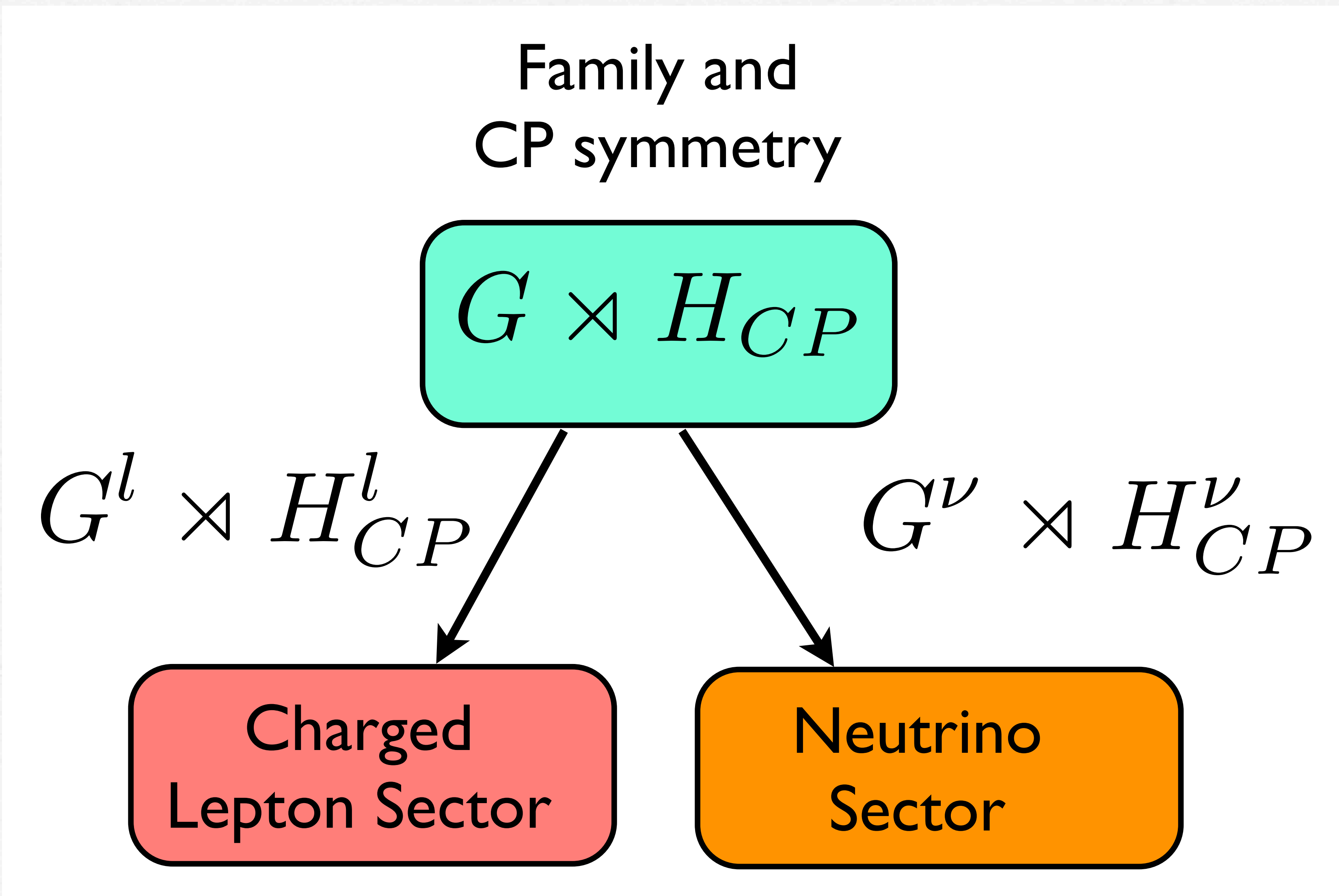
$$\sin\theta_{13} = \frac{r}{\sqrt{2}}$$

0710.0530

SFK,
Antusch,
...

$$s \approx r\cos\delta \qquad a \approx -\frac{r}{2}\cos\delta \qquad a \approx r\cos\delta$$

CP violation



Feruglio,Hagedorn; Holthausen,Lindner Schmidt; Ding,SFK,Luhn,Stuart; Nishi,Xing; Hagedorn,Meroni, Molinaro; Ding,SFK,Neder; Branco, SFK, Varzielas, Chen,...

E.g. in semi-direct models where

$$G^\nu \sim Z_2^S$$

Typically predicts maximal Dirac Phase

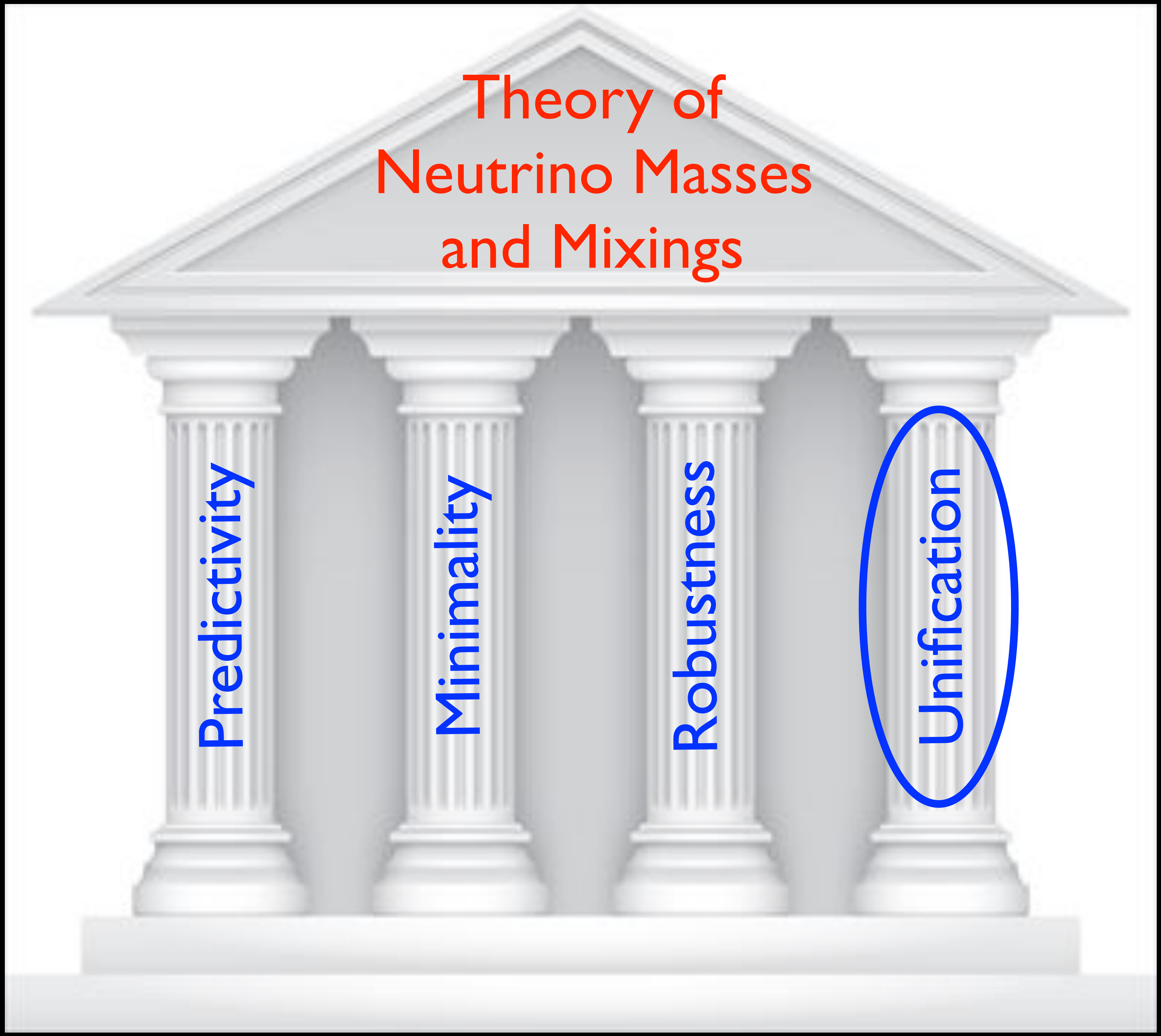
$$\delta = \pm\pi/2$$

$$G^{\nu T} m_\nu G^\nu = m_\nu$$

$$H^{\nu T}_{CP} m_\nu H^\nu_{CP} = m^*_\nu$$

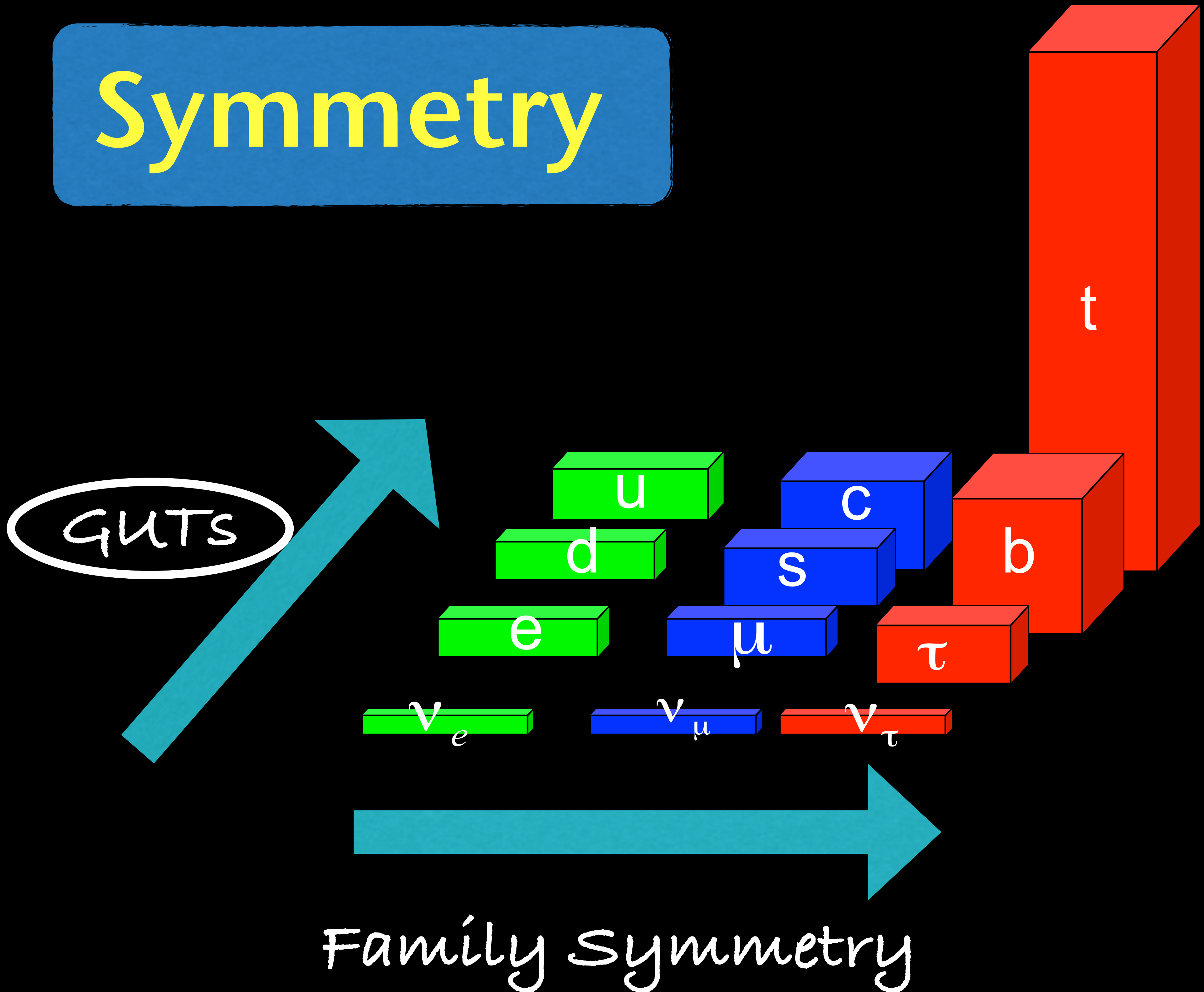
CP involves complex conjugation

From review SFK 1701.04413



Unification

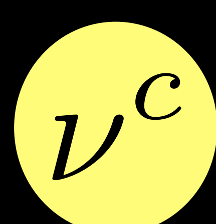
Symmetry



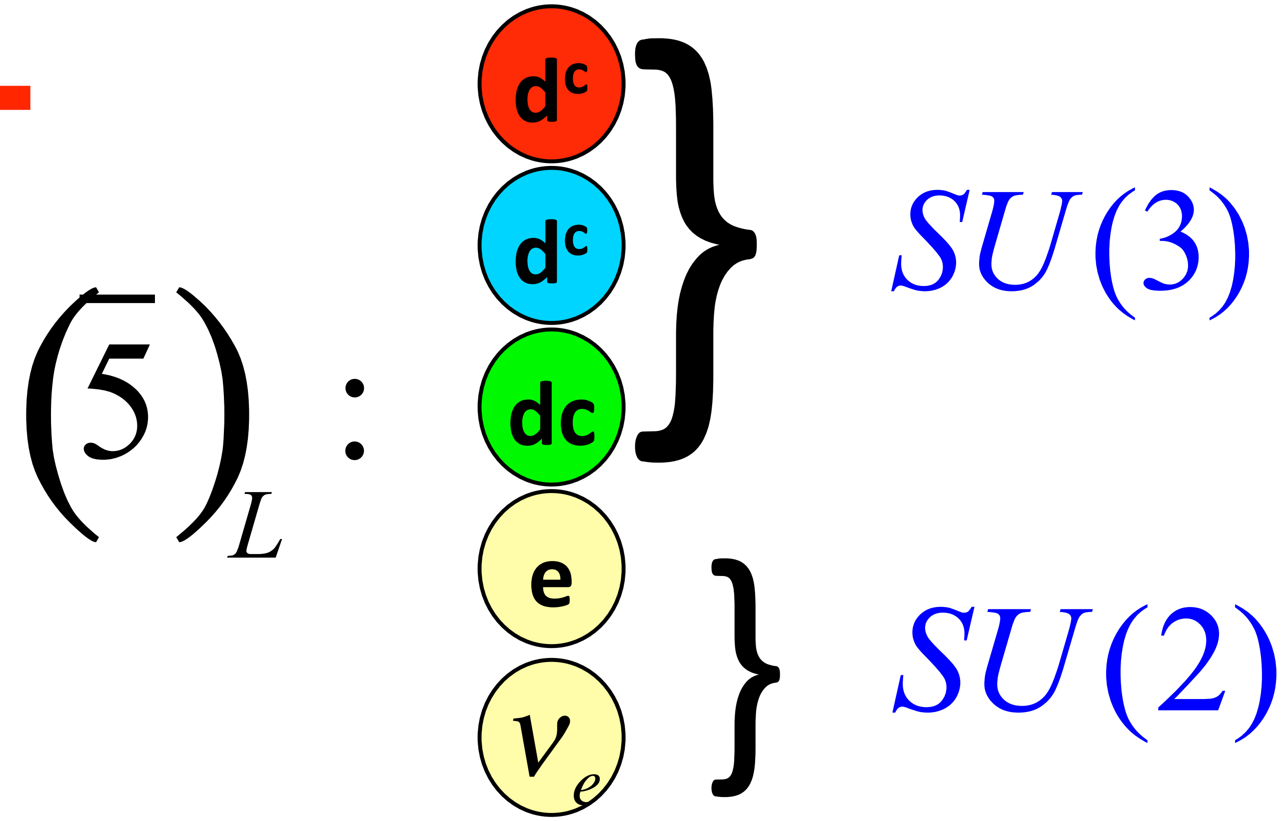
SU(5) GUT

Georgi, Glashow

Right-handed neutrino is singlet

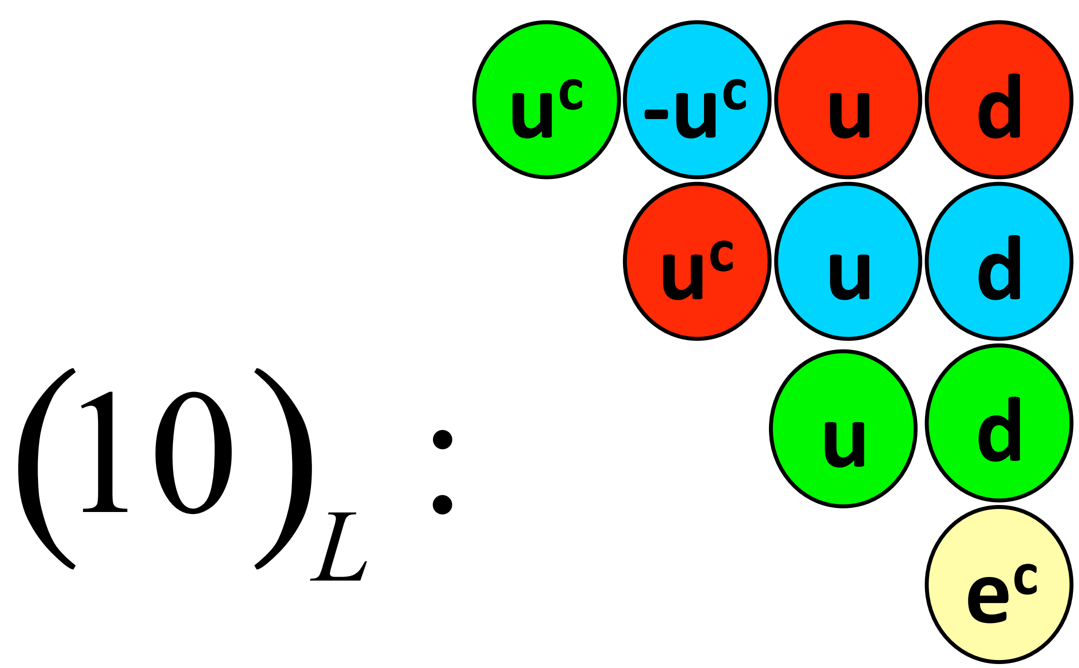


Altarelli, Feruglio



$$\overline{\mathbf{5}} = d^c(\overline{\mathbf{3}}, \mathbf{1}, 1/3) \oplus L(\mathbf{1}, \overline{\mathbf{2}}, -1/2),$$

$$\mathbf{10} = u^c(\overline{\mathbf{3}}, \mathbf{1}, -2/3) \oplus Q(\mathbf{3}, \mathbf{2}, 1/6) \oplus e^c(\mathbf{1}, \mathbf{1}, 1)$$



Fermion Masses in SU(5)

$$\lambda_u H_i 10_{jk} 10_{lm} \epsilon^{\overset{1,\ldots,5}{ijklm}} + \lambda_d \overline{H}^i 10_{ij} \overline{5}^j$$

$$\rightarrow \quad \lambda_u H_u Q u^c + \lambda_d (H_d Q d^c + H_d e^c L)$$

$$\lambda_d = \lambda_e \text{ at the GUT scale}$$

Assuming this relation holds for all 3 families

$$\lambda_d = \lambda_e^T$$

SO(10) GUT

Georgi; Minkowski; Raby;
Mohapatra, Senjanovic;

$\overleftrightarrow{SU(3)}\ \overleftrightarrow{SU(2)}$

$u\ u\ u$ $\quad |++--+\rangle, |+-+-+\rangle, |-++-+\rangle$

$d\ d\ d$ $\quad |++-+-\rangle, |+-++-\rangle, |-+++-\rangle$

$u^c\ u^c\ u^c$ $\quad |--+++\rangle, |-+-++\rangle, |+--++\rangle$

$d^c\ d^c\ d^c$ $\quad |--+--\rangle, |-+---\rangle, |+----\rangle$

ν_e $\quad |----+\rangle$

e $\quad |---+-\rangle$

e^c $\quad |+++--\rangle$

ν^c $\quad |+++++\rangle$

Right-handed neutrino part of 16

Good Motivation for SO(10)

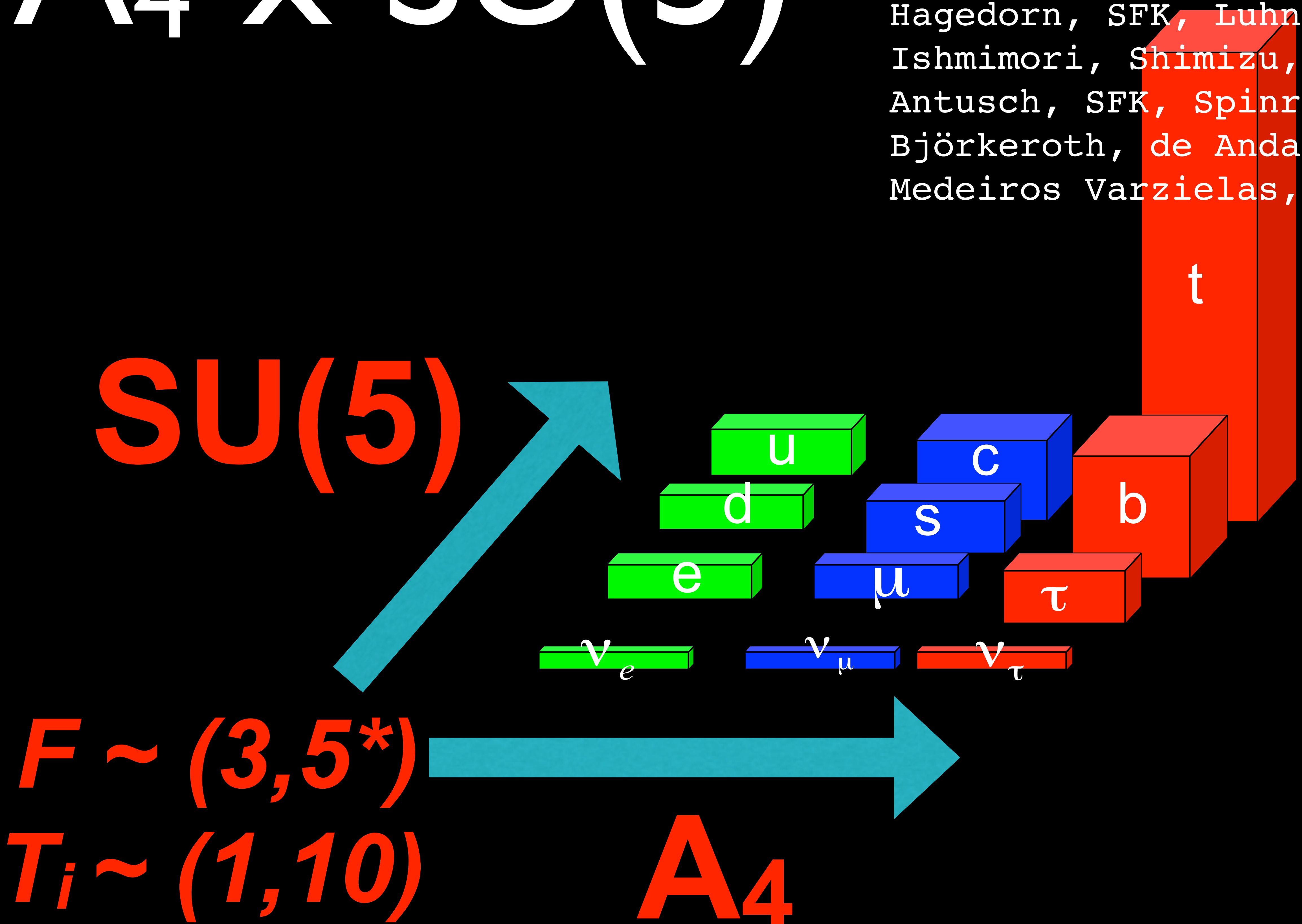
For references see review
SFK 1701.04413

GUTs with Family Symmetry

G_{GUT} / G_{FAM}	$SU(2)_L \times U(1)_Y$	$SU(5)$	PS	$SO(10)$
S_3	[29]			[150]
A_4	[36, 49, 51, 62, 151–154]	[155–158]	[66, 159, 160]	
T'		[161]		
S_4	[31, 49, 51, 154, 163]	[164, 165]	[162]	[166]
A_5	[51, 169]	[170]		
T_7	[171, 172]			
$\Delta(27)$	[173]			[174]
$\Delta(96)$	[175, 176]	[177]		[178]
D_N	[179]			
Q_N	[180]			
other	[181]	[182]	[183]	

A_4 x SU(5)

Callen Volkas;
Cooper, SFK, Luhn;
Meroni, Petcov, Spinrath;
Hagedorn, SFK, Luhn;
Ishmimori, Shimizu, Tanimoto;
Antusch, SFK, Spinrath;
Björkeroth, de Anda, de
Medeiros Varzielas, SFK,...



Björkeroth, de Anda,
de Medeiros Varzielas, SFK
1503.03306, 1505.05504

Field	Representation				
	A_4	SU(5)	$\mathbb{Z}_9$	$\mathbb{Z}_6$	$\mathbb{Z}_4^R$
F	3	$\bar{5}$	0	0	1
T_1	1	10	5	0	1
T_2	1	10	7	0	1
T_3	1	10	0	0	1
N_1^c	1	1	7	3	1
N_2^c	1	1	8	3	1
Γ	1	1	0	3	1
H_5	1	5	0	0	0
$H_{\bar{5}}$	1	$\bar{5}$	2	0	0
H_{24}	$1'$	24	3	0	0
Λ_{24}	$1'$	24	0	0	0
H_{45}	1	45	4	0	2
$H_{\overline{45}}$	1	$\overline{45}$	5	0	0
ξ	1	1	2	0	0
θ_1	1	1	1	3	0
θ_2	1	1	1	4	0
ϕ_e	3	1	0	0	0
ϕ_μ	3	1	3	0	0
ϕ_τ	3	1	7	0	0
ϕ_1	3	1	3	2	0
ϕ_2	3	1	1	3	0
ϕ_3	3	1	3	1	0
ϕ_4	3	1	2	1	0
ϕ_5	3	1	6	2	0
ϕ_6	3	1	5	2	0

Quarks and Leptons

Higgs

Flavons

Fermion Messengers

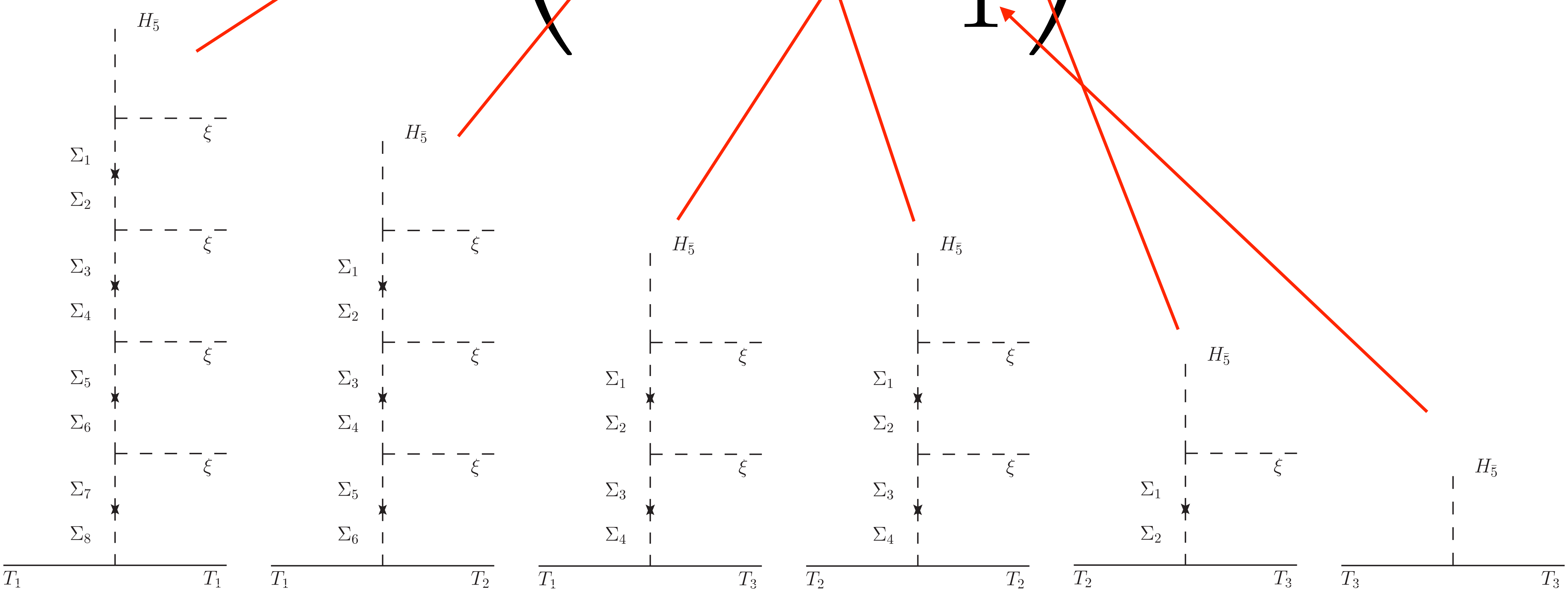
Higgs Messengers

Field	Representation				
	A_4	SU(5)	$\mathbb{Z}_9$	$\mathbb{Z}_6$	$\mathbb{Z}_4^R$
X_1	1	$\bar{5}$	7	0	1
X_2	1	5	2	0	1
X_3	1	$\bar{5}$	6	0	1
X_4	1	5	3	0	1
X_5	$1''$	$\bar{5}$	3	0	1
X_6	$1'$	5	6	0	1
X_7	1	$\bar{5}$	2	0	1
X_8	$1''$	5	7	0	1
X_9	$1'$	$\bar{5}$	0	0	1
X_{10}	$1'$	5	0	0	1
X_{11}	1	$\bar{5}$	1	3	1
X_{12}	1	5	7	5	1
X_{13}	1	$\bar{5}$	2	3	1
X_{14}	1	5	6	5	1
Σ_1	1	$\bar{5}$	7	0	2
Σ_2	1	5	2	0	0
Σ_3	1	$\bar{5}$	5	0	2
Σ_4	1	5	4	0	0
Σ_5	1	$\bar{5}$	3	0	2
Σ_6	1	5	6	0	0
Σ_7	1	$\bar{5}$	1	0	2
Σ_8	1	5	8	0	0
Σ_9	1	$\bar{5}$	8	0	2
Σ_{10}	1	5	1	0	0
Σ_{11}	1	$\bar{5}$	6	0	2
Σ_{12}	1	5	3	0	0
Σ_{13}	1	$\bar{5}$	4	0	2
Σ_{14}	1	5	5	0	0
Σ_{15}	1	$\bar{5}$	2	0	2
Σ_{16}	1	5	7	0	0

Up-type quarks

Froggatt-Nielsen

$$Y^u_{ij} \sim \begin{pmatrix} \tilde{\xi}^4 & \tilde{\xi}^3 & \tilde{\xi}^2 \\ & \tilde{\xi}^2 & \tilde{\xi} \\ & & 1 \end{pmatrix}$$

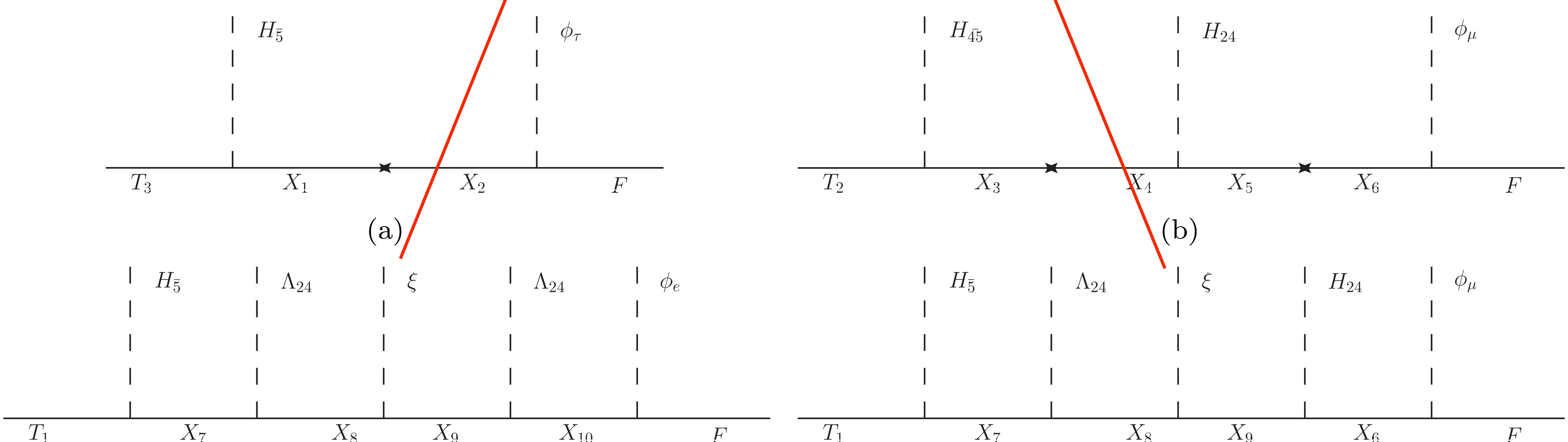


$$\langle \phi_e \rangle = v_e \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \langle \phi_\mu \rangle = v_\mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \langle \phi_\tau \rangle = v_\tau \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Down-type quarks

Charged Leptons

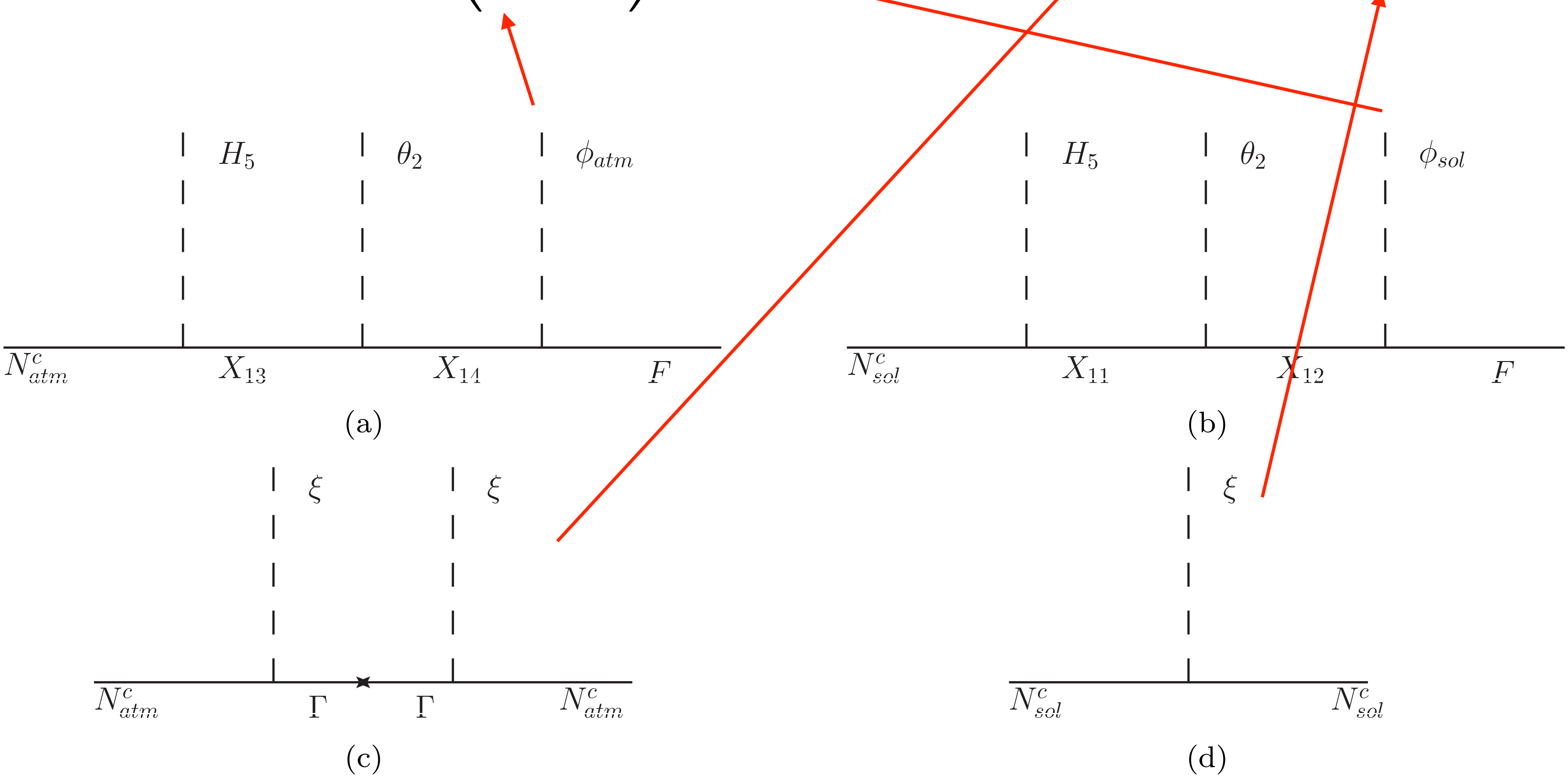
$$Y^d_{LR} \sim Y^e_{RL} \sim \begin{pmatrix} \frac{\langle \xi \rangle v_e}{v^2_{\Lambda_{24}}} & \frac{\langle \xi \rangle v_\mu}{v_{\Lambda_{24}} v_{H_{24}}} & 0 \\ 0 & \frac{v_{H_{24}} v_\mu}{M^2} & 0 \\ 0 & 0 & \frac{v_\tau}{M} \end{pmatrix}$$



Neutrinos with Littlest Seesaw

$$\langle \phi_{\rm atm} \rangle = v_{\rm atm} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \langle \phi_{\rm sol} \rangle = v_{\rm sol} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$m_D = \begin{pmatrix} 0 & b \\ a & 3b \\ a & b \end{pmatrix} \qquad M_R = \begin{pmatrix} M_{\rm atm} & 0 \\ 0 & M_{\rm sol} \end{pmatrix}$$



Summary of A_4 x SU(5)

- Explains quark mass hierarchies, mixing angles and the CP phase.
- Reproduces Littlest Seesaw model predictions
- Z_9 flavour symmetry fixes the phase η to be 2pi/3
- Leptogenesis fixes $M_{atm} \sim 10^{10}$ GeV
- Renormalisable at GUT scale, SU(5) breaking potential, spontaneously broken CP.
- The MSSM is reproduced with R-parity from discrete Z_4^R .
- Doublet-triplet splitting via the Missing Partner mechanism.
- mu term is generated at the correct scale.
- Proton decay is sufficiently suppressed.
- Solves strong CP problem through the Nelson-Barr mechanism .

Questions?

Tutorial Questions

2. Consider a *Dirac neutrino* mass model involving *one* right-handed neutrino $\nu_R^{\rm atm}$ with Yukawa couplings [4],

$$\overline{\nu_R^{\rm atm}}(dL_e + eL_\mu + fL_\tau)H, \tag{7}$$

where $L_e = (\nu_e, e)_L$, etc., H is the Higgs doublet and d, e, f are real Yukawa couplings.

(a) When the Higgs gets a VEV in its first component, explain why this model leads to *one massive Dirac neutrino*, together with *two massless neutrinos.*

(b) If we interpret the massive neutrino as the *atmospheric neutrino*, show that left-handed component can be parametrized in terms of two angles θ_{13} and θ_{23} as

$$\nu_L^{\rm atm} = s_{13}\nu_{eL} + s_{23}c_{13}\nu_{\mu L} + c_{23}c_{13}\nu_{\tau L}. \tag{8}$$

where $\nu_L^{\rm atm}$ is *correctly normalised* ($s_{13} = \sin\theta_{13}$, etc.). Then, by comparing the above parametrisation of $\nu_L^{\rm atm}$ to the third column of the PMNS matrix (with zero CP phase), explain why θ_{13} is the reactor angle and θ_{23} is the atmospheric angle.

$$\overline{\nu_R^{\rm atm}}(dL_e + eL_\mu + fL_\tau)H, \tag{7}$$

$$\nu_L^{\rm atm} = s_{13}\nu_{eL} + s_{23}c_{13}\nu_{\mu L} + c_{23}c_{13}\nu_{\tau L}. \tag{8}$$

(c) Using Eqs.7 and 8, find expressions for the sine of the reactor angle $\sin\theta_{13}$ and the tangent of the atmospheric angle $\tan\theta_{23}$ in terms of the Yukawa couplings d, e, f.

(d) If the solar neutrino is identified as one of the massless neutrinos, explain why the solar angle θ_{12} is not well defined in this model.

3. Consider a *see-saw* neutrino model involving *two* right-handed neutrinos $\nu_R^{\rm sol}$ and $\nu_R^{\rm atm}$ with Yukawa couplings [5],

$$\overline{\nu_R^{\rm sol}}(aL_e + bL_\mu + cL_\tau)H + \overline{\nu_R^{\rm atm}}(dL_e + eL_\mu + fL_\tau)H, \tag{9}$$

and heavy right-handed Majorana masses,

$$M_{\rm sol}\overline{\nu_R^{\rm sol}}(\nu_R^{\rm sol})^c + M_{\rm atm}\overline{\nu_R^{\rm atm}}(\nu_R^{\rm atm})^c. \tag{10}$$

(a) After the Higgs gets a VEV in its first component, write down the Dirac mass matrix m^D_{RL}.

(b) Write down the (diagonal) right-handed neutrino heavy Majorana mass matrix M_{RR}.

(c) Using the see-saw formula, $m^\nu = (m^D_{RL})^T M^{-1}_{RR} m^D_{RL}$, calculate the light effective left-handed Majorana neutrino mass matrix m^ν (i.e. the physical neutrino mass matrix).

(d) Assuming that the determinant of m^ν vanishes (which you may if you wish check by explicit calculation) what is the physical implication of this?

(e) Imposing the constraints $d = 0$ and $e = f$, with $a = b = -c$ known as "constrained sequential dominance" [6], show that the resulting physical neutrino mass matrix m^ν is diagonalised by the tri-bimaximal mixing matrix, $U_{\mathrm{TB}}^T m^\nu U_{\mathrm{TB}}$. What is the physical interpretation of this result if the charged lepton mass matrix is diagonal?

Tutorial Questions

1. The PMNS matrix for Dirac neutrinos is [1],

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}, \quad (1)$$

where $s_{13} = \sin\theta_{13}$, etc.

(a) Show that tri-bimaximal mixing defined by

$$s_{13} = 0,\ s_{12} = \frac{1}{\sqrt{3}},\ s_{23} = \frac{1}{\sqrt{2}}, \quad (2)$$

implies the tri-bimaximal (TB) mixing matrix,

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (3)$$

(b) Consider the reactor, solar and atmospheric parameters r,s,a which parameterise the deviations from tri-bimaximal mixing [2],

$$s_{13} = \frac{r}{\sqrt{2}},\ s_{12} = \frac{(1+s)}{\sqrt{3}},\ s_{23} = \frac{(1+a)}{\sqrt{2}}. \quad (4)$$

By expanding the PMNS mixing matrix to first order in the small parameters r, s, a, it is possible to show (although you do not need to do this) that,

$$U \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1 - \frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+r\cos\delta) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-r\cos\delta) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}r\cos\delta) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}. \quad (5)$$

Verify that for TB mixing $r = s = a = 0$, the mixing matrix reduces to $U_{\rm TB}$.

Show that, for $s \approx 0, \quad a \approx r\cos\delta$, the first column of the mixing matrix approximately corresponds to that of TB mixing (TM1 mixing).

Similarly show that for $s \approx 0, \quad a \approx -(r/2)\cos\delta$, the second column of the mixing matrix approximately corresponds to that of TB mixing (TM2 mixing).

(c) Show that the relations $a \approx r\cos\delta$ and $a \approx -(r/2)\cos\delta$ imply the approximate "atmospheric sum rules" of the form,

$$\theta_{23} - 45^\circ \approx C \times \theta_{13}\cos\delta \tag{6}$$

and find the constant C in each case. [**Hint**: take the sine of both sides of the Eq.6, assuming $\sin\theta_{13} \approx \theta_{13}$, then expand $\sin(\theta_{23} - 45^\circ)$ and use definitions of r, a.]

Then discuss how well these so called "atmospheric sum rules" are satisfied by current data on the atmospheric and reactor mixing angles and how future precision measurements of these angles will fix the CP violating phase δ [3].

(d) If the charged lepton mixing matrix has a Cabibbo-like mixing angle [1],

$$U_e = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

calculate the (1,3), (3,1) and (3,3) elements of PMNS matrix $U = U_e U_{\rm TB}$ (you don't need to calculate the whole matrix). Comparing the absolute value of the (1,3) element to that of the standard parameterisation of the PMS matrix, find s_{13} in terms of s^e_{12} and show that choosing $\theta^e_{12} = \theta_C \approx 13^\circ$ (the Cabibbo angle) gives a reasonable value for the reactor angle [7]. Comparing the absolute value of the (3,1) and (3,3) elements to that of the standard parameterisation of the PMS matrix, find relations between PMNS parameters. By combining and expanding these relations show that they lead to the approximate "solar sum rule",

$$\theta_{12} - 35^\circ \approx \theta_{13}\cos\delta, \tag{8}$$

[**Hint**: take the sine of both sides of the Eq.8, assuming $\sin\theta_{13} \approx \theta_{13}$ as well as $\sin 35^\circ \approx 1/\sqrt{3}$.] Discuss the resulting prediction for the CP phase δ [7].

2. Consider a *Dirac neutrino* mass model involving *one* right-handed neutrino $\nu_R^{\rm atm}$ with Yukawa couplings [4],

$$\overline{\nu_R^{\rm atm}}(dL_e + eL_\mu + fL_\tau)H, \tag{9}$$

where $L_e = (\nu_e, e)_L$, etc., H is the Higgs doublet and d, e, f are real Yukawa couplings.

(a) When the Higgs gets a VEV in its first component, explain why this model leads to *one massive Dirac neutrino*, together with *two massless neutrinos*.

(b) If we interpret the massive neutrino as the *atmospheric neutrino*, show that left-handed component can be parametrized in terms of two angles θ_{13} and θ_{23} as

$$\nu_L^{\rm atm} = s_{13}\nu_{eL} + s_{23}c_{13}\nu_{\mu L} + c_{23}c_{13}\nu_{\tau L}. \tag{10}$$

where $\nu_L^{\rm atm}$ is *correctly normalised* ($s_{13} = \sin\theta_{13}$, etc.). Then, by comparing the above parametrisation of $\nu_L^{\rm atm}$ to the third column of the PMNS matrix (with zero CP phase), explain why θ_{13} is the reactor angle and θ_{23} is the atmospheric angle.

(c) Using Eqs.9 and 10, find expressions for the sine of the reactor angle $\sin\theta_{13}$ and the tangent of the atmospheric angle $\tan\theta_{23}$ in terms of the Yukawa couplings d, e, f.

(d) If the solar neutrino is identified as one of the massless neutrinos, explain why the solar angle θ_{12} is not well defined in this model.

3. Consider a *see-saw* neutrino model involving *two* right-handed neutrinos $\nu_R^{\rm sol}$ and $\nu_R^{\rm atm}$ with Yukawa couplings [5],

$$\overline{\nu_R^{\rm sol}}(aL_e + bL_\mu + cL_\tau)H + \overline{\nu_R^{\rm atm}}(dL_e + eL_\mu + fL_\tau)H, \tag{11}$$

and heavy right-handed Majorana masses,

$$M_{\rm sol}\overline{\nu_R^{\rm sol}}(\nu_R^{\rm sol})^c + M_{\rm atm}\overline{\nu_R^{\rm atm}}(\nu_R^{\rm atm})^c. \tag{12}$$

(a) After the Higgs gets a VEV in its first component, write down the Dirac mass matrix m_{RL}^D.

(b) Write down the (diagonal) right-handed neutrino heavy Majorana mass matrix M_{RR}.

(c) Using the see-saw formula, $m^\nu = (m_{RL}^D)^T M_{RR}^{-1} m_{RL}^D$, calculate the light effective left-handed Majorana neutrino mass matrix m^ν (i.e. the physical neutrino mass matrix).

(d) Assuming that the determinant of m^ν vanishes (which you may if you wish check by explicit calculation) what is the physical implication of this?

(e) Imposing the constraints $d = 0$ and $e = f$, with $a = b = -c$ known as "constrained sequential dominance" [6], show that the resulting physical neutrino mass matrix m^ν is diagonalised by the tri-bimaximal mixing matrix, $U_{\rm TB}^T m^\nu U_{\rm TB}$. What is the physical interpretation of this result if the charged lepton mass matrix is diagonal?

References

[1] http://arxiv.org/abs/arXiv:1301.1340

[2] http://arxiv.org/pdf/0710.0530.pdf

[3] http://arxiv.org/abs/arXiv:1308.4314

[4] http://arxiv.org/pdf/hep-ph/9806440.pdf

[5] http://arxiv.org/pdf/hep-ph/9912492.pdf

[6] http://arxiv.org/abs/hep-ph/0506297

[7] http://arxiv.org/pdf/1205.0506.pdf

Solutions

1. (a) This is simply a matter of substituting the expressions into the PMNS matrix, using $c_{13} = (1 - s_{13}^2)^{1/2}$, etc.

   (b) For $r = s = a = 0$ the mixing matrix reduces to the TB matrix,

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{1}$$

   Assuming $s \approx 0, \quad a \approx r \cos\delta$, we find the TM1 matrix,

$$U_{\rm TM1} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix}. \tag{2}$$

   With $s \approx 0, \quad a \approx -(r/2)\cos\delta$, we find the TM2 matrix,

$$U_{\rm TM2} \approx \begin{pmatrix} - & \frac{1}{\sqrt{3}} & - \\ - & \frac{1}{\sqrt{3}} & - \\ - & -\frac{1}{\sqrt{3}} & - \end{pmatrix}. \tag{3}$$

   (c) Following the hint, one finds,

$$a \approx r\cos\delta \longleftrightarrow \theta_{23} - 45^\circ \approx \sqrt{2}\theta_{13}\cos\delta \tag{4}$$

$$a \approx -(r/2)\cos\delta \longleftrightarrow \theta_{23} - 45^\circ \approx -\frac{\theta_{13}}{\sqrt{2}}\cos\delta \tag{5}$$

   i.e. $C = \sqrt{2}$ and $C = -1/\sqrt{2}$.

   Current data may involve for example $\theta_{23} = 40^\circ - 50^\circ$ and $\theta_{13} = 8^\circ - 9^\circ$, leading to $|\theta_{23} - 45^\circ| \lesssim 5^\circ$ and hence constraints on the two sum rules, which can be solved for $\cos\delta$ in terms of the measured angles. (This is a rather open ended question which the students can discuss in various ways in detail).

(d) If the charged lepton mixing matrix involves a Cabibbo-like mixing, then the PMNS matrix is given by,

$$U_{\rm PMNS} = \begin{pmatrix} c^e_{12} & s^e_{12}e^{-i\delta^e_{12}} & 0 \\ -s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \frac{s^e_{12}}{\sqrt{2}}e^{-i\delta^e_{12}} \\ \cdots & \cdots & \frac{c^e_{12}}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Comparing to the PMNS parametrisation we identify,

$$s_{13} = \frac{s^e_{12}}{\sqrt{2}}\,, \tag{6}$$

$$|s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta}| = \frac{1}{\sqrt{6}}\,, \tag{7}$$

$$c_{13}c_{23} = \frac{1}{\sqrt{2}}. \tag{8}$$

The first equation a reactor angle $\theta_{13} \approx 9.2°$ if $\theta_e \approx \theta_C \approx 13°$ [7]. The second and third equations allow to eliminate θ_{23} to give a new relation between the PMNS parameters, called a solar sum rule, which may be expanded to first order to give the approximate relation,

$$\theta_{12} - 35° \approx \theta_{13}\cos\delta, \tag{9}$$

or,

$$\cos\delta \approx \frac{\theta_{12} - 35°}{\theta_{13}}. \tag{10}$$

This highlights the importance of an accurate measurement of the solar angle in order to predict the CP phase. Current data on the solar and reactor angles seems to predict $\cos\delta \approx 0$ or $\delta \approx \pm 90°$, consistent with the experimental hint for the CP phase $\delta \approx -90°$.

2. (a) Inserting the Higgs VEV, $\nu_R^{\rm atm}$ only couples to one linear combination of left-handed neutrinos,

$$\nu_L^{\rm atm} \propto d\nu_{eL} + e\nu_{\mu L} + f\nu_{\tau L}, \tag{11}$$

and the two orthogonal combinations must therefore be massless because they have no couplings to the single right-handed neutrino.

(b) To check the normalisation we consider the product,

$$\begin{aligned} \nu_L^{\rm atm}.\nu_L^{\rm atm} &= (s_{13}\nu_{eL} + s_{23}c_{13}\nu_{\mu L} + c_{23}c_{13}\nu_{\tau L}).(s_{13}\nu_{eL} + s_{23}c_{13}\nu_{\mu L} + c_{23}c_{13}\nu_{\tau L}) \\ &= s_{13}^2 + (s_{23}c_{13})^2 + (c_{23}c_{13})^2 = 1, \end{aligned} \tag{12}$$

where we have used results like $\nu_{eL}.\nu_{eL} = 1$ and $\nu_{eL}.\nu_{\mu L} = 0$, etc. Comparing to the PMNS matrix, one can see that the third column with zero phase $\delta = 0$ is identical to the parameterisation in Eq.**??**, hence we can identify θ_{13} as the reactor angle and θ_{23} as the atmospheric angle.

(c) Including a normalisation factor we have from Eq.**??**,

$$\nu_L^{\rm atm} = \frac{1}{\sqrt{d^2 + e^2 + f^2}}(d\nu_{eL} + e\nu_{\mu L} + f\nu_{\tau L}). \tag{13}$$

Comparing the coefficients of $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$ to those in the parametrisation in Eq.**??**,

$$\nu_L^{\rm atm} = s_{13}\nu_{eL} + s_{23}c_{13}\nu_{\mu L} + c_{23}c_{13}\nu_{\tau L}, \tag{14}$$

we read-off the results,

$$s_{13} = \frac{d}{\sqrt{d^2 + e^2 + f^2}}, \quad s_{23}c_{13} = \frac{e}{\sqrt{d^2 + e^2 + f^2}}, \quad c_{23}c_{13} = \frac{f}{\sqrt{d^2 + e^2 + f^2}}. \tag{15}$$

Taking the ratio of the last two terms,

$$t_{23} = \frac{e}{f}. \tag{16}$$

(d) The solar neutrino state $\nu_L^{\rm sol}$ is not uniquely specified, since it is degenerate with another massless state, hence the solar mixing angle θ_{12} is not well defined.

3. (a) In the basis, with rows $(\overline{\nu_R^{\rm sol}}, \overline{\nu_R^{\rm atm}})^T$ and columns $(\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})$, the Dirac mass matrix is,

$$m^D_{RL} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}. \tag{17}$$

(b) The Majorana mass matrix with rows $(\overline{\nu_R^{\rm sol}}, \overline{\nu_R^{\rm atm}})^T$ and columns $(\nu_R^{\rm sol}, \nu_R^{\rm atm})$,

$$M_{RR} = \begin{pmatrix} M_{\rm sol} & 0 \\ 0 & M_{\rm atm} \end{pmatrix}. \tag{18}$$

(c) Then by multiplying the matrices we find,

$$m^\nu = (m^D_{RL})^T M^{-1}_{RR} m^D_{RL} = \begin{pmatrix} \frac{a^2}{M_{\rm sol}} + \frac{d^2}{M_{\rm atm}} & \frac{ab}{M_{\rm sol}} + \frac{de}{M_{\rm atm}} & \frac{ac}{M_{\rm sol}} + \frac{df}{M_{\rm atm}} \\ \frac{ab}{M_{\rm sol}} + \frac{de}{M_{\rm atm}} & \frac{b^2}{M_{\rm sol}} + \frac{e^2}{M_{\rm atm}} & \frac{bc}{M_{\rm sol}} + \frac{ef}{M_{\rm atm}} \\ \frac{ac}{M_{\rm sol}} + \frac{df}{M_{\rm atm}} & \frac{bc}{M_{\rm sol}} + \frac{ef}{M_{\rm atm}} & \frac{c^2}{M_{\rm sol}} + \frac{f^2}{M_{\rm atm}} \end{pmatrix}. \tag{19}$$

(d) By explicit calculation, one can check that $\det m^\nu = 0$. Since the determinant of a real symmetric matrix is the product of mass eigenvalues

$$\det m^\nu = m_1 m_2 m_3, \tag{20}$$

one may conclude that one of the masses is zero, which we take to be the lightest one $m_1 = 0$.

(e) Setting $d = 0$ and $e = f$, with $a = b = -c$, one finds,

$$m^\nu = \begin{pmatrix} \frac{a^2}{M_{\rm sol}} & \frac{a^2}{M_{\rm sol}} & \frac{-a^2}{M_{\rm sol}} \\ \frac{a^2}{M_{\rm sol}} & \frac{a^2}{M_{\rm sol}} + \frac{e^2}{M_{\rm atm}} & \frac{-a^2}{M_{\rm sol}} + \frac{e^2}{M_{\rm atm}} \\ \frac{-a^2}{M_{\rm sol}} & \frac{-a^2}{M_{\rm sol}} + \frac{e^2}{M_{\rm atm}} & \frac{a^2}{M_{\rm sol}} + \frac{e^2}{M_{\rm atm}} \end{pmatrix}. \tag{21}$$

By explicit calculation one finds,

$$U^T_{\rm TB} m^\nu U_{\rm TB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3a^2}{M_{\rm sol}} & 0 \\ 0 & 0 & \frac{2e^2}{M_{\rm atm}} \end{pmatrix}. \tag{22}$$

If the charged lepton mass matrix is diagonal, the interpretation is that these constrained couplings lead to TB mixing, with the lightest neutrino mass $m_1 = 0$, the second lightest neutrino identified as the solar neutrino with mass $m_2 = \frac{3a^2}{M_{\rm sol}}$ and the heaviest neutrino identified as the atmospheric neutrino with mass $m_3 = \frac{2a^2}{M_{\rm atm}}$. Note that each of the right-handed neutrinos contributes uniquely to a particular physical neutrino mass. This general feature is known as sequential dominance and the particular example with constrained couplings is known as constrained sequential dominance [5].

References

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