Turbulent polarization in QED and QCD plasma

Andrei Leonidov

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Theory of hadronic matter under extremal conditions, 02.10.2013
The only available model description of (some) HIC data is that by 3+1 weakly viscous hydro:

\[ \partial_{\mu} T_{\mu\nu} = 0 \]

It is assumed that EOS \( p = f(\epsilon) \) exists and is close to that from lattice QCD.

Initial conditions \( T_{\mu\nu}(\tau = \tau_0, \eta, x_\perp) \) are fixed at \( \tau = \tau_0 \sim 0.2 - 0.5 \text{ fm} \).

This is hard to believe!

Initial conditions for hydro are fixed at time of order of the size of the initial overlap zone at LHC energies.

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Do we have a theory?

We do not understand the nature of the early stage of heavy ion collisions, but we have several directions of research, each having their own merits and weaknesses, that look reasonable:

- Quantum field theory at strong coupling
- Color Glass Condensate based description (strong initial classical fields dressed by quantum fluctuations).
- Turbulent gluon matter

This talk focuses on one particular segment of these studies: instabilities/turbulence effects in QGP understood as a system of hard particle modes and soft fields.
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This talk focuses on one particular segment of these studies: instabilities/turbulence effects in QGP understood as a system of hard particle modes and soft fields.
Plan of the talk

Evolution of Weibel instabilities and turbulence in fixed-box anisotropic QGP in the HTL approximation

Evolution of Weibel instabilities in expanding anisotropic QGP in the HTL approximation

Evolution of Weibel instabilities in fixed-box anisotropic QGP beyond the HTL approximation

Anomalous viscosity in turbulent QGP

Turbulence-induced instabilities and damping in weakly turbulent QED and QCD plasma

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Anisotropic static HTL non-Abelian plasma


- Vlasov collisionless evolution

\[ D_\mu (A) F^{\mu \nu} - g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2|p|} p^\mu \frac{\partial f(p)}{\partial p^\beta} W^{\beta} (x; v) \]

\[ [v \cdot D(A)] W^{\beta} (x; v) = F_{\beta \gamma} (A) v^\gamma \]
Anisotropic static HTL non-Abelian plasma

Vlasov collisionless evolution

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Initial distribution

\[ f(p) \sim f_{iso}(p^2 + \xi p_z^2) \]
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- Initial seeds are those for auxiliary fields \( W \):

\[ \langle \mathcal{W}_v^a (0, x) \mathcal{W}_v^b (0, y) \rangle = \delta^{ab} \delta^3_{x, y} \sigma^2 a^3 \]
Vlasov collisionless evolution

\[ D_\mu(A) F^{\mu\nu} \equiv g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2|p|} p^\mu \frac{\partial f(p)}{\partial p^\beta} W_\beta(x; v) \]

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Early dynamics is dominated by the Weibel instabilities
Anisotropic static HTL Abelian plasma: evolution

Turbulent QED and QCD plasma

Andrei Leonidov (LPI Moscow)
Anisotropic static HTL Abelian plasma: evolution

- An approximately linear regime sets in at $tm_\infty \sim 60$
Definition of spectra

\begin{align*}
    f_A(k) &= \frac{k}{N_{\text{dof}} V} \langle A^2(k) \rangle, \\
    f_E(k) &= \frac{1}{N_{\text{dof}} k V} \langle E^2(k) \rangle
\end{align*}

Spectra are calculated in the lattice Coulomb gauge. In the saturated regime, the spectra display Kolmogorov turbulent-like scaling \( f \sim \frac{1}{k^\nu} \) with \( \nu = 2 \).
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Anisotropic static HTL Abelian plasma: spectra

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- In the saturated regime the spectra display Kolmogorov turbulent-like scaling \( f \sim 1/k^\nu \) with \( \nu = 2 \).
Evolution of field spectra for $80 \leq m_\infty t \leq 150$. 
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Initial conditions matched to the CGC description of the initial fields at LHC.
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Initial conditions matched to the CGC description of the initial fields at LHC.

The study describes the QGP evolution for $\tau \lesssim 10$ fm.
Anisotropic expanding HTL non-Abelian plasma: energy

\[ \tau \sim \frac{Q_s \tau}{10} \]

Field energy densities / \( (Q_s^4 / g^2) \)

Total Energy

- No saturation of exponential growth
Rapid buildup of longitudinal pressure
The spectra are exponential, not powerlike.
Vlasov collisionless evolution

\[ p^\mu \left[ \partial_\mu - g q^a F^{a\mu\nu} \partial_p^\nu - g f_{abc} A^b_{\mu} q^c \partial_{q^a} \right] f(x, p, q) = 0 \]

\[ D_\mu F^{\mu\nu} = J^\nu = g \int \frac{d^3p}{(2\pi)^3} dq q^\nu f(t, x, p, q) \]
Vlasov collisionless evolution

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Initial distribution

\[ f(p) = n_g \left( \frac{2\pi}{p_h} \right)^2 \delta(p_z) \exp(-p_T/p_h) \]
Anisotropic bHTL static non-Abelian plasma bHTL


- **Vlasov collisionless evolution**
  \[
  p^\mu \left[ \partial_\mu - g q^a F_{\mu \nu}^a \partial_\nu^p - g f_{abc} A_{\mu}^b q^c \partial_{q^a} \right] f(x, p, q) = 0
  \]
  \[
  D_\mu F^{\mu \nu} = J^\nu = g \int \frac{d^3 p}{(2\pi)^3} dq \; q^\nu f(t, x, p, q)
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  \[
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  \]

- **Initial field amplitudes**
  \[
  \langle A_i^a(x) A_j^b(y) \rangle = \frac{4 \mu^2}{g^2} \delta_{ij} \delta^{ab} \delta(x - y)
  \]
Vlasov collisionless evolution

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Characteristic mass scale

\[ m^2_\infty = g^2 N_c \int \frac{d^3 p}{(2\pi)^3} \frac{f(p)}{|p|} \sim g^2 N_c \frac{n_g}{p_h} \]
Anisotropic bHTL static Abelian plasma: evolution

Weak initial fields $\sim 0.1 m_\perp^4$/$g_4^2$. Slope for $B_\perp^2$ growth is half of the HTL value: nonlinear effects are important!

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Anisotropic bHTL static Abelian plasma: evolution

Weak initial fields \( \sim 0.1m^4_{\infty}/g^2 \)
Anisotropic bHTL static Abelian plasma: evolution

- Weak initial fields $\sim 0.1m_\infty^4/g^2$
- Slope for $B_T^2$ growth is half of the HTL value: nonlinear effects are important!
The time evolution of non-Abelian fields stronger than $\sim m_\infty^4 / g^2$ differs from that in the (effectively Abelian) extreme weak-field limit.
Distribution of fields gets isotropic.
Anisotropic bHTL static non-Abelian plasma: evolution

For strong initial fields no instability effects can be seen.
Anisotropic bHTL static non-Abelian plasma: particle anisotropy

Particle distribution remains strongly anisotropic when instability-related field evolution is already long over.

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Turbulent QED and QCD plasma

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Particle distribution remains strongly anisotropic when instability-related field evolution is already long over.
Experimental studies of "ordinary" plasma have demonstrated that it is practically never observed in the state of textbook thermal equilibrium.

Collective properties of turbulent plasma are markedly different from those of the ordinary equilibrium one. Turbulent plasmas are characterized, in particular, by anomalously low viscosity and conductivity, dominant effects of coherent nonlinear structures on transport properties, etc.
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Realistic description of experimentally observed plasma properties is possible only through taking into account a presence, in addition to thermal excitations, of randomly excited fields. The resulting object was termed turbulent plasma.
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Collective properties of turbulent plasma are markedly different from those of the ordinary equilibrium one. Turbulent plasmas are characterized, in particular, by anomalously low viscosity and conductivity, dominant effects of coherent nonlinear structures on transport properties, etc.
Weakly turbulent plasma is described as a system of hard particles coexisting with weak turbulent fields $F_{\mu\nu}^T$. 

The collisionless Vlasov approximation plasma properties are defined by the following system of equations ($F_R^{\mu\nu}$ is a regular non-turbulent field):

$$p^\mu \left[ \partial^\mu - \text{eq} (F_R^{\mu\nu} + F_T^{\mu\nu}) \partial^\nu \right] f(p, x, q) = 0$$

$$\partial^\mu (F_R^{\mu\nu} + F_T^{\mu\nu}) = j^\nu (x) = e \sum_q \int dP p^\nu q f(p, x, q)$$
• Weakly turbulent plasma is described as a system of hard particles coexisting with weak turbulent fields $F_{\mu\nu}^T$

• In the collisionless Vlasov approximation plasma properties are defined by the following system of equations ($F_{\mu\nu}^R$ is a regular non-turbulent field):

$$p^\mu \left[ \partial_\mu - eq \left( F_{\mu\nu}^R + F_{\mu\nu}^T \right) \frac{\partial}{\partial p_\nu} \right] f(p, x, q) = 0$$

$$\partial^\mu \left( F_{\mu\nu}^R + F_{\mu\nu}^T \right) = j_\nu$$

$$j_\nu(x) = e \sum_{q,s} \int dP p_\nu q f(p, x, q)$$
Weakly turbulent QED plasma: EOM

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    \partial_\mu \left( F_{\mu \nu}^R + F_{\mu \nu}^T \right) &= j_\nu \\
    j_\nu(x) &= e \sum_{q,s} \int dP p_\nu q f(p, x, q)
\end{align*}
\]

• Main effects to follow: collisions of particles with turbulent fields
The origin of anomalous viscosity is in deflection of particles in random turbulent fields.
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\[ \eta = \frac{1}{3} n \bar{p} \lambda_f, \quad \lambda_f = r_m \left\langle \frac{\bar{p}^2}{(\Delta p)^2} \right\rangle \]
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\[ \Delta p = g Q^a B^a r_m \quad \rightarrow \quad \lambda_f = \frac{\bar{p}^2}{g^2 Q^2 \langle B^2 \rangle r_m} \]
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\[ \eta_\Lambda = \frac{n \bar{p}^3}{3g^2 Q^2 \left\langle B^2 \right\rangle r_m} \approx \frac{9}{4} s T^3 \]
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• Anomalous viscosity is a beyond-HTL effect.
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Anomalous viscosity is a beyond-HTL effect.

The effect does essentially depend on the characteristic scale of turbulent field fluctuations $r_m$. 
The ensemble of stochastic turbulent fields is assumed to be Gaussian:

\[ \langle F_{\mu\nu}^T \rangle = 0 \]

\[ \langle F^{T\mu\nu}(x)F^{T\mu'\nu'}(y) \rangle = K^{\mu\nu\mu'\nu'}(x, y) \]
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The simplest case of stochastically stationary and homogeneous case is considered

\[ K^{\mu\nu\mu'\nu'}(x, y) = K^{\mu\nu\mu'\nu'}(|x^0 - y^0|, |x - y|) \]
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\[ K_{\mu\nu \mu'\nu'}(x, y) = K_{\mu\nu \mu'\nu'}(|x^0 - y^0|, |x - y|) \]

In the present study we use

\[ K_{\mu\nu \mu'\nu'}(x) = K_{0 \mu\nu \mu'\nu'} \exp \left[ -\frac{t^2}{2\tau^2} - \frac{r^2}{2a^2} \right] . \]
Compute field-dependent contribution to the distribution function of hard particles $\delta f(p, k, q|F^R, F^T)$

Compute the turbulent polarization (a response to a regular perturbation that depends on turbulent fields):

$$\Pi_{\mu\nu}(k) = \delta \langle j_\mu(k|F^R, F^T) \rangle_{F^T} \delta A^R_{\nu}$$
Compute field-dependent contribution to the distribution function of hard particles $\delta f(p, k, q | F^R, F^T)$

Compute the induced current:

$$\langle j^\mu(k | F^R, F^T) \rangle_{F^T} = e \sum_{q,s} \int dPp^\nu q \langle \delta f(p, k, q | F^R, F^T) \rangle_{F^T}$$
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$$\Pi^{\mu\nu}(k) = \frac{\delta \langle j^\mu(k | F^R, F^T) \rangle_{F^T}}{\delta A^R_{\nu}}$$
Kinetic equation:

\[ f = f^{eq} + G p^\mu F_{\mu\nu} \partial_\nu f, \quad G \equiv \frac{eq}{\varsigma((pk) + \varsigma \epsilon)} \]
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\[ f = f^{eq} + G p^\mu F_{\mu\nu} \partial_p f , \quad G \equiv \frac{eq}{\imath((p k) + \imath \epsilon)} \]

Let us introduce a generic expansion in \( F^R_{\mu\nu} \) (formal expansion in \( \rho \)) and \( F^T_{\mu\nu} \) (formal expansion in \( \tau \)):

\[
\delta f = \sum_{m=0}^\infty \sum_{n=0}^\infty \rho^m \tau^n \delta f_{mn}
\]

\[
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\]
Kinetic equation:

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\delta f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho^m \tau^n \delta f_{mn} \\
F_{\mu\nu}^{\mu\nu} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho^m \tau^n F_{mn}^{\mu\nu}
\]

The leading turbulent contribution for induced current comes from the cubic terms that are of the first order in \( F^{R}_{\mu\nu} \) and of the second order in \( F^{T}_{\mu\nu} \). We are thus interested in computing \( \delta f_{12} \)
Initial distribution $f_{eq}$ is taken to be an isotropic Fermi distribution.
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Relevant contributions to the distribution function:

$$\delta f \simeq \delta f_{\text{HTL}} + \langle \delta f_{12} \rangle_I + \langle \delta f_{12} \rangle_{II}$$
Turbulent polarization: QED plasma

- Initial distribution $f_{eq}$ is taken to be an isotropic Fermi distribution
- Relevant contributions to the distribution function:

$$\delta f \simeq \delta f_{HTL} + \langle \delta f_{12} \rangle_{I} + \langle \delta f_{12} \rangle_{II}$$

- Explicit expressions for these contributions read:

\[
\begin{align*}
\delta f_{HTL} &= Gp_{\mu} F^{\mu\nu}_{10} \partial_{\mu,p} f^{eq} \\
\langle \delta f_{12} \rangle_{I} &= Gp_{\mu} \langle F^{\mu\nu}_{01} \partial_{\nu,p} Gp_{\mu'} F^{\mu'\nu'}_{10} \partial_{\nu',p} Gp_{\rho} F^{\rho\sigma}_{01} \rangle \partial_{\sigma,p} f^{eq} \\
\langle \delta f_{12} \rangle_{II} &= Gp_{\mu} \langle F^{\mu\nu}_{01} \partial_{\nu,p} Gp_{\mu'} F^{\mu'\nu'}_{01} \partial_{\nu',p} Gp_{\rho} F^{\rho\sigma}_{10} \rangle \partial_{\sigma,p} f^{eq}
\end{align*}
\]
Turbulent polarization: diagrams

\[ \delta f_{\text{HTL}} = \langle \delta f_{12} \rangle_\text{I} = \langle \delta f_{12} \rangle_\text{II} \]

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The evolution of the distribution function due to turbulent fields corresponds to $f_{02}$ and is governed by the (Dupree) equation

\[
(p \partial)\langle f \rangle - p_\mu \langle F_{01}^{\mu \nu} \partial_\nu p_\mu' G F_{01}^{\mu' \nu'} \rangle \partial_\nu' \langle f \rangle = 0
\]
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$$\left(p \partial \right)\langle f \rangle - p_\mu \langle F_{01}^{\mu \nu} \partial_\nu p_{\mu'} G F_{01}^{\mu' \nu'} \rangle \partial_\nu \langle f \rangle = 0$$

The above equation does not have stationary solutions.
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\[
(p \partial) \langle f \rangle - p_\mu \langle F_{01}^{\mu \nu} \partial_\nu p_{\mu'} G_{01}^{\mu' \nu'} \rangle \partial_{\nu'} \langle f \rangle = 0
\]

The above equation does not have stationary solutions.

We will assume that turbulent evolution of the distribution function is slower than that of the probe $F_{10}$ and can be neglected.
Generic decomposition of polarization tensor ($l \equiv \sqrt{2(\tau a)}/\sqrt{\tau^2 + a^2}$):

$$\Pi_{ij}(\omega, k\vert l) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\Pi_T(\omega, |k|\vert l) + \frac{k_i k_j}{k^2}\Pi_L(\omega, |k|\vert l)$$
Turbulent polarization: QED plasma

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- Separation into HTL and turbulent components:

\[
\Pi_{L(T)}(\omega, k| l) = \Pi_{L(T)}^{HTL}(\omega, k) + \Pi_{L(T)}^{turb}(\omega, k| l)
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- Generic decomposition of polarization tensor \( (l \equiv \sqrt{2(\tau a)}/\sqrt{\tau^2 + a^2}) \):

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\Pi_{L(T)}(\omega, k \mid l) = \Pi_{L(T)}^{\text{HTL}}(\omega, k) + \Pi_{L(T)}^{\text{turb}}(\omega, k \mid l)
\]

- Gradient expansion of the turbulent contribution:

\[
\Pi_{L(T)}^{\text{turb}}(\omega, |k| \mid l) = \sum_{n=1}^{\infty} \left( \frac{|k| \mid l}{k^2} \right)^n \left[ \phi_{L(T)}^{(n)} \left( \frac{\omega}{|k|} \right) \left\langle E_{\text{turb}}^2 \right\rangle + \chi_{L(T)}^{(n)} \left( \frac{\omega}{|k|} \right) \left\langle B_{\text{turb}}^2 \right\rangle \right]
\]
Hard thermal loops contribution:

\[ \Pi_{L}^{HTL}(\omega, |k|) = -m_{D}^{2}x^{2}\left[1 - \frac{x}{2} L(x)\right] \]

\[ \Pi_{T}^{HTL}(\omega, |k|) = m_{D}^{2}\frac{x^{2}}{2}\left[1 + \frac{1}{2x}(1 - x^{2}) L(x)\right] \]

\[ L(x) \equiv \ln\left|\frac{1 + x}{1 - x}\right| - i\pi\theta(1 - x); \quad m_{D}^{2} = e^{2}T^{2}/3 \]

HTL imaginary part at \( x < 1 \) corresponds to Landau damping
Turbulent polarization: QED plasma

- Transverse polarization:
  \[
  \text{Im} \Pi_T(\omega, k| l) \simeq -\pi m_D^2 \frac{x}{4} (1 - x^2) \theta(1 - x) \\
  + \frac{(|k| l)}{k^2} \left( \langle E^2 \rangle \text{Im} \phi_{I T}(x) + \langle B^2 \rangle \text{Im} \chi_{I T}(x) \right)
  \]

- Longitudinal polarization:
  \[
  \text{Im} \Pi_L(\omega, k| l) \simeq -\pi m_D^2 \frac{x^3}{2} \theta(1 - x) \\
  + \frac{(|k| l)}{k^2} \left( \langle E^2 \rangle \text{Im} \phi_{I L}(x) + \langle B^2 \rangle \text{Im} \chi_{I L}(x) \right)
  \]
The functions $\frac{6\pi\sqrt{\pi}}{e^4} \text{Im} \left[ \phi_T^{(1)}(x) \right]$ (solid line) and $\frac{6\pi\sqrt{\pi}}{e^4} \text{Im} \left[ \chi_T^{(1)}(x) \right]$ (dashed line).

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Spacelike domain $x < 1$, competition between Landau damping and turbulent enhancement for $\Pi_T$:

\[
\Phi(x) = x(1 - x^2) \left[ -4 + 12x^2 + 2x \ln \left| \frac{1 + x}{1 - x} \right| \right]
\]

\[
\frac{4\pi^2}{\sqrt{\pi}} \left( \frac{|k|}{l} \right) k^2 e^2 \langle B^2 \rangle T^2 \Phi(x) > 1
\]
Turbulent polarization: spacelike transverse instability

- Spacelike domain $x < 1$, competition between Landau damping and turbulent enhancement for $\Pi_T$:
- For example, for the purely magnetic instability

$$\text{Im}\Pi_T(\omega, |\mathbf{k}|) = -\pi \frac{e^2 T^2}{12} x(1 - x^2) \left[ 1 - \frac{4}{\pi^2 \sqrt{\pi}} \frac{|\mathbf{k}|}{k^2} \frac{e^2 \langle B^2 \rangle}{T^2} \Phi(x) \right]$$

$$\Phi(x) = \frac{1}{x(1 - x^2)} \left[ \frac{-4 + 12x^2}{3(1 - x^2)} + 2x \ln \left| \frac{1 + x}{1 - x} \right| \right]$$
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- Instability condition

\[
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Instability condition

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Instability condition: numerics
Possible origin of the spacelike transverse instability is in not taking into account the backreaction of the turbulent field on the thermal distribution of hard modes.

The problem of finding the deformed stationary distribution of hard modes corresponding to truly (stochastically) stationary turbulent plasma is very difficult.

Calculations for the anisotropic case are in progress. It is likely that the turbulent instability will act in addition to the Weibel one. If so, there is an interval (here $\xi > 1$ parametrizes anisotropy) $T_e < B < \xi T_e$ in which turbulent instabilities will play a role.
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$$\frac{T}{e} < B < \xi \frac{T}{e}$$

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In the timelike domain $x > 1$ one gets turbulent damping (resonance broadening) of both longitudinal and transverse modes. Because of the absence of HTL timelike damping this is a leading order effect.
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In the spacelike domain $x < 1$ longitudinal modes get additional turbulent damping in addition to the Landau one.

The corresponding damping rates can be calculated from

$$\Gamma^2 = -\text{Im} \Pi(\omega, |k|/l)$$
Turbulent polarization: plasmon properties

- Dispersion relations for plasmons, generic equations

\[ k^2 \left( 1 - \frac{\Pi_L(k^0, |k|)}{\omega^2} \right) \bigg|_{k^0=\omega_L(|k|)} = 0 \]

\[ k^2 - (k^0)^2 + \Pi_T((k^0, |k|) \bigg|_{k^0=\omega_T(|k|)} = 0 \]
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- Dispersion equations for plasmons, \( |k| \ll \omega \) \((y_T(L) = |k| / \omega_{pl T(L)}) \)
  \[ \omega^2_L(|k|)_{turb} = (\omega_{pl L})^2 \left( 1 + \frac{3}{5} y^2_L \right) - \frac{e^4 l^2}{6\pi^2} \left( \frac{24}{5} \langle E^2 \rangle + \frac{64}{15} \langle B^2 \rangle \right) y^2_L + O(y^4_L) \]
  \[ \omega^2_T(|k|)_{turb} = (\omega_{pl T})^2 \left( 1 + \frac{3}{5} y^2_T \right) - \frac{e^4 l^2}{6\pi^2} \left( \frac{24}{7} \langle E^2 \rangle + \frac{32}{15} \langle B^2 \rangle \right) y^2_T + O(y^4_T) \]

- Turbulent corrections to plasma frequencies:
  \[ (\omega_{pl L}^{turb})^2 = \omega_{pl L}^2 - \frac{e^4 l^2}{6\pi^2} \left( \frac{16}{3} \langle E^2 \rangle + \frac{8}{3} \langle B^2 \rangle \right) \]
  \[ (\omega_{pl T}^{turb})^2 = \omega_{pl T}^2 - \frac{e^4 l^2}{6\pi^2} \left( \frac{128}{15} \langle E^2 \rangle + \frac{8}{3} \langle B^2 \rangle \right) \]
The generalization to the non-Abelian case reads

\[ p^\mu \left[ \partial_\mu - gf_{abc} A_\mu^b Q^c \frac{\partial}{\partial Q^a} - gQ_a F_{\mu\nu} \frac{\partial}{\partial p^\nu} \right] = 0, \]

where the fields \( F_{\mu\nu} \) satisfy the Yang-Mills equations

\[ D^\mu F_{\mu\nu}^a = j^a_\nu \]

The main distinction from the Abelian case is the dependence of the distribution function on the color spin \( Q \), where for \( SU(3) \)
\( Q = (Q^1, Q^2, ..., Q^8) \), so that \( f(x, p, Q) \). The components of color spin \( Q = (Q^1, Q^2, ..., Q^8) \) are dynamic variables satisfying the Wong equation

\[ \frac{dQ^a}{d\tau} = -gf^{abc} p^\mu A_\mu^b Q^c \]
Turbulent QCD plasma

Separation of regular and turbulent contributions:

\[
f = f^R + f^T, \quad A^a_\mu = A^{Ra}_\mu + A^{Ta}_\mu, \quad \langle A^{Ta}_\mu \rangle = 0
\]

Gauge fixing making \( \langle A^{Ta}_\mu \rangle = 0 \) gauge invariant:

\[
\delta A^{Ra}_\mu = \partial_\mu \alpha^a + gf^{abc} A^{Rb}_\mu \alpha^c
\]
\[
\delta A^{Tb}_\mu = gf^{abc} A^{Tb}_\mu \alpha^c
\]

This leads to gauge field strength decomposition:

\[
F^a_{\mu\nu} = F^{Ra}_{\mu\nu} + F^{Ta}_{\mu\nu} + \mathcal{F}^{Ta}_{\mu\nu}
\]

where:

\[
F^{Ra}_{\mu\nu} = \partial_\mu A^{Ra}_\nu - \partial_\nu A^{Ra}_\mu + gf^{abc} A^{Rb}_\mu A^{Rc}_\nu
\]
\[
\mathcal{F}^{Ta}_{\mu\nu} = \partial_\mu A^{Ta}_\nu - \partial_\nu A^{Ta}_\mu + gf^{abc} A^{Tb}_\mu A^{Tc}_\nu
\]
\[
F^{Ta}_{\mu\nu} = gf^{abc} \left( A^{Tb}_\mu A^{Rc}_\nu + A^{Rb}_\mu A^{Tc}_\nu \right)
\]
Turbulent QCD plasma

Turbulent contributions to the distribution function:

\[ f^{R(1)} = HTL + I_1 + I_2 \]

where

\[ I_1 = g^3 p^\mu f^{abc} \left\langle A_{\mu}^{Tb} Q^c \frac{\partial}{\partial Q^a} p^{\mu'} \frac{1}{p^\mu \partial_{\mu}} f^{def} A_{\mu'}^{Re} Q^f \frac{\partial}{\partial Q^d} \frac{1}{p^\mu \partial_{\mu}} p^{\mu'} Q_g F_{\mu''\nu} \right\rangle \frac{\partial}{\partial p_\nu} f^{R(0)} \]

\[ I_2 = g^2 p^\mu f^{abc} \left\langle A_{\mu}^{Tb} Q^c \frac{\partial}{\partial Q^a} \left( \frac{1}{p^\mu \partial_{\mu}} \right) p^{\mu'} Q_d F_{\mu'\nu}^{Td} \right\rangle \frac{\partial}{\partial p_\nu} f^{R(0)} \]
\[ \langle A^T_a(x) A^T_b(y) \rangle = G^{ab}_{\mu\nu}(x, y) \]
\[ \langle \mathcal{F}^T_{\mu\nu}(x) U^{ab}(x, y) \mathcal{F}^T_{\mu'\nu'}(y) \rangle = K^{ab}_{\mu\nu;\mu'\nu'}(x, y) \]
\[ K^{T_a}_{\mu\nu;\mu'\nu'}(x, y) = K^{T_a}_{\mu\nu;\mu'\nu'}(x - y) \]
\[ G^{ab}_{\mu\nu} = \delta_{ab} \left[ g_{\mu\nu} g_{\nu 0} \langle A_0^2 \rangle + \frac{1}{3} \hat{\delta}_{\mu\nu} \langle A^2 \rangle \right] \exp \left[ -\frac{r^2}{2a^2} - \frac{t^2}{2\tau^2} \right] \]

Defining \( f^{Ra(1)}(x, p) = \int Q^a d Q \ f^{R(1)}(x, p, Q), \) we get
\[ [(p^\mu \partial_\mu) + p^\gamma] f^{Rl(1)} = \int Q^l d Q \ (HTL + l_1 + l_2), \]
\[ \gamma = g^2 \frac{N^2 - 1}{4N} \sqrt{\pi} l \left[ \langle A_0^2 \rangle + \langle \frac{1}{3} A^2 \rangle \right] \]
Turbulent polarization: non-Abelian contributions

- Leading order contribution:

\[ \Pi_L = m_g^2 x^2 \left[ -1 + x \text{Arcth}[y] - i \frac{\gamma x}{|k|} \frac{1}{1 - y^2} \right] \]

\[ \Pi_T = m_g^2 x \left[ y + (1 - y^2) \text{Arcth}[y] - i \frac{\gamma x}{|k|} (2 - 2y \text{Arcth}[y]) \right] \]

\[ y = \frac{k^0 + i\gamma}{|k|}, \quad \gamma = g^2 \frac{N^2 - 1}{4N} \sqrt{\pi} l \left[ \langle A_0^2 \rangle + \left\langle \frac{1}{3} A^2 \right\rangle \right] \]

- No new contributions to the previously discussed instabilities/damping phenomena
We have several beautiful theoretical schemes, theoretical progress is very rapid
Conclusions

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- Relation of these constructions to experimental data is unclear.
Conclusions

- We have several beautiful theoretical schemes, theoretical progress is very rapid.

- Relation of these constructions to experimental data is unclear.

- What is however clear that the system we are studying is, in contrast to early Universe, expanding so fast that its phenomenology should be sensitive, perhaps even anomalously sensitive, to event-by-event details of its early dynamics.