QUARKS-2018
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Structure of the UV Divergences in Maximally Supersymmetric Theories

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JHEP 1612 (2016) 154, arXiv:1610.05549v2 [hep-th]
Phys.Rev. D95 (2017) no.4, 045006 arXiv:1603.05501 [hep-th]
arXiv:1712.04348 [hep-th], arXiv:1804.08387 [hep-th]

Motivation

<u>Maximal SYM</u>

D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

- Partial or total cancell... ...nces (all bubble an... ...cancel)
- First ... ...ams at D=4+6/L
- ...ual conformal symmetry
- ...ion structure of the integrands

All of them can be obtained from 10dim superstring by compactification on a torus

Bern, Dixon &Co 10
Drummond, Henn, Korchemsky, Sokatchev 10
Arkani-Hamed 12

D=4 N=8 Supergravity

- On-shell finite up to 8 loops
- Similar to higher dim SYM

<u>Object</u>: Helicity Amplitudes on mass shell with arbitrary number of legs and loops

<u>The case:</u> Planar limit $N_c \to \infty,\ g^2_{YM} \to 0$ and $g^2_{YM} N_c$ - fixed

<u>The aim</u>: to get all loop (exact) result

Study of higher dim SYM gives insight into quantum gravity

UV divergences in all Loops

Spinor-helicity formalism: S-matrix elements

D=4 N=4	**No UV div**	**IR div on shell**
D=6 N=2	**UV div from 3 loops**	**No IR div**
D=8 N=1	**UV div from 1 loop**	**No IR div**
D=10 N=1	**UV div from 1 loop**	**No IR div**

All these theories are non-renormalizable by power counting

The coupling g^2 **has dimension** $[g^2] = \frac{1}{M^{D-4}}$

<u>The aim</u>: to get all loop (exact) result for the leading (at least) divs

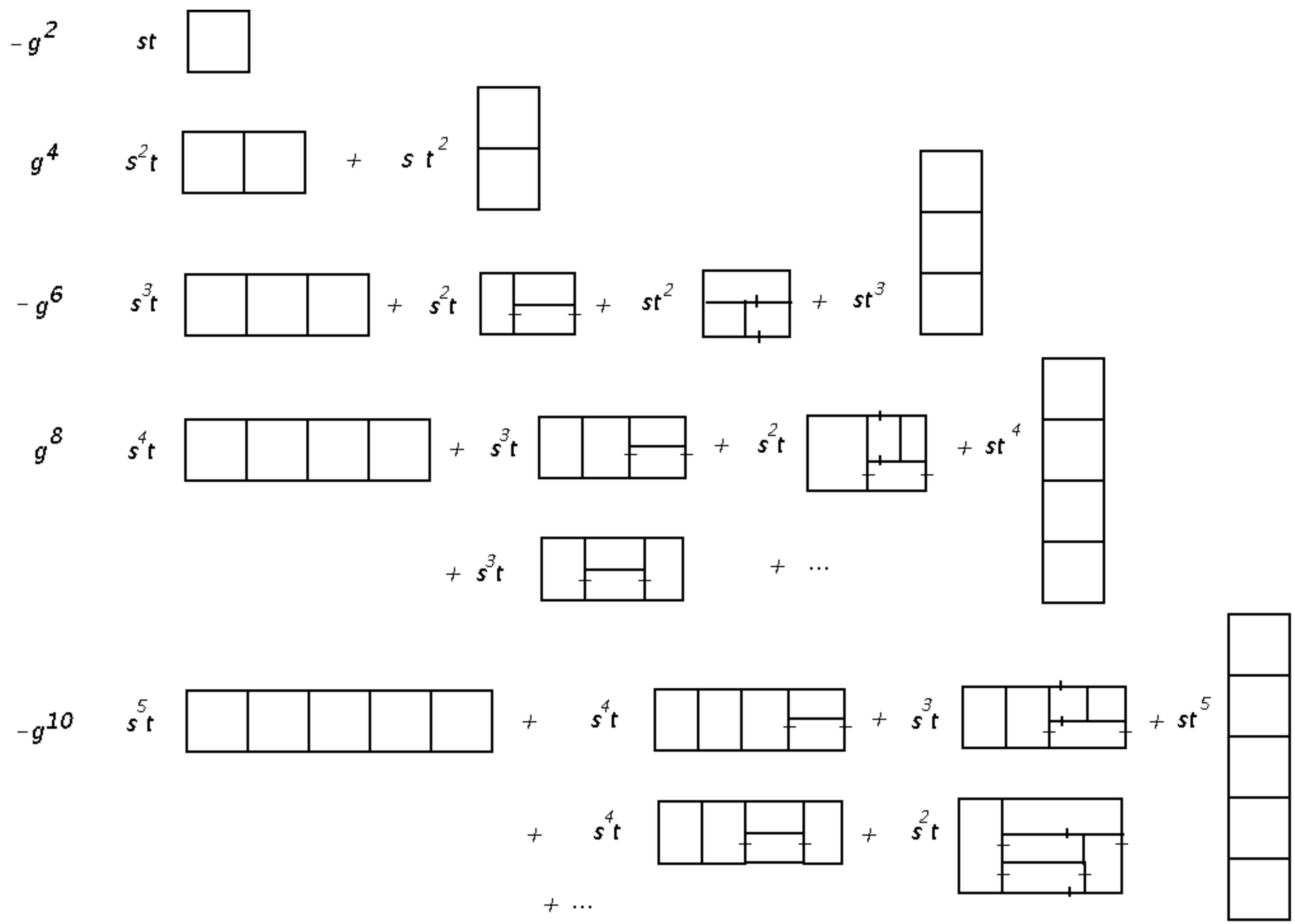
Perturbation Expansion for the 4-point Amplitudes for any D

A_4/A_4^{tree}

No bubbles
No Triangles

First UV div at
L=[6/(D-4)] loops

IR finite



1

2

4

15

60

T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

Universal expansion for any D in maximal SYM due to Dual conformal invariance

Leading Divergences from Generalized «Renormalization Group»

- **In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of $1/\epsilon^n$ in n loops is**

$$\mathcal{R}'G = \sum_n \frac{a_n^{(n)}}{\epsilon^n} \qquad a_n^{(n)} = (a_1^{(1)})^n$$

- **In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation**

$$\mathcal{R}'G = 1 - \sum_\gamma K\mathcal{R}'_\gamma + \sum_{\gamma,\gamma'} K\mathcal{R}'_\gamma K\mathcal{R}'_{\gamma'} - ...,$$

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + ... + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$
$$+ \frac{B_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-1}} + \frac{B_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-1}} + ... + \frac{B_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^{n-1}}$$
$$+\text{lower order terms}$$

Leading pole

SubLeading pole

$A_1^{(n)}, \ B_1^{(n)}$ **1-loop graph**

$B_2^{(n)}$ **2-loop graph**

SubLeading Divergences from Generalized «Renormalization Group»

- **In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation**

All terms like $(log\mu^2)^m/\epsilon^k$ **should cancel**

$$A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n},$$
$$B_n^{(n)} = (-1)^n\left(\frac{2}{n}B_2^{(n)} + \frac{n-2}{n}B_1^{(n)}\right)$$

Leading pole from 1 loop diagrams

SubLeading pole from 2 loop diagrams

$$\mathcal{KR}'G_n = \sum_{k=1}^{n}\left(\frac{A_k^{(n)}}{\epsilon^n} + \frac{B_k^{(n)}}{\epsilon^{n-1}}\right) \equiv \frac{A_n^{(n)'}}{\epsilon^n} + \frac{B_n^{(n)'}}{\epsilon^{n-1}}.$$

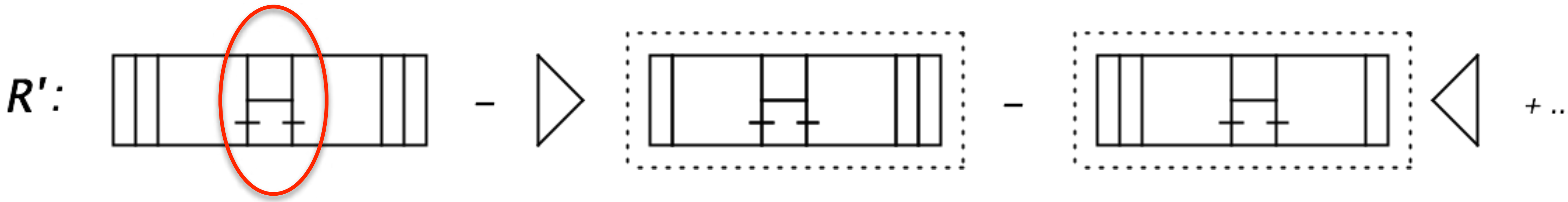
$$A_n^{(n)'} = (-1)^{n+1}A_n^{(n)} = \frac{A_1^{(n)}}{n},$$
$$B_n^{(n)'} = \left(\frac{2}{n(n-1)}B_2^{(n)} + \frac{2}{n}B_1^{(n)}\right)$$

Just like in renormalizable theories one can deduce the leading, subheading, etc divergences from 1, 2, etc diagrams

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



A_n $\qquad$ A_{n-1}

$$nA_n = -A_{n-1} \quad \longrightarrow \quad A_n = (-1)^n \frac{2}{n!} \qquad (-g^2 s)^n$$

Summation

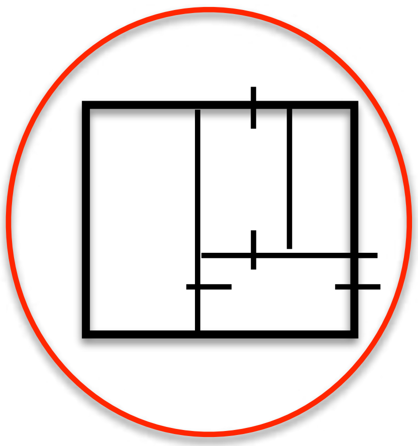
$$\Sigma_L = \sum_{n=3}^{\infty} A_n (-z)^n \qquad z \equiv \frac{g^2 s}{\epsilon}$$

$$\Sigma_L = \frac{2}{z^2}(e^z - 1 - z - \frac{z^2}{2})$$

$$\epsilon \to 0 \qquad \Sigma_L \to \{ \begin{array}{ll} \infty & s > 0 \\ -1 & s < 0 \end{array}$$

R-operation and Recurrence Relation

D=6 N=2



Horizontal boxes + double tennis court

$$nA_n^t = -\frac{1}{3}A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3}A_{n-1}^t$$

$$A_n^t = \frac{(-1)^n}{3^{n-3}}\frac{1}{n!}, \qquad A_n^s = \frac{1}{2}\frac{(-1)^n}{3^{n-3}}\frac{1}{n!} - \frac{1}{2}(-1)^n\frac{1}{n!}$$

$$(-g^2 s)^{n-1}(-g^2 t) \qquad (-g^2 s)^n$$

Summation

$$\Sigma_{L2} = \sum_{n=3}^{\infty} A_n^s(-z)^n + \frac{t}{s}A_n^t(-z)^n \qquad z \equiv \frac{g^2 s}{\epsilon}$$

$$\Sigma_{L2} = \frac{1}{2s^2z^2}\left[27(e^{z/3} - 1 - \frac{z}{3} - \frac{1}{2}\frac{z^2}{9} - \frac{1}{6}\frac{z^3}{27})(1 + 2\frac{t}{s}) - (e^z - 1 - sz - \frac{1}{2}z^2 - \frac{1}{6}z^3)\right]$$

- **Similar relations one can get for all other series**
- **All of them have 1/n! behavior**
- **Number of these series group as n!**

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy \; (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$n \geq 4$

$t' = t(x-y) - sy$

$$S_3 = -s/3, \; T_3 = -t/3$$

Summation

$$\Sigma_k(s,t,z) = \sum_{n=k}^{\infty} (-z)^n S_n(s,t)$$

Diff eqn

$$\frac{d}{dz}\Sigma_4(s,t,z) = 2s \int_0^1 dx \int_0^x dy \; (\Sigma_3(s,t',z) + \Sigma_3(t',s,z))|_{t'=xt+yu}$$

$$\Sigma_4(s,t,z) = \Sigma_3(s,t,z) + S_3(s,t)z^3 \qquad \Sigma(s,t,z) = z^{-2}\Sigma_3(s,t,z)$$

$$\frac{d}{dz}\Sigma(s,t,z) = s - \frac{2}{z}\Sigma(s,t,z) + 2s \int_0^1 dx \int_0^x dy \; (\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=xt+yu}$$

Ladder diagrams (leading divs)

D=8 N=1 **Horizontal boxes** $A_n^{(n)} = s^{n-1} A_n$

$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3$$

$A_1 = 1/6$ **1 loop box**

Summation $$\Sigma_m(z) = \sum_{n=m}^{\infty} A_n(-z)^n$$

$$-\frac{d}{dz}\Sigma_3 = -\frac{2}{4!}\Sigma_2 + \frac{2}{5!}\Sigma_1\Sigma_1.$$

$$\Sigma_3 = \Sigma_1 + A_1 z - A_2 z^2, \quad \Sigma_2 = \Sigma_1 + A_1 z, \quad A_1 = \frac{1}{3!}, \quad A_2 = -\frac{1}{3!4!}$$

$\Sigma_A \equiv \Sigma_1$ **Diff eqn**

$$\frac{d}{dz}\Sigma_A = -\frac{1}{3!} + \frac{2}{4!}\Sigma_A - \frac{2}{5!}\Sigma_A^2$$

$z = g^2 s^2/\epsilon$

$$\Sigma_A(z) = -\sqrt{5/3}\frac{4\tan(z/(8\sqrt{15}))}{1 - \tan(z/(8\sqrt{15}))\sqrt{5/3}} = \sqrt{10}\frac{\sin(z/(8\sqrt{15}))}{\sin(z/(8\sqrt{15}) - z_0)}$$

$$\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \dots)$$

$$z_0 = \arcsin(\sqrt{3/8})$$

All loop Exact Recurrence Relation

D=8 N=1

s-channel term $S_n(s,t)$ **t-channel term** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ \left(S_{n-1}(s,t') + T_{n-1}(s,t')\right)|_{t'=tx+yu}$$

$$+ \quad s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!}\ \frac{d^p}{dt'^p}\left(S_k(s,t') + T_k(s,t')\right) \times$$

$$\times \frac{d^p}{dt'^p}\left(S_{n-1-k}(s,t') + T_{n-1-k}(s,t')\right)|_{t'=-sx}\ (tsx(1-x))^p$$

$$S_1 = \frac{1}{12},\ T_1 = \frac{1}{12}$$

summation $\Sigma_3(s,t,z) = \Sigma_1(s,t,z) - S_2(s,t)z^2 + S_1(s,t)z,\ \Sigma_2(s,t,z) = \Sigma_1(s,t,z) + S_1(s,t)z$

Diff eqn

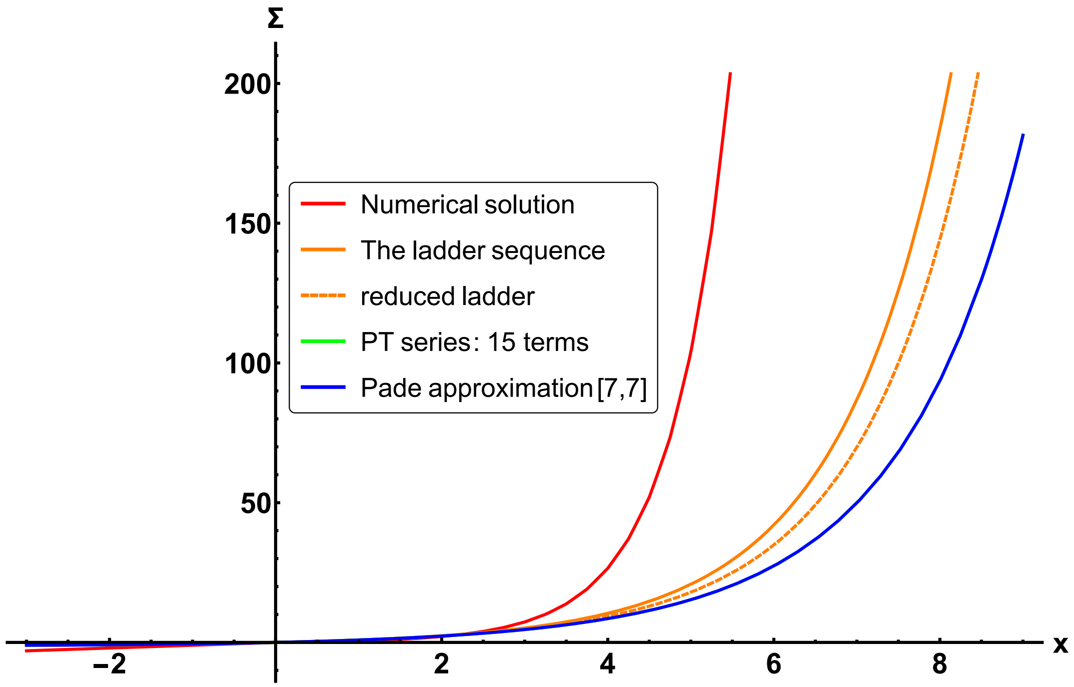
$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ \left(\Sigma(s,t',z) + \Sigma(t',s,z)\right)|_{t'=tx+yu}$$

$$-s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!}\left(\frac{d^p}{dt'^p}\left(\Sigma(s,t',z) + \Sigma(t',s,z)\right)|_{t'=-sx}\right)^2\ (tsx(1-x))^p.$$

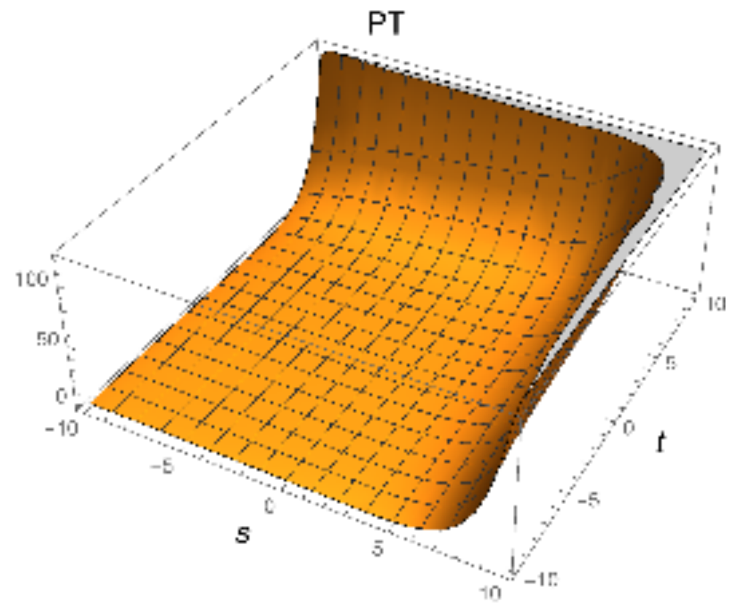
All loop Solution (leading divs)

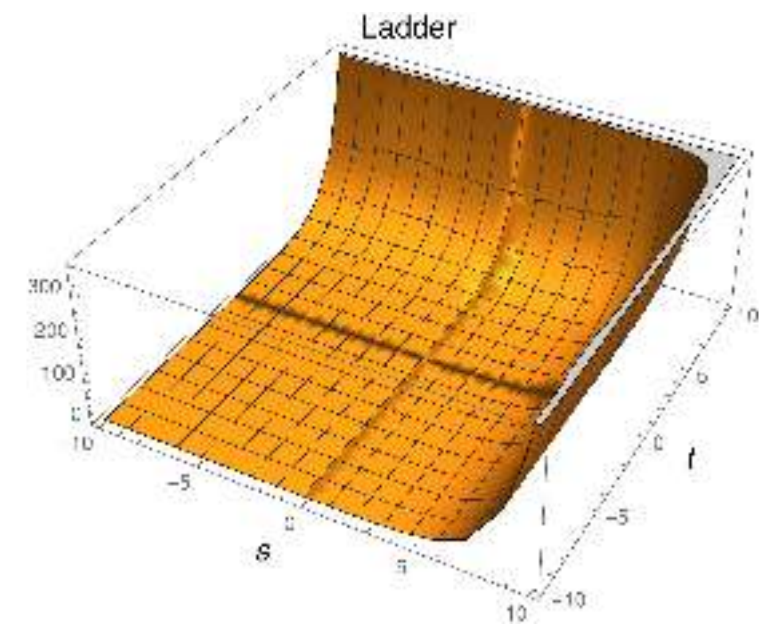
D=6 N=2

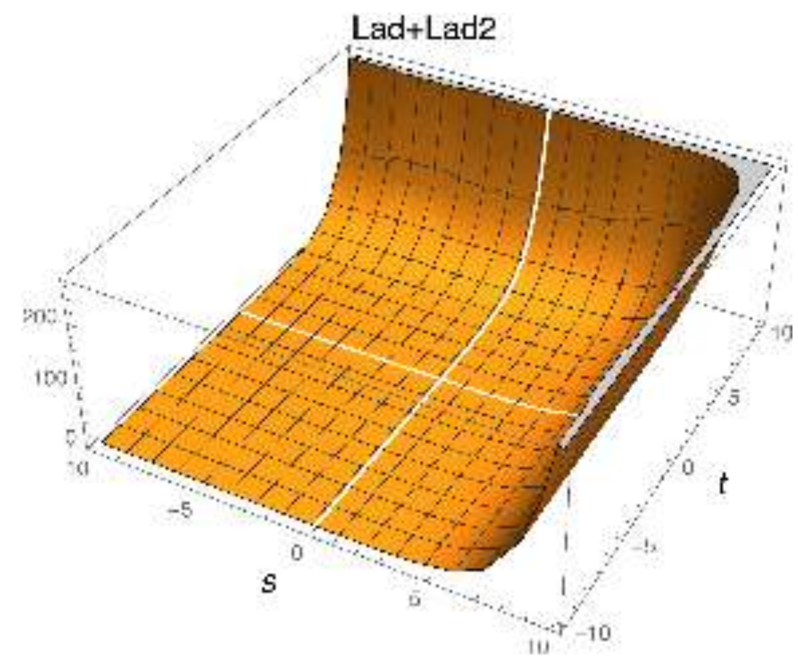
PT (15 terms)

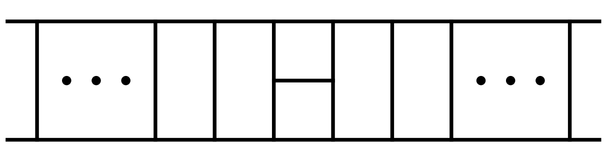


PT and Pade versus ladder for t=s

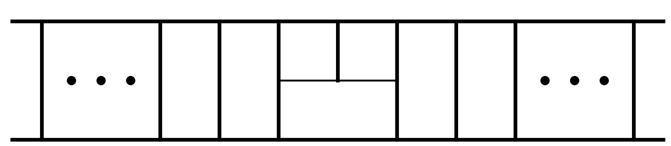








Ladder



Lddder 2

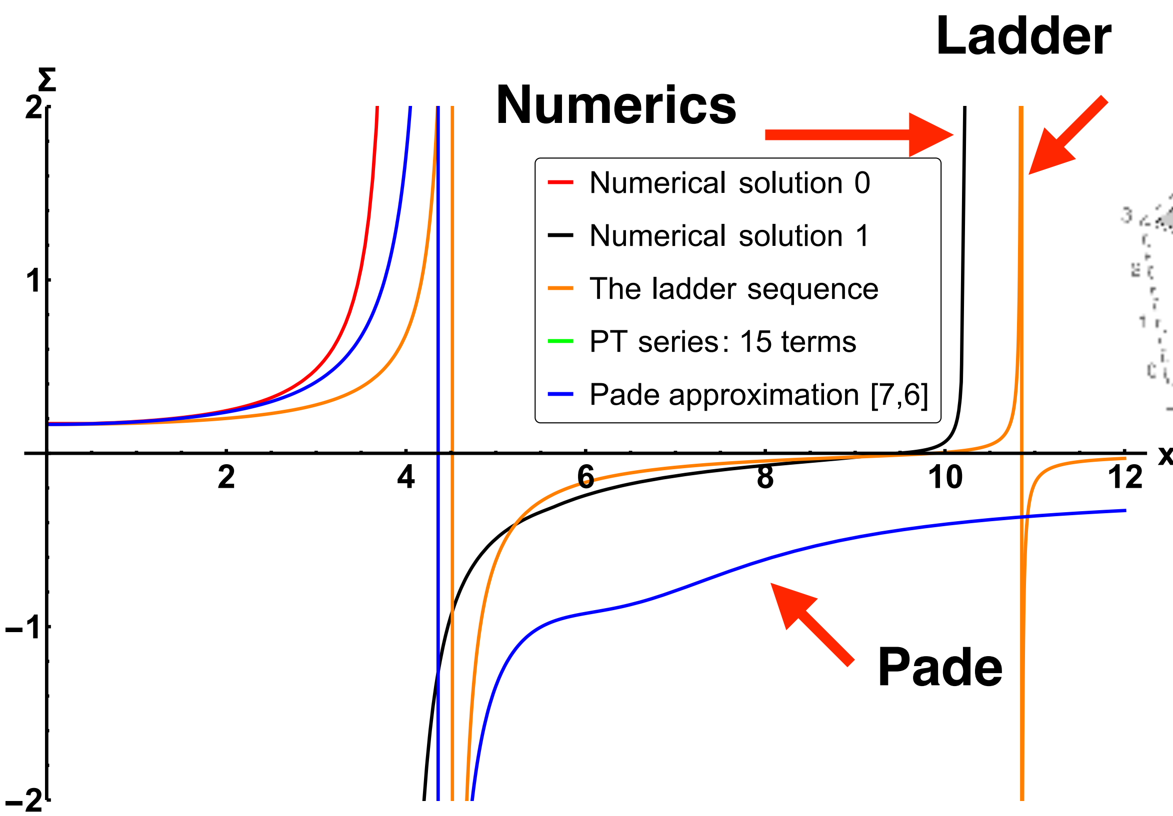
$$\Sigma_L(s,z) = \frac{2}{s^2 z^2}\left(e^{sz} - 1 - sz - \frac{s^2 z^2}{2}\right)$$

$$z = \frac{g^2}{\epsilon}$$

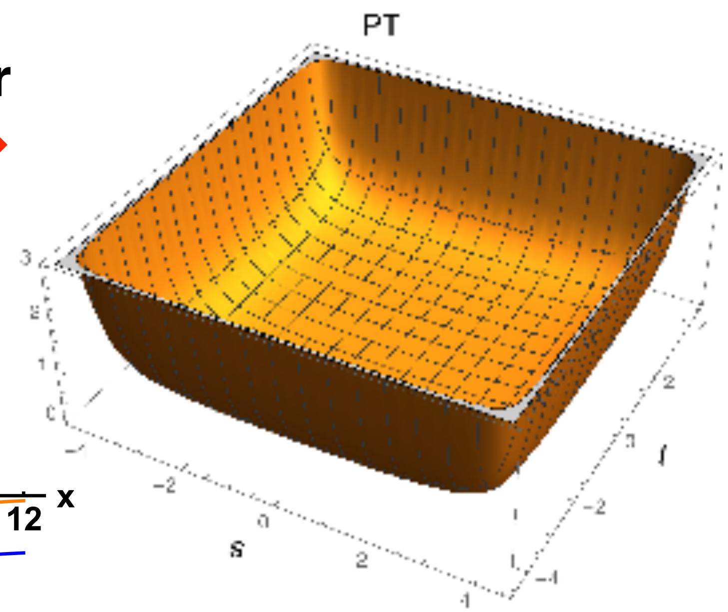
Numerical solution of the full equation is close to the ladder approx

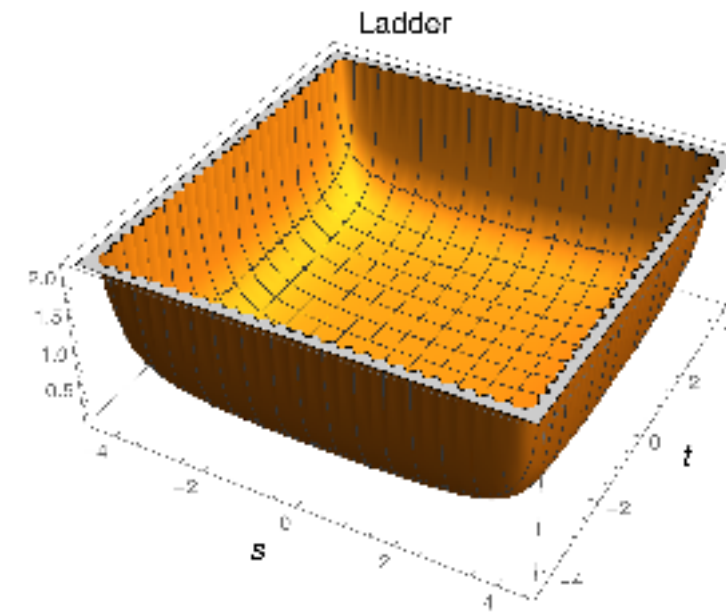
All loop Solution (leading divs)

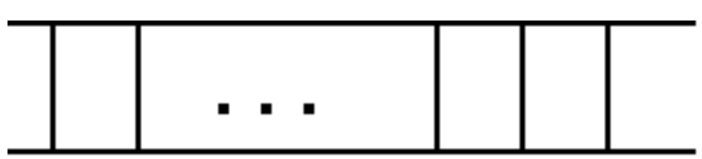
D=8 N=1



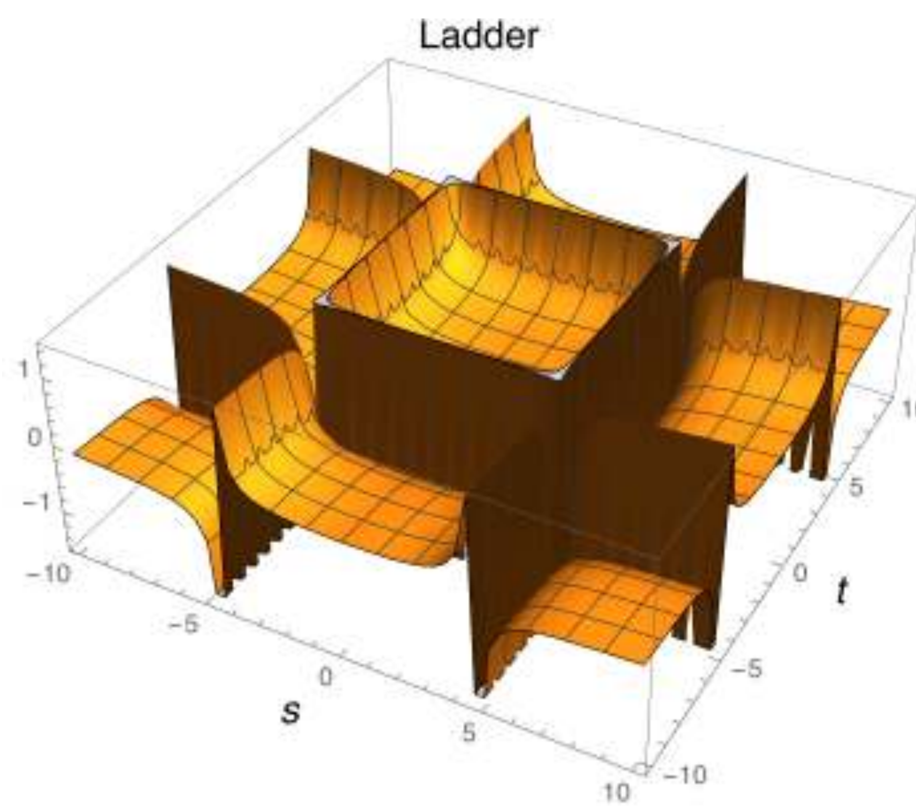
PT and Pade versus ladder for t=s







$$z = \frac{g^2}{\epsilon}$$



$$\Sigma_L(s,z) = -\sqrt{5/3}\,\frac{4\tan(zs^2/(8\sqrt{15}))}{1-\tan(zs^2/(8\sqrt{15}))\sqrt{5/3}}$$

Subleading divergences

$$\Sigma_L(z) + \epsilon\Sigma_{NL}(z) + \epsilon^2\Sigma_{NNL}(z) + \cdots$$

$$\Sigma(z) = \sum_n^{\infty} z^n F_n$$

$$D = 4 \quad N = 4 \quad z = g^2/\epsilon$$
$$D = 6 \quad N = 2 \quad z = g^2 s/\epsilon, z = g^2 t/\epsilon$$
$$D = 8 \quad N = 1 \quad z = g^2 s^2/\epsilon, z = g^2 st/\epsilon, ..$$
$$D = 10 \quad N = 1 \quad z = g^2 s^3/\epsilon, z = g^2 s^2 t/\epsilon, ..$$

D=8 N=1

sLadder case

$$\Sigma_{NL} = s\Sigma_{sB}(z) + t\Sigma_{tB}(z)$$

$$z = \frac{g^2 s^2}{\epsilon}$$

$$\Sigma'_{tB}(z) = \frac{5}{6}\left[e^{z/60}(2\cos(z/30) - \sin(z/30)) - 2\right]$$

$$\Sigma_{tB} = -\frac{1}{36}\left[60 + z + e^{z/60}(-(60+z)\cos(z/30) - 2(-15+z)\sin(z/30))\right]$$

Sum of Ladder diagrams (subleading divs)

$$\Sigma'_{sB} = \sum_{n=2}^{\infty} z^n B'_{sn}$$

$$\frac{d^2\Sigma'_{sB}(z)}{dz^2} + f_1(z)\frac{d\Sigma'_{sB}(z)}{dz} + f_2(z)\Sigma'_{sB}(z) = f_3(z)$$

Diff eqn

$$f_1(z) = -\frac{1}{6} + \frac{\Sigma_A}{15},$$

$$f_2(z) = \frac{1}{80} - \frac{\Sigma_A}{360} + \frac{\Sigma_A^2}{600} + \frac{1}{15}\frac{d\Sigma_A}{dz},$$

$$f_3(z) = \frac{2321}{5!5!2}\Sigma_A + \frac{11}{1800}\Sigma'_{tB} - \frac{47}{5!45}\Sigma_A^2 - \frac{1}{5!72}\Sigma_A\Sigma'_{tB} + \frac{23}{6750}\Sigma_A^3 + \frac{1}{1200}\Sigma_A^2\Sigma'_{tB} - \frac{19}{36}\frac{d\Sigma_A}{dz} - \frac{1}{15}\frac{d\Sigma'_{tB}}{dz} + \frac{23}{225}\frac{d\Sigma_A^2}{dz} + \frac{1}{30}\frac{d(\Sigma_A\Sigma'_{tB})}{dz} - \frac{3}{32}$$

Solution to Diff eqn

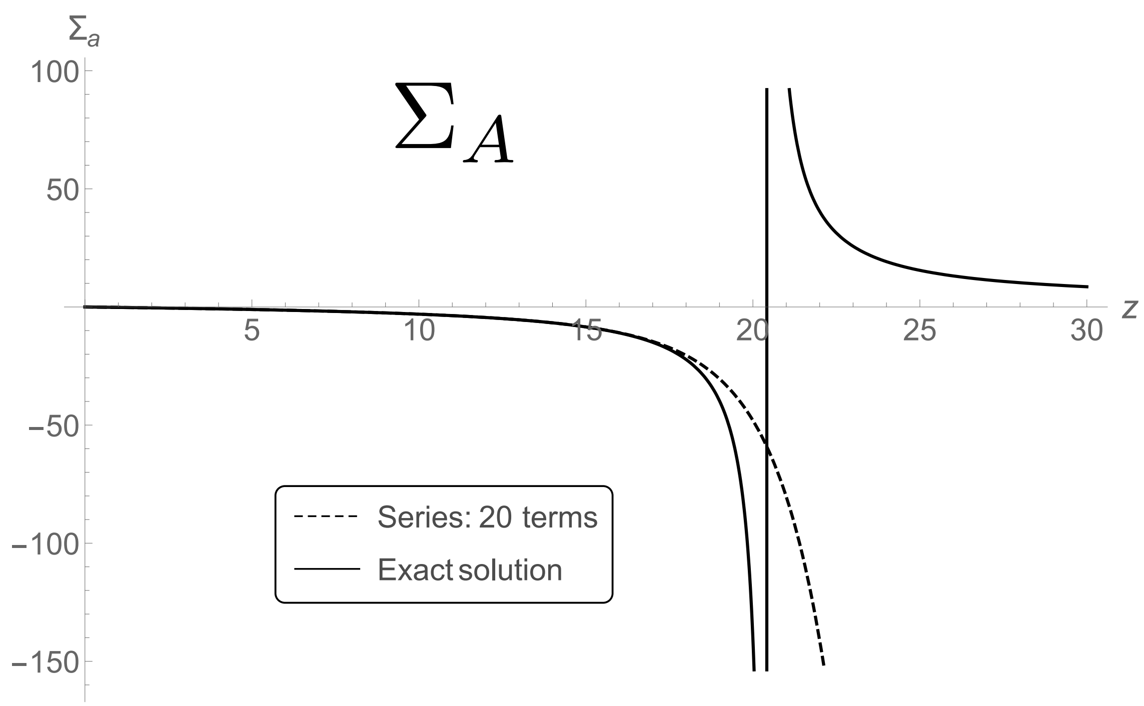
smooth monotonic function

$$\Sigma'_{sB}(z) = \frac{d\Sigma_A}{dz}u(z) \qquad u(z) = \int_0^z dy \int_0^y dx \frac{f_3(x)}{d\Sigma_A(x)/dx}$$

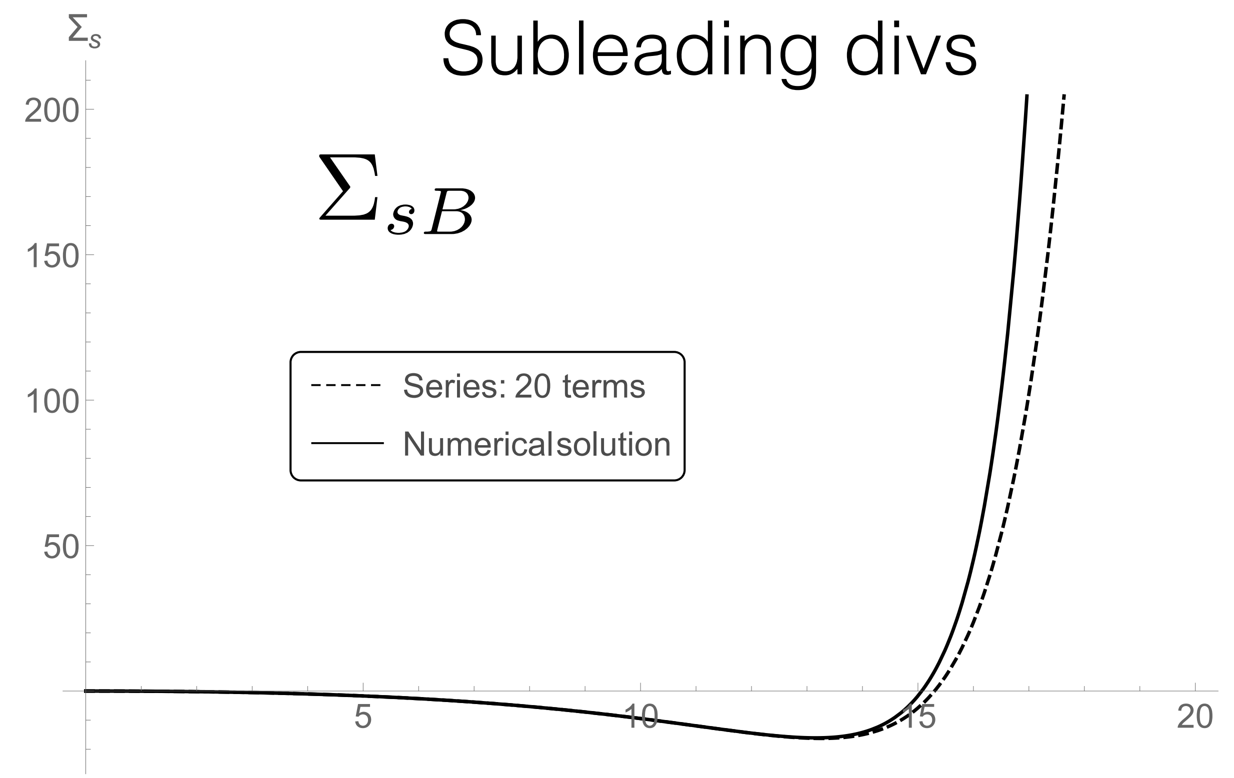
Sum of the Ladder diagrams

solutions

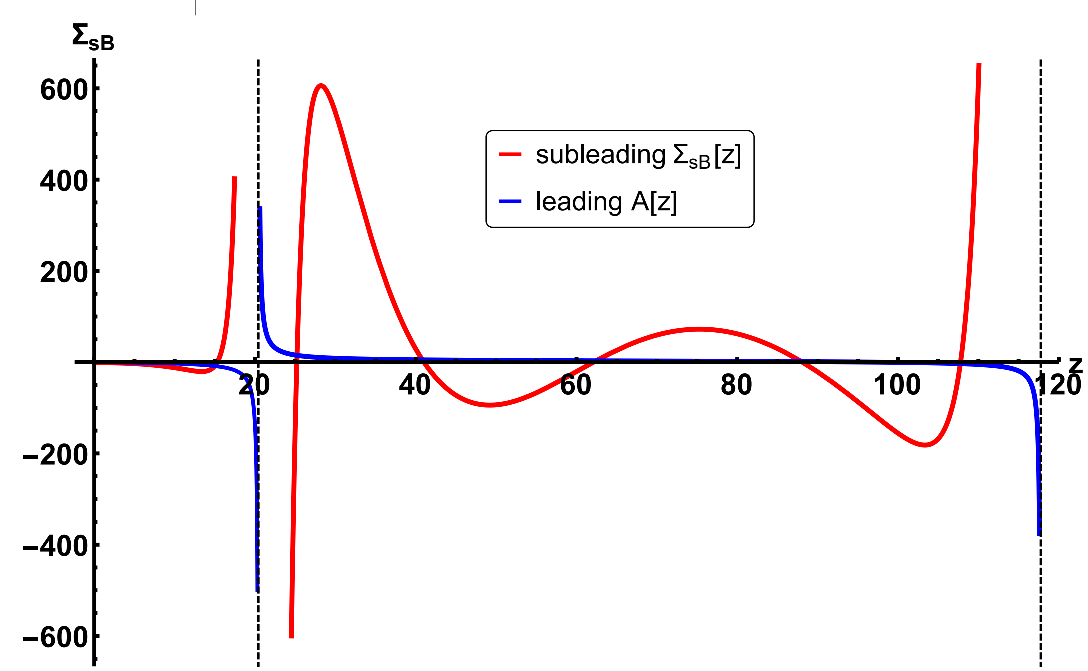
Leading divs



Subleading divs



Infinite number of poles



Infinite number of poles at the same position

Scheme dependence and arbitrariness of subtraction

subleading case

$$A'_1 + B'_{s1} = \frac{1}{6\epsilon}(1 + c_1\epsilon) \qquad \Delta\Sigma'_{sB} = c_1 z \frac{d\Sigma'_A}{dz}. \quad \longrightarrow \quad z \to z(1 + c_1\epsilon).$$

sub-subleading case

$$A'_2 + B'_2 = \frac{s}{3!4!\epsilon^2}\left(1 - \frac{5}{12}\epsilon + 2c_1\epsilon + c_2\epsilon^2\right) \qquad \Delta\Sigma'_{sC} = c_2 z^2 \frac{d\Sigma'_A}{dz}.$$

$$\longrightarrow \quad z \to z(1 + c_1\epsilon) + z^2 c_2 \epsilon^2.$$

$$\Delta\Sigma'_{sC} = -c_1^2 \frac{z}{4!}\left(\frac{d\Sigma'_A}{dz} - 12\frac{d^2\Sigma'_A}{dz^2}\right) \quad \longrightarrow \quad z \to z(1 + c_1\epsilon) + z^2(c_2 + c_1^2/4!)\epsilon^2$$

Scheme dependence and arbitrariness of subtraction

sub-subleading case

linear term

$$A_2' + B_2' = \frac{s}{3!4!\epsilon^2}\left(1 - \frac{5}{12}\epsilon + 2c_1\epsilon + c_2\epsilon^2\right)$$

new contribution from subleading term

$$\Delta\Sigma_{sC}'(3-loop) = -\frac{719 c_1 s^2}{1036800\epsilon}$$

$$\Sigma_{sB}'(3-loop) = -\frac{71 s^2}{345600\epsilon^2}$$

$$\Sigma_{sB}^{\prime\,trunc}(3-loop) = -\frac{719 s^2}{3110400\epsilon^2}$$

the source of a problem

$$\Delta\Sigma_{sC}'(3-loop) = c_1 z \frac{d\Sigma_{sB}^{\prime\,trunc}}{dz}(3-loop)$$

$$z \to z(1+c_1\epsilon) + z^2(c_2 - c_1^2/4!)\epsilon^2 + z^3 c_1^3/6!\epsilon^3 - z^4 c_1^4/4!6!\epsilon^4 + ...$$

Kinematically dependent renormalization

- **R-operation is equivalent to**

renormalizable theories

nonrenormalizable theories

$$\bar{A}_4 = Z_4(g^2)\bar{A}_4^{bare}|_{g^2_{bare} -> g^2 Z_4}$$

$$g^2_{bare} = \mu^\epsilon Z_4(g^2)g^2.$$

$$Z = 1 - \sum_i KR'G_i$$

simple multiplication

operator multiplication

$$Z = 1 + \frac{g^2}{\epsilon} + g^4(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}) + ...$$

$$Z = 1 + \frac{g^2}{\epsilon}st + g'^4 st(\frac{s^2+t^2}{\epsilon^2} + \frac{s^2+st+t^2}{\epsilon}) + ...$$

scheme dependence

scheme dependence

$$g^2 = zg'^2, \quad z = 1 + g'^2 c_1 + g'^4 c_2 + ...$$

$$g^2 = zg'^2, \quad z = 1 + g'^2 stc_1 + g'^4 st(s^2+t^2)c_2 + ...$$

Kinematically dependent renormalization

operator kinematically dependent renormalization

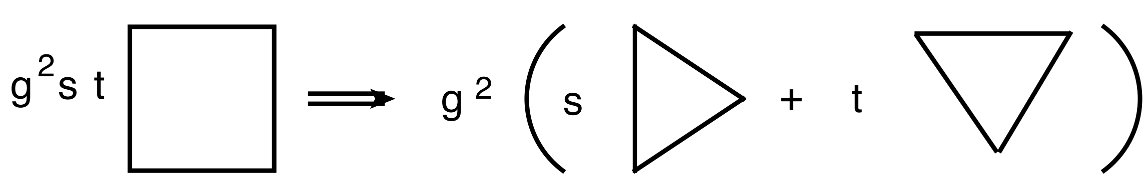
at 2 loops

$$\bar{A}_4 = 1 - \frac{g_B^2 st}{3!\epsilon} - \frac{g_B^4 st}{3!4!}\left(\frac{s^2+t^2}{\epsilon^2} + \frac{27/4s^2 + 1/3st + 27/4t^2}{\epsilon}\right) + \ldots$$

$$\bar{A}_4 = Z_4(g^2)\bar{A}_4^{bare}|_{g_{bare}^2 -> g^2 Z_4}$$

$$Z_4 = 1 + \frac{g^2 st}{3!\epsilon} + \frac{g^4 st}{3!4!}\left(-\frac{s^2+t^2}{\epsilon^2} + \frac{5/12s^2 + 1/3st + 5/12t^2}{\epsilon}\right)$$

$$g_B^2 = g^2(1 + \frac{g^2}{3!\epsilon})$$



this is operator action!

Conclusions

- **The UV divergences for the on-shell scattering amplitudes DO NOT CANCEL in any given order of PT**

- **The recurrence relations allow one to calculate the leading UV divergences in ALL orders of PT algebraically starting from 1 loop**

- **The recurrence relations allow one to calculate the sub leading UV divergences in ALL orders of PT algebraically starting from 1 and 2 loops**

- **This procedure apparently continues the same way for all divergences just like in renormalizable theories**

Conclusions cont'd

- **The sum of the leading UV divergences to ALL orders obeys the nonlinear integro-differential equation**

- **The numerical solution indicates that solution to the full equation seems to behave like the ladder approximation**

- **There is no simple limit when** $\epsilon \to +0$

- **This means that one cannot simply remove the UV divergence and non-renormalizability of a theory is not improved when summing the infinite series**

Conclusions cont'd

- **The structure of UV divergences in non-renormalizable theories essentially copies that of renormalizable ones**
- **The main difference is that the renormalization constant depends on kinematics and acts like an operator rather than simple multiplication**
- **As a result, one can construct the higher derivative theory that gives the finite scattering amplitudes with a single arbitrary coupling g defined in PT within the given renormalization scheme.**
- **Transition to another scheme is performed by the action on the amplitude of a finite renormalization operator z that depends on kinematics.**