Lattice Gauge Theory

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Introduction

- Lattice QCD provides a non-perturbative framework to compute relations between SM parameters and experimental quantities from first principles

- Formulate QCD on a euclidean space-time lattice with spacing a and volume $L^3 \cdot T$

$$a^{-1} \sim \Lambda_{\mathrm{UV}} \quad \text{(gauge invariant)}$$

$$\langle\Omega\rangle = Z^{-1} \int [dU][d\overline{\psi}][d\psi]\, \Omega\, e^{-S_G - S_F}$$

$\rightarrow$ allows for stochastic evaluation of $\langle\Omega\rangle$

- Ideally want to use lattice QCD as a phenomenological tool, e.g.

$$\left.\begin{array}{l} f_\pi \\ m_{K^+} \end{array}\right\} \mapsto \left\{\begin{array}{l} \alpha_s(M_Z) \\ (m_{\mathrm{u}} + m_{\mathrm{s}})(2\,\mathrm{GeV}) \end{array}\right.$$

- However, realistic simulations of lattice QCD are hard. . .

(1) Lattice artefacts (cutoff effects)

$$\langle\Omega\rangle^{\text{lat}} = \langle\Omega\rangle^{\text{cont}} + O(a^p)$$

$$a^{-1} \approx 1 \Leftrightarrow 4\,\text{GeV} \quad \Leftrightarrow \quad a \approx 0.2 \Leftrightarrow 0.05\,\text{fm}$$

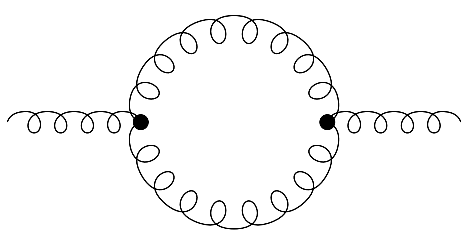
→ need to extrapolate to continuum limit: $a \to 0$
→ choose better discretisations: avoid small p.

(2) Inclusion of dynamical quark effects:

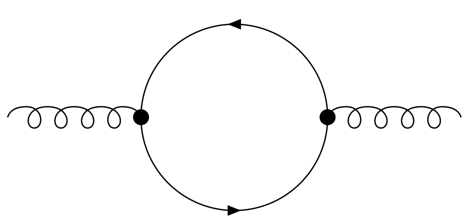
$$Z = \int [dU]\, e^{-S_G[U]} \prod_f \det\left(D^{\text{lat}} + m_f\right)$$

$\det(D^{\text{lat}} + m_f) = 1$: Quenched Approximation

→ neglect quark loops in the evaluation of $\langle\Omega\rangle$.



cheap



expensive

Scale ambiguity in the quenched approximation:

$$a^{-1}\,[\mathrm{MeV}] = \frac{Q\,[\mathrm{MeV}]}{(aQ)}, \quad Q = f_\pi,\, m_N,\, m_\rho,\, \sqrt{\sigma}, \ldots$$

(3) Restrictions on quark masses:

$$a \ll {m_\mathrm{q}}^{-1} \ll L$$

→ cannot simulate u, d and b quarks directly
→ need to control extrapolations in m_q

(4) Chiral symmetry breaking:

Nielsen & Ninomiya (1979): exact chiral symmetry cannot be realised at non-zero lattice spacing

→ impossible to separate chiral and continuum limits

Outline:

I. The lattice Dirac operator

   – Improved actions
   – Exact chiral symmetry on the lattice

II. Simulations with dynamical quarks

   – Light hadron spectrum, quenched & unquenched

III. Light quark masses

   – Non-perturbative renormalisation
   – Recent estimates

IV. Glueballs & heavy hybrids

V. Omissions

VI. Summary

I. The lattice Dirac operator

- Massless free fermions:

$$S_F = a^4 \sum_{x,y} \overline{\psi}(x)\, D(x \Leftrightarrow y) \psi(y)$$

(a) $D(x \Leftrightarrow y)$ is local
(b) $\widetilde{D}(p) = i\gamma_\mu p_\mu + O(ap^2)$
(c) $\widetilde{D}(p)$ is invertible for $p \neq 0$
(d) $\gamma_5 D + D\gamma_5 = 0$

→ Nielsen & Ninomiya: (a)–(d) do not hold simultaneously

→ either left with doublers
or chiral symmetry broken explicitly

- Staggered (Kogut-Susskind) fermions

- Wilson fermions

$$D_W = \tfrac{1}{2}\gamma_\mu \left(\nabla_\mu + \nabla^*_\mu\right) \Leftrightarrow \tfrac{1}{2} a \nabla^*_\mu \nabla_\mu$$

- degeneracy fully lifted; chiral symmetry broken
- leading cutoff effects of order $a \Rightarrow p = 1$

$O(a)$ improved Wilson fermions

- Can one find discretisation for Wilson fermions with reduced cutoff effects?

- Symanzik improvement: Remove leading lattice artefacts by adding a counterterm to the Wilson-Dirac operator (Sheikholeslami-Wohlert,. . .)

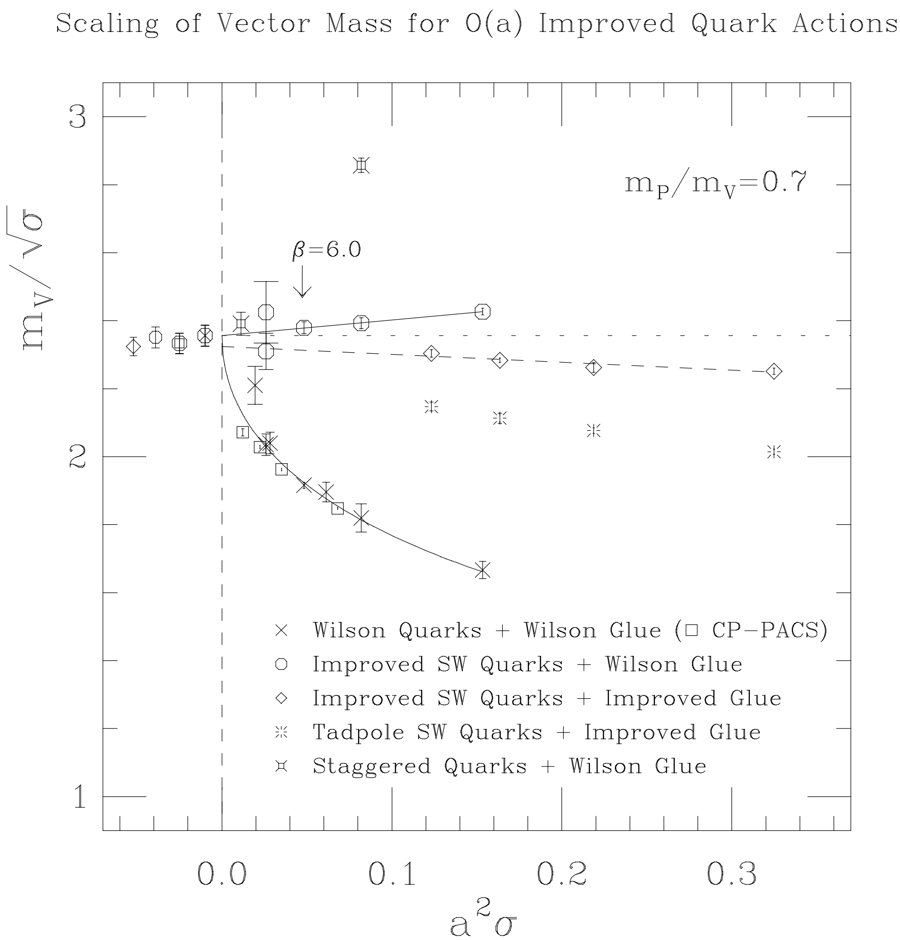
$$D_{SW} = D_W + m_0 + \frac{ia}{4} c_{\rm sw} \sigma_{\mu\nu} F_{\mu\nu}(x)$$

- Fix $c_{\rm sw}(g_0)$ through suitable improvement condition

  → leading lattice artefacts are $O(a^2)$ in spectral quantities

- For $n_f = 0$ and $n_f = 2$, $c_{\rm sw}$ has been determined non-perturbatively M. Lüscher et al. 1997; R. Edwards et al. 1997; Jansen & Sommer 1998.

- Scaling of $m_{\rm V}$ for $m_{\rm PS}/m_{\rm V} = 0.7$ (T. Klassen 1998)



- Non-perturbatively improved Wilson action: scaling behaviour is consistent with residual a^2 artefacts

- Can push Symanzik improvement to higher orders (Alford, Klassen, Lepage 1998)

- Chiral symmetry remains broken

Exact chiral symmetry on the lattice

- Chiral symmetry breaking may be tolerated in many applications of QCD

- Chiral symmetry is crucial for non-perturbative formulation of
  - EW theory
  - SUSY

- Proposals to circumvent the NN Theorem:
  – Domain Wall Fermions (Kaplan 1992; Shamir, Furman 1993)
  – Overlap Formalism (Narayanan, Neuberger, 1993–98)
  – "Perfect Actions" (Hasenfratz, Niedermayer, . . ., 1993–98)
  – Ginsparg-Wilson relation (Ginsparg & Wilson 1982, Lüscher 1998–99)

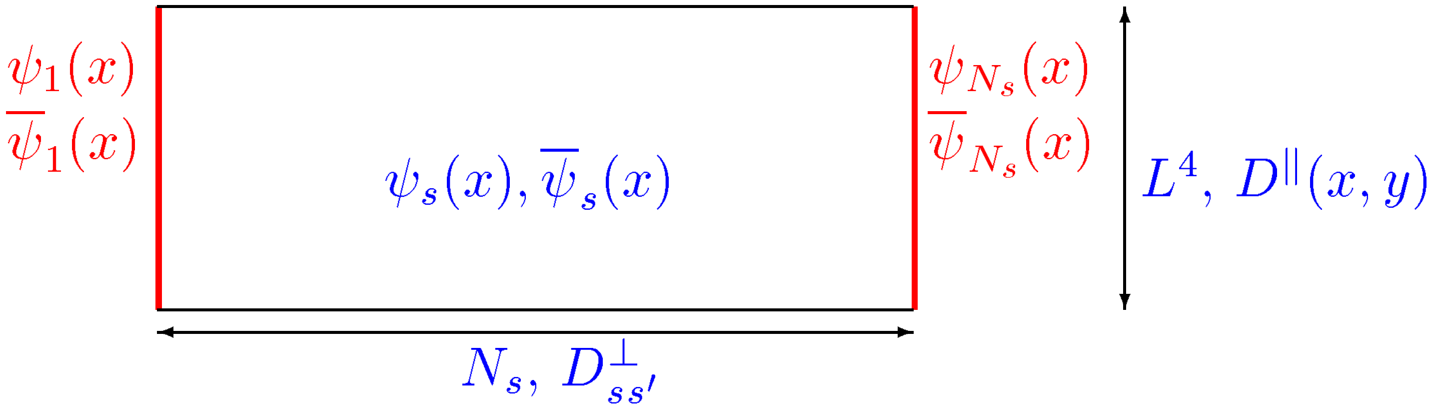
- See also:

T. Blum, Lattice 98

M. Lüscher
H. Neuberger } Lattice 99

Domain Wall Fermions (Shamir & Furman)

- Introduce discrete extra (5^{th}) dimension:



$$D^{\mathrm{DWF}}_{ss'}(x,y) = D^{||}(x,y)\delta_{ss'} + \delta(x \Leftrightarrow y)D^{\perp}_{ss'}$$

$D^{||}(x,y)$: Wilson-Dirac operator with mass $\Leftrightarrow M$.
$D^{\perp}_{ss'}$: contains Dirac mass m.

- For $m = 0,\ N_s \to \infty$:

  - no fermion doublers
  - chiral modes are trapped in 4-dim. domain walls at either end

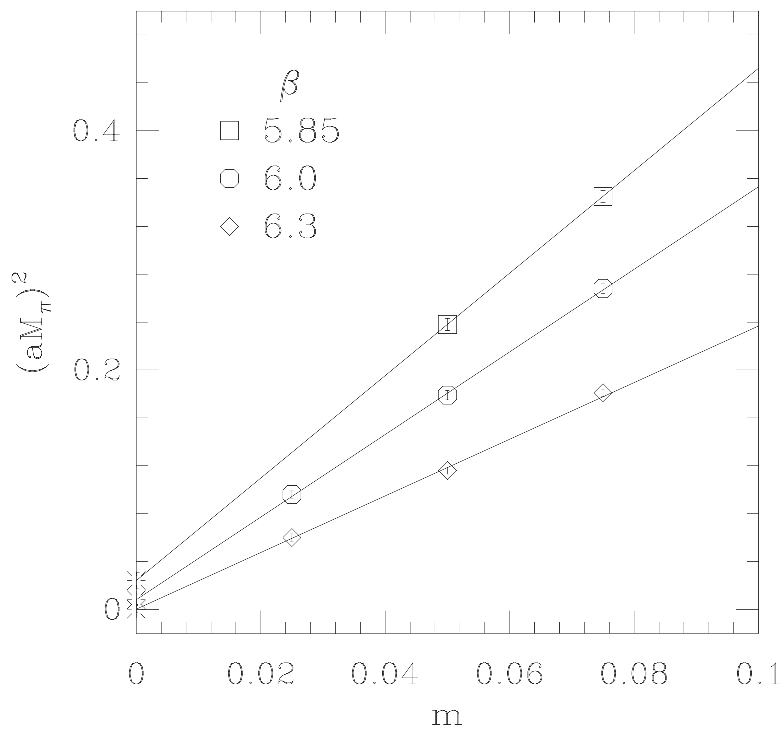
- In practice: work at finite N_s

  $\to$ decoupling of chiral modes not exact
  – terms which break chiral symmetry are exponentially suppressed

• Pion mass:

$$(am_\pi)^2 = C(am + ae^{-\gamma N_s})$$

Blum, Soni, Wingate, hep-lat/9902016, $N_s = 10$



→ Can realise (almost exact) chiral symmetry at non-zero a at the expense of simulating 5-dim. theory

– $N_s \approx 10 \Leftrightarrow 30$ may be sufficient?

Ginsparg-Wilson Relation

- Replace condition $\gamma_5 D + D\gamma_5 = 0$ by the weaker relation (Ginsparg & Wilson 1982)

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

- If D satisfies GWR, then this implies an exact global symmetry (Lüscher 1998)

$$\delta\psi = \gamma_5(1 \Leftrightarrow \tfrac{1}{2}aD)\psi, \quad \delta\overline{\psi} = \overline{\psi}(1 \Leftrightarrow \tfrac{1}{2}aD)\gamma_5$$

→ "lattice chiral symmetry"

→ Flavour-singlet case: Ward identities have correct anomaly

- Construction of gauge theories with local chiral symmetry (Abelian and non-Abelian) relies on GWR (Lüscher 1998–99)

Overlap Formalism (Narayanan, Neuberger)

- Reinterpretation of the domain wall proposal; Overlap operator (Neuberger 1998):

$$D_N = \frac{1}{a}\left(1 \Leftrightarrow A\left(A^\dagger A\right)^{-1/2}\right), \quad A = 1 \Leftrightarrow aD_W$$

D_W : massless Wilson-Dirac operator

→ satisfies GW-relation

Witten's SU(2) anomaly on the lattice

- Witten 1982: SU(2) gauge theory coupled to a single left-handed fermion is mathematically inconsistent:

$$\mathcal{Z} = \int [dA] e^{-S_G[A]} \int [d\overline{\psi}][d\psi]_{\text{Weyl}}\, e^{-\overline{\psi} D \psi}$$
$$= \int [dA] \left(\det D[A]\right)^{1/2} e^{-S_G[A]}$$

- There is a non-trivial gauge transformation $U(x)$ such that

$$(\det D[A])^{1/2} = \Leftrightarrow (\det D[A^U])^{1/2}$$

- Expectation value w.r.t. $\mathcal{Z}$ is indeterminate: $\langle \Omega \rangle =$"0/0"

- D hermitian: eigenvalues are real and come in pairs, $\pm\lambda$

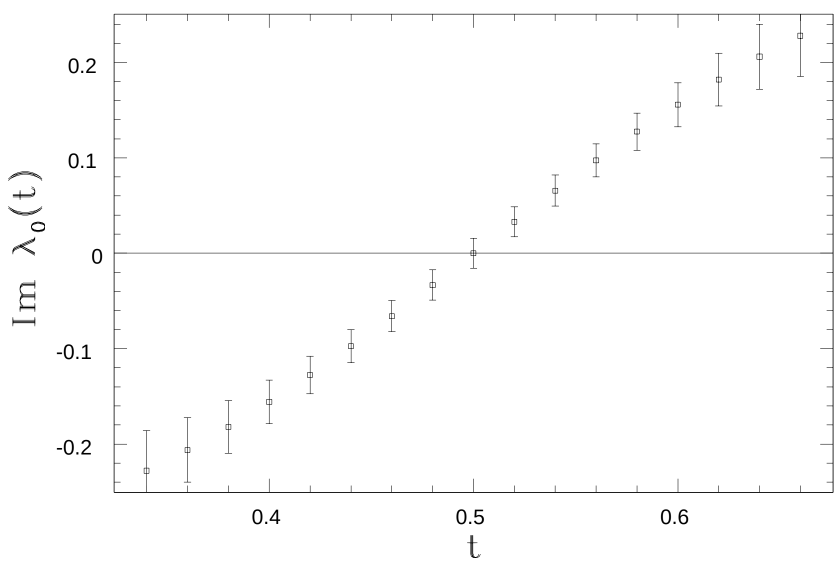
- Vary gauge field smoothly from A_μ to A_μ^U :

$$A_\mu(t) = (1 \Leftrightarrow t)\, A_\mu + t A_\mu^U, \quad t \in [0,1]$$

- Atiyah-Singer: number of eigenvalues that become negative as t is varied from 0 to 1 is odd.

- Compute the eigenvalues of the overlap operator D_N in a lattice simulation, as A_μ is smoothly turned into A_μ^U (O. Bär, Lattice 99)

- D_N : eigenvalues come in complex conjugate pairs; expect level crossing for $\mathrm{Im}\lambda$



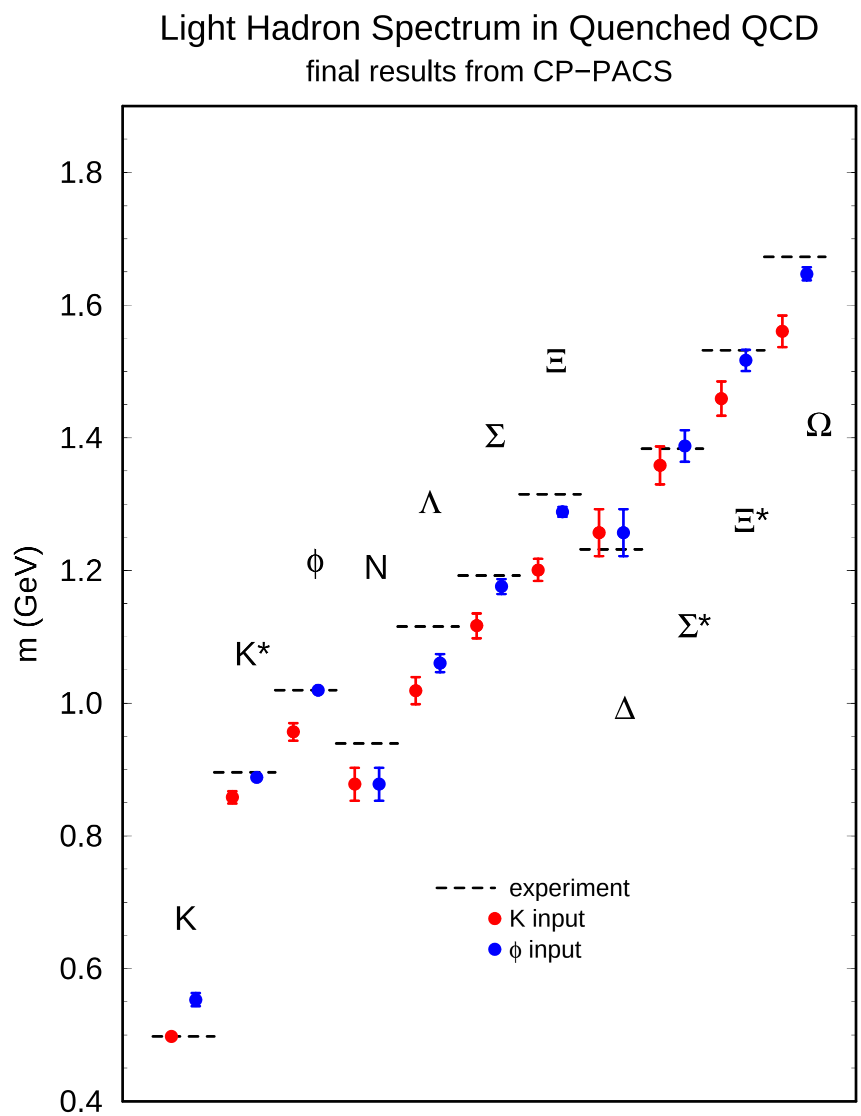
- Crossing observed for lowest eigenvalue

→ Numerical proof of Witten's SU(2) anomaly

- Only possible if lattice Dirac operator has the correct chiral properties

II. Simulations with dynamical quarks

- Quenched light hadron spectrum deviates from experiment (CP-PACS Collaboration, hep-lat/9904012):



$m_{K^*} \Leftrightarrow m_K$ too small by $10 \Leftrightarrow 16\%$ $(4 \Leftrightarrow 6\sigma)$

$m_N/m_\rho = 1.143 \pm 0.033$ (Exp.: 1.22, 2.5σ)

- Quenched QCD describes the light hadron spectrum at the level of 10%.

- Can sea quark effects account for the deviation of the quenched spectrum from experiment?

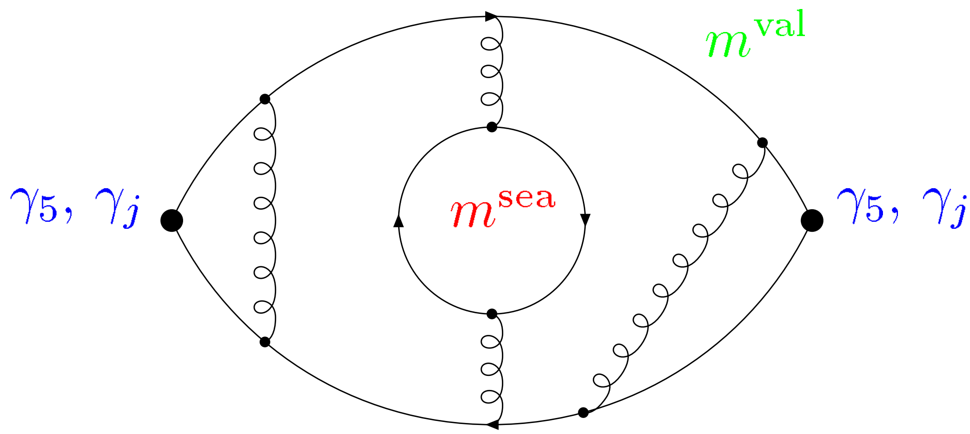
- O(a) Improvement even more important:
  - expensive to simulate very small lattice spacings
  - need to separate lattice artefacts from sea quark effects

- Recent simulations with $(n_f = 2)$

Collaboration		GFlops	Gluons	Quarks
CU/BNL/Riken	US	~ 250	Wilson	DWF
CP-PACS	J	~ 300	Iwasaki	SW, tad
UKQCD	UK	~ 28	Wilson	SW, n.p.
SESAM/TχL	D/I	~ 14	Wilson	Wilson
MILC	US	$\gtrsim 7.3$	LW	staggered

$\rightarrow$ identify two light flavours with physical u, d quarks

- Observables in full QCD with $n_f = 2$ flavours:
  Example: meson propagator:



- Sea quarks are still relatively heavy:
  $m^{\rm sea} = m^{\rm val}$:

$$\frac{m_{\rm PS}}{m_{\rm V}} = \begin{cases} 0.67 \Leftrightarrow 0.86 & \text{UKQCD} \\ 0.6 \Leftrightarrow 0.8 & \text{CP-PACS} \end{cases}, \quad \left(\frac{m_\pi}{m_\rho} = 0.169\right)$$

- Study the dependence of observables on $m^{\rm sea}$;
  Extrapolate in $m^{\rm sea}$ to physical m_π/m_ρ

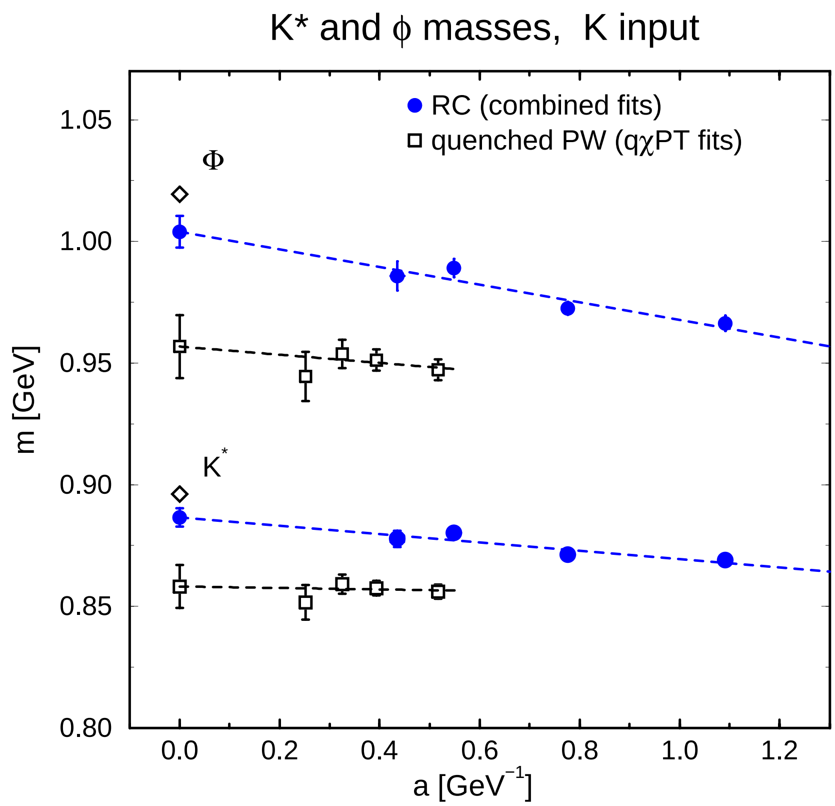
- Incorrect n_f for Kaon physics:

$\rightarrow$ choose $m^{\rm val} > m^{\rm sea}$
and identify $m_s = m^{\rm val}$, $m_{u,d} = m^{\rm sea}$

$\rightarrow$ "partially quenched"

Sea quark effects in the light hadron spectrum:

- Study the continuum limit of hadron masses in "partially quenched" QCD
  CP-PACS, (T. Kaneko, Lattice 99)



- Mesons: discrepancy with experimental spectrum diminished when sea quarks are "switched on"

- Quantify sea quark effects in the continuum limit

- Can remaining differences be blamed on
  - incorrect $n_f = 2$ for strange hadrons?
  - continuum extrapolations?
  - extrapolations in the quark masses?

III. Light Quark Masses

- Recent progress:
  - controlling lattice artefacts (improvement)
  - matching lattice results to $\overline{\rm MS}$ scheme
  - results from simulations using DWF's
  - estimating sea quark effects

- Current quark masses:

$$\begin{aligned} m_K^2 f_K &= (\overline{m}_{\rm u} + \overline{m}_{\rm s})\langle 0|\overline{u}\gamma_5 s|K^+\rangle \\ (\overline{u}\gamma_5 s)_{\overline{\rm MS}} &= Z_{\rm P}(g_0, a\mu)\,(\overline{u}\gamma_5 s)_{\rm lat} \\ Z_P(g_0, a\mu) &= 1 + \frac{g_0^2}{4\pi}\left\{\frac{2}{\pi}\ln(a\mu) + C\right\} + O(g_0^4) \end{aligned}$$

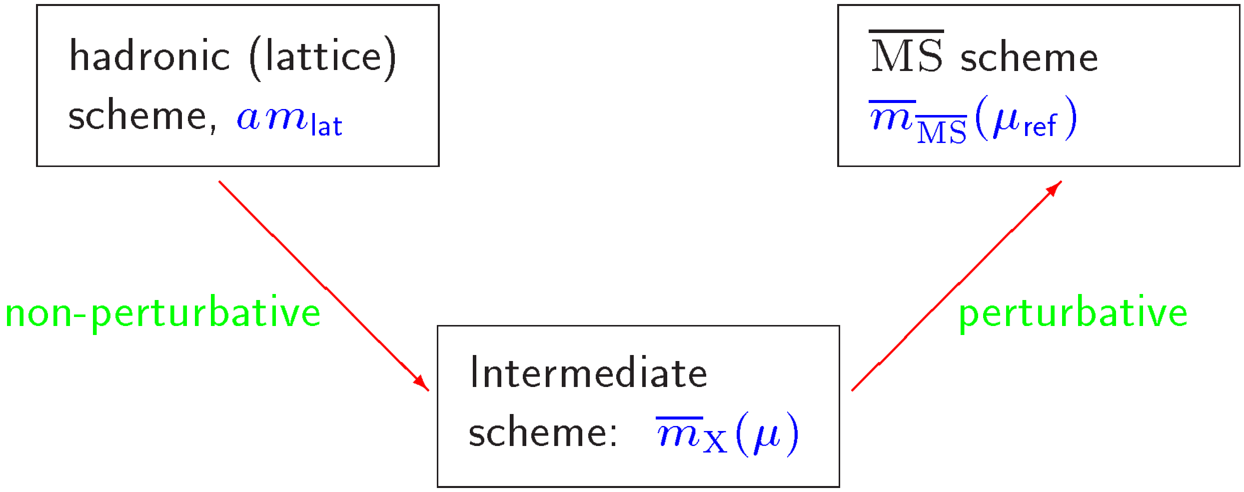
- lattice perturbation theory does not converge well

- scale dependence complicates the situation

→ no controlled perturbative relation between hadronic (lattice) scheme and $\overline{\rm MS}$

→ develop non-perturbative matching techniques

- Introduce an intermediate renormalisation scheme:



(1) Regularisation Independent (Martinelli, Sachrajda,...)

$$Z_{\rm P}(g_0, a\mu)\ \langle q(\mu)\,|(\overline{q}\gamma_5 q)_{\rm lat}|\,q(\mu)\rangle_{\rm G.F.} = 1$$

$\overline{\rm MS}$ and RI schemes related through continuum perturbation theory

(2) Schrödinger Functional scheme (Lüscher et al.)

Finite volume scheme; impose renormalisation condition at scale $\mu = 1/L$:

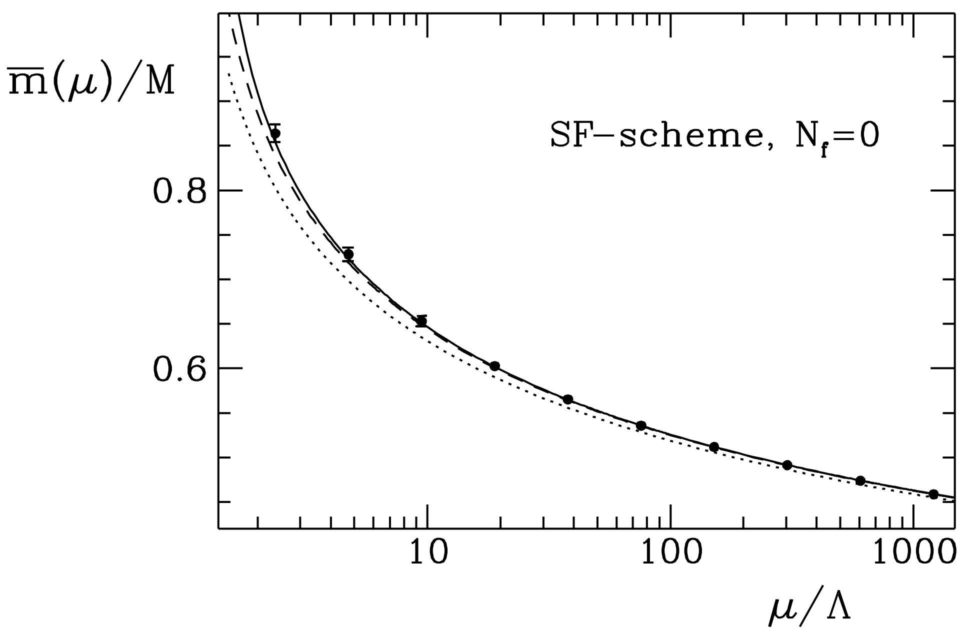
$$Z_{\rm P}(g_0, a\mu_0), \quad \mu_0 = 1/L_0 \simeq 275\,\text{MeV}$$

$Z_{\rm P}(g_0, a\mu_0)$ computed non-perturbatively for $a \approx 0.1 \Leftrightarrow 0.045\,{\rm fm}$ for $O(a)$ improved Wilson action

- Compute $\overline{m}_{\mathsf{SF}}(\mu)/M$ for $\mu_0 \leq \mu \lesssim 200\,\mathsf{GeV}$

  M : renormalisation group invariant quark mass

- Scale evolution of $\overline{m}_{\mathsf{SF}}(\mu)/M$ (quenched QCD)
  (Capitani, Lüscher, Sommer, HW, Nucl. Phys. B544 (1999) 669)



- For $\mu_0 = 275\,\mathsf{MeV}$:

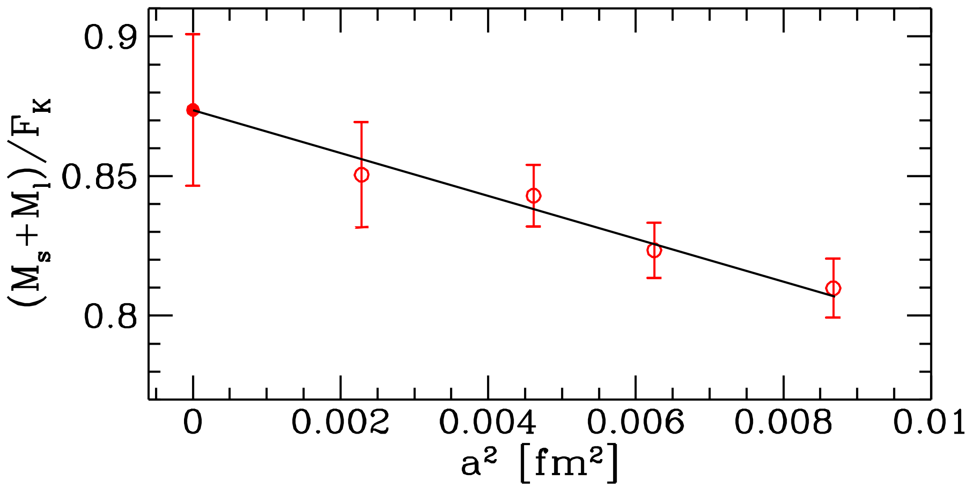
$$\frac{M}{\overline{m}_{\mathrm{SF}}}(\mu_0) = 1.157(15)$$

- Combine with matching factor $Z_{\mathrm{P}}(g_0, a\mu_0)$ and lattice matrix elements

$$\frac{(M_s + M_l)}{f_K} = \frac{M}{\overline{m}_{\mathrm{SF}}(\mu_0)} \frac{m_K^2}{Z_{\mathrm{P}}(g_0, a\mu_0)\langle 0|P|K\rangle^{\mathsf{lat}}} + O(a^2)$$

$(M_l = (M_u + M_d)/2)$

- Continuum extrapolation of $(M_s + M_l)/f_K$:
  (Garden, Heitger, Sommer, HW, hep-lat/9906013)



- Continuum limit; $(f_K = 160(2)\,\text{MeV})$:

$$M_s + M_l = 140 \pm 5\,\text{MeV}$$

- Convert into $\overline{m}_s^{\overline{\text{MS}}}(2\,\text{GeV})$ using

$$M_s/M_l = 24.4 \pm 1.5 \qquad \text{(Chiral P.T.)}$$

$$\overline{m}^{\overline{\text{MS}}}(\mu)/M = 0.7208 \;\text{ at }\; \mu = 2\,\text{GeV}$$

→ Final result: (all errors, except quenching)

$$\overline{m}_s^{\overline{\text{MS}}}(2\,\text{GeV}) = 97 \pm 4\,\text{MeV}$$

$\pm 10\%$ uncertainty due to scale ambiguity in the quenched approximation.

- Other recent results in the quenched approximation ($\overline{\rm MS}$ scheme, $\mu_{\rm ref} = 2\,{\rm GeV}$)

Collab.	m_l	m_s	$a \to 0$	$Z_{\rm P}^{NP}$	Impr.
BNL/Riken/CU†	5.1(4)	129(10)	×	$\surd_{\rm RI}$	DWF
QCDSF†	4.40(9)	104(3)	$\surd$	$\surd_{\rm SF}$	$\surd$
CP-PACS	4.55(18)	115(2)	$\surd$	×	×
JLQCD	4.23(29)	106(7)	$\surd$	$\surd_{\rm RI}$	stagg.
Blum, Soni & Wingate		95(26)	$\surd$	×	DWF
Becirevic et al.	4.5(4)	111(12)	×	$\surd_{\rm RI}$	$\surd$

†: preliminary

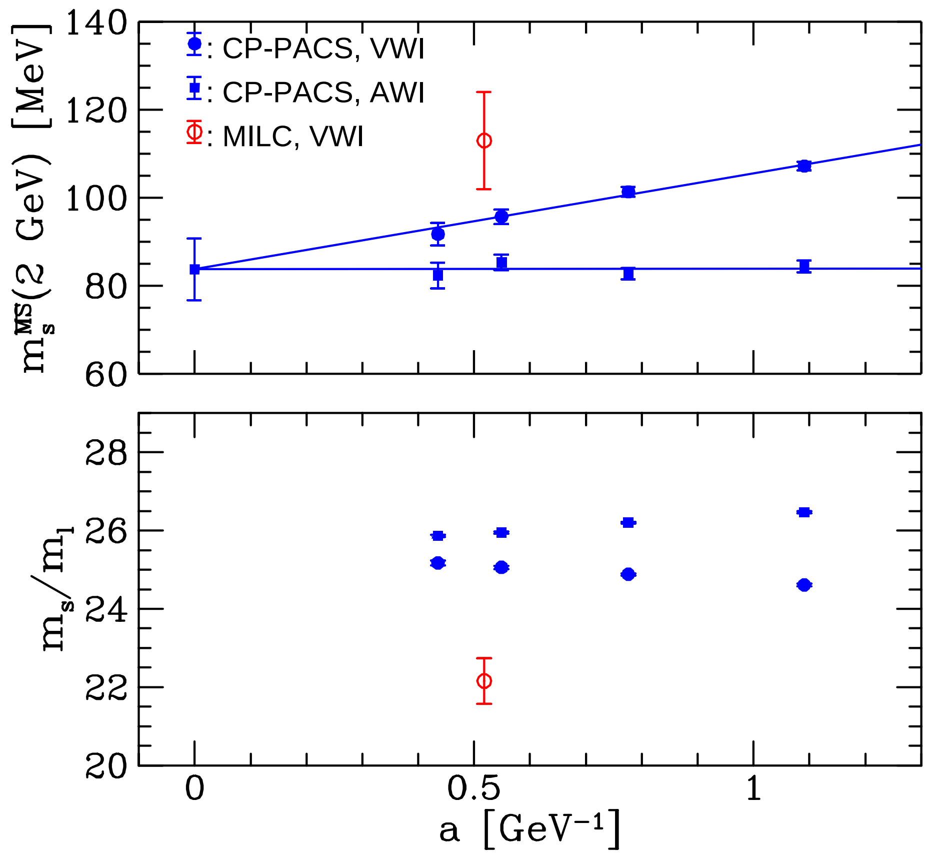
Caution:

- Systematic errors not estimated uniformly

- Conversion into physical units using different quantities

Sea quark effects in m_l, m_s:

- Last year: very large effects due to dynamical quarks?
- 1999: better separation of lattice artefacts and sea quark effects
- Recent results for $n_f = 2$ by CP-PACS and MILC Collaborations (partially quenched)



- Light quark masses decrease for $n_f = 2$
- CP-PACS: $m_s/m_l \approx 25$ in continuum limit
- Further exploration of systematics required ($m^{\rm sea}$-dependence, a-effects, renormalisation)

IV. Glueballs and Heavy Hybrids

- Idea: use anisotropic lattices: $a_t \ll a_s$

- Exponential decay of correlation function governed by $a_t M$:

$$\sum_{\vec{x}} \langle O(\vec{x},t) O^\dagger(0) \rangle \stackrel{t \gg 0}{\gtrsim} e^{-(a_t M)(t/a_t)}$$

- For $a_t \ll a_s$: slow exponential fall-off, whilst preserving large spatial volumes

- Typically: $a_s \approx 0.2 \Leftrightarrow 0.4\,\mathrm{fm}$
  $\xi \equiv a_s/a_t = 3 \Leftrightarrow 5$

→ Cutoff effects in spatial lattice spacing may be large; use Symanzik improvement to reduce leading lattice artefacts

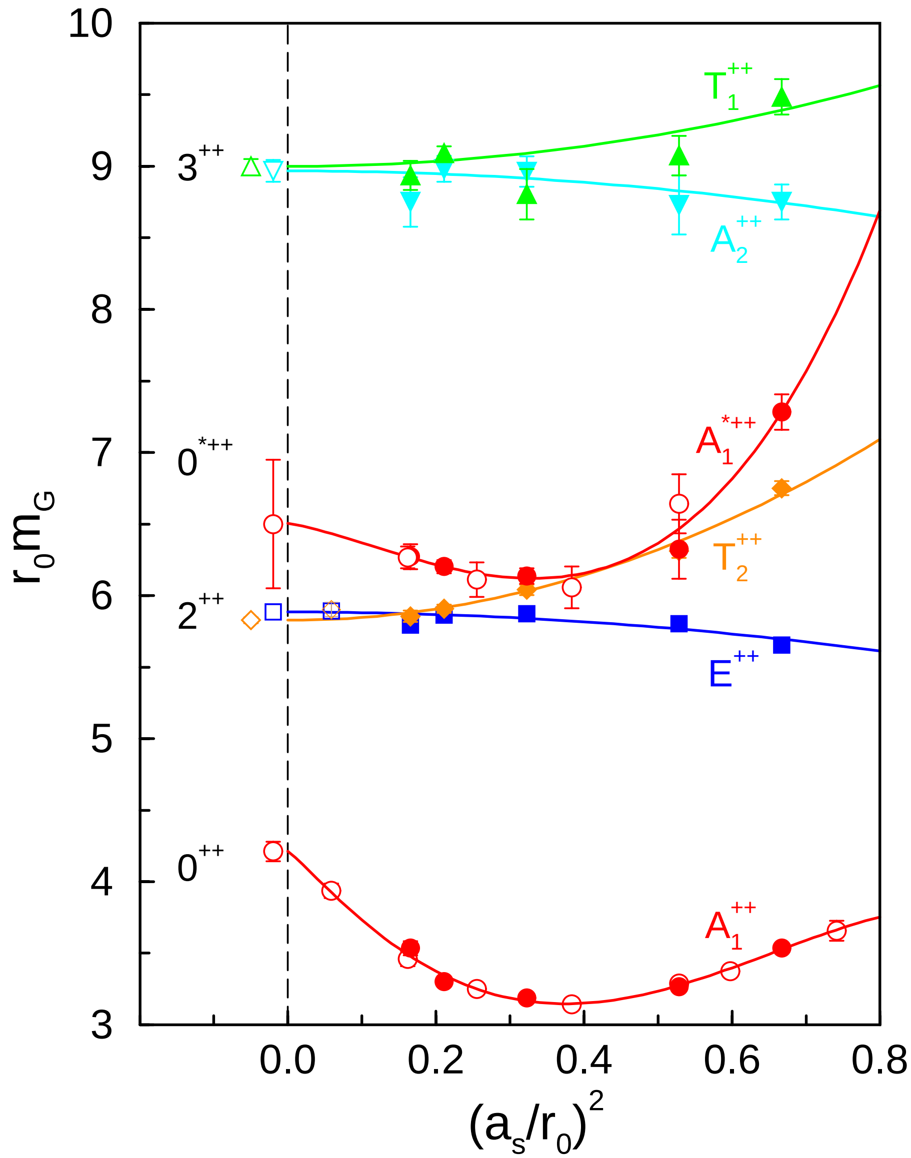
- Recent results:
  - Glueball spectrum (below $4\,\mathrm{GeV}$)
  - Quarkonia and heavy hybrids

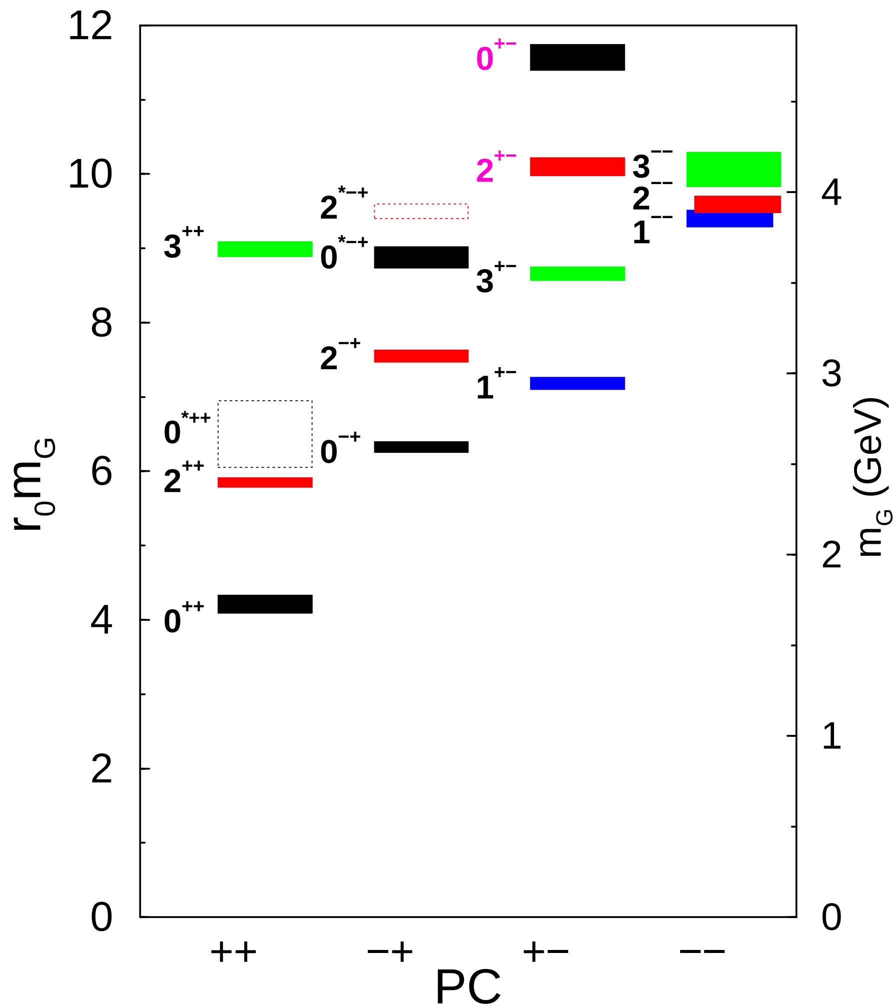
Glueball spectrum: (quenched QCD)
(Morningstar & Peardon, hep-lat/9901004)

- Anisotropic, $O(a_s^2)$ improved gluon lattice action; improvement coefficients fixed perturbatively
$\xi = a_s/a_t = 3,\ 5$

→ leading cutoff effects: $O(g^2 a_s^2, a_s^4, a_t^2)$

- Glueball operators constructed from representations of the octahedral group: A_1, A_2, E, T_1, T_2





$$\left.\begin{array}{l} r_0\, m_{0^{++}} = 4.21(12) \\ r_0\, m_{2^{++}} = 5.85(6) \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} 4.33(5) \\ 6.0(1) \end{array}\right.$$ M. Teper hep-th/9812187

$$(r_0^{-1} \approx 395\,\text{MeV})$$

- Very comprehensive study of glueballs
- If a_s too large: complicated model function for continuum extrapolation may cause loss in precision (c.f. 0^{++})
- So far: no effects of dynamical quarks, glueball-meson mixing

Quarkonia and heavy hybrids:

- Use similar techniques to study $\bar{b}b, \bar{c}c, \bar{b}gb, \bar{c}gc$ states

$\rightarrow$ use anisotropic lattice gluon action

$\rightarrow$ treat heavy quarks non-relativistically:

$$M_Q \gg p_{\text{cut}} = a_s^{-1}$$

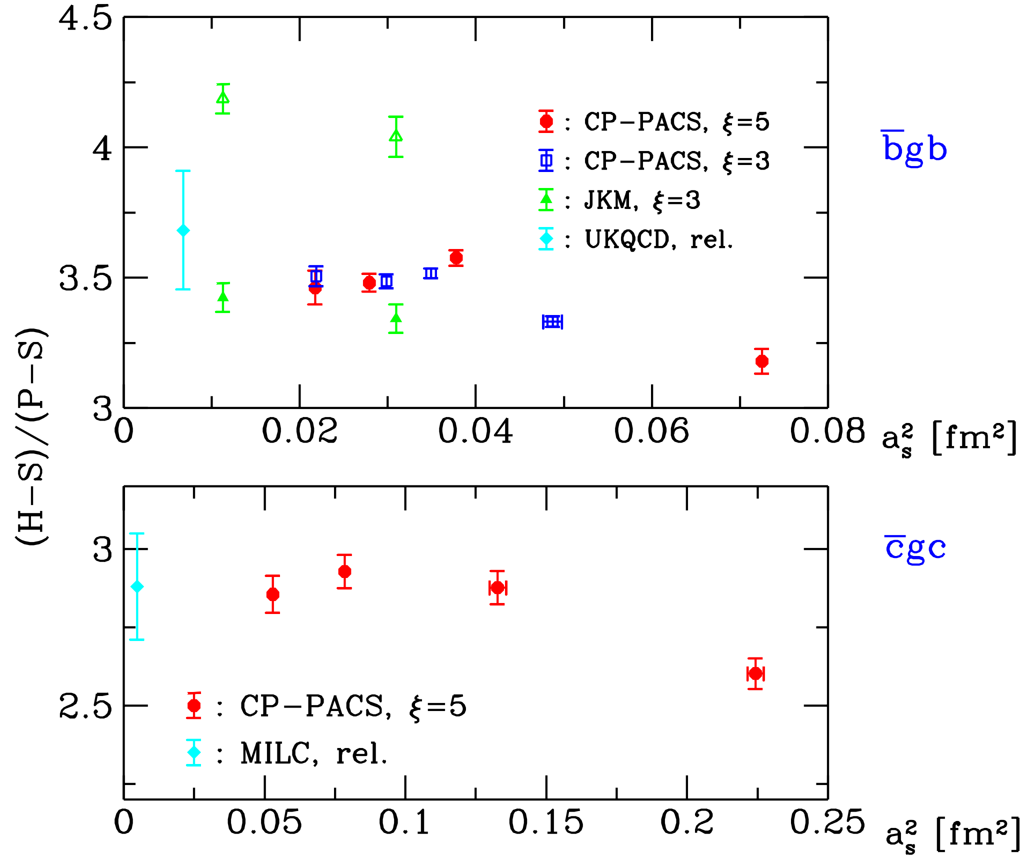
- Discretised, effective QCD action (NRQCD)

- Lattice spacing: acts as UV regulator and as relativistic cutoff

$\rightarrow$ Continuum limit $a_s \rightarrow 0$ cannot be taken

$\rightarrow$ Rely on “window” in a_s where NRQCD works and cutoff effects are small

- Recent results:
  CP-PACS, Phys. Rev. Lett. 82 (1999) 4396
  (T. Manke, EPS99)
  Juge, Kuti & Morningstar, Phys. Rev. Lett. 82 (1999) 4400
- Compute S, P waves
  $H_1(1^{--})$, $H_2(1^{++})$, $H_3(0^{++})$ hybrids



- Results for lowest $\bar{b}gb$ hybrid:

$$\delta(H_1 \Leftrightarrow S) = \begin{cases} 1.542(8)\ \text{GeV} & \text{(CP-PACS)} \\ 1.49(2)(5)\ \text{GeV} & \text{(JKM)} \end{cases}$$

V. Omissions

- Weak matrix elements: $\Delta S = 2, \Delta I = 1/2, \Delta I = 3/2$
  - non-perturbative renormalisation
  - chiral properties: domain wall fermions
    (A. Soni, EPS99)

- Heavy-light decay constants: $f_B, f_{B_s}, f_D, \ldots$
  $B^0 \Leftrightarrow \overline{B}^0$ mixing; (S. Hashimoto, Lattice 99)
  - quenched estimates stabilised; sea quark effects increase results by $15 \Leftrightarrow 20\%$?
    (H. Shanahan, EPS99)
  - study different formulations of heavy quarks (NRQCD,. . .)

- Flavour-singlet amplitudes
  - η, η' mass, πN sigma term,. . .
  - sea quark contributions to disconnected diagrams

- Structure functions (R. Petronzio, Lattice 99)
  - non-perturbative renormalisation: RI, SF

- QCD at finite temperature and density
  (F. Karsch, Lattice 99)

VI. Summary

- Significant progress in formulation of chiral symmetry at non-zero lattice spacing

  Chiral gauge theories can be put on the lattice in a consistent way

- Lattice simulations are becoming more refined:
  - Symanzik improvement
  - Non-perturbative renormalisation
  - Anisotropic lattices
  - Simulations with dynamical quarks

- Quenched approximation works surprisingly well;

  Effects of dynamical quarks are significant