

Bogoliubov's Quasiaverages, Broken Symmetry and Quantum Statistical Physics

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Abstract

The development and applications of the method of quasiaverages developed by N. N. Bogoliubov to quantum statistical physics and to quantum solid state theory and, in particular, to quantum theory of magnetism, were analyzed. The problem of finding the ferromagnetic, antiferromagnetic and superconducting *symmetry broken* solutions of the correlated lattice fermion models was discussed within the irreducible Green functions method. A unified scheme for the construction of generalized mean fields (elastic scattering corrections) and self-energy (inelastic scattering) in terms of the Dyson equation was generalized in order to include the *source fields*. The interrelation of the Bogoliubov's idea of quasiaverages and the concepts of symmetry breaking and quantum protectorate was discussed briefly in the context of quantum statistical physics. The idea of quantum protectorate reveals the essential difference in the behavior of the complex many-body systems at the low-energy and high-energy scales. It was shown that the role of symmetry (and the breaking of symmetries) in combination with the degeneracy of the system was reanalyzed and essentially clarified within the framework of the method of quasiaverages. The complementary notion of quantum protectorate might provide distinctive signatures and good criteria for a hierarchy of energy scales and the appropriate emergent behavior.

Keywords: Symmetry principles; breaking of symmetries; statistical physics and condensed matter physics; Bogoliubov's quasiaverages; quantum protectorate; emergence; quantum theory of magnetism; theory of superconductivity.

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1 Introduction

It is well known that symmetry principles play a crucial role in physics.^{1,2} The theory of symmetry is a basic tool for understanding and formulating the fundamental notions of physics.³ According to F. Wilczek,⁴ "the primary goal of fundamental physics is to discover profound concepts that illuminate our understanding of nature". It is known that symmetry is a driving force in the shaping of physical theory; moreover, the primary lesson of physics of last century is that the secret of nature is symmetry. Every symmetry leads to a conservation law; the well known examples are the conservation of energy, momentum and electrical charge. A variety of other conservation laws can be deduced from symmetry or invariance properties of the corresponding Lagrangian or Hamiltonian of the system. According to Noether theorem, every continuous symmetry transformation under which the Lagrangian of a given system remains invariant implies the existence of a conserved function.² Many fundamental laws of physics in addition to their detailed features possess various symmetry properties. These symmetry properties lead to certain constraints and regularities on the possible properties of matter. Thus the principles of symmetries belong to the **underlying principles** of physics.

It is known that when the Hamiltonian of a system is invariant under a symmetry operation, but the ground state is not, the symmetry of the system can be spontaneously broken.⁵ Symmetry breaking is termed *spontaneous* when there is no explicit term in a Lagrangian which manifestly breaks the symmetry. Symmetries and breaking of symmetries play an important role in statistical physics, quantum field theory, physics of elementary particles, etc.^{6,7}

In physics, spontaneous symmetry breaking occurs when a system that is symmetric with respect to some symmetry group goes into a vacuum state that is not symmetric. When that happens, the system no longer appears to behave in a symmetric manner. It is a phenomenon that naturally occurs in many situations. The mechanism of spontaneous symmetry breaking is usually understood as the mechanism responsible for the occurrence of asymmetric states in quantum systems in the thermodynamic limit and is used in various field of quantum physics.⁸ The intriguing mechanism of spontaneous symmetry breaking is a unifying concept that lie at the basis of most of the recent developments in theoretical physics, from statistical mechanics to many-body theory and to elementary particles theory.^{6,7}

It should be stressed that symmetry implies degeneracy. The greater the symmetry, the greater the degeneracy. The study of the degeneracy of the energy levels plays a very important role in quantum physics. It is of importance to emphasize that when spontaneous symmetry breaking takes place, the ground state of the system is degenerate. Substantial progress in the understanding of the broken symmetry concept was connected with Bogoliubov's fundamental ideas on quasiaverages.⁹⁻¹⁸ Studies of degenerate systems led Bogoliubov in 1960-61 to the formulation of **the method of quasiaverages**. This method has proved to be a universal tool for systems whose ground states become unstable under small perturbations. Thus the role of symmetry (and the breaking of symmetries) in combination with the degeneracy of the system was reanalyzed and essentially clarified by N. N. Bogoliubov in 1960-1961. He invented and formulated a powerful innovative idea of *quasiaverages* in statistical mechanics.^{9,10,13,15,16,18} The very elegant work of N. N. Bogoliubov¹⁰ has been of great importance for a deeper understanding of phase transitions, superfluidity and superconductivity, quantum theory of magnetism¹⁹ and other fields of equilibrium and nonequilibrium statistical mechanics.^{10-13,15,16,18,20-23} The concept of quasiaverages is indirectly related to the theory of phase transition. The instability of thermodynamic averages with respect to perturbations of the Hamiltonian by a breaking of the invariance with respect to a certain group of transformations means that in the system transition to an extremal state occurs. The mathematical apparatus of the method of quasiaverages includes the Bogoliubov theorem^{10,13,18,24} on singularities of type $1/q^2$ and the Bogoliubov inequality for Green and cor-

relation functions as a direct consequence of the method. It includes algorithms for establishing non-trivial estimates for equilibrium quasiaverages, enabling one to study the problem of ordering in statistical systems and to elucidate the structure of the energy spectrum of the underlying excited states. Thus the Bogoliubov's idea of *quasiaverages* is an essential conceptual advance of modern physics.

Many-particle systems where the interaction is strong have often complicated behavior, and require nonperturbative approaches to treat their properties. Such situations often arise in condensed matter systems. Electrical, magnetic and mechanical properties of materials are *emergent collective behaviors* of the underlying quantum mechanics of their electrons and constituent atoms. A principal aim of solid state physics and materials science is to elucidate this emergence. A full achievement of this goal would imply the ability to engineer a material that is optimum for any particular application. The current understanding of electrons in solids uses simplified but workable picture known as the Fermi liquid theory. This theory explains why electrons in solids can often be described in a simplified manner which appears to ignore the large repulsive forces that electrons are known to exert on one another. There is a growing appreciation that this theory probably fails for entire classes of possibly useful materials and there is the suspicion that the failure has to do with unresolved competition between different possible emergent behaviors. It is appropriate to note here that the emergent properties of matter were analyzed and discussed by R. Laughlin and D. Pines^{25,26} from a general point of view (see also Ref.²⁷). They introduced a unifying idea of *quantum protectorate*. This concept belongs also to the underlying principles of physics. The idea of quantum protectorate reveals the essential difference in the behavior of the complex many-body systems at the low-energy and high-energy scales. The existence of two scales, low-energy and high-energy, in the description of physical phenomena is used in physics, explicitly or implicitly.

From the other hand, it was recognized for many years that the strong correlation in solids exist between the motions of various particles (electrons and ions, i.e. the fermion and boson degrees of freedom) which arise from the Coulomb forces. The most interesting objects are metals and their compounds. They are invariant under the translation group of a crystal lattice and have lattice vibrations as well as electron degrees of freedom. There are many evidences for the importance of many-body effects in these systems. Within the semi-phenomenological theory it was suggested that the low-lying excited states of an interacting Fermi gas can be described in terms of a set of "*independent quasiparticles*". However, this was a phenomenological approach and did not reveal the nature of relevant interactions. An alternative way of viewing quasiparticles, more general and consistent, is through the Green function scheme of many-body theory.^{19,22,28,29} It becomes clear that only a thorough experimental and theoretical investigation of quasiparticle many-body dynamics of the many particle systems can provide the answer on the relevant microscopic picture.²² In our works, we discussed the microscopic view of a dynamic behavior of various interacting many-body systems on a lattice.^{22,30-40} A comprehensive description of transition and rare-earth metals and alloys and other materials (as well as efficient predictions of properties of new materials) is possible only in those cases, when there is an adequate quantum-statistical theory based on the information about the electron and crystalline structures. The main theoretical problem of this direction of research, which is the essence of the quantum theory of magnetism, is investigations and improvements of quantum-statistical models describing the behavior of the complex compounds and materials in order to take into account the main features of their electronic structure, namely, their dual "band-atomic" nature.²² The construction of a consistent theory explaining the electronic structure of these substances encounters serious difficulties when trying to describe the collectivization-localization duality in the behavior of electrons. This problem appears to be extremely important, since its solution gives us a key to understanding magnetic, electronic, and other properties of this diverse group of substances. The author of the

present paper investigated the suitability of the basic models with strong electron correlations and with a complex spectrum for an adequate and correct description of the dual character of electron states.²² A universal mathematical formalism was developed for this investigation.^{22,38} It takes into account the main features of the electronic structure and allows one to describe the true quasiparticle spectrum, as well as the appearance of the magnetically ordered, superconducting, and dielectric (or semiconducting) states. With a few exceptions, diverse physical phenomena observed in compounds and alloys of transition and rare-earth metals, cannot be explained in the framework of the mean-field approximation, which overestimates the role of inter-electron correlations in computations of their static and dynamic characteristics. The circle of questions without a precise and definitive answer, so far, includes such extremely important (not only from a theoretical, but also from a practical point of view) problems as the adequate description of quasiparticle dynamics for quantum-statistical models in a wide range of their parameter values. The source of difficulties here lies not only in the complexity of calculations of certain dynamic properties (such as, the density of states, electrical conductivity, susceptibility, electron-phonon spectral function, the inelastic scattering cross section for slow neutrons), but also in the absence of a well-developed method for a consistent quantum-statistical analysis of a many-particle interaction in such systems. A self-consistent field approach was used in the papers^{22,30-40} for description of various dynamic characteristics of strongly correlated electronic systems. It allows one to consistently and quite compactly compute quasiparticle spectra for many-particle systems with strong interaction taking into account damping effects. The correlation effects and quasiparticle damping are the determining factors in analysis of the normal properties of high-temperature superconductors, heavy fermion compounds, etc. We also formulated a general scheme for a theoretical description of electronic properties of many-particle systems taking into account strong inter-electron correlations.^{22,38} The scheme is a synthesis of the method of two-time temperature Green's functions^{19,29} and the diagram technique. An important feature of this approach is a clear-cut separation of the elastic and inelastic scattering processes in many-particle systems (which is a highly nontrivial task for strongly correlated systems). As a result, one can construct a correct basic approximation in terms of generalized mean fields (the elastic scattering corrections), which allows one to describe magnetically ordered or superconducting states of the system. The residual correlation effects, which are the source of quasiparticle damping, are described in terms of the Dyson equation with a formally exact representation for the mass operator. In the present paper we will discuss some applications of the symmetry principles to quantum and statistical physics and quantum solid state theory in the light of our results on quasiparticle many-body dynamics.

2 Bogoliubov's Quasiaverages in Statistical Mechanics

In the work of N. N. Bogoliubov "Quasiaverages in Problems of Statistical Mechanics" the innovative notion of *quasiaverage*¹⁰ was introduced and applied to various problem of statistical physics. In particular, quasiaverages of Green's functions constructed from ordinary averages, degeneration of statistical equilibrium states, principle of weakened correlations, and particle pair states were considered. In this framework the $1/q^2$ -type properties in the theory of the superfluidity of Bose and Fermi systems, the properties of basic Green functions for a Bose system in the presence of condensate, and a model with separated condensate were analyzed.

The method of quasiaverages is a constructive workable scheme for studying systems with spontaneous symmetry breakdown. A quasiaverage is a thermodynamic (in statistical mechanics) or vacuum (in quantum field theory) average of dynamical quantities in a specially modified averaging procedure, enabling one to take into account the effects of the influence of state degeneracy of the system. The method gives the so-called macro-objectivation of the degeneracy in the domain

of quantum statistical mechanics and in quantum physics. In statistical mechanics, under spontaneous symmetry breakdown one can, by using the method of quasiaverages, describe macroscopic observable within the framework of the microscopic approach.

In considering problems of findings the eigenfunctions in quantum mechanics it is well known that the theory of perturbations should be modified substantially for the degenerate systems. In the problems of statistical mechanics we have always the degenerate case due to existence of the additive conservation laws. The traditional approach to quantum statistical mechanics¹⁸ is based on the unique canonical quantization of classical Hamiltonians for systems with finitely many degrees of freedom together with the ensemble averaging in terms of traces involving a statistical operator ρ . For an operator \mathcal{A} corresponding to some physical quantity A the average value of A will be given as

$$\langle A \rangle_H = \text{Tr} \rho A; \quad \rho = \exp^{-\beta H} / \text{Tr} \exp^{-\beta H}, \quad (2.1)$$

where H is the Hamiltonian of the system, $\beta = 1/kT$ is the reciprocal of the temperature.

The core of the problem lies in establishing the existence of a thermodynamic limit (such as $N/V = \text{const}$, $V \rightarrow \infty$, $N = \text{number of degrees of freedom}$, $V = \text{volume}$) and its evaluation for the quantities of interest. Thus in the statistical mechanics the average $\langle A \rangle$ of any dynamical quantity A is defined in a single-valued way. In the situations with degeneracy the specific problems appear. In quantum mechanics, if two linearly independent state vectors (wavefunctions in the Schroedinger picture) have the same energy, there is a degeneracy. In this case more than one independent state of the system corresponds to a single energy level. If the statistical equilibrium state of the system possesses lower symmetry than the Hamiltonian of the system (i.e. the situation with the symmetry breakdown), then it is necessary to supplement the averaging procedure (2.1) by a rule forbidding irrelevant averaging over the values of macroscopic quantities considered for which a change is not accompanied by a change in energy. This is achieved by introducing quasiaverages, that is, averages over the Hamiltonian $H_{\nu\vec{e}}$ supplemented by infinitesimally-small terms that violate the additive conservations laws $H_{\nu\vec{e}} = H + \nu(\vec{e} \cdot \vec{M})$, ($\nu \rightarrow 0$). Thermodynamic averaging may turn out to be unstable with respect to such a change of the original Hamiltonian, which is another indication of degeneracy of the equilibrium state. According to Bogoliubov,¹⁰ the quasiaverage of a dynamical quantity A for the system with the Hamiltonian $H_{\nu\vec{e}}$ is defined as the limit

$$\asymp A \asymp = \lim_{\nu \rightarrow 0} \langle A \rangle_{\nu\vec{e}}, \quad (2.2)$$

where $\langle A \rangle_{\nu\vec{e}}$ denotes the ordinary average taken over the Hamiltonian $H_{\nu\vec{e}}$, containing the small symmetry-breaking terms introduced by the inclusion parameter ν , which vanish as $\nu \rightarrow 0$ after passage to the thermodynamic limit $V \rightarrow \infty$. It is important to note that in this equation limits cannot be interchanged. Thus the existence of degeneracy is reflected directly in the quasiaverages by their dependence upon the arbitrary unit vector \vec{e} . It is also clear that

$$\langle A \rangle = \int \asymp A \asymp d\vec{e}. \quad (2.3)$$

According to definition (2.3), the ordinary thermodynamic average is obtained by extra averaging of the quasiaverage over the symmetry-breaking group. Thus to describe the case of a degenerate state of statistical equilibrium quasiaverages are more convenient, more physical, than ordinary averages.¹⁸ The latter are the same quasiaverages only averaged over all the directions \vec{e} .

It is necessary to stress, that the starting point for Bogoliubov's work¹⁰ was an investigation of additive conservation laws and selection rules, continuing and developing the approach by P. Curie for derivation of selection rules for physical effects (see also Ref.⁶). Bogoliubov demonstrated that in the cases when the state of statistical equilibrium is degenerate, as in the case of the Heisenberg ferromagnet, one can remove the degeneracy of equilibrium states with respect to the group of

spin rotations by including in the Hamiltonian H an additional noninvariant term $\nu M_z V$ with an infinitely small ν . For the Heisenberg ferromagnet the ordinary averages must be invariant with regard to the spin rotation group. The corresponding quasiaverages possess only the property of covariance. Thus the quasiaverages do not follow the same selection rules as those which govern ordinary averages, due to their invariance with regard to the spin rotation group. It is clear that that the unit vector \vec{e} , i.e., the direction of the magnetization \vec{M} vector, characterizes the degeneracy of the considered state of statistical equilibrium. In order to remove the degeneracy one should fix the direction of the unit vector \vec{e} . It can be chosen to be along the z direction. Then all the quasiaverages will be the definite numbers. This is the kind that one usually deals with in the theory of ferromagnetism.

The value of the quasi-average (2.2) may depend on the concrete structure of the additional term $\Delta H = H_\nu - H$, if the dynamical quantity to be averaged is not invariant with respect to the symmetry group of the original Hamiltonian H . For a degenerate state the limit of ordinary averages (2.3) as the inclusion parameters ν of the sources tend to zero in an arbitrary fashion, may not exist. For a complete definition of quasiaverages it is necessary to indicate the manner in which these parameters tend to zero in order to ensure convergence.¹⁵ On the other hand, in order to remove degeneracy it suffices, in the construction of H , to violate only those additive conservation laws whose switching lead to instability of the ordinary average. Thus in terms of quasiaverages the selection rules for the correlation functions^{11,18} that are not relevant are those that are restricted by these conservation laws.

By using H_ν , we define the state $\omega(A) = \langle A \rangle_\nu$ and then let ν tend to zero (after passing to the thermodynamic limit).^{10,11,18} If all averages $\omega(A)$ get infinitely small increments under infinitely small perturbations ν , this means that the state of statistical equilibrium under consideration is nondegenerate.^{10,11,18} However, if some states have finite increments as $\nu \rightarrow 0$, then the state is degenerate. In this case, instead of ordinary averages $\langle A \rangle_H$, one should introduce the quasiaverages (2.2), for which the usual selection rules do not hold.

The method of quasiaverages is directly related to the principle weakening of the correlation^{10,11,18} in many-particle systems. According to this principle, the notion of the weakening of the correlation, known in statistical mechanics,^{10,11,18} in the case of state degeneracy must be interpreted in the sense of the quasiaverages.¹¹

The quasiaverages may be obtained from the ordinary averages by using the cluster property which was formulated by Bogoliubov.¹¹ This was first done when deriving the Boltzmann equations from the chain of equations for distribution functions, and in the investigation of the model Hamiltonian in the theory of superconductivity.^{9,10,13,16,18} To demonstrate this let us consider averages (quasiaverages) of the form

$$F(t_1, x_1, \dots, t_n, x_n) = \langle \dots \Psi^\dagger(t_1, x_1) \dots \Psi(t_j, x_j) \dots \rangle, \quad (2.4)$$

where the number of creation operators Ψ^\dagger may be not equal to the number of annihilation operators Ψ . We fix times and split the arguments $(t_1, x_1, \dots, t_n, x_n)$ into several clusters $(\dots, t_\alpha, x_\alpha, \dots), \dots, (\dots, t_\beta, x_\beta, \dots)$. Then it is reasonably to assume that the distances between all clusters $|x_\alpha - x_\beta|$ tend to infinity. Then, according to the cluster property, the average value (2.4) tends to the product of averages of collections of operators with the arguments $(\dots, t_\alpha, x_\alpha, \dots), \dots, (\dots, t_\beta, x_\beta, \dots)$

$$\lim_{|x_\alpha - x_\beta| \rightarrow \infty} F(t_1, x_1, \dots, t_n, x_n) = F(\dots, t_\alpha, x_\alpha, \dots) \dots F(\dots, t_\beta, x_\beta, \dots). \quad (2.5)$$

For equilibrium states with small densities and short-range potential, the validity of this property can be proved.¹⁸ For the general case, the validity of the cluster property has not yet been proved. Bogoliubov formulated it not only for ordinary averages but also for quasiaverages, i.e., for anomalous averages, too. It works for many important models, including the models of superfluidity^{13,18}

and superconductivity^{13,18,41,42} (see also Refs.⁴³⁻⁴⁵).

To illustrate this statement consider Bogoliubov's theory of a Bose-system with separated condensate, which is given by the Hamiltonian^{13,18}

$$H_\Lambda = \int_\Lambda \Psi^\dagger(x) \left(-\frac{\Delta}{2m}\right) \Psi(x) dx - \mu \int_\Lambda \Psi^\dagger(x) \Psi(x) dx + \frac{1}{2} \int_{\Lambda^2} \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Phi(x_1 - x_2) \Psi(x_2) \Psi(x_1) dx_1 dx_2. \quad (2.6)$$

This Hamiltonian can be written also in the following form

$$H_\Lambda = H_0 + H_1 = \int_\Lambda \Psi^\dagger(q) \left(-\frac{\Delta}{2m}\right) \Psi(q) dq + \frac{1}{2} \int_{\Lambda^2} \Psi^\dagger(q) \Psi^\dagger(q') \Phi(q - q') \Psi(q') \Psi(q) dq dq'. \quad (2.7)$$

Here, $\Psi(q)$, and $\Psi^\dagger(q)$ are the operators of annihilation and creation of bosons. They satisfy the canonical commutation relations

$$[\Psi(q), \Psi^\dagger(q')] = \delta(q - q'); \quad [\Psi(q), \Psi(q')] = [\Psi^\dagger(q), \Psi^\dagger(q')] = 0. \quad (2.8)$$

The system of bosons is contained in the cube A with the edge L and volume V . It was assumed that it satisfies periodic boundary conditions and the potential $\Phi(q)$ is spherically symmetric and proportional to the small parameter. It was also assumed that, at temperature zero, a certain macroscopic number of particles having a nonzero density is situated in the state with momentum zero. The operators $\Psi(q)$, and $\Psi^\dagger(q)$ can be represented in the form

$$\Psi(q) = a_0/\sqrt{V}; \quad \Psi^\dagger(q) = a_0^\dagger/\sqrt{V}, \quad (2.9)$$

where a_0 and a_0^\dagger are the operators of annihilation and creation of particles with momentum zero. To explain the phenomenon of superfluidity, one should calculate the spectrum of the Hamiltonian, which is quite a difficult problem. Bogoliubov suggested the idea of approximate calculation of the spectrum of the ground state and its elementary excitations based on the physical nature of superfluidity. His idea consists of a few assumptions. The main assumption is that at temperature zero the macroscopic number of particles (with nonzero density) has the momentum zero. Therefore, in the thermodynamic limit, the operators a_0/\sqrt{V} and a_0^\dagger/\sqrt{V} commute

$$\lim_{V \rightarrow \infty} [a_0/\sqrt{V}, a_0^\dagger/\sqrt{V}] = \frac{1}{V} \rightarrow 0 \quad (2.10)$$

and are c -numbers. Hence, the operator of the number of particles $N_0 = a_0^\dagger a_0$ is a c -number, too. It is worth noting that the Hamiltonian (2.7) is invariant under the gauge transformation $\tilde{a}_k = \exp(i\varphi) a_k$, $\tilde{a}_k^\dagger = \exp(-i\varphi) a_k^\dagger$, where φ is an arbitrary real number. Therefore, the averages $\langle a_0/\sqrt{V} \rangle$ and $\langle a_0^\dagger/\sqrt{V} \rangle$ must vanish. But this contradicts to the assumption that a_0/\sqrt{V} and a_0^\dagger/\sqrt{V} must become c -numbers in the thermodynamic limit. In addition it must be taken into account that $a_0^\dagger a_0/V = N_0/V \neq 0$ which implies that $a_0/\sqrt{V} = N_0 \exp(i\alpha)/\sqrt{V} \neq 0$ and $a_0^\dagger/\sqrt{V} = N_0 \exp(-i\alpha)/\sqrt{V} \neq 0$, where α is an arbitrary real number. This contradiction may be overcome if we assume that the eigenstates of the Hamiltonian are degenerate and not invariant under gauge transformations, i.e., that a spontaneous breaking of symmetry takes place.

Thus the averages $\langle a_0/\sqrt{V} \rangle$ and $\langle a_0^\dagger/\sqrt{V} \rangle$, which are nonzero under spontaneously broken gauge invariance, are called anomalous averages or *quasiaverages*. This innovative idea of Bogoliubov

penetrate deeply into the modern quantum physics. The systems with spontaneously broken symmetry are studied by use of the transformation of the operators of the form

$$\Psi(q) = a_0/\sqrt{V} + \theta(q); \quad \Psi^\dagger(q) = a_0^\dagger/\sqrt{V} + \theta^*(q), \quad (2.11)$$

where a_0/\sqrt{V} and a_0^\dagger/\sqrt{V} are the numbers first introduced by Bogoliubov in 1947 in his investigation of the phenomenon of superfluidity.^{10,13,18} The main conclusion was made that for the systems with spontaneously broken symmetry, the quasiaverages should be studied instead of the ordinary averages. It turns out that the long-range order appears not only in the system of Bose-particles but also in all systems with spontaneously broken symmetry. Bogoliubov's papers outlined above anticipated the methods of investigation of systems with spontaneously broken symmetry for many years.

As mentioned above, in order to explain the phenomenon of superfluidity, Bogoliubov assumed that the operators a_0/\sqrt{V} and a_0^\dagger/\sqrt{V} become c -numbers in the thermodynamic limit. This statement was rigorously proved in the papers by Bogolyubov and some other authors. Bogolyubov's proof was based on the study of the equations for two-time Green's functions²⁹ and on the assumption that the cluster property holds. It was proved that the solutions of equations for Green's functions for the system with Hamiltonian (2.7) coincide with the solutions of the equations for the system with the same Hamiltonian in which the operators a_0/\sqrt{V} and a_0^\dagger/\sqrt{V} are replaced by numbers. These numbers should be determined from the condition of minimum for free energy. Since all the averages in both systems coincide, their free energies coincide, too.

It is worth noting that the validity of the replacement of the operators a_0 and a_0^\dagger by c -numbers in the thermodynamic limit was confirmed in the numerous subsequent publications of various authors. Thus Bogoliubov's 1947 analysis of the many-body Hamiltonian by means of a c -number substitution for the most relevant operators in the problem, the zero-momentum mode operators, was justified rigorously. Since the Bogoliubov's 1947 analysis is one of the key developments in the theory of the Bose gas, especially the theory of the low density gases currently at the forefront of experiment,⁴³⁻⁴⁵ this result is of importance for the legitimation of that theory. Additional arguments were given in study, where the Bose-Einstein condensation and spontaneous $U(1)$ symmetry breaking were investigated on the basis of Bogoliubov's truncated Hamiltonian H_B for a weakly interacting Bose system, and adding a $U(1)$ symmetry breaking term $\sqrt{V}(\lambda a_0 + \lambda^* a_0^\dagger)$ to H_B . It was shown also, by using the coherent state theory and the mean-field approximation rather than the c -number approximations, that the Bose-Einstein condensation occurs if and only if the $U(1)$ symmetry of the system is spontaneously broken. The real ground state energy and the justification of the Bogoliubov c -number substitution were given by solving the Schrodinger eigenvalue equation and using the self-consistent condition. Thus the Bogoliubov c -number substitutions were fully correct and the symmetry breaking causes the displacement of the condensate state.

The concept of quasiaverages was introduced by Bogoliubov on the basis of an analysis of many-particle systems with a degenerate statistical equilibrium state. Such states are inherent to various physical many-particle systems.^{13,18} Those are liquid helium in the superfluid phase, metals in the superconducting state, magnets in the ferromagnetically ordered state, liquid crystal states, the states of superfluid nuclear matter, etc. (for a review, see Refs.^{22,46}). In case of superconductivity, the source $\nu \sum_k v(k)(a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger + a_{-k\downarrow} a_{k\uparrow})$ was inserted in the BCS-Bogoliubov Hamiltonian, and the quasiaverages were defined by use of the Hamiltonian H_ν . In the general case, the sources are introduced to remove degeneracy. If infinitesimal sources give infinitely small contributions to the averages, then this means that the corresponding degeneracy is absent, and there is no reason to insert sources in the Hamiltonian. Otherwise, the degeneracy takes place, and it is removed by the sources. The ordinary averages can be obtained from quasiaverages by averaging with respect to the parameters that characterize the degeneracy.

N. N. Bogoliubov, Jr.¹⁵ considered some features of quasiaverages for model systems with four-fermion interaction. He discussed the treatment of certain three-dimensional model systems which can be solved exactly. For this aim a new effective way of defining quasiaverages for the systems under consideration was proposed.

Peletminskii and Sokolovskii⁴⁷ have found general expressions for the operators of the flux densities of physical variables in terms of the density operators of these variables. The method of quasiaverages and the expressions found for the flux operators were used to obtain the averages of these operators in terms of the thermodynamic potential in a state of statistical equilibrium of a superfluid liquid.

Vozyakov⁴⁸ reformulated the theory of quantum crystals in terms of quasiaverages. He analyzed a Bose system with periodic distribution of particles which simulates an ensemble in which the particles cannot be regarded as vibrating independently about a position of equilibrium lattice sites. With allowance for macroscopic filling of the states corresponding to the distinguished symmetry, a calculation was made of an excitation spectrum in which there exists a collective branch of gapless type.

Peregoudov⁴⁹ discussed the effective potential method, used in quantum field theory to study spontaneous symmetry breakdown, from the point of view of Bogoliubov's quasiaveraging procedure. It was shown that the effective potential method is a disguised type of this procedure. The catastrophe theory approach to the study of phase transitions was discussed and the existence of the potentials used in that approach was proved from the statistical point of view. It was shown that in the case of broken symmetry, the nonconvex effective potential is not a Legendre transform of the generating functional for connected Green's functions. Instead, it is a part of the potential used in catastrophe theory. The relationship between the effective potential and the Legendre transform of the generating functional for connected Green's functions is given by Maxwell's rule. A rigorous rule for evaluating quasiaveraged quantities within the framework of the effective potential method was established.

N. N. Bogoliubov, Jr. with M. Yu. Kovalevsky and co-authors⁵⁰ developed a statistical approach for solving the problem of classification of equilibrium states in condensed media with spontaneously broken symmetry based on the quasiaverage concept. Classification of equilibrium states of condensed media with spontaneously broken symmetry was carried out. The generators of residual and spatial symmetries were introduced and equations of classification for the order parameter has been found. Conditions of residual symmetry and spatial symmetry were formulated. The connection between these symmetry conditions and equilibrium states of various media with tensor order parameter was found out. An analytical solution of the problem of classification of equilibrium states for superfluid media, liquid crystals and magnets with tensor order parameters was obtained. Superfluid 3He , liquid crystals, quadrupolar magnetics were considered in detail. Possible homogeneous and heterogeneous states were found out. Discrete and continuous thermodynamic parameters, which define an equilibrium state, allowable form of order parameter, residual symmetry, and spatial symmetry generators were established. This approach, which is alternative to the well-known Ginzburg-Landau method, does not contain any model assumptions concerning the form of the free energy as functional of the order parameter and does not employ the requirement of temperature closeness to the point of phase transition. For all investigated cases they found the structure of the order parameters and the explicit forms of generators of residual and spatial symmetries. Under the certain restrictions they established the form of the order parameters in case of spins 0, 1/2, 1 and proposed the physical interpretation of the studied degenerate states of condensed media.

3 Quantum Protectorate

The "quantum protectorate" concept was formulated in paper.²⁵ Its inventors, R. Laughlin and D. Pines, discussed the most fundamental principles of matter description in the widest sense of this word. They formulated their main thesis: emergent physical phenomena, which are regulated by higher physical principles, have a certain property, typical for these phenomena only. This property is their insensitivity to microscopic description. For instance, the crystalline state is the simplest known example of a quantum protectorate, a stable state of matter whose generic low-energy properties are determined by a higher organizing principle and nothing else. There are many other examples.²⁵ These quantum protectorates, with their associated emergent behavior, provide us with explicit demonstrations that the underlying microscopic theory can easily have no measurable consequences whatsoever at low energies. **The nature of the underlying theory is unknowable until one raises the energy scale sufficiently to escape protection.** The existence of two scales, the low-energy and high-energy scales, relevant to the description of magnetic phenomena was stressed by the author of the present work in the papers,^{22,51,52} which were devoted to comparative analysis of localized and band models of quantum theory of magnetism. It was shown there, that the low-energy spectrum of magnetic excitations in the magnetically-ordered solid bodies corresponds to a hydrodynamic pole ($\vec{k}, \omega \rightarrow 0$) in the generalized spin susceptibility χ , which is present in the Heisenberg, Hubbard, and the combined $s - d$ model. In the Stoner band model the hydrodynamic pole is absent, there are no spin waves there. At the same time, the Stoner single-particle's excitations are absent in the Heisenberg model's spectrum. The Hubbard model with narrow energy bands contains both types of excitations: the collective spin waves (the low-energy spectrum) and Stoner single-particle's excitations (the high-energy spectrum). This is a big advantage and flexibility of the Hubbard model in comparison to the Heisenberg model. The latter, nevertheless, is a very good approximation to the realistic behavior in the limit $\vec{k}, \omega \rightarrow 0$, the domain where the hydrodynamic description is applicable, that is, for long wavelengths and low energies. The quantum protectorate concept was applied to the quantum theory of magnetism by the the present author in paper,⁵² where a criterion of applicability of models of the quantum theory of magnetism to description of concrete substances was formulated. The criterion is based on the analysis of the model's low-energy and high-energy spectra. There the detailed analysis was carried out of the idea of quantum protectorate²⁵ in the context of quantum theory of magnetism.⁵² It was suggested that the difficulties in the formulation of quantum theory of magnetism at the microscopic level, that are related to the choice of relevant models, can be understood better in the light of the quantum protectorate concept.⁵² We argued that the difficulties in the formulation of adequate microscopic models of electron and magnetic properties of materials are intimately related to dual, **itinerant** and **localized** behavior of electrons.⁵¹ We formulated a criterion of what basic picture describes best this dual behavior. The main suggestion was that quasiparticle excitation spectra might provide distinctive signatures and good criteria for the appropriate choice of the relevant model. A broad class of the problems of condensed matter physics^{22,23} in the fields of the magnetism and superconductivity of complex materials were reconsidered in relation to these ideas.

4 Irreducible Green Functions Method

It was shown above that it becomes clear that only a thorough experimental and theoretical investigation of quasiparticle many-body dynamics of the many particle systems can provide the answer on the relevant microscopic picture. Many-particle systems where the interaction is strong have often complicated behavior, and require nonperturbative approaches to treat their properties. There are many different approaches to construction of generalized mean-field approximations;

however, all of them have a special-case character. The method of irreducible Green functions (IGF) allows one to tackle this problem in a more systematic fashion. In order to clarify this statement let us consider briefly the main ideas of the IGF approach that allows one to describe completely quasiparticle spectra with damping in a very natural way. When working with infinite hierarchies of equations for Green functions the main problem is finding the methods for their efficient decoupling, with the aim of obtaining a closed system of equations, which determine the Green functions. A decoupling approximation must be chosen individually for every particular problem, taking into account its character. This "individual approach" is the source of critique for being too not transparent, which sometimes appear in the papers using the causal Green functions and diagram technique. However, the ambiguities are also present in the diagram technique, when the choice of an appropriate approximation is made there. The decision, which diagrams one has to sum up, is obvious only for a narrow range of relatively simple problems. In the paper⁵³ devoted to Bose-systems, and in the papers by the author of present work^{30,31,38} devoted to Fermi systems it was shown that for a wide range of problems in statistical mechanics and theory of condensed matter one can outline a fairly systematic recipe for constructing approximate solutions in the framework of irreducible Green's functions method. Within this approach one can look from a unified point of view at the main problems of fundamental characters arising in the method of two-time temperature Green functions. The method of irreducible Green functions is a useful reformulation of the ordinary Bogoliubov-Tyablikov method of equations of motion.^{19,29} We reformulated the two-time Green functions method^{19,30,31,38} to the form which is especially adjusted to correlated fermion systems on a lattice and systems with complex spectra. A very important concept of the whole method is the *generalized mean fields* (GMFs), as it was formulated in.^{22,38} These GMFs have a complicated structure for a strongly correlated case and complex spectra, and are not reduced to the functional of mean densities of the electrons or spins when one calculates excitation spectra at finite temperatures.

To clarify the foregoing, let us consider a retarded Green function of the form¹⁹

$$G^r = \langle\langle A(t), A^\dagger(t') \rangle\rangle = -i\theta(t-t')\langle[A(t)A^\dagger(t')]_\eta\rangle, \eta = \pm 1. \quad (4.1)$$

As an introduction to the concept of IGF, let us describe the main ideas of this approach in a symbolic and simplified form. To calculate the retarded Green function $G(t-t')$, let us write down the equation of motion for it

$$\omega G(\omega) = \langle[A, A^\dagger]_\eta\rangle + \langle\langle[A, H]_- | A^\dagger\rangle\rangle_\omega. \quad (4.2)$$

Here we use the notation $\langle\langle A(t), A^\dagger(t') \rangle\rangle$ for the time-dependent Green function and $\langle\langle A | A^\dagger \rangle\rangle_\omega$ for its Fourier transform.¹⁹ The notation $[A, B]_\eta$ refers to commutation and anticommutation, depending on the value of $\eta = \pm$. The essence of the method is as follows.³⁸ It is based on the notion of the "IRREDUCIBLE" parts of Green functions (or the irreducible parts of the operators, A and A^\dagger , out of which the Green function is constructed) in terms of which it is possible, without recourse to a truncation of the hierarchy of equations for the Green functions, to write down the exact Dyson equation and to obtain an exact analytic representation for the self-energy operator. By definition, we introduce the irreducible part (**ir**) of the Green function

$${}^{(ir)}\langle\langle[A, H]_- | A^\dagger\rangle\rangle = \langle\langle[A, H]_- - zA | A^\dagger\rangle\rangle. \quad (4.3)$$

The unknown constant z is defined by the condition (or constraint)

$$\langle\langle[A, H]_-^{(ir)}, A^\dagger\rangle\rangle_\eta = 0, \quad (4.4)$$

which is an analogue of the orthogonality condition in the Mori formalism.^{54,55} Let us emphasize that due to the complete equivalence of the definition of the irreducible parts for the Green

functions $\langle\langle [A, H]_- | A^\dagger \rangle\rangle^{(ir)}$ and operators $\langle\langle [A, H]_- \rangle\rangle^{(ir)} \equiv ([A, H]_-)^{(ir)}$ we will use both the notation freely ($\langle\langle [A, H]_- | B \rangle\rangle^{(ir)}$ is the same as $\langle\langle (A)^{(ir)} | B \rangle\rangle$). A choice one notation over another is determined by the brevity and clarity of notation only. From the condition (4.4) one can find

$$z = \frac{\langle\langle [A, H]_-, A^\dagger \rangle\rangle_\eta}{\langle\langle [A, A^\dagger] \rangle\rangle_\eta} = \frac{M_1}{M_0}. \quad (4.5)$$

Here M_0 and M_1 are the zeroth and first order moments of the spectral density. Therefore, the irreducible Green functions are defined so that they cannot be reduced to the lower-order ones by any kind of decoupling. It is worth noting that the term "irreducible" in a group theory means a representation of a symmetry operation that cannot be expressed in terms of lower dimensional representations. Irreducible (or connected) correlation functions are known in statistical mechanics. In the diagrammatic approach, the irreducible vertices are defined as graphs that do not contain inner parts connected by the G^0 -line. With the aid of the definition (4.3) these concepts are expressed in terms of retarded and advanced Green functions. The procedure extracts all relevant (for the problem under consideration) mean-field contributions and puts them into the generalized mean-field Green function which is defined here as

$$G^0(\omega) = \frac{\langle\langle [A, A^\dagger] \rangle\rangle_\eta}{(\omega - z)}. \quad (4.6)$$

To calculate the IGF $\langle\langle [A, H]_-(t), A^\dagger(t') \rangle\rangle$ in (4.2), we have to write the equation of motion for it after differentiation with respect to the second time variable t' . The condition of orthogonality (4.4) removes the inhomogeneous term from this equation and is a very crucial point of the whole approach. If one introduces the irreducible part for the right-hand side operator, as discussed above for the "left" operator, the equation of motion (4.2) can be exactly rewritten in the following form:

$$G = G^0 + G^0 P G^0. \quad (4.7)$$

The scattering operator P is given by

$$P = (M_0)^{-1} \langle\langle [A, H]_- | [A^\dagger, H]_- \rangle\rangle^{(ir)} (M_0)^{-1}. \quad (4.8)$$

The structure of equation (4.8) enables us to determine the self-energy operator M by analogy with the diagram technique

$$P = M + M G^0 P. \quad (4.9)$$

We used here the notation M for self-energy (mass operator in quantum field theory). From the definition (4.9) it follows that the self-energy operator M is defined as a proper (in the diagrammatic language, "connected") part of the scattering operator $M = (P)^P$. As a result, we obtain the exact Dyson equation for the thermodynamic double-time Green functions

$$G = G^0 + G^0 M G. \quad (4.10)$$

The difference between P and M can be regarded as two different solutions of two integral equations (4.7) and (4.10). However, from the Dyson equation (4.10) only the full GF is seen to be expressed as a formal solution of the form

$$G = [(G^0)^{-1} - M]^{-1}. \quad (4.11)$$

Equation (4.11) can be regarded as an alternative form of the Dyson equation (4.10) and the *definition* of M provides that the generalized mean-field GF G^0 is specified. On the contrary, for the scattering operator P , instead of the property $G^0 G^{-1} + G^0 M = 1$, one has the property

$$(G^0)^{-1} - G^{-1} = P G^0 G^{-1}.$$

Thus, the *very functional form* of the formal solution (4.11) precisely determines the difference between P and M .

Thus, by introducing irreducible parts of GF (or irreducible parts of the operators, out of which the GF is constructed) the equation of motion (4.2) for the GF can exactly be (but using the orthogonality constraint (4.4)) transformed into the Dyson equation for the double-time thermal GF (4.10). This result is very remarkable because the traditional form of the GF method does not include this point. Notice that all quantities thus considered are exact. Approximations can be generated not by truncating the set of coupled equations of motions but by a specific approximation of the functional form of the mass operator M within a self-consistent scheme expressing M in terms of the initial GF

$$M \approx F[G].$$

Different approximations are relevant to different physical situations. The projection operator technique has essentially the same philosophy. But with using the constraint (4.4) in our approach we emphasize the fundamental and central role of the Dyson equation for calculation of single-particle properties of many-body systems. The problem of reducing the whole hierarchy of equations involving higher-order GFs by a coupled nonlinear set of integro-differential equations connecting the single-particle GF to the self-energy operator is rather nontrivial. A characteristic feature of these equations is that besides the single-particle GF they involve also higher-order GF. The irreducible counterparts of the GFs, vertex functions, serve to identify correctly the self-energy as

$$M = G_0^{-1} - G^{-1}.$$

The integral form of the Dyson equation (4.10) gives M the physical meaning of a nonlocal and energy-dependent effective single-particle potential. This meaning can be verified for the exact self-energy using the diagrammatic expansion for the causal GF.

It is important to note that for the retarded and advanced GFs, the notion of the proper part $M = (P)^p$ is symbolic in nature.^{22,38} In a certain sense, it is possible to say that it is defined here by analogy with the irreducible many-particle T -matrix. Furthermore, by analogy with the diagrammatic technique, we can also introduce the proper part defined as a solution to the integral equation (4.9). These analogues allow us to better understand the formal structure of the Dyson equation for the double-time thermal GF, but only in a symbolic form. However, because of the identical form of the equations for GFs for all three types (advanced, retarded, and causal), we can convert our calculations to causal GF at each stage of calculations and, thereby, confirm the substantiated nature of definition (4.9). We therefore should speak of an analogy of the Dyson equation. Hereafter, we drop this stipulating, since it does not cause any misunderstanding. In a sense, the IGF method is a variant of the Gram-Schmidt orthogonalization procedure.³⁸

It should be emphasized that the scheme presented above gives just a general idea of the IGF method. A more exact explanation why one should not introduce the approximation already in P , instead of having to work out M , is given below when working out the application of the method to specific problems.

The general philosophy of the IGF method is in the separation and identification of elastic scattering effects and inelastic ones. This latter point is quite often underestimated, and both effects are mixed. However, as far as the right definition of quasiparticle damping is concerned, the separation of elastic and inelastic scattering processes is believed to be crucially important for many-body systems with complicated spectra and strong interaction.

From a technical point of view, the elastic GMF renormalizations can exhibit quite a nontrivial structure. To obtain this structure correctly, one should construct the full GF from the complete algebra of relevant operators and develop a special projection procedure for higher-order GFs, in

accordance with a given algebra. Then a natural question arises how to select the relevant set of operators $\{A_1, A_2, \dots, A_n\}$ describing the "relevant degrees of freedom". The above consideration suggests an intuitive and heuristic way to the suitable procedure as arising from an infinite chain of equations of motion (4.2). Let us consider the column

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix},$$

where

$$A_1 = A, \quad A_2 = [A, H], \quad A_3 = [[A, H], H], \dots, \quad A_n = [\dots [A, \underbrace{H \dots H}_n] \dots].$$

Then the most general possible Green function can be expressed as a matrix

$$\hat{G} = \langle\langle \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} | \begin{pmatrix} A_1^\dagger & A_2^\dagger & \dots & A_n^\dagger \end{pmatrix} \rangle\rangle$$

This generalized Green function describes the one-, two-, and n -particle dynamics. The equation of motion for it includes, as a particular case, the Dyson equation for single-particle Green function, and the Bethe-Salpeter equation which is the equation of motion for the two-particle Green function and which is an analogue of the Dyson equation, etc. The corresponding reduced equations should be extracted from the equation of motion for the generalized Green function with the aid of special techniques such as the projection method and similar techniques. This must be a final goal towards a real understanding of the true many-body dynamics. At this point, it is worthwhile to underline that the above discussion is a heuristic scheme only, but not a straightforward recipe. The specific method of introducing the IGFs depends on the form of operators A_n , the type of the Hamiltonian, and conditions of the problem. Here a sketchy form of the IGF method was presented. The aim was to introduce the general scheme and to lay the groundwork for generalizations. We demonstrated in^{22,38} that the IGF method is a powerful tool for describing the quasiparticle excitation spectra, allowing a deeper understanding of elastic and inelastic quasiparticle scattering effects and the corresponding aspects of damping and finite lifetimes. In a certain sense, it provides a clear link between the equation-of-motion approach and the diagrammatic methods due to derivation of the Dyson equation. Moreover, due to the fact that it allows the approximate treatment of the self-energy effects on a final stage, it yields a systematic way of the construction of approximate solutions.

5 Effective and Generalized Mean Fields

The most common technique for studying the subject of interacting many-particle systems is to use the mean field theory. This approximation was especially popular in the theory of magnetism.^{19,22,38} To calculate the susceptibility and other characteristic functions of a system of localized magnetic moments, with a given interaction Hamiltonian, the approximation, termed the "molecular field approximation" was used widely. However, it is not an easy task to give the formal unified definition what the mean field is. In a sense, the mean field is the umbrella term for a variety of theoretical methods of reducing the many-particle problem to the single-particle

one. Mean field theory, that approximates the behavior of a system by ignoring the effect of fluctuations and those spin correlations which dominate the collective properties of the ferromagnet usually provides a starting and estimating point only, for studying phase transitions. The mean field theories miss important features of the dynamics of a system. The main intention of the mean field theories, starting from the works of J. D. van der Waals and P. Weiss, is to take into account the cooperative behavior of a large number of particles. It is well known that earlier theories of phase transitions based on the ideas of van der Waals and Weiss lead to predictions which are qualitatively at variance with results of measurements near the critical point. Other variants of simplified mean field theories such as the Hartree-Fock theory for electrons in atoms, lead to discrepancies of various kinds too. It is therefore natural to analyze the reasons for such drawbacks of earlier variants of the mean field theories.

A number of effective field theories which are improved versions of the "molecular field approximation" were proposed. In our papers^{22,30,31,38} we stressed a specificity of strongly correlated many-particle systems on a lattice contrary to continuum (uniform) systems. The earlier concepts of molecular field were described in terms of a functional of mean magnetic moments (in magnetic terminology) or mean particle densities. The corresponding mean-field functional $F[\langle n \rangle, \langle S^z \rangle]$ describes the *uniform* mean field. Actually, the Weiss model was not based on discrete "spins" as is well known, but the uniformity of the mean internal field was the most essential feature of the model. In the modern language, one should assume that the interaction between atomic spins S_i and its neighbors is equivalent to a mean (or molecular) field, $M_i = \chi_0[h_i^{(ext)} + h_i^{(mf)}]$ and that the molecular field $h_i^{(mf)}$ is of the form $h_i^{(mf)} = \sum_j J(R_{ji})\langle S_j \rangle$ (above T_c). Here h^{ext} is an applied conjugate field, χ_0 is the response function, and $J(R_{ji})$ is an interaction. In other words, the mean field approximation reduces the many-particle problem to a single-site problem in which a magnetic moment at any site can be either parallel or antiparallel to the total magnetic field composed of the applied field and the molecular field. The average interaction of i neighbors was taken into account only, and the fluctuations were neglected. One particular example, where the mean field theory works relatively well is the homogeneous structural phase transitions; in this case the fluctuations are confined in phase space. The next important step was made by L. Neel. He conjectured that the Weiss internal field might be either positive or negative in sign. In the latter case, he showed that below a critical temperature (Neel temperature) an ordered arrangement of equal numbers of oppositely directed atomic moments could be energetically favorable. This new magnetic structure was termed antiferromagnetism. It was conjectured that the two-sublattice Neel (classical) ground state is formed by local staggered internal mean fields.

There is a number of the "correlated effective field" theories, that tend to repair the limitations of simplified mean field theories. The remarkable and ingenious one is the Onsager "reaction field approximation". He suggested that the part of the molecular field on a given dipole moment which comes from the reaction of neighboring molecules to the instantaneous orientation of the moment should not be included into the effective orienting field. This "reaction field" simply follows the motion of the moment and thus does not favor one orientation over another (for details see Refs.^{22,38}).

It is known⁵⁶ that mean-field approximations, for example the molecular field approximation for a spin system, the Hartree-Fock approximation and the BCS-Bogoliubov approximation for an electron system are universally formulated by the Bogoliubov inequality:

$$-\beta^{-1} \ln(\text{Tr} e^{-\beta H}) \leq -\beta^{-1} \ln(\text{Tr} e^{-\beta H^{mf}}) + \frac{\text{Tr} e^{-\beta H^{mf}}(H - H^{mf})}{\text{Tr} e^{-\beta H^{mf}}}. \quad (5.1)$$

Here F is the free energy, and H^{mf} is a "trial" or a "mean field" approximating Hamiltonian. This inequality gives the upper bound of the free energy of a many-body system. It is important

to emphasize that the BCS-Bogoliubov theory of superconductivity^{12,41,42} was formulated on the basis of a trial Hamiltonian which consists of a quadratic form of creation and annihilation operators, including "anomalous" (off-diagonal) averages. The functional of the mean field (for the superconducting single-band Hubbard model) is of the following form:⁵⁷

$$\Sigma_{\sigma}^c = U \begin{pmatrix} \langle a_{i-\sigma}^{\dagger} a_{i-\sigma} \rangle & -\langle a_{i\sigma} a_{i-\sigma} \rangle \\ -\langle a_{i-\sigma}^{\dagger} a_{i\sigma}^{\dagger} \rangle & -\langle a_{i\sigma}^{\dagger} a_{i\sigma} \rangle \end{pmatrix}. \quad (5.2)$$

The "anomalous" off-diagonal terms fix the relevant BCS-Bogoliubov vacuum and select the appropriate set of solutions. From the point of view of quantum many-body theory, the problem of adequate introduction of mean fields for system of many interacting particles can be most consistently investigated in the framework of the IGF method. A correct calculation of the quasiparticle spectra and their damping, particularly, for systems with a complicated spectrum and strong interaction²² reveals, that the generalized mean fields can have very complicated structure which cannot be described by a functional of the mean-particle density.

To illustrate the actual distinction of description of the generalized mean field in the equation-of-motion method for the double-time Green functions, let us compare the two approaches, namely, that of Tyablikov¹⁹ and of Callen.⁵⁸ We shall consider the Green function $\langle\langle S^+ | S^- \rangle\rangle$ for the isotropic Heisenberg model

$$H = -\frac{1}{2} \sum_{ij} J(i-j) \vec{S}_i \vec{S}_j. \quad (5.3)$$

The equation of motion for the spin Green function is of the form

$$\begin{aligned} \omega \langle\langle S_i^+ | S_j^- \rangle\rangle_{\omega} = \\ 2\langle S^z \rangle \delta_{ij} + \sum_g J(i-g) \langle\langle S_i^+ S_g^z - S_g^+ S_i^z | S_j^- \rangle\rangle_{\omega}. \end{aligned} \quad (5.4)$$

The Tyablikov decoupling expresses the second-order Green function in terms of the first (initial) Green function:

$$\langle\langle S_i^+ S_g^z | S_j^- \rangle\rangle = \langle S^z \rangle \langle\langle S_i^+ | S_j^- \rangle\rangle. \quad (5.5)$$

This approximation is an RPA-type; it does not lead to the damping of spin wave excitations

$$E(q) = \sum_g J(i-g) \langle S^z \rangle \exp[i(\vec{R}_i - \vec{R}_g) \vec{q}] = 2\langle S^z \rangle (J_0 - J_q). \quad (5.6)$$

The reason for this is rather transparent. This decoupling does not take into account the *inelastic* magnon-magnon scattering processes. In a sense, the Tyablikov approximation consists of approximating the commutation relations of spin operators to the extent of replacing the commutation relation $[S_i^+, S_j^-]_- = 2S_i^z \delta_{ij}$ by $[S_i^+, S_j^-]_- = 2\langle S^z \rangle \delta_{ij}$.

Callen⁵⁸ has proposed an improved decoupling approximation in the method of Tyablikov in the following form:

$$\langle\langle S_g^z S_f^+ | B \rangle\rangle \rightarrow \langle S^z \rangle \langle\langle S_f^+ | B \rangle\rangle - \alpha \langle S_g^- S_f^+ \rangle \langle\langle S_g^+ | B \rangle\rangle.$$

Here $0 \leq \alpha \leq 1$. To clarify this point, it should be reminded that for spin 1/2 (the procedure was generalized by Callen to an arbitrary spin), the spin operator S^z can be written as $S_g^z = S - S_g^- S_g^+$ or $S_g^z = \frac{1}{2}(S_g^+ S_g^- - S_g^- S_g^+)$. It is easy to show that

$$S_g^z = \alpha S + \frac{1-\alpha}{2} S_g^+ S_g^- - \frac{1+\alpha}{2} S_g^- S_g^+.$$

The operator $S_g^- S_g^+$ represents the deviation of $\langle S^z \rangle$ from S . In the low-temperature region, this deviation is small, and $\alpha \sim 1$. Similarly, the operator $\frac{1}{2}(S_g^+ S_g^- - S_g^- S_g^+)$ represents the deviation of $\langle S^z \rangle$ from 0. Thus, when $\langle S^z \rangle$ approaches to zero, one can expect that $\alpha \sim 0$. Thus, in this way, it is possible to obtain a correction to the Tyablikov decoupling with either a positive or negative sign, or no correction at all, or any intermediate value, depending on the choice of α . The above Callen arguments are not rigorous, for, although the difference in the operators $S^+ S^-$ and $S^- S^+$ is small if $\langle S^z \rangle \sim 0$, each operator makes a contribution of the order of S , and it is each operator which is treated approximately, not the difference. There are some other drawbacks of the Callen decoupling scheme. Nevertheless, the Callen decoupling was the first conceptual attempt to introduce the interpolation decoupling procedure. Let us note that the choice of $\alpha = 0$ over the entire temperature range is just the Tyablikov decoupling (5.5).

The energy spectrum for the Callen decoupling is given by

$$E(q) = 2\langle S^z \rangle ((J_0 - J_q) + \frac{\langle S^z \rangle}{NS^2} \sum_k [J(k) - J(k - q)] N(E(k))). \quad (5.7)$$

Here $N(E(k))$ is the Bose distribution function $N(E(k)) = [\exp(E(k)\beta) - 1]^{-1}$. This is an implicit equation for $N(E(k))$, involving the unknown quantity $\langle S^z \rangle$. For the latter an additional equation is given.⁵⁸ Thus, both these equations constitute a set of coupled equations which must be solved self-consistently for $\langle S^z \rangle$. This formulation of the Callen decoupling scheme displays explicitly the tendency of the improved description of the mean field. In a sense, it is possible to say that the Callen work dates really the idea of the generalized mean field within the equation-of-motion method for double-time GFs, however, in a semi-intuitive form. The next essential steps were made by Plakida⁵³ for the Heisenberg ferromagnet and by Kuzemsky³⁰ for the Hubbard model. Later many approximate schemes for decoupling the hierarchy of equations for GF were proposed, improving the Tyablikov's and Callen's decouplings. Various approaches generalizing the random phase's approximation in the ferromagnetism theory for wide ranges of temperature were considered in paper by Czachor and Holas.⁵⁹ As was mentioned above, the correct definition of *generalized mean fields* depends on the condition of the problem, the strength of interaction, the choice of relevant operators, and on the symmetry requirements. The most important conclusion to be drawn from the present consideration is that the GMF, in principle, can have quite a nontrivial structure and cannot be reduced to the mean-density functional only.

6 Quasiaverages and Irreducible Green Functions Method

In condensed matter physics, the symmetry is important in classifying different phases and understanding the phase transitions between them. There is an important distinction between the case where the broken symmetry is continuous (e.g. translation, rotation, gauge invariance) or discrete (e.g. inversion, time reversal symmetry).²² The Goldstone theorem states that when a continuous symmetry is spontaneously broken and the interactions are short ranged a collective mode (excitation) exists with a gapless energy spectrum (i.e. the energy dispersion curve starts at zero energy and is continuous). Acoustical phonons in a crystal are prime examples of such so-called gapless Goldstone modes. Other examples are the Bogoliubov sound modes in (charge neutral) Bose condensates⁴³⁻⁴⁵ and spin waves (magnons) in ferro- and antiferromagnets. N. N. Bogoliubov and then Y. Nambu in their works shown that the general features of superconductivity are in fact model independent consequences of the spontaneous breakdown of electromagnetic gauge invariance. It is important to emphasize that the BCS-Bogoliubov theory of superconductivity^{12, 41, 42} was formulated on the basis of a trial Hamiltonian which consists of a quadratic form of creation and annihilation operators, including "anomalous" (off-diagonal) averages.¹³ The strong-coupling

BCS-Bogoliubov theory of superconductivity was formulated for the Hubbard model in the localized Wannier representation in Refs.^{57,60,61} Therefore, instead of the algebra of the normal state's operator $a_{i\sigma}, a_{i\sigma}^\dagger$ and $n_{i\sigma}$, for description of superconducting states, one has to use a more general algebra, which includes the operators $a_{i\sigma}, a_{i\sigma}^\dagger, n_{i\sigma}$ and $a_{i\sigma}a_{i-\sigma}, a_{i\sigma}^\dagger a_{i-\sigma}^\dagger$. The relevant generalized one-electron Green function will have the following form:^{38,57,60}

$$G_{ij}(\omega) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} \langle\langle a_{i\sigma} | a_{j\sigma}^\dagger \rangle\rangle & \langle\langle a_{i\sigma} | a_{j-\sigma} \rangle\rangle \\ \langle\langle a_{i-\sigma}^\dagger | a_{j\sigma}^\dagger \rangle\rangle & \langle\langle a_{i-\sigma}^\dagger | a_{j-\sigma} \rangle\rangle \end{pmatrix}. \quad (6.1)$$

As it was discussed in Refs.,^{22,38} the off-diagonal (anomalous) entries of the above matrix select the vacuum state of the system in the BCS-Bogoliubov form, and they are responsible for the presence of anomalous averages. For treating the problem we follow the general scheme of the irreducible Green functions method.^{22,38} In this approach we start from the equation of motion for the Green function $G_{ij}(\omega)$ (normal and anomalous components)

$$\sum_j (\omega \delta_{ij} - t_{ij}) \langle\langle a_{j\sigma} | a_{i'\sigma}^\dagger \rangle\rangle = \delta_{ii'} + \quad (6.2)$$

$$U \langle\langle a_{i\sigma} n_{i-\sigma} | a_{i'\sigma}^\dagger \rangle\rangle + \sum_{nj} V_{ijn} \langle\langle a_{j\sigma} u_n | a_{i'\sigma}^\dagger \rangle\rangle,$$

$$\sum_j (\omega \delta_{ij} + t_{ij}) \langle\langle a_{j-\sigma}^\dagger | a_{i'\sigma}^\dagger \rangle\rangle = \quad (6.3)$$

$$-U \langle\langle a_{i-\sigma}^\dagger n_{i\sigma} | a_{i'\sigma}^\dagger \rangle\rangle + \sum_{nj} V_{jin} \langle\langle a_{j-\sigma}^\dagger u_n | a_{i'\sigma}^\dagger \rangle\rangle.$$

The irreducible Green functions are introduced by definition

$${}^{(ir)} \langle\langle a_{i\sigma} a_{i-\sigma}^\dagger a_{i-\sigma} | a_{i'\sigma}^\dagger \rangle\rangle_\omega = \langle\langle a_{i\sigma} a_{i-\sigma}^\dagger a_{i-\sigma} | a_{i'\sigma}^\dagger \rangle\rangle_\omega - \quad (6.4)$$

$$-\langle n_{i-\sigma} \rangle G_{11} + \langle a_{i\sigma} a_{i-\sigma} \rangle \langle\langle a_{i-\sigma}^\dagger | a_{i'\sigma}^\dagger \rangle\rangle_\omega,$$

$${}^{(ir)} \langle\langle a_{i\sigma}^\dagger a_{i\sigma} a_{i-\sigma}^\dagger | a_{i'\sigma}^\dagger \rangle\rangle_\omega = \langle\langle a_{i\sigma}^\dagger a_{i\sigma} a_{i-\sigma}^\dagger | a_{i'\sigma}^\dagger \rangle\rangle_\omega -$$

$$-\langle n_{i\sigma} \rangle G_{21} + \langle a_{i\sigma}^\dagger a_{i-\sigma}^\dagger \rangle \langle\langle a_{i\sigma} | a_{i'\sigma}^\dagger \rangle\rangle_\omega.$$

The self-consistent system of superconductivity equations follows from the Dyson equation^{22,38,57}

$$\hat{G}_{ii'}(\omega) = \hat{G}_{ii'}^0(\omega) + \sum_{jj'} \hat{G}_{ij}^0(\omega) \hat{M}_{jj'}(\omega) \hat{G}_{j'i'}(\omega). \quad (6.5)$$

The mass operator $M_{jj'}(\omega)$ describes the processes of inelastic electron scattering on lattice vibrations. The elastic processes are described by the quantity (5.2). Thus the "anomalous" off-diagonal terms fix the relevant BCS-Bogoliubov vacuum and select the appropriate set of solutions. The functional of the generalized mean field for the superconducting single-band Hubbard model is of the form Σ_σ^c . A remark about the BCS-Bogoliubov mean-field approach is instructive. Speaking in physical terms, this theory involves a condensation correctly, in spite that such a condensation cannot be obtained by an expansion in the effective interaction between electrons. Other mean field theories, *e.g.* the Weiss molecular field theory and the van der Waals theory of the liquid-gas transition are much less reliable. The reason why a mean-field theory of the superconductivity in the BCS-Bogoliubov form is successful would appear to be that the main correlations in metal are governed by the extreme degeneracy of the electron gas. The correlations due to the pair condensation, although they have dramatic effects, are weak (at least in the ordinary superconductors) in comparison with the typical electron energies, and may be treated in an average way

with a reasonable accuracy. It should be emphasized that the high-temperature superconductors, discovered two decades ago, motivated an intensification of research in superconductivity, not only because applications are promising, but because they also represent a new state of matter that breaks certain fundamental symmetries. These are the broken symmetries of gauge (superconductivity), reflection (d -wave superconducting order parameter), and time-reversal (ferromagnetism). Superconductivity and antiferromagnetism are both the spontaneously broken symmetries. The question of symmetry breaking within the localized and band models of antiferromagnets was studied by the author of this work in Refs.^{22,32,37,51} It has been found there that the concept of spontaneous symmetry breaking in the band model of magnetism is much more complicated than in the localized model. In the framework of the band model of magnetism one has to additionally consider the so called anomalous propagators of the form

$$\begin{aligned} \text{FM} : G_{fm} &\sim \langle\langle a_{k\sigma}; a_{k-\sigma}^\dagger \rangle\rangle, \\ \text{AFM} : G_{afm} &\sim \langle\langle a_{k+Q\sigma}; a_{k+Q'\sigma'}^\dagger \rangle\rangle. \end{aligned}$$

In the case of the band antiferromagnet the ground state of the system corresponds to a spin-density wave (SDW), where a particle scattered on the internal inhomogeneous periodic field gains the momentum $Q - Q'$ and changes its spin: $\sigma \rightarrow \sigma'$. The long-range order parameters are defined as follows:

$$\text{FM} : m = 1/N \sum_{k\sigma} \langle a_{k\sigma}^\dagger a_{k-\sigma} \rangle, \quad (6.6)$$

$$\text{AFM} : M_Q = \sum_{k\sigma} \langle a_{k\sigma}^\dagger a_{k+Q-\sigma} \rangle. \quad (6.7)$$

It is important to stress, that the long-range order parameters here are functionals of the internal field, which in turn is a function of the order parameter. Thus, in the cases of rotation and translation invariant Hamiltonians of band ferro- and antiferromagnetics one has to add the following infinitesimal sources removing the degeneracy:

$$\text{FM} : \nu \mu_B H_x \sum_{k\sigma} a_{k\sigma}^\dagger a_{k-\sigma}, \quad (6.8)$$

$$\text{AFM} : \nu \mu_B H \sum_{kQ} a_{k\sigma}^\dagger a_{k+Q-\sigma}. \quad (6.9)$$

Here, $\nu \rightarrow 0$ after the usual in statistical mechanics infinite-volume limit $V \rightarrow \infty$. The ground state in the form of a spin-density wave was obtained for the first time by Overhauser. There, the vector \vec{Q} is a measure of inhomogeneity or translation symmetry breaking in the system. The analysis performed by various authors showed that the antiferromagnetic and more complicated states (for instance, ferrimagnetic) can be described in the framework of a generalized mean-field approximation.³⁷ In doing that we have to take into account both the normal averages $\langle a_{i\sigma}^\dagger a_{i\sigma} \rangle$ and the anomalous averages $\langle a_{i\sigma}^\dagger a_{i-\sigma} \rangle$. It is clear that the anomalous terms break the original rotational symmetry of the Hubbard Hamiltonian. Thus, the generalized mean-field's approximation for the antiferromagnet has the following form³⁷ $n_{i-\sigma} a_{i\sigma} \simeq \langle n_{i-\sigma} \rangle a_{i\sigma} - \langle a_{i-\sigma}^\dagger a_{i\sigma} \rangle a_{i-\sigma}$. A self-consistent theory of band antiferromagnetism³⁷ was developed by the author of this work using the method of the irreducible Green functions.^{22,38} The following definition of the irreducible Green functions was used:

$$\begin{aligned} {}^{ir} \langle\langle a_{k+p\sigma} a_{p+q-\sigma}^\dagger a_{q-\sigma} | a_{k\sigma}^\dagger \rangle\rangle_\omega &= \langle\langle a_{k+p\sigma} a_{p+q-\sigma}^\dagger a_{q-\sigma} | a_{k\sigma}^\dagger \rangle\rangle_\omega - \\ &\delta_{p,0} \langle n_{q-\sigma} \rangle G_{k\sigma} - \langle a_{k+p\sigma} a_{p+q-\sigma}^\dagger \rangle \langle\langle a_{q-\sigma} | a_{k\sigma}^\dagger \rangle\rangle_\omega. \end{aligned} \quad (6.10)$$

The algebra of relevant operators must be chosen as follows $((a_{i\sigma}, a_{i\sigma}^\dagger, n_{i\sigma}, a_{i\sigma}^\dagger a_{i-\sigma})$. The corresponding initial GF will have the following matrix structure

$$\mathcal{G}_{AFM} = \begin{pmatrix} \langle\langle a_{i\sigma} | a_{j\sigma}^\dagger \rangle\rangle & \langle\langle a_{i\sigma} | a_{j-\sigma}^\dagger \rangle\rangle \\ \langle\langle a_{i-\sigma} | a_{j\sigma}^\dagger \rangle\rangle & \langle\langle a_{i-\sigma} | a_{j-\sigma}^\dagger \rangle\rangle \end{pmatrix}.$$

The off-diagonal terms select the vacuum state of the band's antiferromagnet in the form of a spin-density wave. With this definition, one introduces the so-called anomalous (off-diagonal) Green functions which fix the relevant vacuum and select the proper symmetry broken solutions. The theory of the itinerant antiferromagnetism³⁷ was formulated by using sophisticated arguments of the irreducible Green functions method in complete analogy with our description of the Heisenberg antiferromagnet at finite temperatures.³² For the two-sublattice antiferromagnet we used the matrix Green function of the form

$$\hat{G}(k; \omega) = \begin{pmatrix} \langle\langle S_{ka}^+ | S_{-ka}^- \rangle\rangle & \langle\langle S_{ka}^+ | S_{-kb}^- \rangle\rangle \\ \langle\langle S_{kb}^+ | S_{-ka}^- \rangle\rangle & \langle\langle S_{kb}^+ | S_{-kb}^- \rangle\rangle \end{pmatrix}. \quad (6.11)$$

Here, the Green functions on the main diagonal are the usual or normal Green functions, while the off-diagonal Green functions describe contributions from the so-called anomalous terms, analogous to the anomalous terms in the BCS-Bogoliubov superconductivity theory. The anomalous (or off-diagonal) average values in this case select the vacuum state of the system precisely in the form of the two-sublattice Neel state.²²

7 Conclusions

In the present work we shown that the development and improvement of the methods of quantum statistical mechanics still remains quite an important direction of research.^{22,23} In particular, the Bogoliubov's method of quasiaverages gives the deep foundation and clarification of the concept of broken symmetry.⁶² It makes the emphasis on the notion of a degeneracy and plays an important role in equilibrium statistical mechanics of many-particle systems. According to that concept, infinitely small perturbations can trigger macroscopic responses in the system if they break some symmetry and remove the related degeneracy (or quasidegeneracy) of the equilibrium state. As a result, they can produce macroscopic effects even when the perturbation magnitude is tend to zero, provided that happens after passing to the thermodynamic limit.

We have discussed the theory of the correlation effects for many-particle interacting systems using the ideas of quasiaverages for interacting electron and spin systems on a lattice. The workable and self-consistent *irreducible Green functions* approach to the decoupling problem for the equation-of-motion method for double-time temperature Green functions has been presented. The main advantage of the formalism consists in the clear separation of the elastic scattering corrections (generalized mean fields) and inelastic scattering effects (damping and finite lifetimes). These effects could be self-consistently incorporated in a general and compact manner. Using the IGF method, it is possible to obtain a closed self-consistent set of equations determining the relevant Green functions and self-energy. These equations give a general microscopic description of correlation effects. Moreover, this approach gives the workable scheme for the definition of relevant generalized mean fields written in terms of appropriate correlators. This picture of interacting many-particle systems on a lattice is far richer and gives more possibilities for the analysis of phenomena which can actually take place. In this sense the approach we described produces more advanced physical picture of the quasiparticle many-body dynamics.

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