Collinear limits of amplitudes in gauge theory

A new way to calculate
the
Altarelli - Parisi Kernels
?

A different approach to calculate the Altarelli-Parisi kernels ?

state of the art

existing method
($\rightsquigarrow$ axial gauge + Mandelstam Leibrandt)

it is believed that in principle
NN calculation is possible

"stand alone" calculation
$\rightsquigarrow$ difficult to check individual parts

<u>new method</u>: use splitting amplitudes

advantages:

- ⊕ second method/calculation
- ⊕ parts of the calculation are "quasi physical" $\rightsquigarrow$ phys. appl., checks
- ⊕ calculation uses no particular gauge

Some notation / tools

colour ordered amplitudes:

generators SU(N)

$$\mathcal{A}_n^{\text{tree}}(\{k_i,\lambda_i,a_i\}) = g_s^{n-2} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}[T^{a_{\sigma(1)}}\dots T^{a_{\sigma(n)}}] \cdot A_n^{\text{tree}}(\sigma(1^{\lambda_1},\dots n^{\lambda_n}))$$

colour ordered ampl.
(gauge independent)

$$\mathcal{A}_n^{\text{1-loop}}(\{k_i\lambda_i a_i\}) = g_s^{n} \sum_{J} n_J \sum_{c=1}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;c}} Gr_{n;c}(\sigma)\, A_{n;c}^{[J]}(\sigma)$$

$$Gr_{n;\sigma}(1,2,\dots,n) = \mathrm{Tr}[T^{a_1}\dots T^{a_{c-1}}]\,\mathrm{Tr}[T^{a_c}\dots T^{a_n}]$$

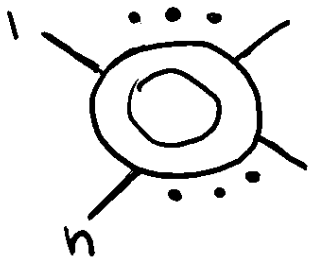
Leading colour:

$$Gr_{n;1} = N_c\,\mathrm{Tr}[T^{a_1}\dots T^{a_n}]$$

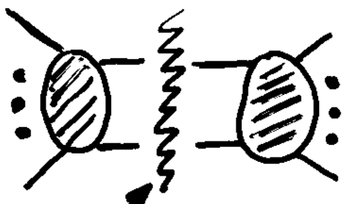
(tree factor)

cut-technique

loop-amplitudes can be obtained from tree amplitudes



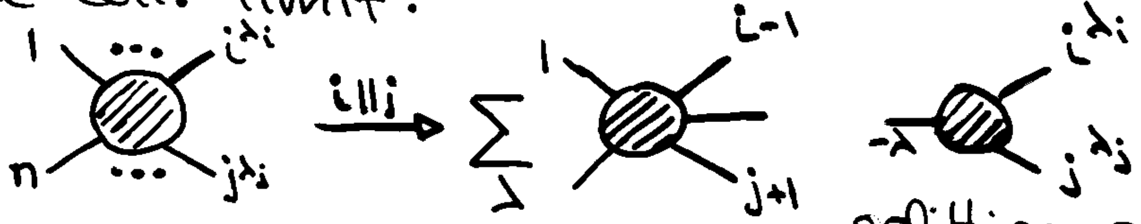
$$= \sum_{\text{cuts}}$$



cut promoted to a loop integral

Factorization of amplitudes at LO

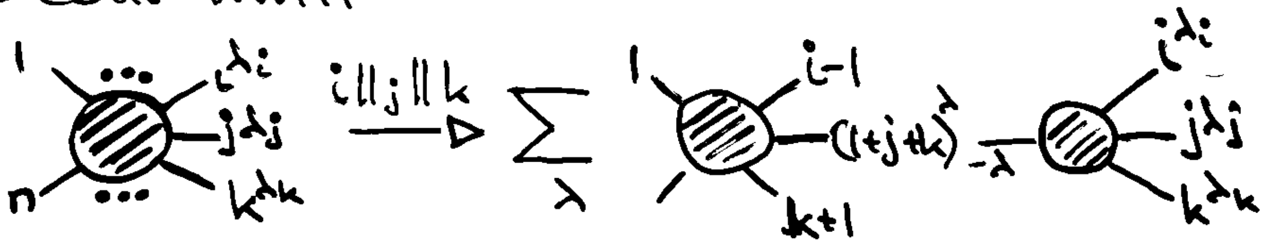
double-coll. limit:



splitting ampl

$\mathrm{Split}^{\text{tree}}_{\lambda}(i^{\lambda_i}, j^{\lambda_j})$

triple-coll. limit



splitting ampl.

$\mathrm{Split}^{\text{tree}}_{\lambda}(i^{\lambda_i}, j^{\lambda_j}, k^{\lambda_k})$

can be proven
by Koba-Nielsen ampl.
or Giele-Berends rec. relation

splitting amplitudes: universal + gauge indep.

Example:

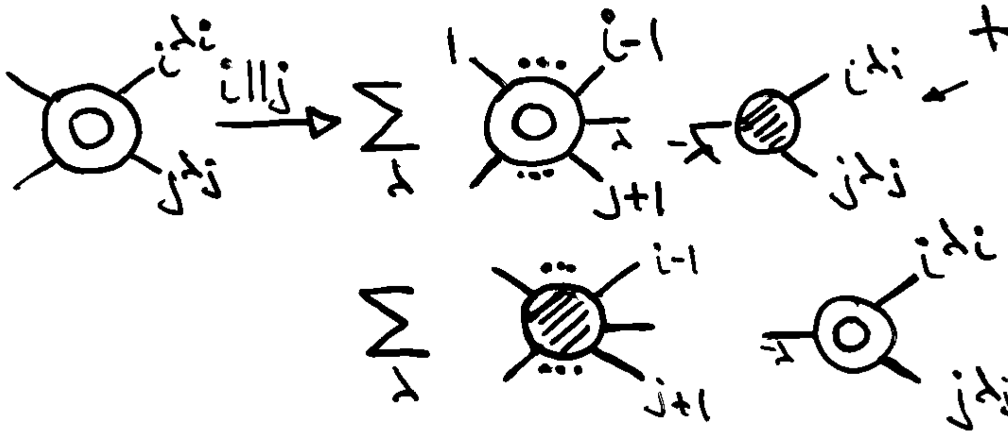
$$\mathrm{Split}_+(i^-, j^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle ij\rangle}$$

with $k_i \approx z(k_i + k_j)$ in the collinear limit

$$\langle ij\rangle = \langle i^- | j^+\rangle$$

spinor product

Factorization at one-loop



tree-splitting amplitude

1-loop splitt. ampl.

$\text{Splitt}^{\text{1-loop}}(i,j)$ can be derived from the study of $A_5^{\text{1-loop}}$

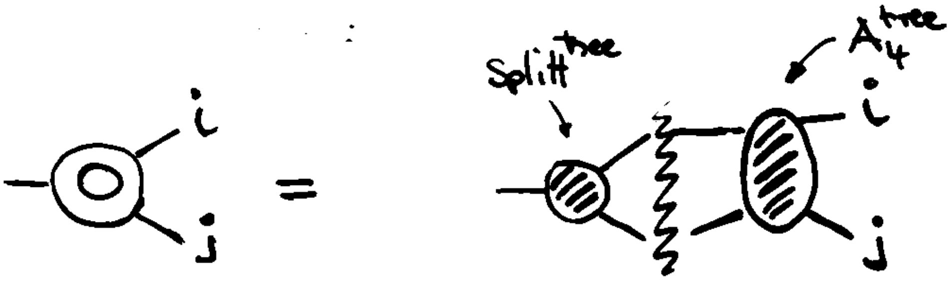
Recently:

Proof of factorization to all orders

and

concrete formula to calculate higher order splitting amplitudes

Basic idea: use cut-technique



calculating the cut to all orders in $\varepsilon = \frac{d-4}{2}$ yields $\mathrm{Splitt}^{1\text{-loop}}(i,j)$

to demonstrate feasibility:

recalculation of all the splitt. ampl. relevant in QC

⇝ proof that new method works

⇝ results are valid to all orders in ε*)
(checked by SUSY Ward-Identities).

* needed for N: LO

Some Results

$$\mathrm{Split}^{\text{1-loop}}(a,b;z)$$

$$= \frac{1}{2}\left(\frac{\mu^2}{-s_{ab}}\right)^{\varepsilon}\left[z\, f_1(z) + (1-z)\cdot f_1(1-z) - 2\cdot f_2\right]\cdot \mathrm{Split}^{\text{tree}}(a,b;z)$$

$$+ \frac{1}{\sqrt{2}}\,\frac{(1-\varepsilon\delta)\cdot\varepsilon^2\cdot f_2}{(1-2\varepsilon)(1-\varepsilon)(3-2\varepsilon)}\left(\frac{\mu^2}{-s_{ab}}\right)\Big(\big((k_a-k_b)\cdot\varepsilon_p\big)\cdot\big(s_{ab}\cdot(\varepsilon_a\cdot\varepsilon_b)-2(k_b\varepsilon_a)(k_a\cdot\varepsilon_b)\big)\Big)$$

$$f_1(z) = \frac{2}{\varepsilon^2}c_\Gamma\left[-\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)z^{-1-\varepsilon}(1-z)^{\varepsilon} - \frac{1}{z} + \frac{(1-z)^{\varepsilon}}{z}\,{}_2F_1(\varepsilon,\varepsilon,1+\varepsilon;z)\right], \quad f_2 = -\frac{1}{\varepsilon^2}c_\Gamma$$

- only two indep. functions f_1, f_2
- valid in FDH ($\delta=0$) and CDR ($\delta=1$)
- formal polarization vectors
- compact formula
- no new functions appear in $\underset{\substack{\uparrow\\ g\to q\bar{q}}}{\mathrm{Split}^{\text{1-loop}}(q,\bar{q})}$, $\underset{\substack{\uparrow\\ g\to gg}}{\mathrm{Split}^{\text{1-loop}}(g,g)}$, ...

Factorization* in the final state

for simplicity

$$\frac{d\sigma(e^+e^- \to H+X)}{dx} \sim \sum_i C_i \otimes D_{i\to H}$$

x = energy fraction

hard scattering coefficient

frag. function

to derive the relation

$$\mathrm{Split}^{\mathrm{tree}} \leftrightarrow P^{LO}_{g\to g}(z)$$

replace all hadronic quantities by partonic ones

→ both sides of the eq. contain coll. singularities

→ matching the singularities yields relation between

$$\mathrm{Split}^{\mathrm{tree}} \leftrightarrow P^{LO}_{g\to g}(z)$$

*) factorization of cross section!

Example: $P^{LO}_{g\to g}$, use $\gamma^* \to Q\bar{Q}g + X$ as specific reaction

$$RS\Big|_{\alpha_s^1,\,\text{sing}} \sim -\frac{1}{\varepsilon}\; \underbrace{P^{LO}_{g\to g}}_{D_{g\to g}} \otimes \int dLips(Q\bar{Q}g)\; \Big|\quad\Big|^2 \cdot \delta(x-x_1)$$

$$LS\Big|_{\alpha_s^2,\,\text{sing}} \sim \int dLips(\bar{Q}Qg)\; 2\,\mathrm{Re}\Big[\quad^{*} \cdot \quad\Big] \cdot \delta(x-x_1)\Big|_{\text{sing}}$$

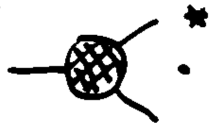
$$+ \int dLips(Q\bar{Q}g_1g_2) \cdot \Big|\quad\Big|^2 \big[\delta(x-x_1) + \delta(x-x_2)\big]\Big|_{\text{sing}}$$

$$= \delta(1-z)\text{-Term} + \underbrace{\int d\sigma_{ab} \sum \big|Split^{tree}\big|^2}_{\frac{1}{\varepsilon}} \otimes \int dLips(Q\bar{Q}g)\; \Big|\quad\Big|^2$$

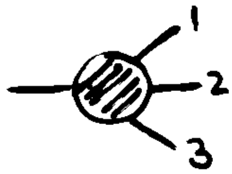
$\rightsquigarrow$ $P^{LO}_{g\to g}(z)$ together with the $\delta(1-z)$ - Term and the + prescription of $\frac{1}{1-z}$

Ingredients for NLO AP kernels

(time-like)

- $\mathrm{Re}\Big[$  $^{*}\cdot$  $\Big]$

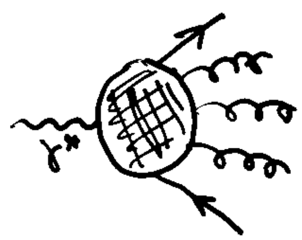
- $P^{LO}_{g\to g} \otimes P^{LO}_{g\to g}$

- $\int d\mathrm{Lips}(k_1,k_2,k_3)\Big|_{\mathrm{coll.}} \Big|$  $\Big|^2$

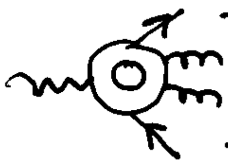
$$\left[\delta(z-z_1)+\delta(z-z_2)+\delta(z-z_3)\right]$$

"double-unresolved"

↳ from the coll. sing. in

$\int d\mathrm{Lips}(Q\bar{Q}g_1g_2g_3)\ \Big|$  $\Big|^2 \left[\delta(z-z_1)+\dots\right]$

and

$\int d\mathrm{Lips}(Q\bar{Q}g_1g_2)\ \mathrm{Re}\Big[$  $\cdot$  $\Big]$

$$
\begin{aligned}
P_{abc\to G} = 8 \times \{ & \\
& \frac{(1-\epsilon)}{s_{ab}^2 s_{abc}^2}\frac{(z_b s_{abc} - (1-z_c)s_{bc})^2}{(1-z_c)^2} + \frac{2(1-\epsilon)s_{bc}}{s_{ab}s_{abc}^2} + \frac{3(1-\epsilon)}{2s_{abc}^2} \\
+ \quad & \frac{1}{s_{ab}s_{abc}}\left(\frac{(1-z_c(1-z_c))^2}{z_c z_a(1-z_a)} - 2\frac{z_b^2 + z_b z_c + z_c}{1-z_c} + \frac{z_b z_a - z_b^2 z_c - 2}{z_c(1-z_c)} + 2\epsilon\frac{z_b}{1-z_c}\right) \\
+ \quad & \frac{1}{2s_{ab}s_{bc}}\left(3z_b^2 - 2\frac{(2-z_a+z_a^2)(z_b^2+z_a(1-z_a))}{z_c(1-z_c)} + \frac{1}{z_c z_a} + \frac{1}{(1-z_c)(1-z_a)}\right)\} \\
+ \quad & (s_{ab} \leftrightarrow s_{bc}, z_a \leftrightarrow z_c)
\end{aligned}
$$

$$
d^d R_{\text{triple}-\text{coll.}} \sim dz_a dz_c ds_{abc} ds_{ab} ds_{bc} \Theta(s_{min} - s_{abc})(-\Delta_4^c)^{-\frac{1}{2}-\epsilon}
$$

$$
\Delta_4^c = ((1-z_a-z_c)s_{ac} - z_a s_{bc} - z_c s_{ab})^2 - 4 z_a z_c s_{ab} s_{bc}
$$

$$\int dz_a dz_b dz_c ds_{abc} ds_{ab} ds_{bc} \delta(1 - z_a - z_b - z_c)\Theta(s_{min} - s_{abc})$$

$$(-\Delta_4^c)^{-\frac{1}{2}-\epsilon}\, P_{G\leftarrow abc}\left[\delta(z - z_a) + \delta(z - z_b) + \delta(z - z_c)\right]$$

$\sim\Bigg($ [redacted] $10\frac{(z^2 - z + 1)^2}{z(1-z)}$ ← must be cancelled !

$+$ [redacted] $\Big(-4\frac{2z^4 - 2z^3 + 5z^2 - 5z + 2}{z(1-z)}\ln(z) - 20\frac{(z^2 - z + 1)^2}{z(1-z)}\ln(1-z)$

$$+\frac{1}{3}\frac{-114z + 135z^2 - 120z^3 + 55 + 55z^4}{z(1-z)}\Big) + \frac{z^4 + 10z^3 - 3z^2 - 8z + 1}{z(1-z)}\ln(z)^2$$

$$+8(1 + 2z)\,\mathrm{Li}_2(z) - 2\frac{(z^2 + z + 1)^2}{z(1+z)}S_2(z) + 20\frac{(z^2 - z + 1)^2}{z(1-z)}\ln(1-z)^2$$

$$-\frac{2}{3}\frac{(-114z + 135z^2 - 120z^3 + 55 + 55z^4)}{z(1-z)}\ln(1-z)$$

$$-\frac{1}{3}\frac{91z^2 - 11z + 88}{z}\ln(z) + 16\frac{(z^2 - z + 1)^2}{z(1-z)}\ln(z)\ln(1-z)$$

$$-\frac{2}{3}\pi^2\frac{3z^4 - 10z^3 + 11z^2 - 4z + 3}{z(1-z)} - \frac{1}{18}\frac{(115z^2 + 49 - 30z)}{1-z} + \frac{67}{19}\frac{2(z^2 - z + 1)^2}{z(1-z)}\Bigg)\frac{1}{\varepsilon}$$

Conclusion
Outlook

- collinear limits of amplitudes well understood: $\text{Split}^{\text{2-loop}}_\lambda(a,b;z)$
  $\rightarrow$ explicit formula for $\text{Split}^{\text{1-loop}}_\lambda(a,b,c,z_a,z_b,z_c)$ 2-loop
- with explicit formula: calculation of $\text{Split}^{\text{tree}}(a,b,z)$ is possible
- find the approach to calculate the Altarelli-Parisi kernels

$\rightsquigarrow$ a lot of work to do