$$j_1(z) = i[\alpha e^{-i\pi\varphi/2} X(z) - \beta e^{i\pi\varphi/2} X^{-1}(z)],$$
$$j_2(z) = \operatorname{ctg}(\pi\varphi/2)[\alpha e^{i\pi\varphi/2} X(z) + \beta e^{-i\pi\varphi/2} X^{-1}(z)], \quad (12)$$

where α and β are arbitrary real constants and the function $X(z)$ is given by Eq. (11).

Proof. Let us consider an arbitrary solution of problem (3) that satisfies identity (7) by Lemma 2 for some fixed value of the constant $C = C_0$. We can assume that $C_0 \neq 0$; otherwise, the conclusion of the theorem follows from Lemma 3. Together with this solution we consider the one-parameter family of solutions of problem (3)

$$j(z;\alpha) = \{j_1(z) - \alpha j_{11}(z),\ z \in S_1;\ j_2(z) - \alpha j_{21}(z),\ z \in S_2\}, \quad (13)$$

where $j_{11}(\dot{z})$ and $j_{21}(\dot{z})$ are given by Eqs. (9) and satisfy identity (7) for $C = 0$. By Lemma 2, the solution (13) satisfies identity (7) for $C = C(\alpha)$. The direct substitution of (13) in identity (7) with allowance for Eq. (9) yields

$$2\alpha e^{i\pi(1-\varphi)/2} X(z)[j_1(z) - i\operatorname{tg}(\pi\varphi/2)\overline{j_2(\bar{z})}] = C_0 - C(\alpha).$$

Since $j(z)$ is postulated not to satisfy the homogeneous identity (7), the left-hand side of the latter equation is a nonzero constant; consequently, an α_0 exists such that $C(\alpha_0) = 0$, and by Lemma 3 the corresponding function (13) $j(z; \alpha_0)$ differs from the function (10) only by a certain real multiplier β.

Theorem 2. Problem (1), (2) is unconditionally uniquely solvable, and its solution is given by Eqs. (12), where the constants are expressed as follows:

$$\alpha = (I_x - I_y)/2\Lambda, \quad \beta = (I_x + I_y)/2\Lambda,$$
$$\Lambda = \frac{h}{4\sqrt{\pi}K}\Gamma\left(\frac{\varphi}{2}\right)\Gamma\left(\frac{1-\varphi}{2}\right). \quad (14)$$

We prove this theorem by direct calculations, substituting Eqs. (12) in the boundary conditions (2).

Remark. It is readily established on the basis of Eqs. (12) and (14) that the solution of the boundary-value problem (1), (2) satisfies identity (7), where

$$C = 4\alpha\beta = \frac{I_x^2 - I_y^2}{\Lambda^2}.$$

It follows from this result, in particular, that the above-formulated particular solutions (9) and (10) correspond to the cases $I_x = \pm I_y$, i.e., to an external current transmitted along the diagonals of the investigated field.

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The method of irreducible Green's functions in condensed-media theory

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Advanced and retarded thermodynamic two-time Green's function were introduced in the statistical mechanics of quantum systems by N. N. Bogolyubov and S. V. Tyablikov. [1] In contrast to causal Green's functions, they can be extended into the complex plane. These convenient analytical properties contributed to a situation in which the method of thermodynamic two-time Green's functions has found broad application in statistical mechanics. [2-4] A system of coupled equations, together with the corresponding spectral representations, is used to find advanced and retarded Green's functions. As other authors have indicated, [1-3] in the treatment of this system of equations the main problem is to develop procedures for effective decoupling of the system of equations in order to obtain closed equations for determining the Green's functions. The fact that such an approximation must be selected on an individual basis, depending on the nature of the problem, [1] sometimes is criticized for its unsystematic nature in works that utilize Green's functions and diagramming. However, in the diagram approach there also is ambiguity in the generation of the required approximation, and the answer to the question which diagrams must be summed is obvious only for a small range of relatively simple problems.

In this paper we will show that for a wide range of various problems of statistical mechanics and condensed-media theory a fairly systematic method can be drawn up for constructing approximation solutions on the basis of the method of irreducible Green's functions. [4-15] In our approach we will discuss from a unified standpoint the fundamental questions that arise in the construction of approximation solutions on the basis of the method of irreducible Green's functions. We will be dealing with Green's functions of the type

$$G^r(t,t') = \langle\langle A(t), B(t')\rangle\rangle = -i\theta(t-t')\langle[A(t)B(t')]_\eta\rangle, \quad \eta = \pm. \quad (1)$$

It has been noted elsewhere [1] that the chains of equations obtained for Eq. (1) is identical for not

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only advanced and retarded Green's functions, but also causal Green's functions. The method of irreducible Green's functions [4-7] is a useful reformulation of the conventional method of the Bogolyubov–Tyablikov equations of motion. The constructive idea was that in calculating the one-particle characteristics of a system it would be convenient first to represent Green's function (1) as a formal solution of the Dyson equation in order to perform the necessary decoupling of multiparticle correlation functions in the mass operator, since by so doing the decoupling procedure could be conditionally monitored by analogy with the diagram approach. The method of irreducible Green's functions [4-15] is closely related to the Mori–Zwanzig projection method,[8] which essentially follows from the idea of N. N. Bogolyubov of "concise description" of the system.[2,3] In this approach the infinite chain of coupled equations for the correlation functions reduces to several relatively simple equations, in which "effective" allowance is made for the significant information on the system that determines the particular features of the given problem. It is essential to emphasize that when the method of irreducible Green's functions is used, the structure of the resultant solution depends significantly on the order of the equations for the Green's functions in which the irreducible parts are singled out. This in turn determines the nature of the approximate solutions that we build on the basis of the exact representation.

To clarify the foregoing, let us consider the retarded Green's function $G = \langle\langle A(t), A^+(t')\rangle\rangle$. Let us introduce into the equation of motion

$$\omega G(\omega) = \langle [A, A^+]_+ \rangle + \langle\langle [A, H]_- | A^+ \rangle\rangle \tag{2}$$

the by definition irreducible (ir) Green's function

$$^{ir}\langle\langle [A, H]_- | A^+ \rangle\rangle_\omega \equiv \langle\langle [A, H]_- - \alpha A | A^+ \rangle\rangle_\omega, \tag{3}$$

where the quantity α is determined from the condition

$$\langle [[A, H]^{ir}_-, A^+]_+ \rangle \equiv 0. \tag{4}$$

From Eq. (4) we find

$$\alpha = \langle [[A, H]_- A^+]_+ \rangle (\langle [A, A^+]_+ \rangle)^{-1} = M_1 (M_0)^{-1}. \tag{5}$$

Here M_0 and M_1 are the zeroth and first order moments of the spectral density.[2,3] A Green's function that cannot be reduced by decoupling to a Green's function of lower order is said to be irreducible. Irreducible correlation functions are well known in statistical physics; in the diagram method irreducible vertices are a set of graphs that cannot be cut along a single line. Definition (3) translates these concepts into the language of retarded and advanced Green's functions. We classify all renormalizations of the average field, which are separated by means of Eq. (3), as Green's functions in the generalized average-field approximation (GAFA):

$$G^0 = \langle [A, A^+]_+ \rangle (\omega - \alpha)^{-1}. \tag{6}$$

To calculate Green's function (3), we use the procedure [5] of differentiation with respect to the second time t'. In the resultant equation we isolate the irreducible part by analogy with Eq. (3). As a result, we find

$$G(\omega) = G^0(\omega) + G^0(\omega) P(\omega) G^0(\omega). \tag{7}$$

Here we have introduced the scattering operator

$$P = (M_0)^{-1}\, {}^{ir}\langle\langle [A, H]_- | ([A, H]_-)^* \rangle\rangle^{ir} (M_0)^{-1}. \tag{8}$$

The structure of Eq. (7) enables us to determine the mass operator M, in complete analogy with the diagram technique:

$$P = M + MG^0P. \tag{9}$$

As a result, we obtain the exact Dyson equation (as yet no decouplings have been performed) for the thermodynamic two-time Green's functions:

$$G = G^0 + G^0MG; \quad M = (P)^P. \tag{10}$$

The mass operator M is expressed according to Eq. (9) in terms of the "proper" part (or in the diagram technique, the bound part) of the multiparticle irreducible Green's function and describes possible inelastic-scattering processes that lead to damping and to additional renormalization of the frequency of the self-consistent quasiparticle excitations. Strictly speaking, definition (9) is symbolic in nature; of course, because of the identical form of the equations for the Green's functions for all three types, we can convert in each stage of calculations to casual Green's functions and confirm the substantiated nature of definition (9). We therefore should speak of an analog of the Dyson equation. Hereafter we will drop this stipulation, since it will not cause any misunderstanding. We wish to emphasize that the method presented above for introducing irreducible Green's functions is only a general scheme. The specific method of introducing irreducible parts (we may legitimately speak of irreducible operators) depends on the form of operators A, the type of Hamiltonian, and the conditions of the problem. The effectiveness of this approach has been demonstrated [9-15] for describing the normal and superconductive properties of systems with strong coupling and a complex type of electronic spectrum. Let us illustrate this through some examples. Let us consider the Hubbard model as our first example. Let us write the Hamiltonian for it in the form

$$H = \sum_{ij\sigma} t_{ij} a^+_{i\sigma} a_{j\sigma} + \frac{1}{2} U \sum_{i\sigma} n_{i\sigma} n_{i-\sigma}, \tag{11}$$

This Hamiltonian depends on two parameters: the effective bandwidth w and the repulsive energy U of electrons. As the ratio of these two quantities is altered, radical changes in the type of metal–insulator transition occur in the system, and so forth. The case of a very strong but finite electron correlation poses the greatest difficulty. Introducing the auxiliary operators [9]

$$d_{i\alpha\sigma} = n^\alpha_{i-\sigma} a_{i\sigma} \quad (\alpha = \pm), \quad n^+_{i\sigma} = n_{i\sigma}, \quad n^-_{i\sigma} = (1 - n_{i\sigma}),$$

we represent the one-particle Green's function as

$$G_{ij\sigma}(\omega) = \langle\langle a_{i\sigma} | a^+_{j\sigma} \rangle\rangle_\omega = \sum_{\alpha\beta} \langle\langle d_{i\alpha\sigma} | d^+_{j\beta\sigma} \rangle\rangle_\omega = \sum_{\alpha\beta} G^{\alpha\beta}_{ij\sigma}(\omega).$$

Using relations (2)-(10), we derive the exact Dyson equation, whose mass operator has the form

$$\bar{M}_{q\sigma}(\omega) = \bar{\Phi}^{-1} \{ \sum_{em} t_{ie} t_{mj}\, {}^{ir}\langle\langle \bar{D}_{ie} | \bar{D}^+_{mj} \rangle\rangle^{ir}_\omega \}_q \bar{\Phi}^{-1}. \tag{12}$$

In our approach we have for Eq. (6) a nonlocal expression, now in the generalized average-field approximation:

$$G_{0\sigma}(q, \omega) \approx \frac{n_{-\sigma}}{\omega - E_- - n_{-\sigma} W^-_{q,-\sigma}} + \frac{(1 - n_{-\sigma})}{\omega - E_+ - (1 - n_{-\sigma}) W^+_{q,-\sigma}}. \tag{13}$$

We thus have obtained in its most complete version the explicit form of the average fields, which are described by the correlators

$$W^{\pm} \sim \{(a^{+}_{i-\sigma} n^{\pm}_{i\sigma} a_{j-\sigma}) + (a_{i-\sigma} n^{\mp}_{i\sigma} a^{+}_{j-\sigma}) +$$

$$+ (n^{\pm}_{j-\sigma} n^{\pm}_{i-\sigma}) + (a_{i\sigma} a^{+}_{i-\sigma} a_{j-\sigma} a^{+}_{j\sigma}) - (a_{i\sigma} a_{i-\sigma} a^{+}_{j-\sigma} a^{+}_{j\sigma})\}. \quad (14)$$

Solution (13) is more general than the "Hubbard III" solution and the solution by the curl method.[9] The "Hubbard I" solution is a particular case of relation (14) which corresponds to the approximation $W^{\pm} \sim \langle n_{j-\sigma} n_{i-\sigma} \rangle \approx n^2_{-\sigma}$. It is worth noting that in our approach the irreducibility of average fields (14) to functionals of the average electron density in the atomic limit is demonstrated exactly. In general, average fields may have an extremely nontrivial structure that is hard to determine for various independent considerations. For example, in the superconductivity theory of N. N. Bogolyubov,[2] the average fields should include anomalous pairings. For example, for model (11) from Ref. 10 we have

$$i^{r}\langle\langle a_{i\sigma} n_{i-\sigma} \mid a^{+}_{j\sigma} \rangle\rangle = \langle\langle a_{i\sigma} n_{i-\sigma} \mid a^{+}_{j\sigma} \rangle\rangle - \langle n_{i-\sigma} \rangle \langle\langle a_{i\sigma} \mid a^{+}_{j\sigma} \rangle\rangle +$$

$$+ \langle a_{i\sigma} a_{i-\sigma} \rangle \langle\langle a^{+}_{i-\sigma} \mid a^{+}_{j\sigma} \rangle\rangle. \quad (15)$$

Definition (15) made it possible to perform a consistent derivation of the equations of superconductivity in the strong-coupling approximation for transition metals[10,11] and strongly disordered binary alloys.[12] To understand more clearly this fundamental proposition regarding the complex structure of the average fields, let us investigate the problem of the magnetic polaron. Usually we consider. The s–d-exchange model of magnetic semiconductors. In contrast to the scattering regime,[13] in calculating the one-electron Green's function by using Eqs. (2)-(10) we will take account of the possible formation of a magnetopolar state at finite temperatures.[14] The expression for the spectrum of a magnetic polaron has the form

$$E_{k\sigma} = \epsilon^{0}_{k\sigma} + I^{2}_{s-d} N^{-1} \Psi_{k\sigma}(E_{k\sigma}),$$

$$\Psi_{k\sigma}(\omega) = \sum_{q} \left\{ \frac{\langle S^{-\sigma}_{-q} S^{\sigma}_{q} \rangle}{[1 - I\Lambda_{k\sigma}(\omega)](\omega + z_{\sigma}\omega_{q} - \epsilon^{0}_{k+q,-\sigma})} \right.$$

$$\left. + \frac{[1 + I\Lambda_{k\sigma}(\omega)]\langle (S^{z}_{-q})^{ir} (S^{z}_{q})^{ir} \rangle}{[1 - I\Lambda_{k\sigma}(\omega)](\omega - \epsilon^{0}_{k+q,\sigma})} \right\}. \quad (16)$$

At a zero temperature the solution which we obtained reduces to the classical Shastry–Mattis result.[14]

The approach developed here is closely related to the fundamental ideas of N. N. Bogolyubov regarding broken symmetry and quasiaverages.[2,3] In essence, any method of introducing average fields should be consistent with the nature of the broken symmetry of the system. A vivid example of this assertion is the unsolved problem of giving a microscopic description of antiferromagnetism,[15] for which a method of constructing a consistent theory can be outlined on the basis of this approach. Furthermore, as we see from relations (12)-(16), the method which we have developed makes it possible to calculate in a straightforward manner the spectra of quasiparticle excitations and damping of these excitations in the case where the system has a complex multibranch spectrum.

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