TRUNCATED MOMENTS OF PARTON DISTRIBUTIONS

TAMPERE – EPS/HEP 99
16/7/1999.

TRUNCATED MOMENTS : an (efficient) method to evolve parton distributions in Mellin space, with the desired precision, without relying on extrapolations of the data to the $x \to 0$ region.

DEFINITION : for a parton distribution $f(x,\mu^2)$:

$$f_N(x_0,\mu^2) \equiv \int_{x_0}^{1} dx \; x^{N-1} f(x,\mu^2)$$

$$\Rightarrow \; f_N(0,\mu^2) = f_N(\mu^2)$$ (the usual Mellin moment).

NOTE : This truncation is consistent with evolution (there is no loss of information), since AP evolution is directional ($\partial f(x,\mu^2)/\partial\mu^2$ is given in terms of $f(y,\mu^2)$ with $y > x$). A similar truncation at $x_1 < 1$ would not share this property.

A walk through conventional AP evolution in Mellin space

$\boxed{\text{NON-SINGLET}}$:

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x,\mu^2) = \frac{\alpha_s(\mu)}{2\pi} \int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}, \alpha_s(\mu)\right) q(y,\mu^2)$$

- $$P_{qq}(z,\mu) = P_{qq}^{(0)}(z) + \frac{\alpha_s(\mu)}{2\pi} P_{qq}^{(1)}(z) + \ldots$$
- $$q_N(\mu^2) \equiv \int_0^1 dx\, x^{N-1} q(x,\mu^2) \quad ; \quad \gamma_N(\alpha_s(\mu)) \equiv \int_0^1 dx\, x^{N-1} P_{qq}(x,\alpha_s(\mu))$$

$$\Rightarrow \quad \mu^2 \frac{\partial}{\partial \mu^2} q_N(\mu^2) = \frac{\alpha_s(\mu)}{2\pi} \gamma_N(\alpha_s(\mu))\, q_N(\mu^2)$$

- NLO solution :

$$\frac{q_N(\mu^2)}{q_N(\mu_0^2)} = \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right]^{\gamma_N^{(0)}/b_0} \left(1 + \frac{b_0 \gamma_N^{(1)} - b_1 \gamma_N^{(0)}}{2\pi b_0^2} \left(\alpha_s(\mu_0) - \alpha_s(\mu)\right)\right)$$

$\boxed{\text{SINGLET}}$:

$$P_{qq} \longrightarrow \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \quad ; \quad q \longrightarrow \begin{pmatrix} q_s \\ g \end{pmatrix}$$

- Diagonalize $\gamma^{(0)}_{N,ij}$: $$R_N \gamma_N^{(0)} R_N^{-1} = \mathrm{diag}\left(\hat{\gamma}^{(0)}_{i,N}\right)$$
- Rotate $\gamma^{(1)}_{N,ij}$: $$R_N \gamma_N^{(1)} R_N^{-1} \equiv \Gamma_N^{(1)}$$
- NLO solution :

$$\hat{f}_{i,N}(\mu^2) \equiv (R_N)_i{}^j f_{j,N}(\mu^2) = \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right]^{\hat{\gamma}^{(0)}_{i,N}/b_0} \left(1 - \frac{\hat{\gamma}^{(0)}_{i,N}\, b_1}{2\pi b_0^2} \left(\alpha_s(\mu_0) - \alpha_s(\mu)\right)\right) \times \hat{f}_{i,N}(\mu_0^2) +$$

$$+ \sum_j \frac{\Gamma^{(1)}_{ij,N}}{2\pi\left(\hat{\gamma}^{(0)}_{i,N} - \hat{\gamma}^{(0)}_{j,N} + b_0\right)} \left[\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{\hat{\gamma}^{(0)}_{i,N}/b_0} \alpha_s(\mu_0) - \left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{\hat{\gamma}^{(0)}_{j,N}/b_0} \alpha_s(\mu)\right] \times \hat{f}_{j,N}(\mu$$

TRUNCATED MOMENTS : EVOLUTION

(non singlet)

The AP equation for $q_N(x_0, \mu^2)$ becomes:

$$\mu^2 \frac{\partial}{\partial \mu^2} q_N(x_0, \mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \int_{x_0}^{1} dy \; y^{N-1} \, q(y, \mu^2) \; G_N\left(\frac{x_0}{y}, \alpha_s(\mu)\right)$$

and is NOT diagonalized because of the y dependence in the function $G_N(x_0/y, \alpha_s(\mu))$:

$$G_N(x) \equiv \int_x^1 dz \; z^{N-1} P_{qq}(z)$$

$$\Rightarrow \quad G_N(0) = \gamma_N$$

However :

- Each truncated moment is coupled only to moments with LARGER n.
- The series of couplings of moment N to moment $N+K$ decreases rapidly with K.
- The exact evolution of the Nth truncated moment can be approximated at will by coupling it only with a finite number of higher moments.

In fact :

$$G_N\left(\frac{x_0}{y}\right) = \sum_{n=0}^{\infty} \frac{g_N^{(n)}(x_0)}{n!} (y-1)^n$$

$$g_N^{(n)}(x_0) = \frac{\partial^n}{\partial y^n} G_N\left(\frac{x_0}{y}\right)\Big|_{y=1}$$

The expansion has radius of convergence $1-x_0$ because of the presence of + distributions in $P(z)$. It can be truncated at a given term, say the Mth.

Approximating $G_N\left(\frac{x_0}{y}\right)$ with the first M terms of its Taylor expansion around [illegible], and changing order of $\int$, one finds:

$$\mu^2 \frac{\partial}{\partial \mu^2}\, q_N(x_0,\mu^2) = \frac{\alpha_s(\mu)}{2\pi} \sum_{K=0}^{M} C_{K,N}^{(M)}(x_0)\; q_{N+K}(x_0,\mu^2)$$

where
$$C_{K,N}^{(M)}(x_0) = \sum_{p=K}^{M} \frac{(-1)^{K+p}\, g_N^{(p)}(x_0)}{K!\,(p-K)!}$$

One gets a solvable system of $M+1$ linear differential equations for the moments $q_N, \dots, q_{N+M}$ PROVIDED it makes sense to consider a decreasing number of terms in the Taylor expansion of G_{N+K} for increasing K.
($M+1$ terms for q_N, $M+1-K$ terms for q_{N+K})

The system of equations has then an UPPER TRIANGULAR matrix of coefficients.

To verify that the approximation makes sense, one can consider the effect of the truncation [illegible] on the AP equation, given by

$$R(N,M;x_0,\mu^2) \equiv \frac{1}{\mathcal{N}} \int_{x_0}^{1} dy\, y^{N-1}\, q(y,\mu^2) \left[G_N\left(\frac{x_0}{y}\right) - \sum_{K=0}^{M} C_{K,N}^{(M)}(x_0)\, y^K \right]$$

$$\mathcal{N} = \int_{x_0}^{1} dy\, y^{N-1}\, q(y,\mu^2)\, G_N\left(\frac{x_0}{y}\right)$$

$R(N,M;x_0,\mu^2)$ is a RAPIDLY DECREASING function of N for all reasonable (small) values of x_0. In fact $G_N\left(\frac{x_0}{y}\right)$ is very nearly constant for small x_0, except for the logarithmic singularities at $y=x_0$, suppressed (increasingly with N) by the factor y^{N-1}.

x_0	0.01	0.03	0.1	0.01	0.03	0.1
N	LO			NLO		
2	$6.3\,10^{-3}$	$3.3\,10^{-2}$	$1.5\,10^{-1}$	$3.5\,10^{-3}$	$2.7\,10^{-2}$	$2.0\,10^{-1}$
3	$1.0\,10^{-4}$	$1.7\,10^{-3}$	$3.0\,10^{-2}$	$6.3\,10^{-5}$	$2.8\,10^{-3}$	$3.3\,10^{-2}$
4	$1.7\,10^{-6}$	$8.6\,10^{-5}$	$5.1\,10^{-3}$	$1.1\,10^{-6}$	$6.9\,10^{-5}$	$5.5\,10^{-3}$
5	$2.7\,10^{-8}$	$4.1\,10^{-6}$	$8.3\,10^{-4}$	$1.8\,10^{-8}$	$3.3\,10^{-6}$	$8.7\,10^{-4}$

Table 1: The percentage error function $R(N, M; x_0, Q^2)$ defined in Eq. (10), computed from the LO and NLO contributions to the nonsinglet splitting function in the DIS scheme, with $M = 5 - N$, the values of N and x_0 shown, $Q^2 = 2.56$ GeV^2 and nonsinglet quark distribution from the CTEQ4D parton set [12].

SOLUTION OF THE EVOLUTION EQUATION

(NLO, NS)

The equation for the evolution of the N.th moment is of the form:

$$\frac{dq_K}{dt} = \frac{\alpha_s(\mu)}{2\pi} \sum_{L=N_0}^{N_0+M} \left[C^{(0)}_{KL} + \frac{\alpha_s(\mu)}{2\pi} C^{(1)}_{KL} \right] q_L$$

with $t = \ln \frac{\mu^2}{\mu_0^2}$, $N_0 \le K, L \le N_0 + M$, while $C^{(0)}$, $C^{(1)}$ are UPPER TRIANGULAR $(M+1)\times(M+1)$ matrices depending on x.

NOTE:

- Because of the smallness of the error function R, for $N_0 \gtrsim 2$, $M \sim 4-5$ is sufficient.
- The implementation of the evolution is greatly simplified by the nice properties of triangular matrices.

LO SOLUTION Diagonalize $C^{(0)}$ (the eigenvalues ARE the diagonal elements $C^{(0)}_{KK}$).

$$\hat{q}_K \equiv \sum_{L=N_0}^{N_0+M} R_{KL}\, q_L \quad \Rightarrow \quad \hat{q}_K(\mu) = \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right]^{C^{(0)}_{KK}/b_0} \hat{q}_K(\mu_0)$$

NLO SOLUTION Rotate $C^{(1)}$ with the matrix R used to diagonalize $C^{(0)}$.

$$\hat{D} \equiv R\, C^{(1)} R^{-1} \quad \Rightarrow$$

$$\Rightarrow \quad \hat{q}_K(\mu) = \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right]^{C^{(0)}_{KK}/b_0} \left[1 - \frac{C^{(0)}_{KK}\, b_1}{2\pi\, b_0^2} \left(\alpha_s(\mu_0) - \alpha_s(\mu) \right) \right] \hat{q}_K(\mu_0) -$$

$$- \sum_L \frac{\hat{D}_{KL}}{2\pi \left(C^{(0)}_{KK} - C^{(0)}_{LL} + b_0 \right)} \left[\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{C^{(0)}_{LL}/b_0} \alpha_s(\mu) - \left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{C^{(0)}_{KK}/b_0} \alpha_s(\mu_0) \right] \hat{q}_L(\mu_0)$$

NOTE: No need to compute determinants to diagonalize a triangular matrix. Recursion relations exist.

SUMMARY

- The AP equation can be diagonalised [illegible] [illegible], (a[illegible]. [illegible]) a truncated Mellin transform $(x_0 < x < 1)$.

- The approximation converges fast and [illegible] efficient implementation. For $x_c \lesssim 0.1$ and $N \gtrsim 2$ the evolution can be solved analytically for arbitrary x_c.

- No extrapolations of the data to $x \to 0$ are needed. [illegible] quite [illegible]

- The formalism can be extended to the singlet sector (work in progress with [illegible] and [illegible])

  There are encounters "black [illegible]" ma[illegible]

- Applications : determination of x_c from ex[illegible]
  gluon distribution
  assessment of uncertainties
  ⋮

.... one more tool in the toolbox ....