



Multipurpose Renormalization Group as a consequence of locality of QFT

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The foundation:

- Quantum field theory is formulated for all types of interactions independently on renormalizability
- R-operation equally works for NR theories and leads to local counter terms resulting in finite amplitudes
- Local Quantum Field Theory obeying the requirements of causality, unitarity and analyticity has a remarkable property: after applying the R-operation all UV divergent structures are local in coordinate space or are at most polynomial in momentum space (Bogoliubov-Parasiuk theorem).
- This is true for any local QFT irrespectively of its renormalizability or non-renormalizability
- All these statements lead to relations between the subsequent orders of PT resulting in the RG equations which aimed on the summation of infinite series of PT for the asymptotics of the Green functions, amplitudes, potential, etc

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Renormalization

BPHZ R-operation

$$RG = (1 - K)R'G$$

Incomplete R-operation

A.N.Vasiliev, Green Book

$$R'G = 1 - \sum_{\gamma} G/\gamma K R' \gamma - \sum_{\gamma, \gamma'} G/\gamma \gamma' K R' \gamma K R' \gamma' - \dots$$

K-operation

$K\gamma$ Singular part of γ

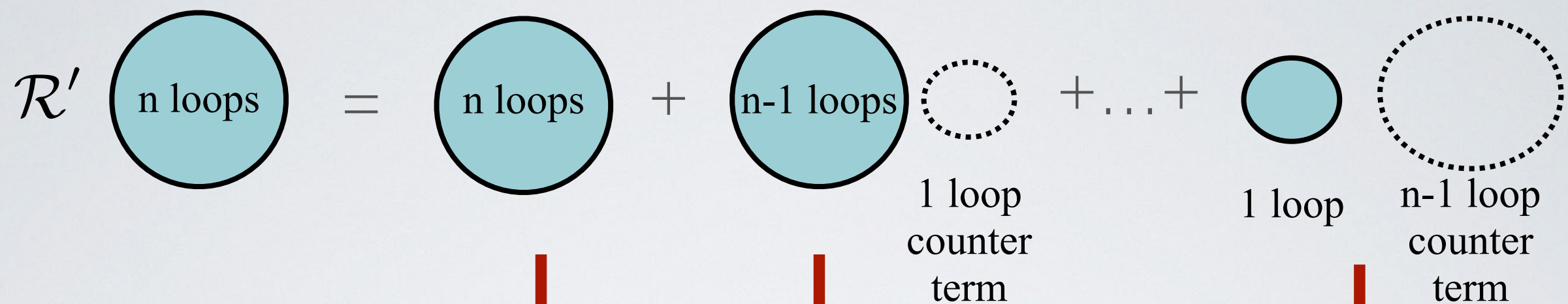
G/γ Graph G with subgraph γ shrunken to the point

Bogolyubov-Parasiuk Theorem: In any local quantum field theory to get the UV finite S-matrix one has to introduce local counter terms to the Lagrangian in each order of perturbation theory - R-operation

$$\mathcal{L} \Rightarrow \mathcal{L} + \Delta\mathcal{L}$$

In non-renormalizable case the BP theorem is still valid and the counter terms are also local (contain a finite number of derivatives or at maximum are polynomial over momenta)

BPHZ R-operation



$$\begin{aligned} \mathcal{R}' G_n = & \frac{A_n^{(n)} (\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)} (\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \dots + \frac{A_1^{(n)} (\mu^2)^\epsilon}{\epsilon^n} \\ & + \frac{B_n^{(n)} (\mu^2)^{n\epsilon}}{\epsilon^{n-1}} + \frac{B_{n-1}^{(n)} (\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-1}} + \dots + \frac{B_1^{(n)} (\mu^2)^\epsilon}{\epsilon^{n-1}} \\ & + \frac{C_n^{(n)} (\mu^2)^{n\epsilon}}{\epsilon^{n-2}} + \frac{C_{n-1}^{(n)} (\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-2}} + \dots + \frac{C_1^{(n)} (\mu^2)^\epsilon}{\epsilon^{n-2}} \\ & + \text{lower pole terms,} \end{aligned}$$

$A_k^{(n)}$ $B_k^{(n)}$ $C_k^{(n)}$ $(\mu^2)^{k\epsilon}$ terms appear after subtraction of (n-k) loop counter terms

BPHZ R-operation

Bogoliubov-Parasiuk Theorem: (Locality)

$R'G_n$ is local, i.e. terms like $\log^k \mu^2 / \epsilon^m$ should cancel for any k and m

- Due to locality all higher order divergences are related to the lower ones

$$A_n^{(n)} = (-1)^{n+1} \frac{A_1^{(n)}}{n}, \quad \text{One loop}$$

$$B_n^{(n)} = (-1)^n \left(\frac{2}{n} B_2^{(n)} + \frac{n-2}{n} B_1^{(n)} \right), \quad \text{One and two loops}$$

$$C_n^{(n)} = (-1)^{n+1} \left(\frac{3}{n} C_3^{(n)} + \frac{2(n-3)}{n} C_2^{(n)} + \frac{(n-2)(n-3)}{2n} C_1^{(n)} \right). \quad \text{One, two and three loops}$$

$$\mathcal{KR}'G_n = \sum_{k=1}^n \left(\frac{A_k^{(n)}}{\epsilon^n} + \frac{B_k^{(n)}}{\epsilon^{n-1}} + \frac{C_k^{(n)}}{\epsilon^{n-2}} + \dots \right) \equiv \frac{A_n^{(n)'}}{\epsilon^n} + \frac{B_n^{(n)'}}{\epsilon^{n-1}} + \frac{C_n^{(n)'}}{\epsilon^{n-2}} + \dots$$

$$A_n^{(n)'} = (-1)^{n+1} A_n^{(n)} = \frac{A_1^{(n)}}{n}, \quad \text{One loop}$$

$$B_n^{(n)'} = \left(\frac{2}{n(n-1)} B_2^{(n)} + \frac{2}{n} B_1^{(n)} \right), \quad \text{One and two loops}$$

$$C_n^{(n)'} = \left(\frac{2}{(n-1)(n-2)} \frac{3}{n} C_3^{(n)} + \frac{2}{n-1} \frac{3}{n} C_2^{(n)} + \frac{3}{n} C_1^{(n)} \right). \quad \text{One, two and three loops}$$

The Local Counter Terms

Consequence:

Leading divergences:

$$A_n^{(n)'} = \frac{A_1^{(n)}}{n}$$

Coefficients of $1/\epsilon^n$



The leading divergences are governed by 1 loop diagrams!

SubLeading divergences:

$$B_n^{(n)'} = \left(\frac{2}{n(n-1)} B_2^{(n)} + \frac{2}{n} B_1^{(n)} \right) \text{ Coefficients of } 1/\epsilon^{n-1}$$

The sub leading divergences are governed by 2 loop diagrams!

Coefficient of the leading logarithms $\log^n \mu^2$

$$\bar{A}_n^{(n)} = A_n^{(n)} = (-1)^{n+1} \frac{A_1^{(n)}}{n}$$

One loop

Coefficient of subleading logarithms $\log^{n-1} \mu^2$

$$\bar{B}_n^{(n)} = (-1)^n \left(\frac{2}{n-1} B_2^{(n)} + B_1^{(n)} \right)$$

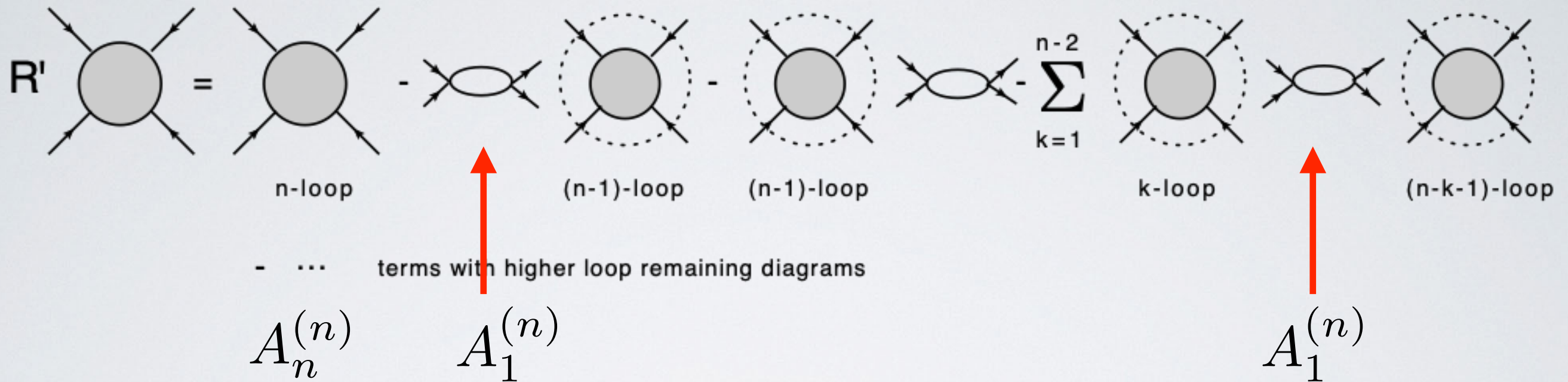
One and two loops

The Recurrence Relations

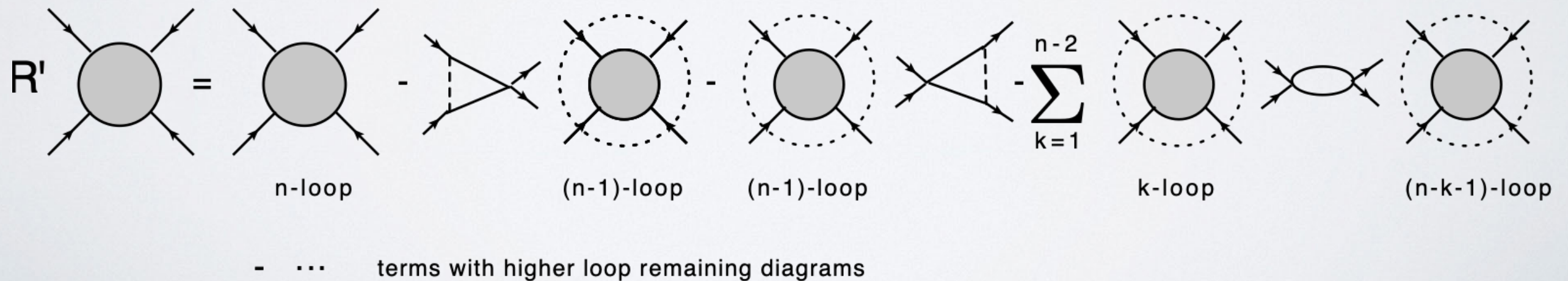
- These properties allow one to write down the recurrence relations connecting the subsequent orders of the counterterms and to evaluate them algebraically without calculating the diagrams. This can be done in renormalizable and non-renormalizable theories. The difference is a more complicated structure of these relations in NR case.
- These recurrence relations can be promoted to the RG equations for the scattering amplitudes, effective potential, etc which sum up the leading divergences (logarithms) and to find out the high energy/field behaviour

Leading divergences in Scattering Amplitudes

Quartic vertices



Cubic vertices



D=6 N=2**S-channel** $S_n(s, t)$ **T-channel** $T_n(s, t)$ $T_n(s, t) = S_n(t, s)$ **Exact all-loop recurrence relation** $S_3 = -s/3, T_3 = -t/3$

$$nS_n(s, t) = -2s \int_0^1 dx \int_0^x dy (S_{n-1}(s, t') + T_{n-1}(s, t'))$$

 $n \geq 4$ $t' = t(x - y) - sy$ **D=8 N=1****S-channel** $S_n(s, t)$ **T-channel** $T_n(s, t)$ $T_n(s, t) = S_n(t, s)$ **Exact all-loop recurrence relation** $S_1 = \frac{1}{12}, T_1 = \frac{1}{12}$

$$\begin{aligned}
 nS_n(s, t) = & -2s^2 \int_0^1 dx \int_0^x dy y(1-x) (S_{n-1}(s, t') + T_{n-1}(s, t'))|_{t'=tx+yu} \\
 + & s^4 \int_0^1 dx x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \times \\
 & \times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t'))|_{t'=-sx} (tsx(1-x))^p
 \end{aligned}$$

Recurrence Relation for the Scattering Amplitude

Kazakov,20

$$n \text{ } \langle A_n \rangle = -2 \triangle \langle A_{n-1} \rangle - \sum_{k=1}^{n-2} \langle A_k \rangle \bigcirc \langle A_{n-1-k} \rangle$$

- This is the general recurrence relation that reflects the locality of the counter terms in any theory
- In renormalizable theories A_n is a constant and this relation is reduced to the algebraic one
- In non-renormalizable theories for the scattering amplitudes A_n depends on kinematics and one has to integrate through the one loop diagrams
- For the effective potential A_n depends on the fields and one has to differentiate

Taking the sum $\sum_n A_n (-z)^n = A(z)$ one can transform the recurrence relation into integro-diff equation

$$\boxed{\frac{d}{dz} A(z) = b_0 \left\{ -1 - 2 \int_{\Delta} A(z) - \int_{\bigcirc} A^2(z) \right\}} \quad \frac{d}{dz} = \frac{d}{d \log \mu^2}$$

This is the generalized RG equation valid in any (even non-renormalizable) theory!

Examples:

- Maximally supersymmetric gauge theory in $D=6,8,10$ dimensions SYM_D
- Scalar field theory in $D=4,6,8,10$ dimensions ϕ_D^4
- Four-fermion interaction in $D=4$ See talk by A. Borlakov for sensation results!
- Supersymmetric Wess-Zumino model with quartic superpotential in $D=4$ Φ_4^4

These are the toy models for (super) gravity - our aim

RG Equation

SYM_D

D=6 N=2

$$\Sigma(s, t, z) = z^{-2} \sum_{n=3}^{\infty} (-z)^n S_n(s, t)$$

$$\frac{d}{dz} \Sigma(s, t, z) = s - \frac{2}{z} \Sigma(s, t, z) + 2s \int_0^1 dx \int_0^x dy (\Sigma(s, t', z) + \Sigma(t', s, z))|_{t'=xt+yu}$$

Linear equation

D=8 N=1

$$\Sigma(s, t, z) = \sum_{n=1}^{\infty} (-z)^n S_n(s, t)$$

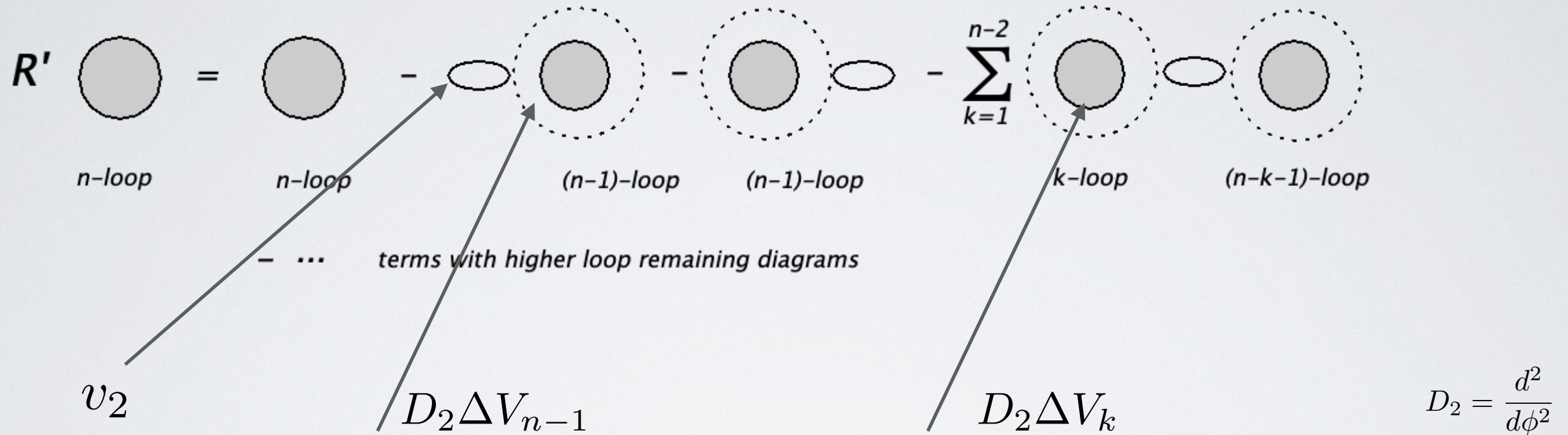
$$\begin{aligned} \frac{d}{dz} \Sigma(s, t, z) = & -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy y(1-x) (\Sigma(s, t', z) + \Sigma(t', s, z))|_{t'=tx+yu} \\ & -s^4 \int_0^1 dx x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!} \left(\frac{d^p}{dt'^p} (\Sigma(s, t', z) + \Sigma(t', s, z))|_{t'=-sx} \right)^2 (tsx(1-x))^p. \end{aligned}$$

Non-linear equation

Recurrence relations for the leading poles for effective potential

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Action of R' -operation on divergent diagram



$$n\Delta V_n = \frac{1}{2}v_2 D_2\Delta V_{n-1} + \frac{1}{4} \sum_{k=1}^{n-2} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 2 \quad \Delta V_1 = \frac{1}{4}v_2^2$$

$$n\Delta V_n = \frac{1}{4} \sum_{k=0}^{n-1} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 1, \quad \Delta V_0 = V_0$$

RG pole equation for arbitrary potential

$$\Sigma(z, \phi) = \sum_{n=0}^{\infty} (-z)^n \Delta V_n(\phi) \quad z = \frac{g}{\epsilon}$$

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RG pole equation

$$\frac{d\Sigma}{dz} = -\frac{1}{4} (D_2 \Sigma)^2 \quad \Sigma(0, \phi) = V_0(\phi)$$

This a non-linear partial differential equation!

Effective potential

See talk by V. Filippov for curved space

$$V_{eff}(g, \phi) = g \Sigma(z, \phi) \Big|_{z \rightarrow -\frac{g}{16\pi^2} \log g v_2 / \mu^2} \quad v_2(\phi) \equiv \frac{d^2 V_0(\phi)}{d\phi^2}$$

RG Equations in Subleading Order Scattering Amplitude

$$\frac{dS(s, t, z)}{dz} = s(S_1 + T_1) \otimes (S_1 + T_1) \quad \text{Coefficients of } 1/\epsilon^n$$

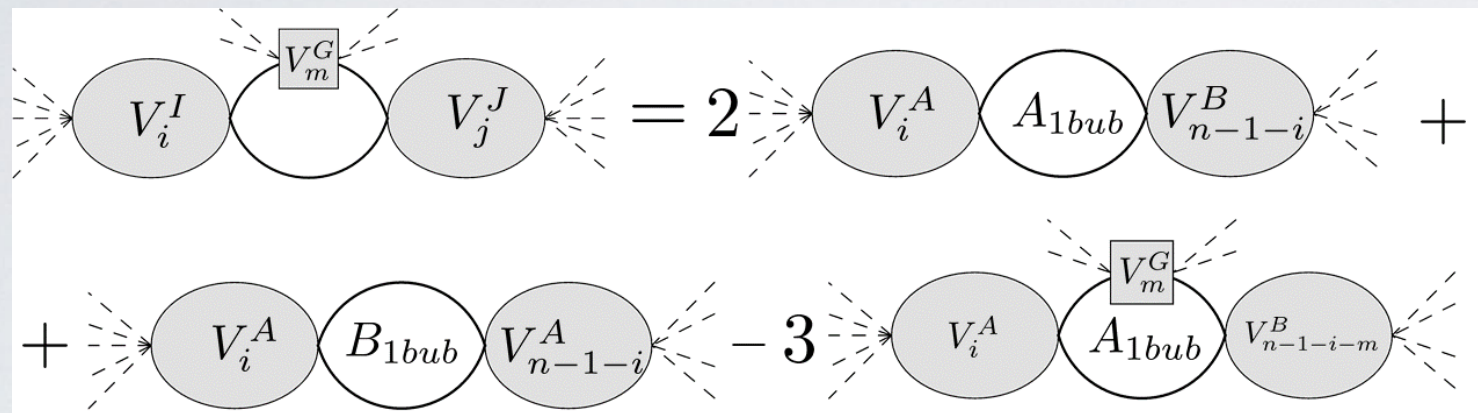
$$\begin{aligned} \frac{d^2 S_2(s, t, z)}{dz^2} = & s \frac{d}{dz} (S_1 + T_1) \otimes (S_2 + T_2) \\ & + s^2 (S_1 + T_1) \otimes (S_2 + T_2) \otimes (S_1 + T_1) \\ & + s (S_1 + T_1) \otimes \otimes (S_1 + T_1) \end{aligned} \quad \begin{array}{l} \text{Coefficients of} \\ 1/\epsilon^{n-1} \end{array}$$

- ◆ Equation for the subleading order function as well as for all the subsequent orders is always linear!
- ◆ This seems to be in contradiction with the usual RG equation but is not!

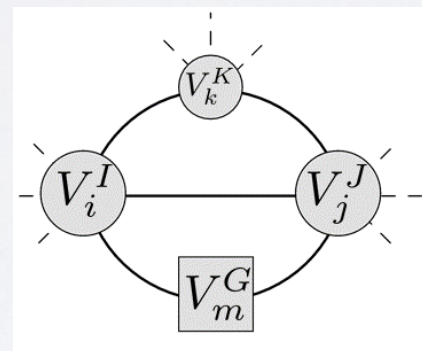
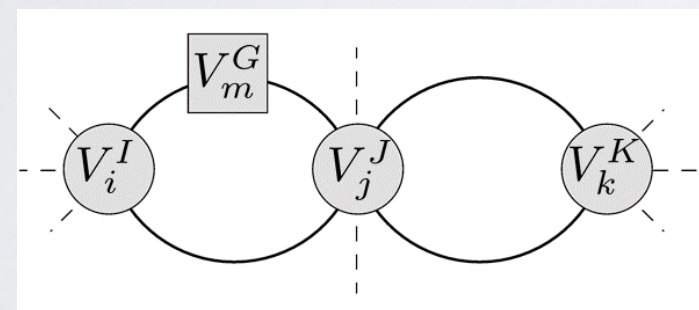
RG Equations in Subleading Order

Effective potential

$$B_n^{(n)'} = \left(\frac{2}{n(n-1)} B_2^{(n)} + \frac{2}{n} B_1^{(n)} \right)$$

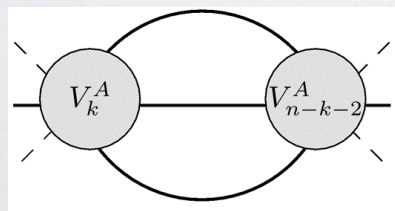


Contribution to $B_1^{(n)}$



Contribution to $B_2^{(n)}$

See talk by R. Iakhibbaev



Contribution to kinetic term

RG Equations in Subleading Order

Effective potential

$$\frac{\partial \Sigma_A}{\partial z} = -\frac{1}{4}(D_2 \Sigma_A)^2$$

Σ_A - leading order

Σ_B - subleading order

Σ_G - kinetic term

$$\begin{aligned} \frac{\partial^2 \Sigma_B}{\partial z^2} = & -2 \frac{1}{4} \frac{\partial}{\partial z} \left\{ 2 D_2 \Sigma_A D_2 \Sigma_B - (D_2 \Sigma_A)^2 D_2 \Sigma_G \right\} - \\ & -2 \left\{ \frac{1}{8} (D_3 \Sigma_A)^2 D_2 \Sigma_B - 2 \frac{1}{8} (D_3 \Sigma_A)^2 D_2 \Sigma_A D_2 \Sigma_B + 2 \frac{1}{8} D_3 \Sigma_A D_3 \Sigma_B D_2 \Sigma_A + \right. \\ & + 2 \frac{1}{8} D_3 \Sigma_A D_3 \Sigma_B D_2 \Sigma_A - \frac{1}{8} (D_3 \Sigma_A)^2 D_2 \Sigma_A + \frac{1}{8} (D_2 \Sigma_A)^2 D_4 \Sigma_B - \\ & \left. - 2 \frac{1}{8} (D_2 \Sigma_A)^2 D_4 \Sigma_A D_2 \Sigma_G + 2 \frac{1}{8} D_2 \Sigma_A D_2 \Sigma_B D_4 \Sigma_A \right\} \end{aligned}$$

$$D_n \equiv \frac{\partial^n}{\partial \phi^n}$$

$$\frac{\partial^2 \Sigma_G}{\partial z^2} = \frac{1}{12} (D_3 \Sigma_A)^2$$

- ◆ Equation for the subleading order function as well as for all the subsequent orders is always linear!
- ◆ This seems to be in contradiction with the usual RG equation which is non-linear, however, the proper form of the usual RG equation is also linear!!

RG Equations for the Logarithms

Coefficient of the leading logarithms $\log^n \mu^2$

$$\bar{A}_n^{(n)} = A_n^{(n)} = (-1)^{n+1} \frac{A_1^{(n)}}{n}$$

Coefficient of subleading logarithms $\log^{n-1} \mu^2$

$$\bar{B}_n^{(n)} = (-1)^n \left(\frac{2}{n-1} B_2^{(n)} + B_1^{(n)} \right)$$

Coefficient of subsubleading logarithms

$$\bar{C}_n^{(n)} = (-1)^{n-1} \left(\frac{3}{n-2} C_3^{(n)} + 2C_2^{(n)} + \frac{n-1}{2} C_1^{(n)} \right)$$

- ◆ Differential equation for the logarithms in all orders are always linear!
- ◆ They contain the known rhs which is calculated from higher order linear differential equation

Resume

- **The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones**
- **Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations for the leading, subleading, etc divergences in all loops divergences starting from one-, two-, three-, etc loop diagrams**
- **The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories**
- **The RG equations allow one to sum up the leading (subleading, etc) divergences in all loops and define the high-energy/field behaviour**
- **The RG equations for the subleading poles, etc are always linear differential equation of increasing order**
- **The RG equations for logarithms are always first order differential equation with known rhs**



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