# Amplitudes in D=6 N=(1,1) SYM Theory


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# Motivation

**Maximal SYM**

**D=4 N=4**
**D=6 N=2**
**D=10 N=1**

- **Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)**
- **First UV divergent diagrams at D=4+6/L**
- **Conformal or dual conformal symmetry**
- **Common structure of the integrands**

Bern, Dixon &Co 10

Drummond, Henn, Korchemsky, Sokatchev

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$$\mathcal{M}_n \equiv \frac{A_n}{A_n^{tree}} = 1 + \sum_{L=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

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$$\mathcal{M}_n(\epsilon) = \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(\frac{\gamma_{cusp}^{(l)}}{(l\epsilon)^2} + \frac{2G_0^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon} + \frac{1}{4}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \gamma_{cusp}^{(l)} F_n^{(1)}(0) + C(g)\right]$$

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$$M_4^{(1-loop)}(\epsilon) = A_4^{(1-loop)}/A_4^{(tree)} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\left[\frac{1}{\epsilon^2}\left((\frac{\mu^2}{s})^{\epsilon} + (\frac{\mu^2}{-t})^{\epsilon}\right) - \frac{1}{2}\log^2\left(\frac{s}{-t}\right) - \frac{\pi^2}{3}\right] + \mathcal{O}(\epsilon)$$

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**D=6**

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**D=6** $\quad [g^2] \sim \frac{1}{M^2}$

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## D=4 N=4

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## D=6

$$[g^2] \sim \frac{1}{M^2}$$

**Toy model for gravity**

# Color decomposition & Spinor helicity formalism

## Color ordered amplitude

$$\mathcal{A}_n^{a_1\dots a_n}(p_1^{\lambda_1}\dots p_n^{\lambda_n}) = \sum_{\sigma\in S_n/Z_n} Tr[\sigma(T^{a_1}\dots T^{a_n})]A_n(\sigma(p_1^{\lambda_1}\dots p_n^{\lambda_n})) + \mathcal{O}(1/N_c)$$

Planar Limit $N_c \to \infty,\ g_{YM}^2 \to 0$ and $g_{YM}^2 N_c$ - fixed

*Cheung, O'Connell 09, Bern&Co 10*

## Spinor helicity formalism

Momentum $p^\mu,\ p^2 = 0,\ \mu = 0,...,5$ $\boxed{SO(5,1)}$

$\boxed{SU^*(4)} \to \lambda^{Aa}$

$$p_{AB} = p_\mu(\sigma^\mu)_{AB},\ p^{AB} = p^\mu(\bar{\sigma}_\mu)^{AB}$$

$$\boxed{p^{AB} = \lambda^{Aa}\lambda_a^B,\ p_{AB} = \tilde{\lambda}_A^{\dot{a}}\tilde{\lambda}_{B\dot{a}}}$$

Little group in D=6: $SO(4) \simeq SU(2)\times \boxed{SU(2)}$

Lorentz invariant structures: $\lambda(i)^{Aa}\tilde{\lambda}(j)_A^{\dot{a}} \doteq \langle i_a|j_{\dot{a}}] = [j_{\dot{a}}|i_a\rangle$

$$\langle 1_a 2_b 3_c 4_d\rangle \doteq \epsilon_{ABCD}\lambda_1^{Aa}\lambda_2^{Bb}\lambda_3^{Cc}\lambda_4^{Dd} \qquad [1_{\dot{a}}2_{\dot{b}}3_{\dot{c}}4_{\dot{d}}] \doteq \epsilon^{ABCD}\tilde{\lambda}_{A,1}^{\dot{a}}\tilde{\lambda}_{B,2}^{\dot{b}}\tilde{\lambda}_{C,3}^{\dot{c}}\tilde{\lambda}_{D,4}^{\dot{d}}$$

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## Spinor helicity formalism

**Helicity is no longer conserved in D=6!**

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# Superfield formalism in D=6

$\mathcal{N}=(1,1)$ D=6 on-shell superspace $=\{\lambda^A_a,\tilde\lambda^{\dot a}_A,\eta^I_a,\overline{\eta}_{I'\dot a}\}$

N=(1,1) on-shell states

$$\Phi^{--},\Phi^{-+},\Phi^{+-},\Phi^{++},$$
$$\Psi^{-a},\Psi^{+a},\bar\Psi^{-\dot a},\bar\Psi^{+\dot a},$$
$$A^{a\dot a}$$

Harmonic superspace
(*Dennen, Huang, Siegel 10*)

$$\frac{SU(2)_R}{U(1)}\times\frac{SU(2)_R}{U(1)}$$

$$u^{\mp}_I \text{ and } \overline{u}^{\pm I'}$$

$$q^{\mp A} = u^{\mp}_I q^{AI},\ \overline{q}^{\pm}_A = u^{\pm I'}\overline{q}_{AI'},$$
$$\eta^{\mp}_a = u^{\mp}_I\eta^I_a,\ \overline{\eta}^{\pm}_{\dot a} = u^{\pm I'}\overline{\eta}_{I'\dot a},$$

$$\{q^{AI},q^{BJ}\}=p^{AB}\epsilon^{IJ}$$
$$\{\bar q_{AI'},\bar q_{BJ'}\}=p_{AB}\epsilon_{I'J'}$$

$$p^{AB}=\sum_i^n\lambda^{Aa}_i\lambda^B_{a,i},\quad q^A=\sum_i^n\lambda^{A,i}_a\eta^a_i,\quad \overline{q}_A=\sum_i^n\tilde\lambda^{\dot a}_{A,i}\overline{\eta}_{\dot a,i}$$

$$\{\lambda^A_a,\tilde\lambda^{\dot a}_A,\eta^-_a,\overline{\eta}^+_{\dot a}\}$$

The full amplitude

$$A_n(\{\lambda^A_a,\tilde\lambda^{\dot a}_A,\eta_a,\overline{\eta}_{\dot a}\})=\delta^6(p^{AB})\delta^4(q^A)\delta^4(\overline{q}_A)\mathcal{P}_n(\{\lambda^A_a,\tilde\lambda^{\dot a}_A,\eta_a,\overline{\eta}_{\dot a}\})$$

The delta function always factorizes!

Polynomial of degree
2n-8 in Grassmannian variables

Grassmannian delta function is defined as:

$$\delta^4(q^A)=\frac{1}{4!}\epsilon_{ABCD}\delta(\sum_i^n q^A_i)\delta(\sum_k^n q^A_k)\delta(\sum_i^n q^A_l)\delta(\sum_k^n q^A_p)$$

Tree level amplitude: n=4

$$A_4^{(0)}=-ig^2_{YM}\delta^6(p^{AB})\frac{\delta^4(q^A)\delta^4(\overline{q}_A)}{st}\qquad \mathcal{P}_4=-i/st$$

In components

$$A_4^{(0)}(1_{a\dot a}2_{b\dot b}3_{c\dot c}4_{d\dot d})=-ig^2_{YM}\frac{\langle 1_a2_b3_c4_d\rangle[1_{\dot a}2_{\dot b}3_{\dot c}4_{\dot d}]}{st}$$

# Perturbation Expansion for the Amplitudes

$A_4/A_4^{tree}$

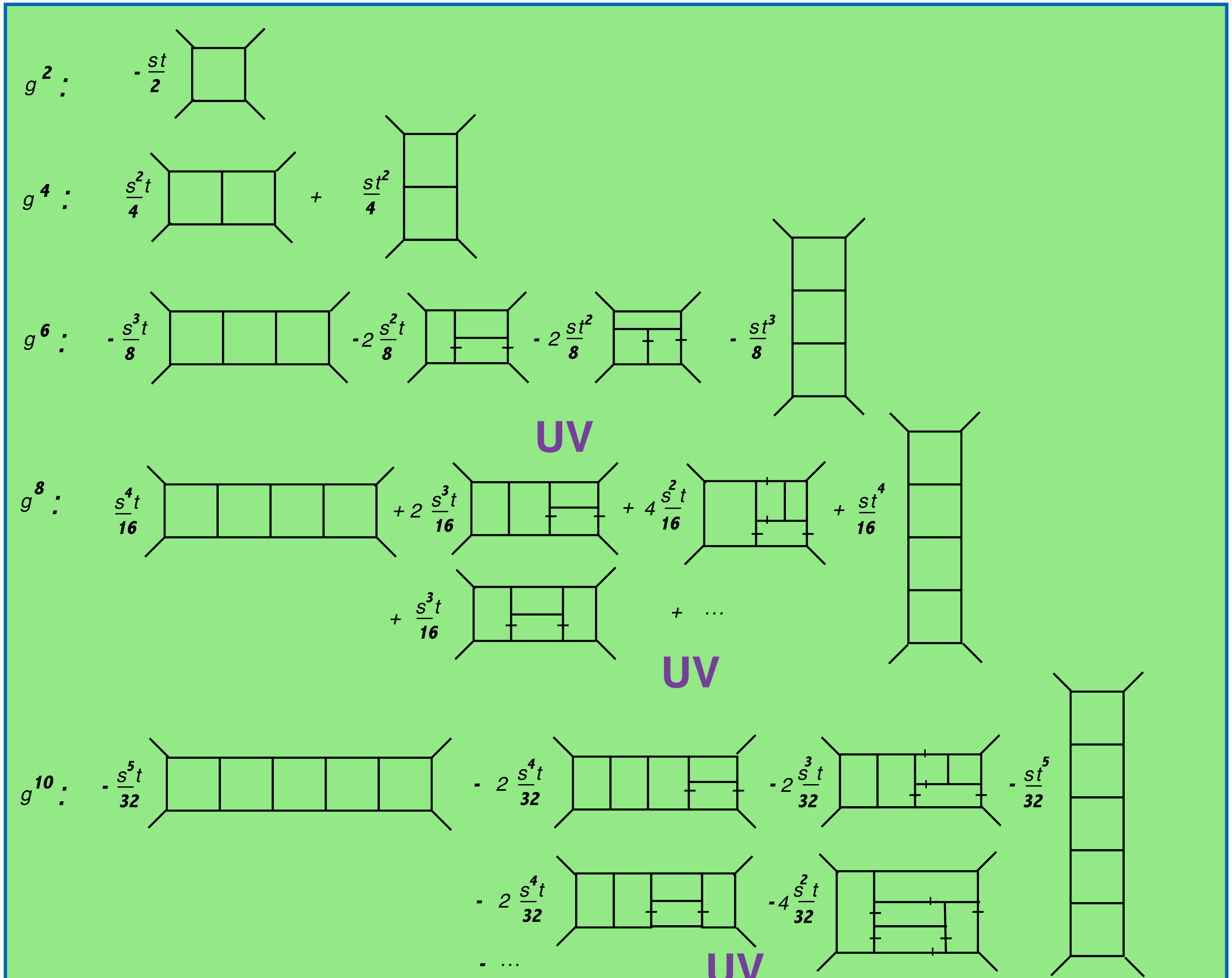
**No bubbles**
**No Triangles**

**First UV div at three loops**

**D=4+6/L**

$$[g^2] \sim \frac{1}{M^2}$$

**IR finite**

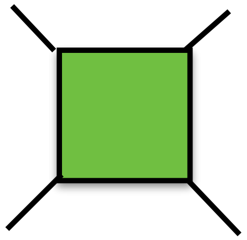



1
2
4
15
60

**Universal expansion for any D in maximal SYM**

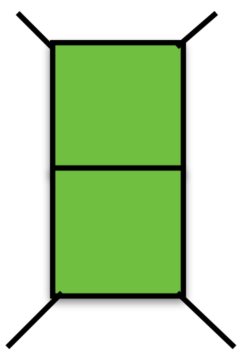
# Perturbation Expansion for the Amplitudes

**Exact calculation**



$$p_i^2 = 0, \;\; m = 0$$

$$B_1(s,t) = \frac{\pi^3}{(2\pi)^6}\frac{b_2(x)}{s+t}, \quad b_2(x) = \frac{L^2(x)+\pi^2}{2}, \quad L(x) \doteq \log(x), \quad x = \frac{t}{s}$$



$$B_2(s,t) = \left(\frac{\pi^3}{(2\pi)^6}\right)^2 \left(\frac{b_4(x)}{t} + \frac{b_3(x)}{s+t}\right)$$

Anastasiou, Tausk, Tejeda-Yeomans , 00
Bork,Kazakov,Vlasenko, 13

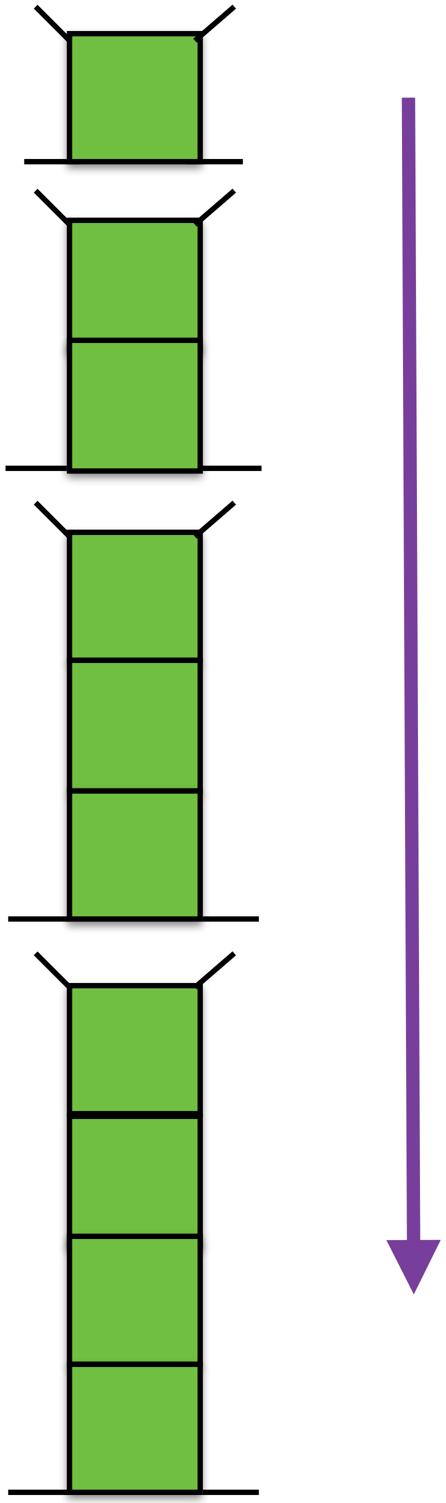
$$\begin{aligned} b_4(x) \;\; &= \;\; \left(2\zeta_3 - 2Li_3(-x) - \frac{\pi^2}{3}L(x)\right)L(1+x) + \left(\frac{1}{2}L(x) + \frac{\pi^2}{2}\right)L^2(1+x) \\ &+ \;\; \left(2L(x)L(1+x) - \frac{\pi^2}{3}\right)Li_2(-x) + 2L(x)S_{1,2}(-x) - 2S_{2,2}(-x) \end{aligned}$$

$$b_3(x) = -2\zeta_3 + \frac{\pi^2}{3}L(x) - \left(L(x)+\pi^2\right)L(1+x) - 2L(x)Li_2(-x) + 2Li_3(-x)$$

**Regge Limit** $\;\; s \to \infty, \;\; t < 0, \;\; fixed$

$$B_1(s,t) \sim \frac{1}{2}L^2(x) \qquad\qquad B_2(s,t) \sim \frac{1}{12}L^4(x)$$

# Perturbation Expansion for the Amplitudes



**Leading Logarithms**

**UV finite**

**Regge Limit** $s \to \infty, \quad t < 0, \; fixed$

$$B_n(t,s) \simeq \frac{1}{s}\frac{L^{2n}(x)}{n!(n+1)!}, \qquad L \equiv \log(s/t)$$

Bork,Kazakov,Vlasenko, 13

$$\left.\frac{A_4}{A_4^{(0)}}\right|_{L.L.} = \sum_{n=0}^{\infty} \frac{(-g^2t/2)^n L^{2n}(x)}{n!(n+1)!}, \quad \text{where} \quad g^2 \equiv \frac{g_{YM}^2 N_c}{64\pi^3}.$$

$$\sum_{n=0}^{\infty} \frac{(-g^2t/2)^n L^{2n}(x)}{n!(n+1)!} = \frac{I_1(2y)}{y}, \qquad y \equiv \sqrt{g^2|t|/2}\; L(x)$$

$$\left.\frac{A_4}{A_4^{(0)}}\right|_{L.L.} \sim \left(\frac{s}{t}\right)^{\alpha(t)-1}$$

!

**Regge behaviour**

**Exact for** $N_c \to \infty$

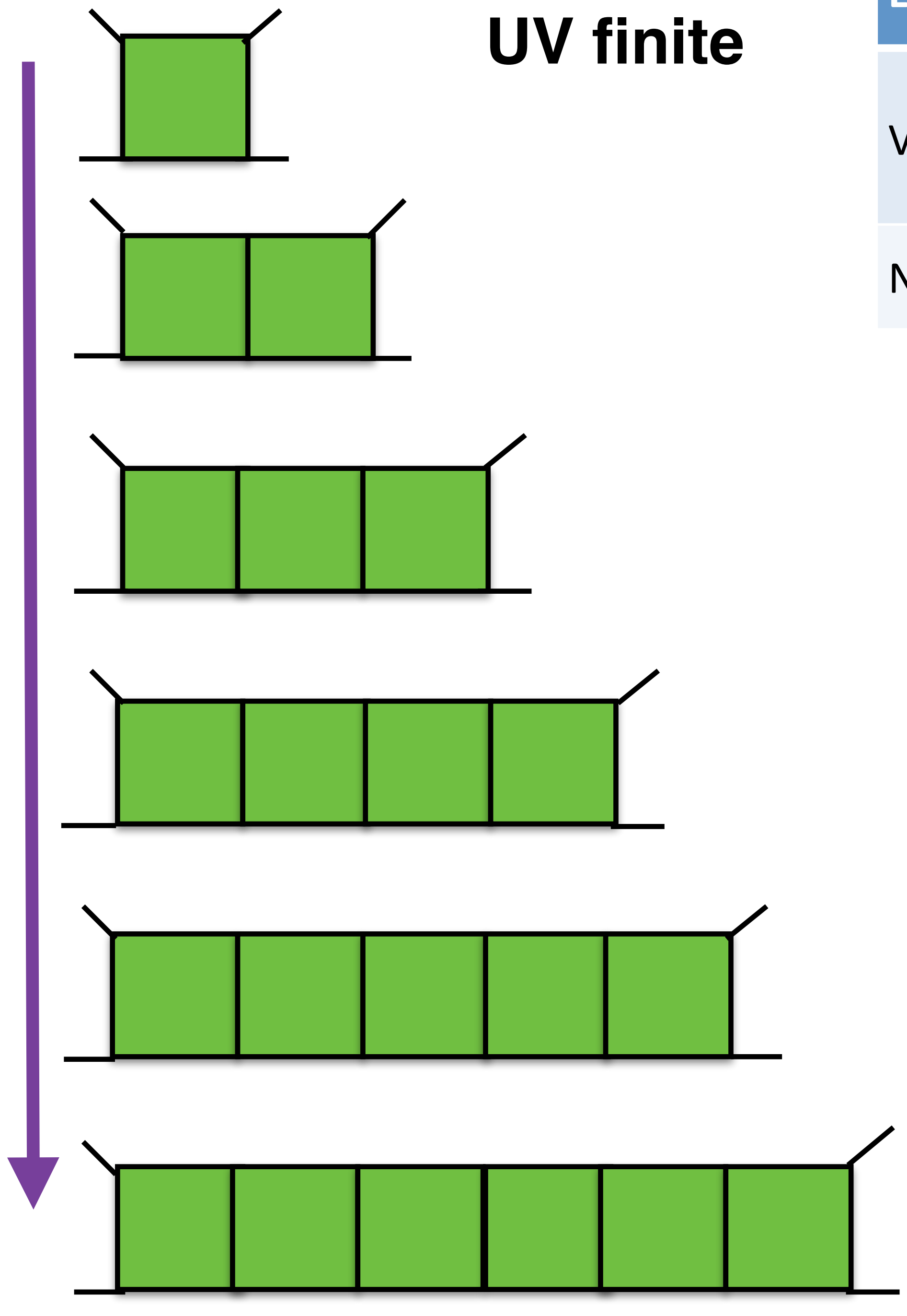
$$\alpha(t) = 1 + 2\sqrt{g^2|t|/2} = 1 + \sqrt{\frac{g_{YM}^2 N_c |t|}{32\pi^3}}$$

# Perturbation Expansion for the Amplitudes

$$B_n(s,t) = \frac{1}{s}\left(C_n + O(t/s)\right), \quad n \geq 2$$

Kazakov, 14

**Leading Powers**



**UV finite**

| Loops | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Values | | | | | | |
| Numerics | | | | | | |

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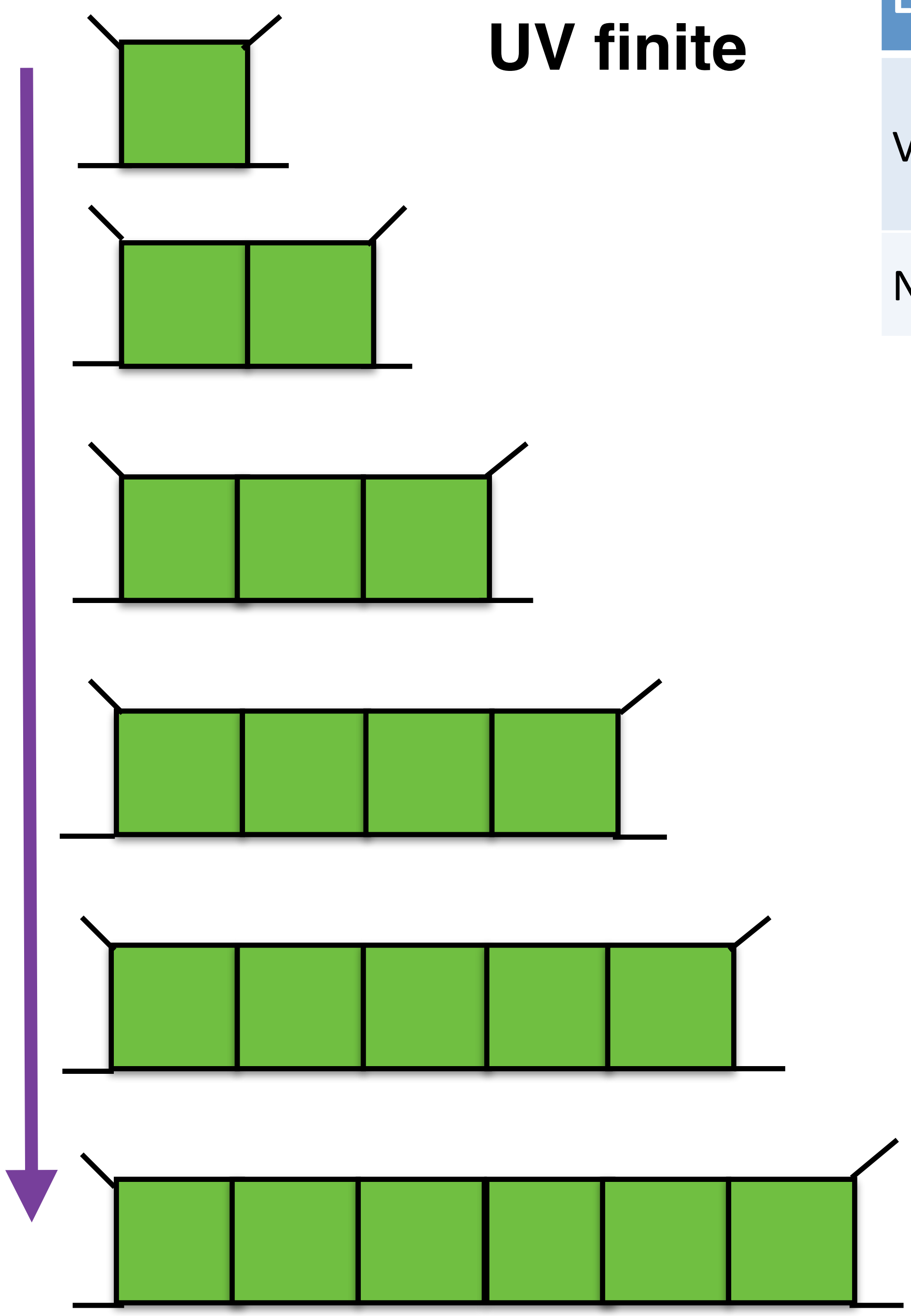

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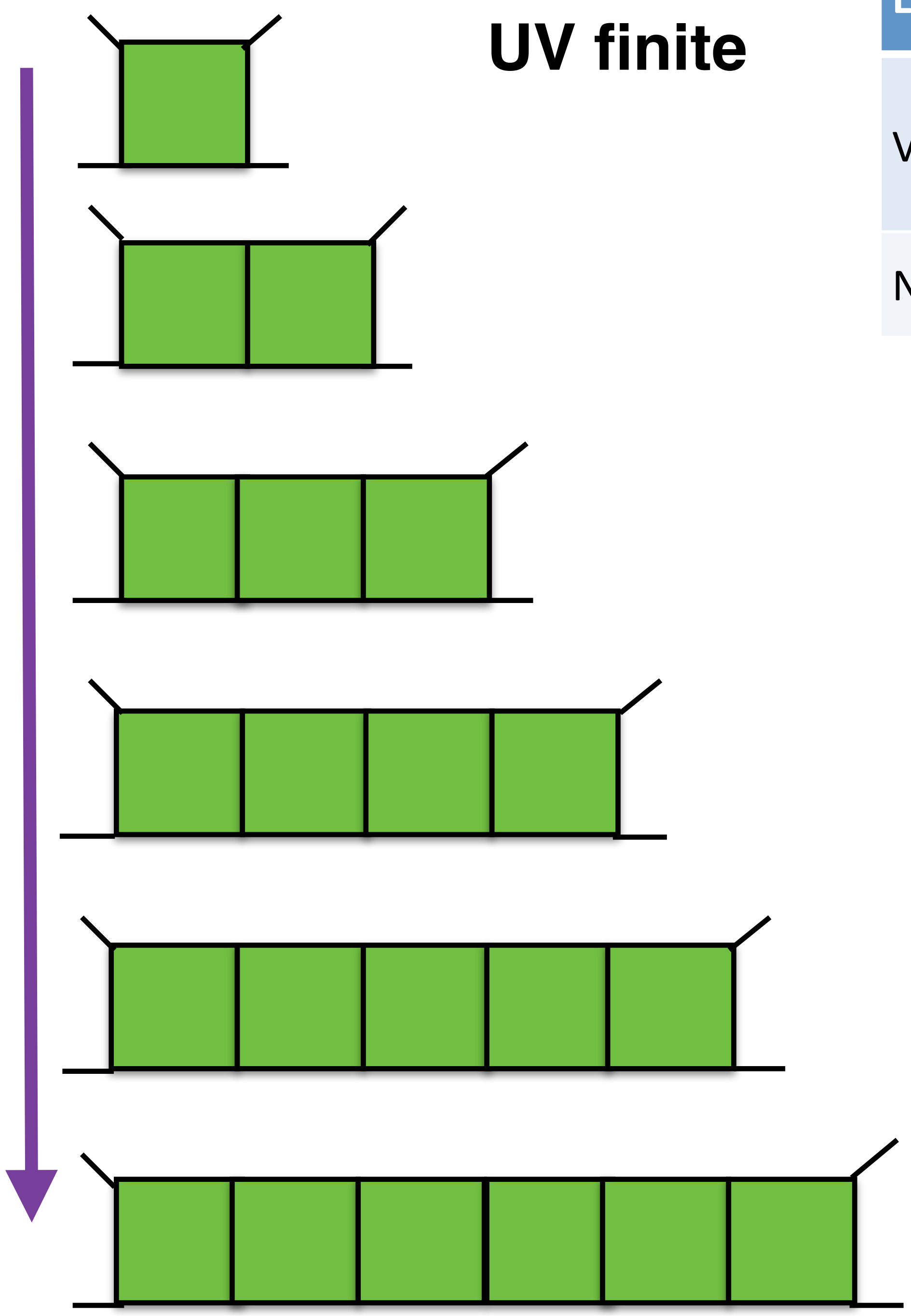

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Kazakov, 14

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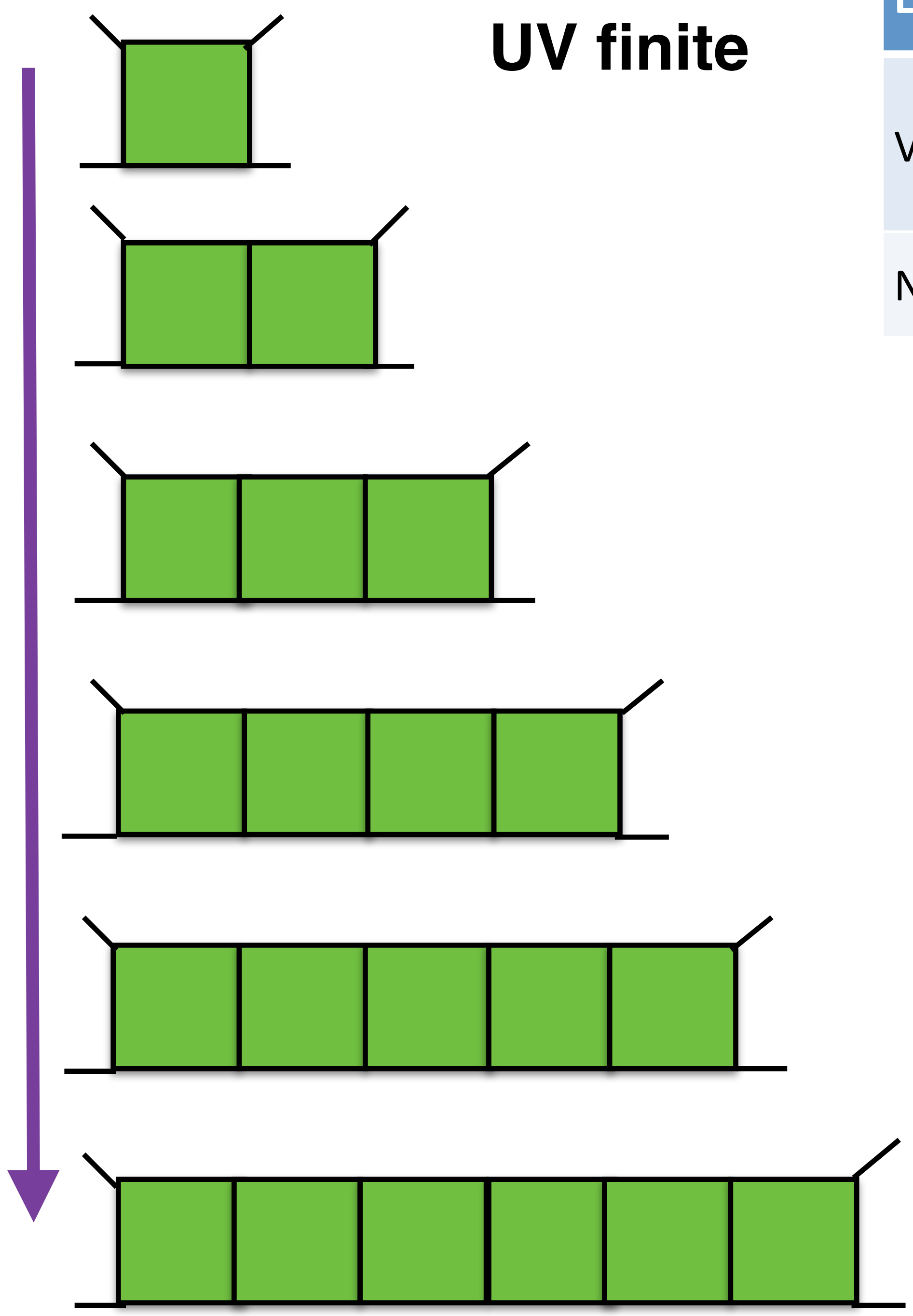

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|---|---|---|---|---|---|---|
| Values | $\frac{\pi^2}{2}$ | $\frac{\pi^2}{3}$ | $-\pi^2 + \frac{31\pi^6}{1890} - 8\zeta_3 + 4\zeta_3^2$ | | | |
| Numerics | | | | | | |

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Kazakov, 14

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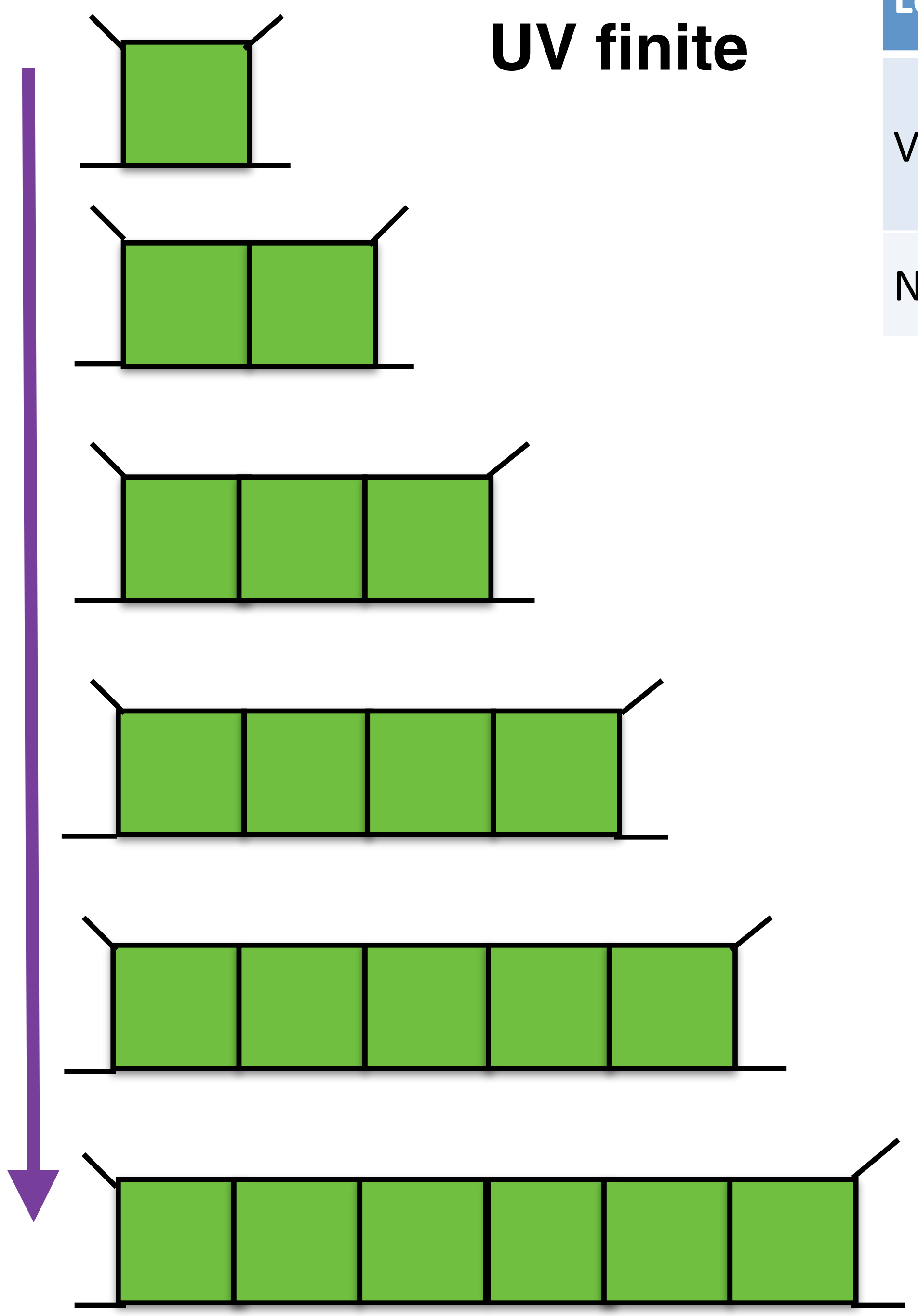

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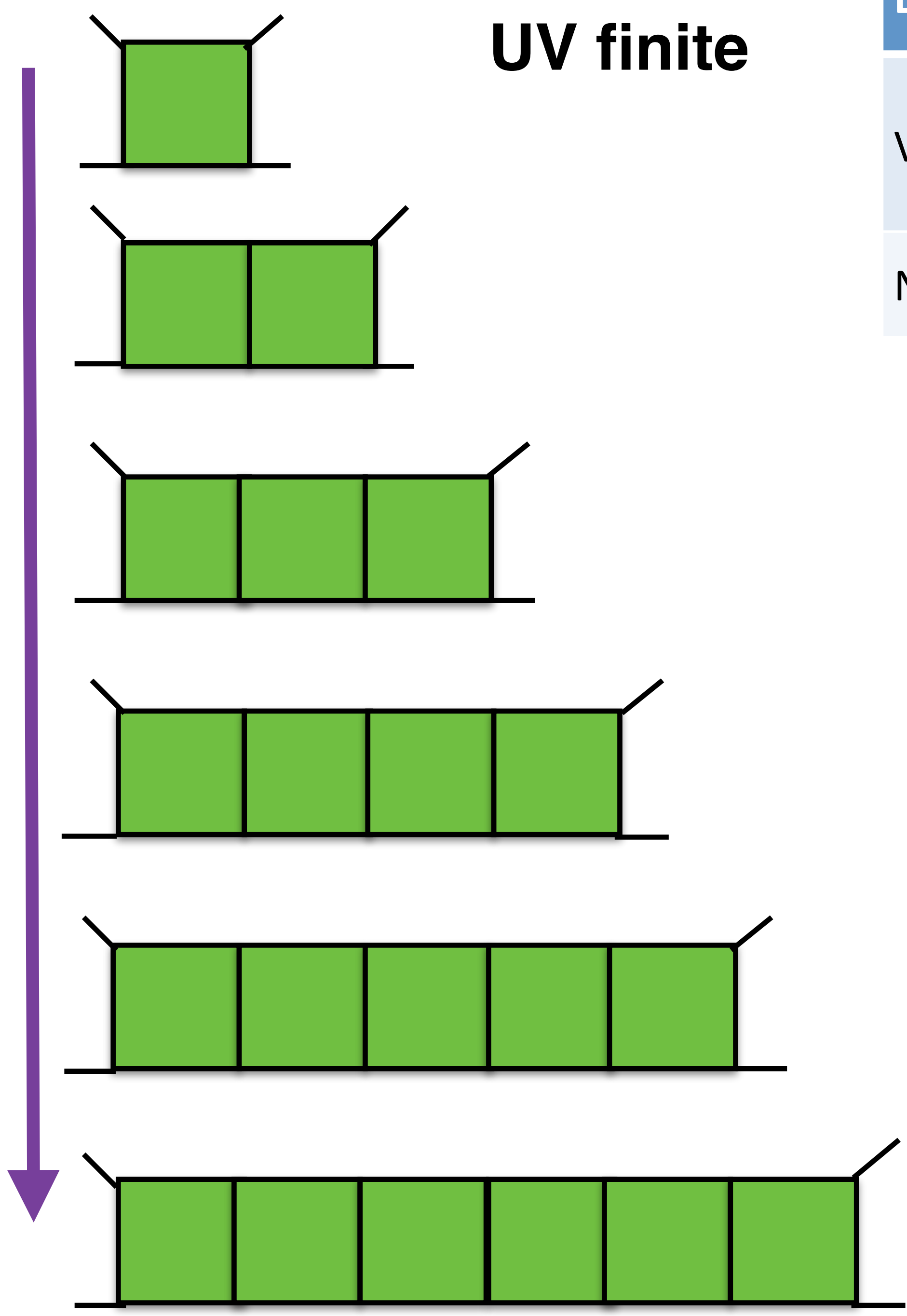

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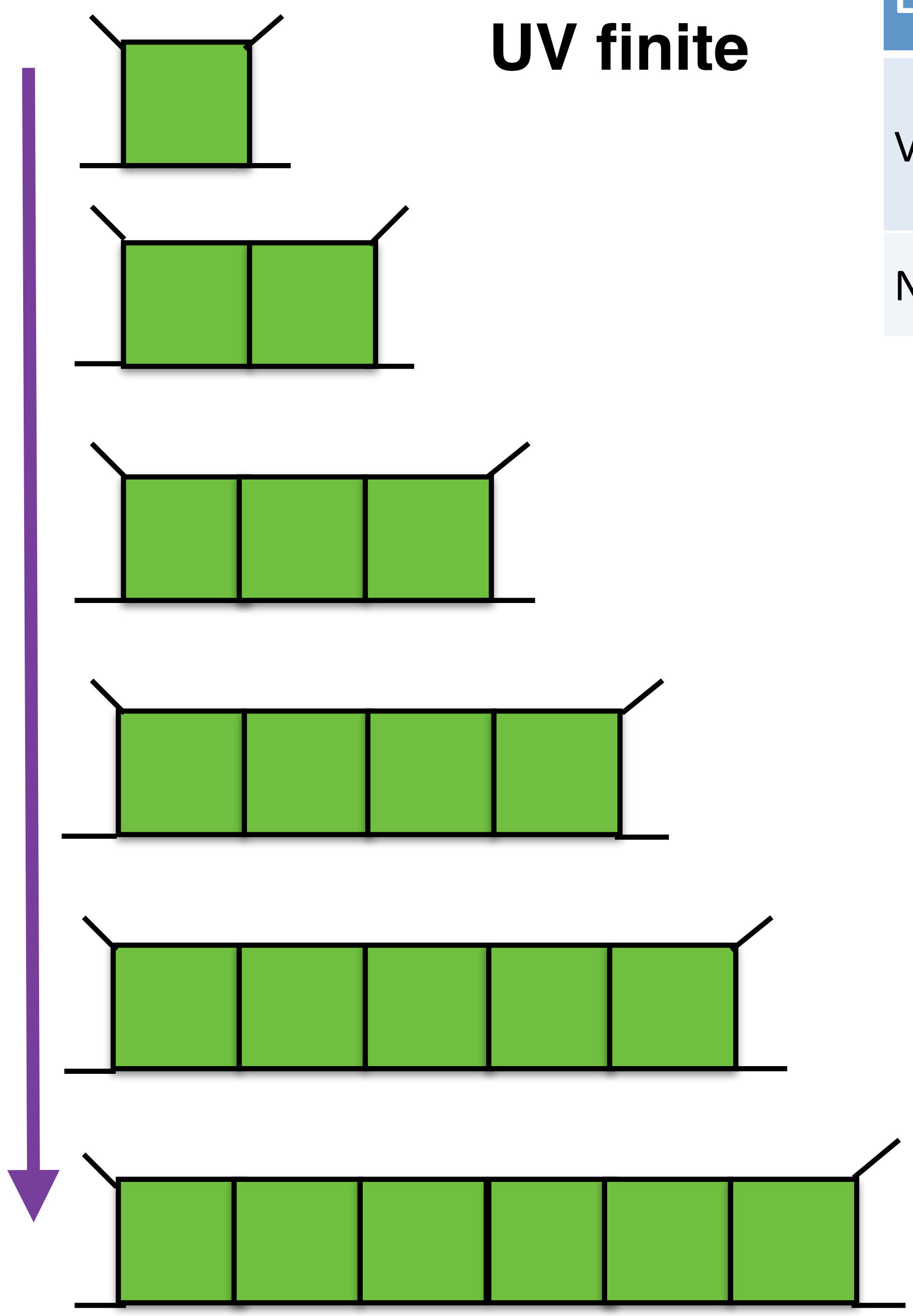

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| Numerics | 4.93 | 3.29 | 2.06 | | | |

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Kazakov, 14

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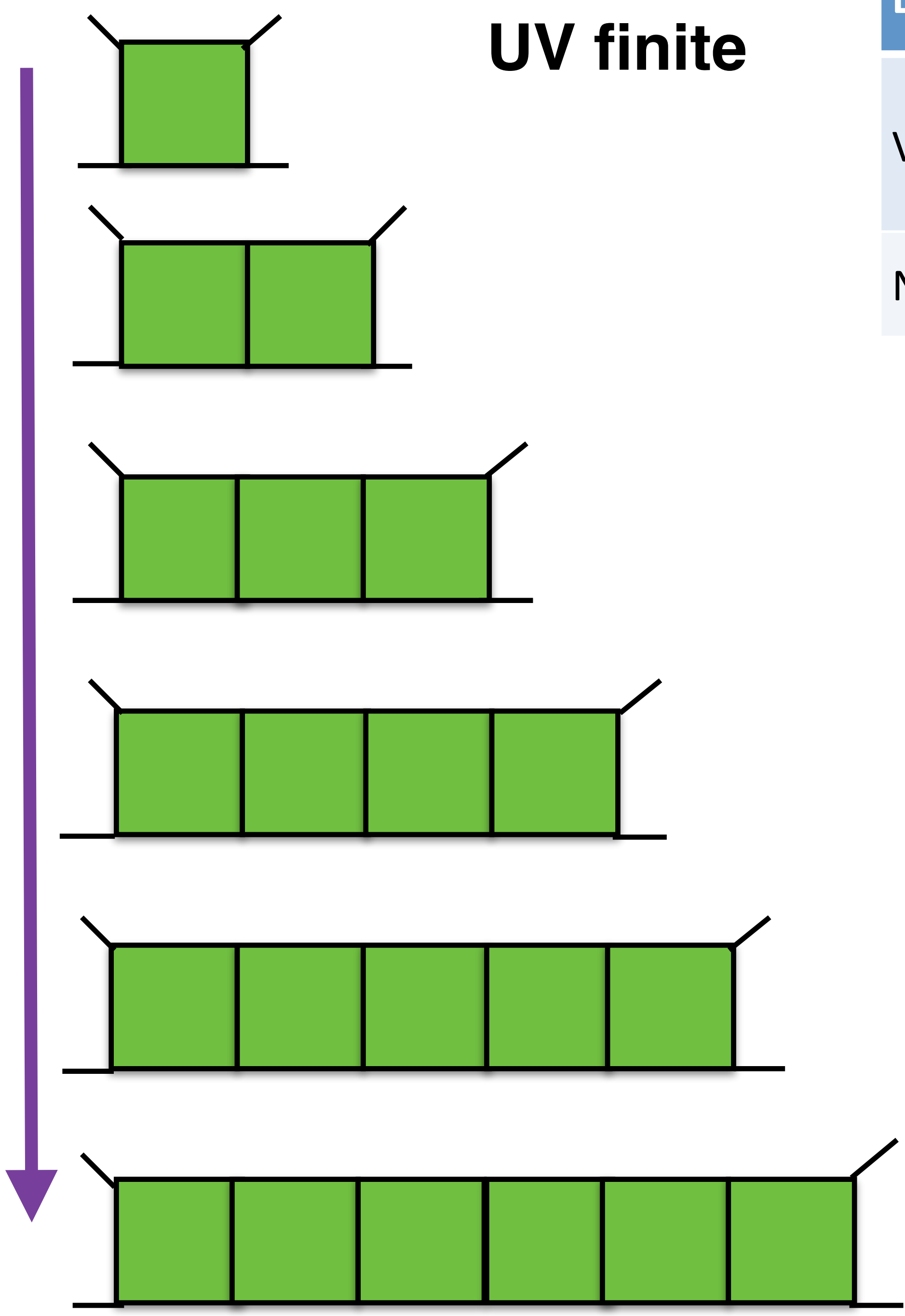

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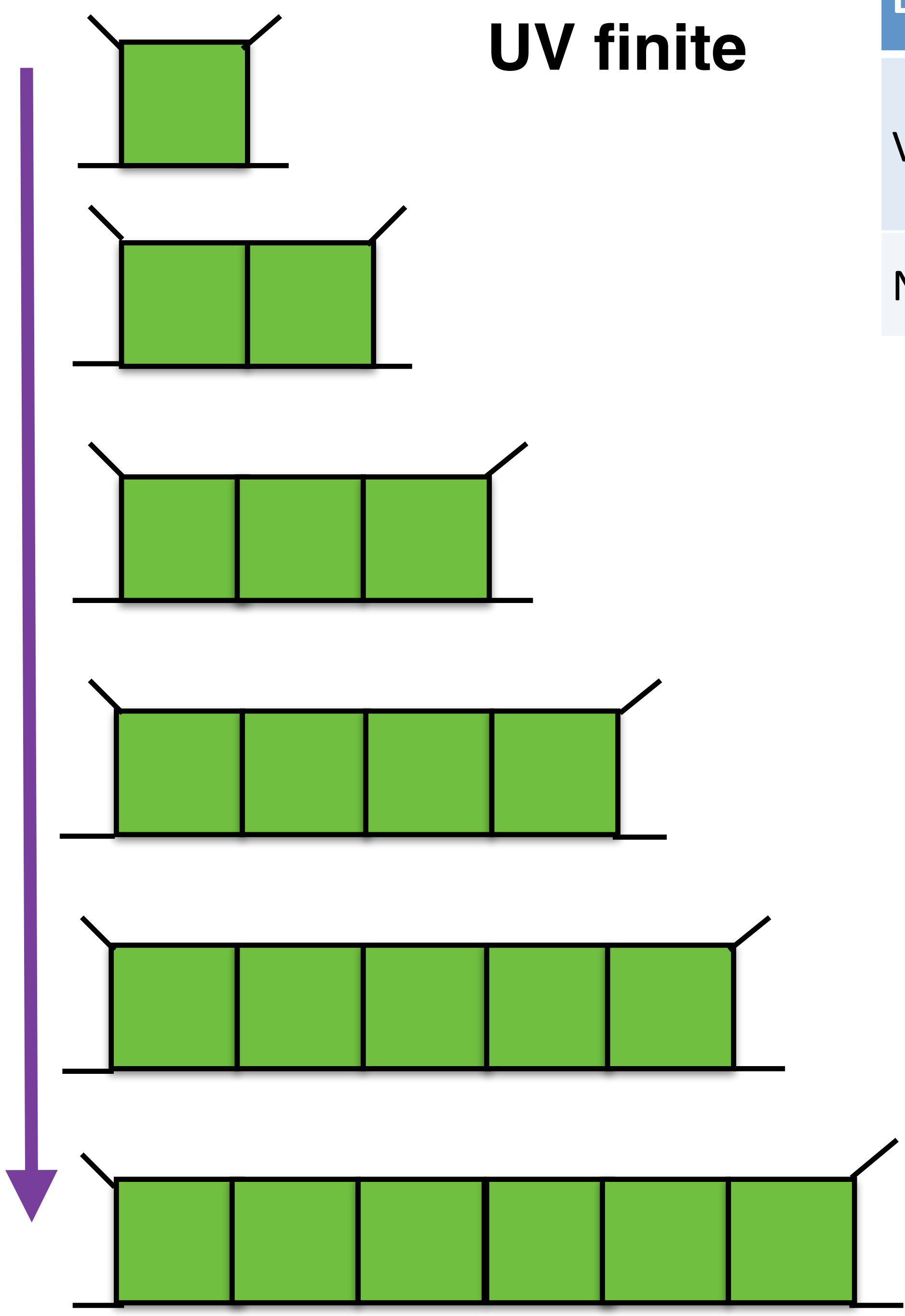


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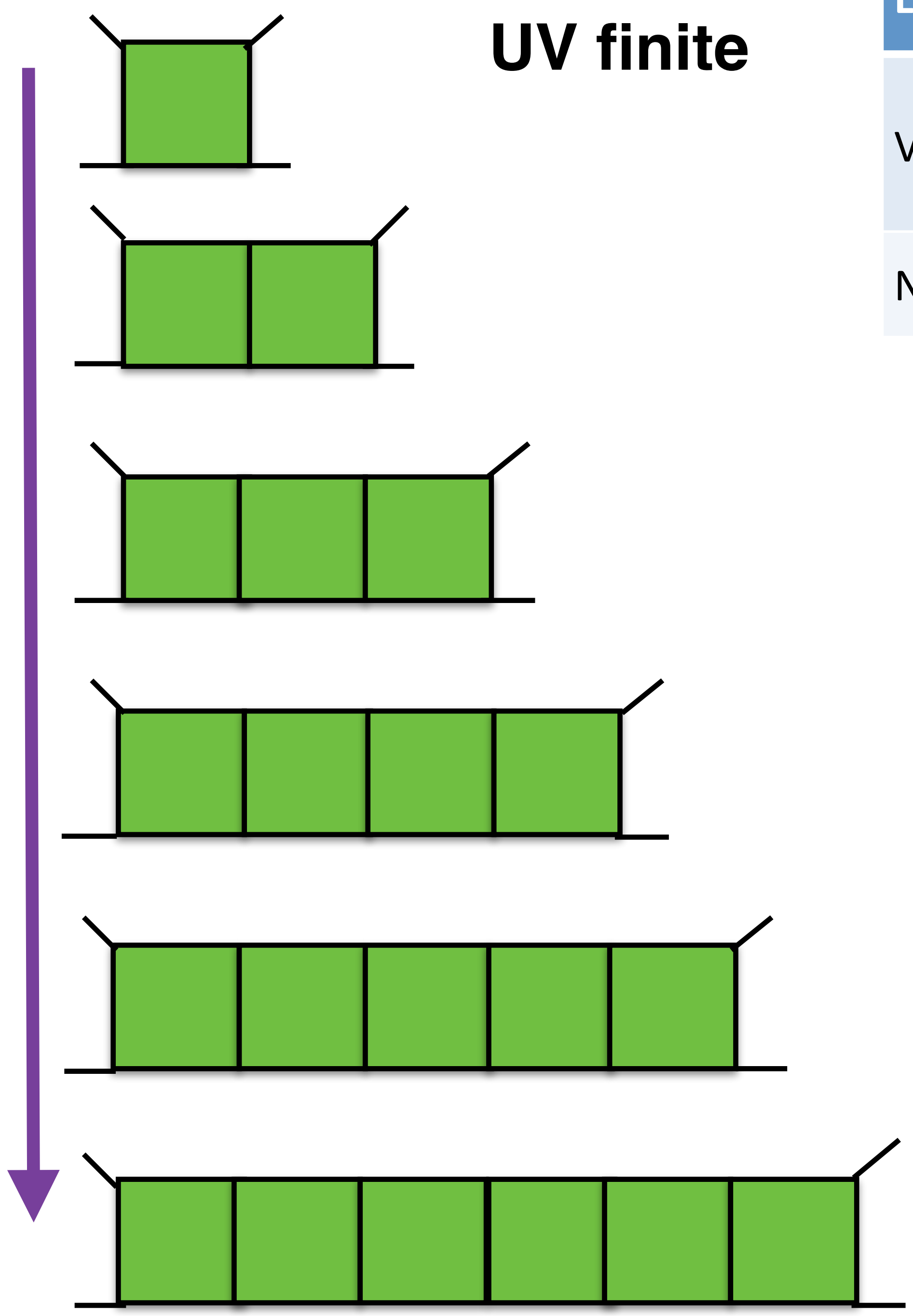

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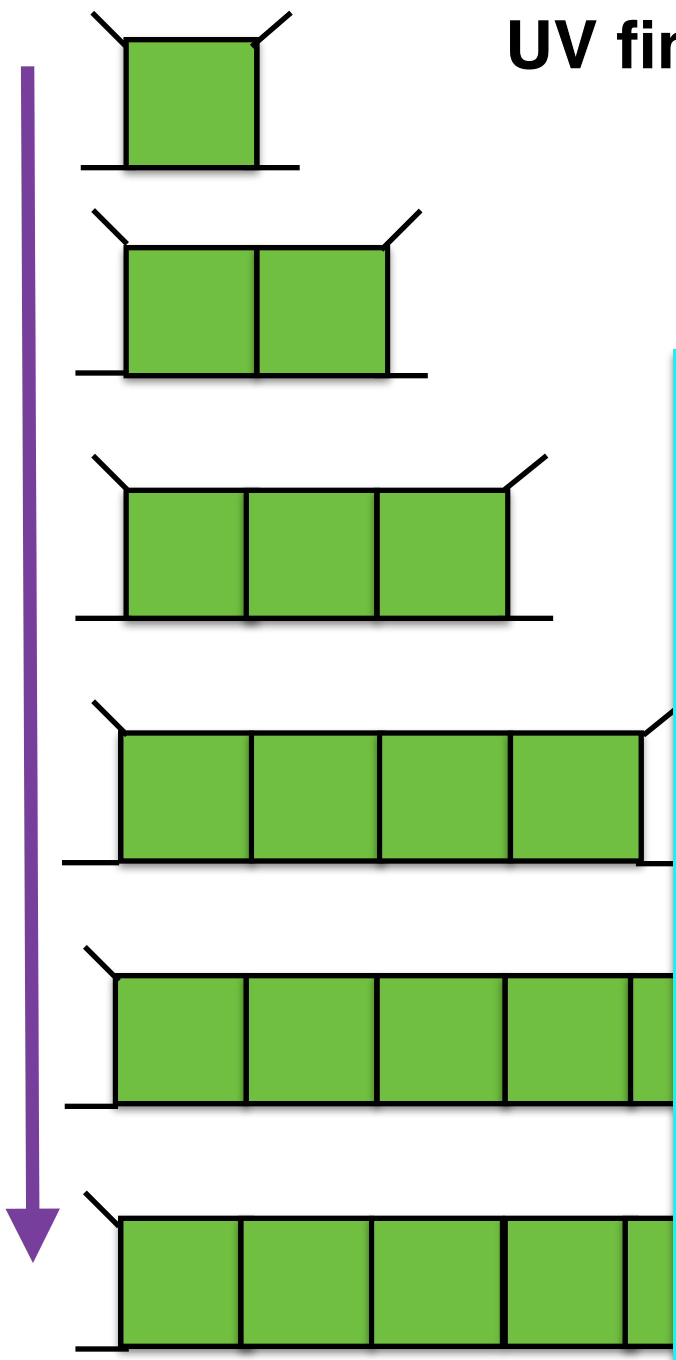

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Panzer,14

$$c_2 = 2\zeta_2,$$

$$c_3 = 4\zeta_3^2 + \tfrac{124}{35}\zeta_2^3 - 8\zeta_3 - 6\zeta_2,$$

$$c_4 = -56\zeta_7 - 32\zeta_2\zeta_5 + 32\zeta_3^2 + \tfrac{8}{5}\zeta_3\left(4\zeta_2^2 - 15\right) + \tfrac{992}{35}\zeta_2^3 - 8\zeta_2^2 - 18\zeta_2,$$

$$c_5 = 56\zeta_7(\zeta_3 - 5) + 26\zeta_5^2 + 4\zeta_5(8\zeta_2\zeta_3 + 35\zeta_3 - 40\zeta_2 - 49) + \tfrac{4}{5}\zeta_3^2\left(140 - 25\zeta_2 - 4\zeta_2^2\right)$$
$$+ 8\zeta_3\left(7\zeta_2 + 4\zeta_2^2 - 14\right) - \tfrac{1168}{385}\zeta_2^5 - \tfrac{24}{7}\zeta_2^4 + \tfrac{496}{5}\zeta_2^3 + 4\zeta_2\left(2\zeta_{3,5} - 21\right) + 20\zeta_{3,5} + 4\zeta_{3,7},$$

$$c_6 = \tfrac{18864}{35}\zeta_2^3 + 336\zeta_{3,5} - 12\zeta_9(20\zeta_2 + 161) + \tfrac{8}{5}\zeta_7\left(104\zeta_2^2 + 35\zeta_2 + 840\zeta_3 - 1120\right)$$
$$+ 624\zeta_5^2 + \tfrac{16}{35}\zeta_5\left(1680\zeta_2\zeta_3 - 3675 - 12\zeta_2^3 - 2240\zeta_2 + 490\zeta_2^2 + 5145\zeta_3\right)$$
$$+ 96\left(\zeta_2^2 + \zeta_{3,7}\right) - \tfrac{48}{5}\zeta_3^2\left(35\zeta_2 + 8\zeta_2^2 - 60\right) - \tfrac{32}{5}\zeta_3\left(105 - 32\zeta_2^2 + 3\zeta_2^3 - 75\zeta_2\right)$$
$$+ 24\zeta_2\left(8\zeta_{3,5} - 21\right) - \tfrac{28032}{385}\zeta_2^5 - \tfrac{288}{5}\zeta_2^4 - 1320\zeta_{11}.$$

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Kazakov, 14

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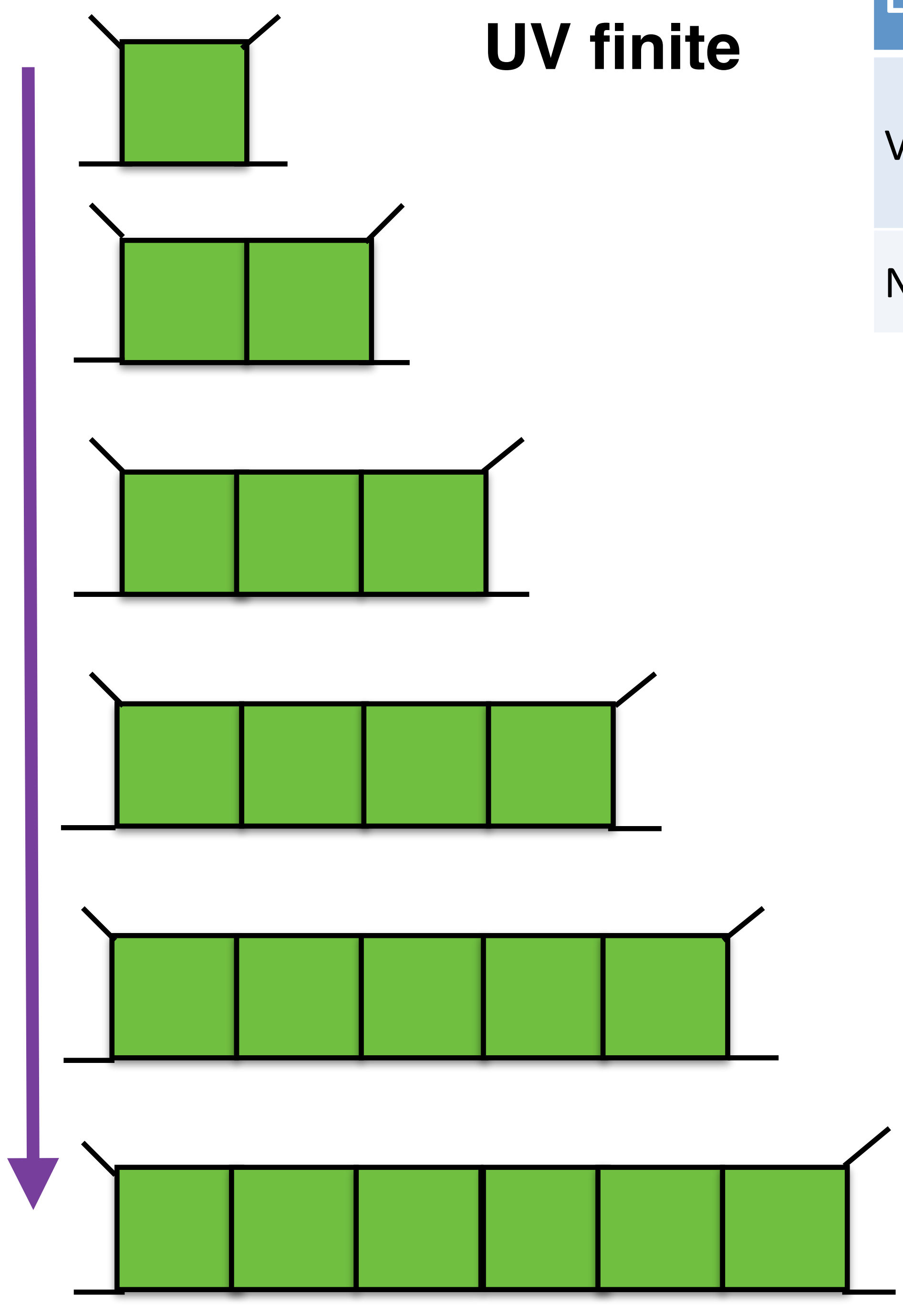

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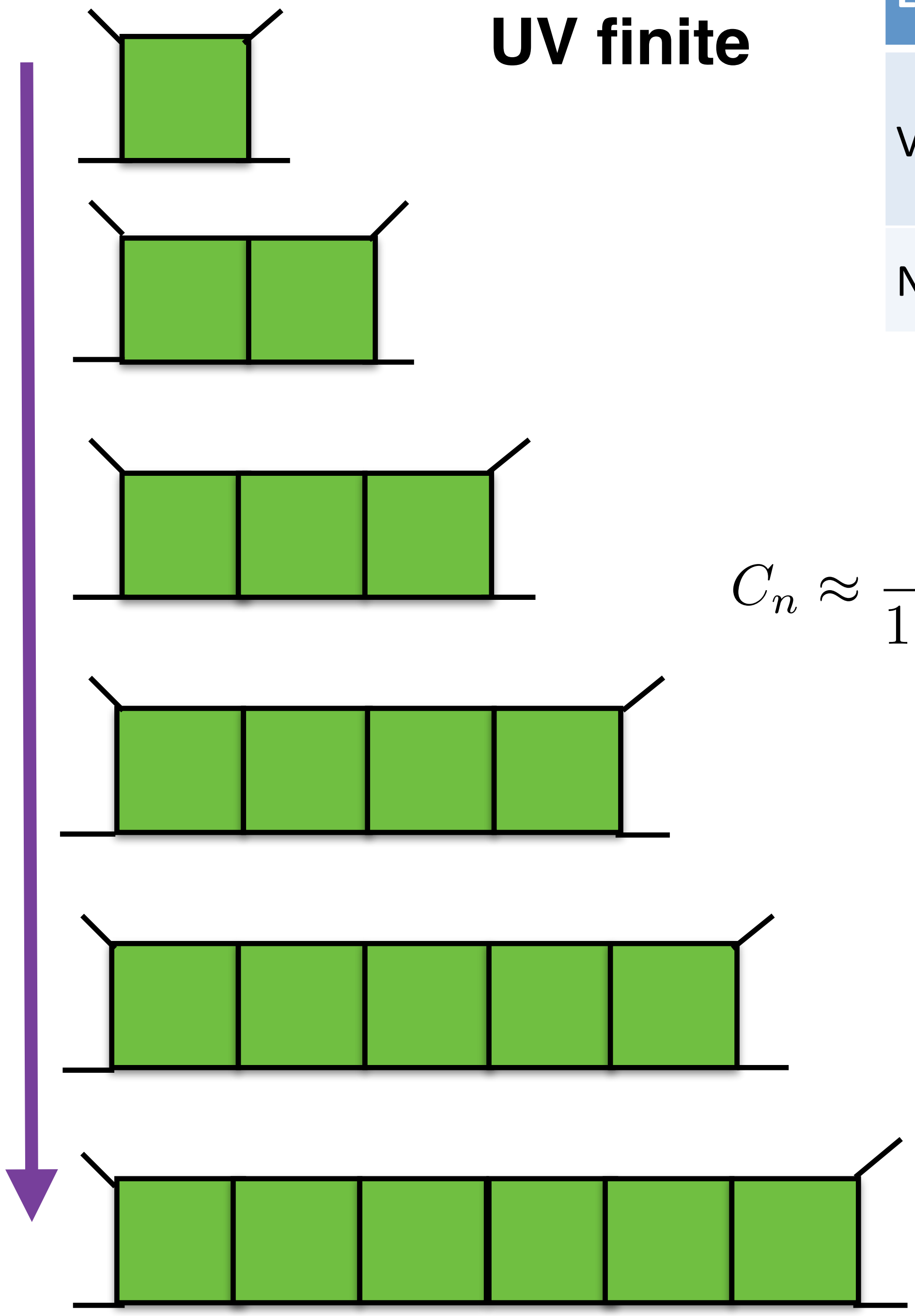

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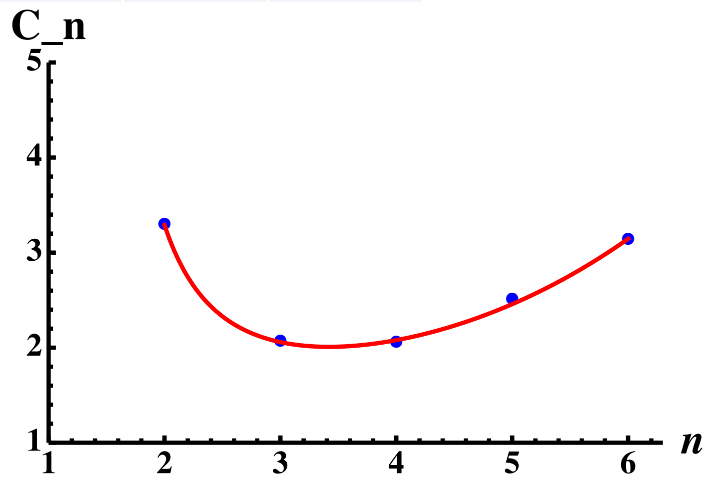

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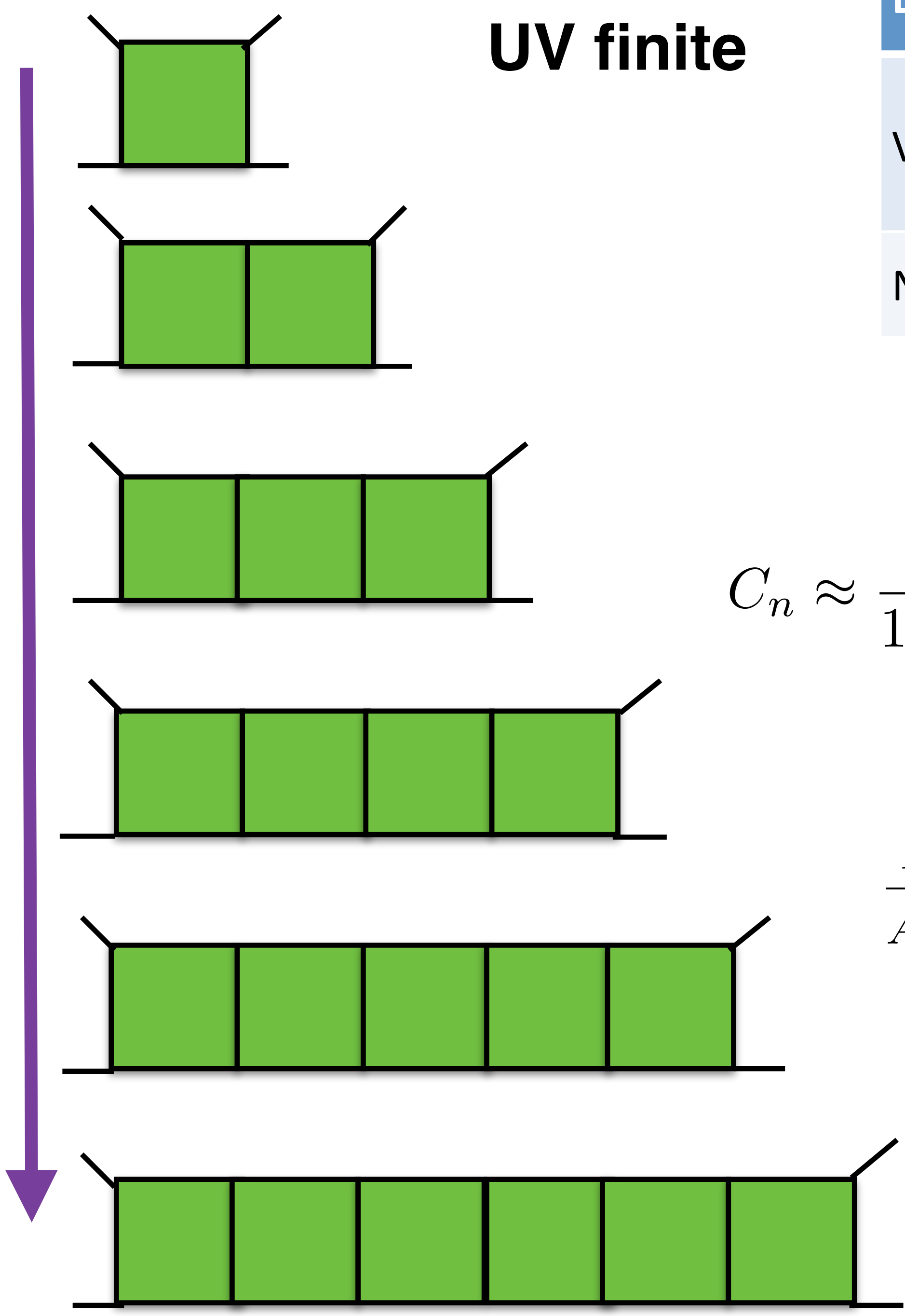

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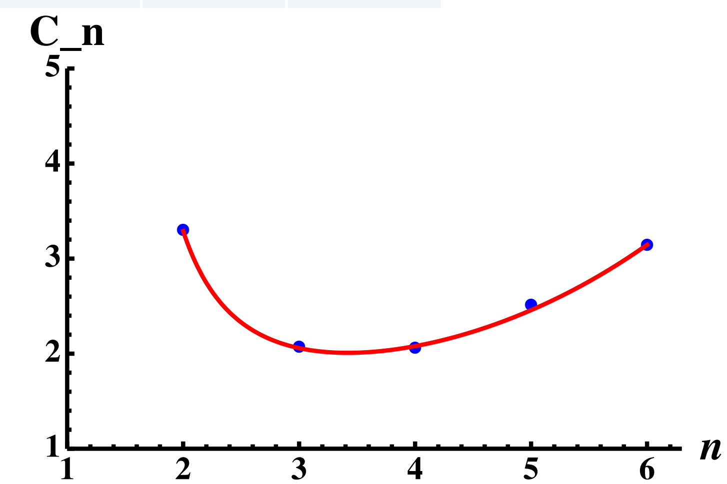


$$\left.\frac{A_4}{A_4^{(0)}}\right|_{L.P.} \approx -\frac{g^2 t}{2}\left[\frac{\pi^2}{2} - \sum_{n=2}^{\infty} 0.76\frac{(-g^2 s/2)^{n-1}(\pi^2/6)^n}{n-4/3}\right]$$

$$= -\frac{g^2 t}{2}\frac{\pi^2}{2}\left[1 - 0.76\frac{g^2 s\pi^2}{24}\,{}_2F_1(1,\frac{2}{3},\frac{5}{3},-\frac{g^2 s\pi^2}{12})\right] \approx -\frac{g^2 t}{2}\frac{\pi^2}{2}\left[1 - \left(\frac{g^2 s}{2}\right)^{1/3}\right].$$

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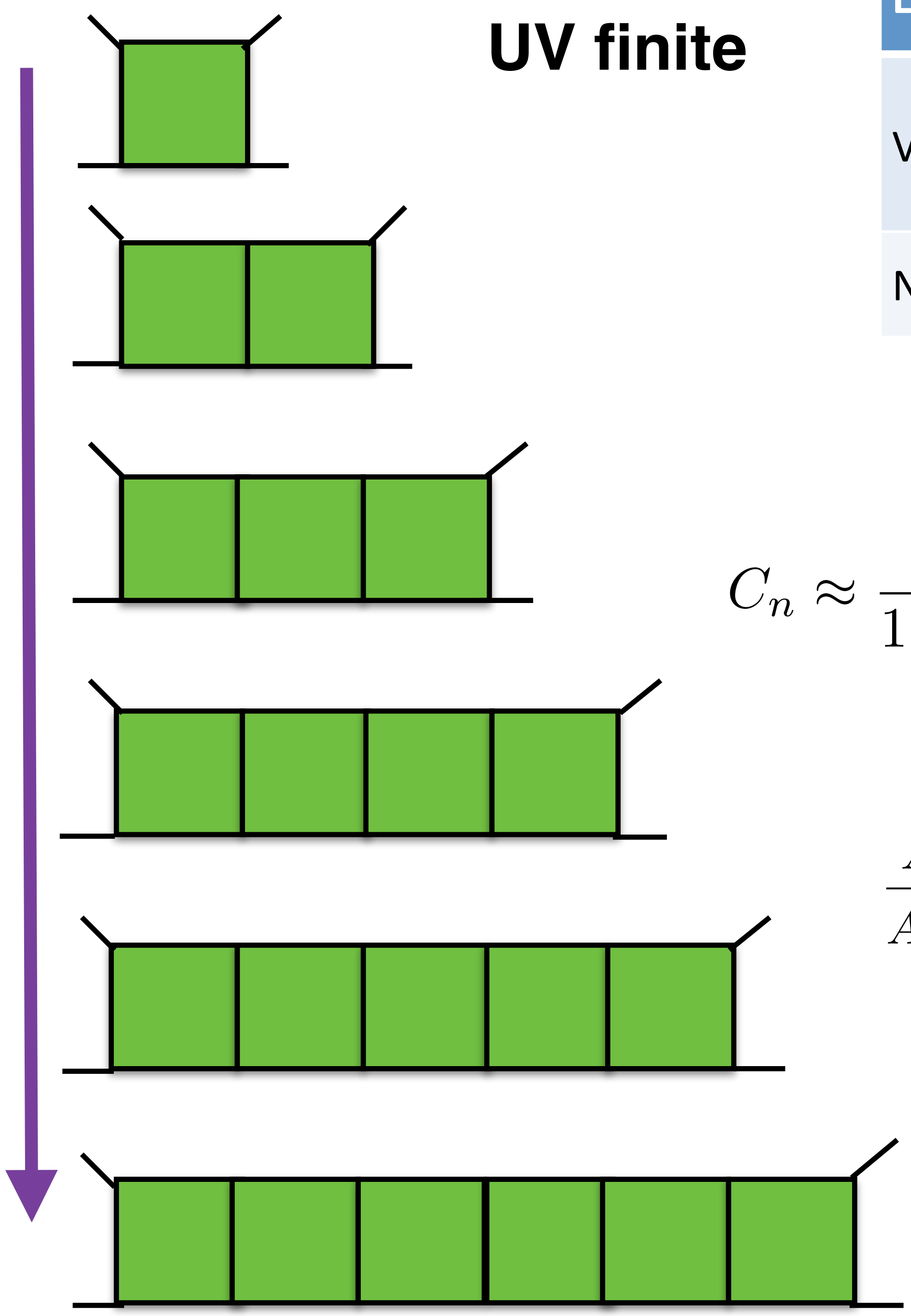

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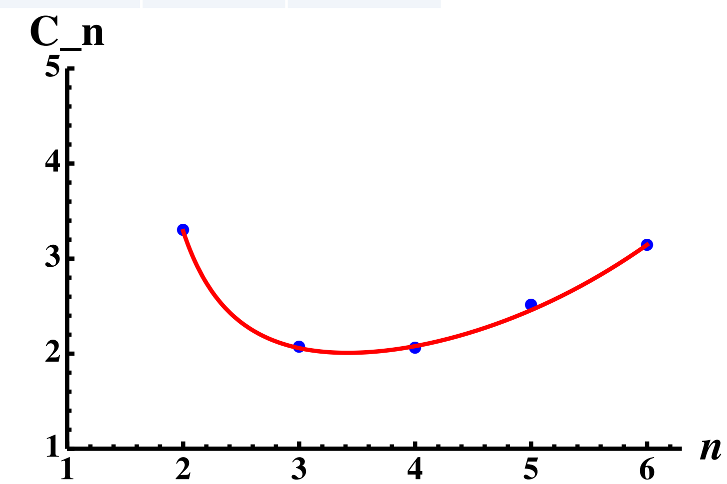


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!

# Perturbation Expansion for the Amplitudes

## Leading Divergences

| Loops | Combinatorics | Divergence |
|---|---|---|
| 3 | $(-g^2 s/2)^3 \quad 2t/s$ | $1/6\epsilon$ |
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**Geom progression !?**

$$\frac{A_4}{A_4^{(0)}}\Bigg|_{Leading\ Div.} = 2\frac{t}{s}\sum_{n=1}^{\infty}\left(-\frac{g^2 s}{2}\right)^{n+2}\left(\frac{1}{6\epsilon}\right)^{n} = 2\frac{t}{s}\left(-\frac{g^2 s}{2}\right)^{2}\frac{\frac{-g^2 s}{12\epsilon}}{1+\frac{g^2 s}{12\epsilon}}$$

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$\frac{1}{\epsilon}$

$\frac{1}{\epsilon^2}$

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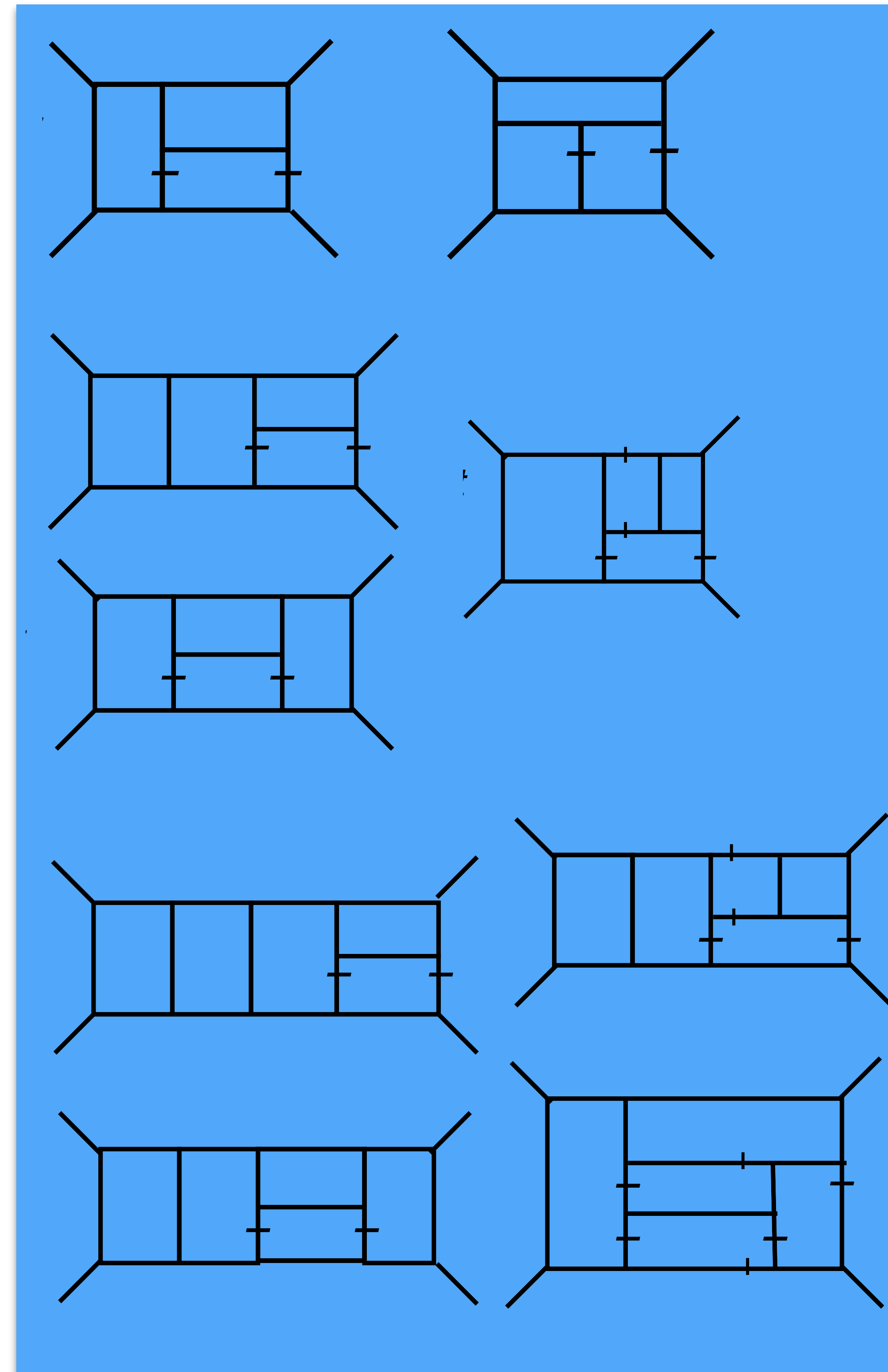

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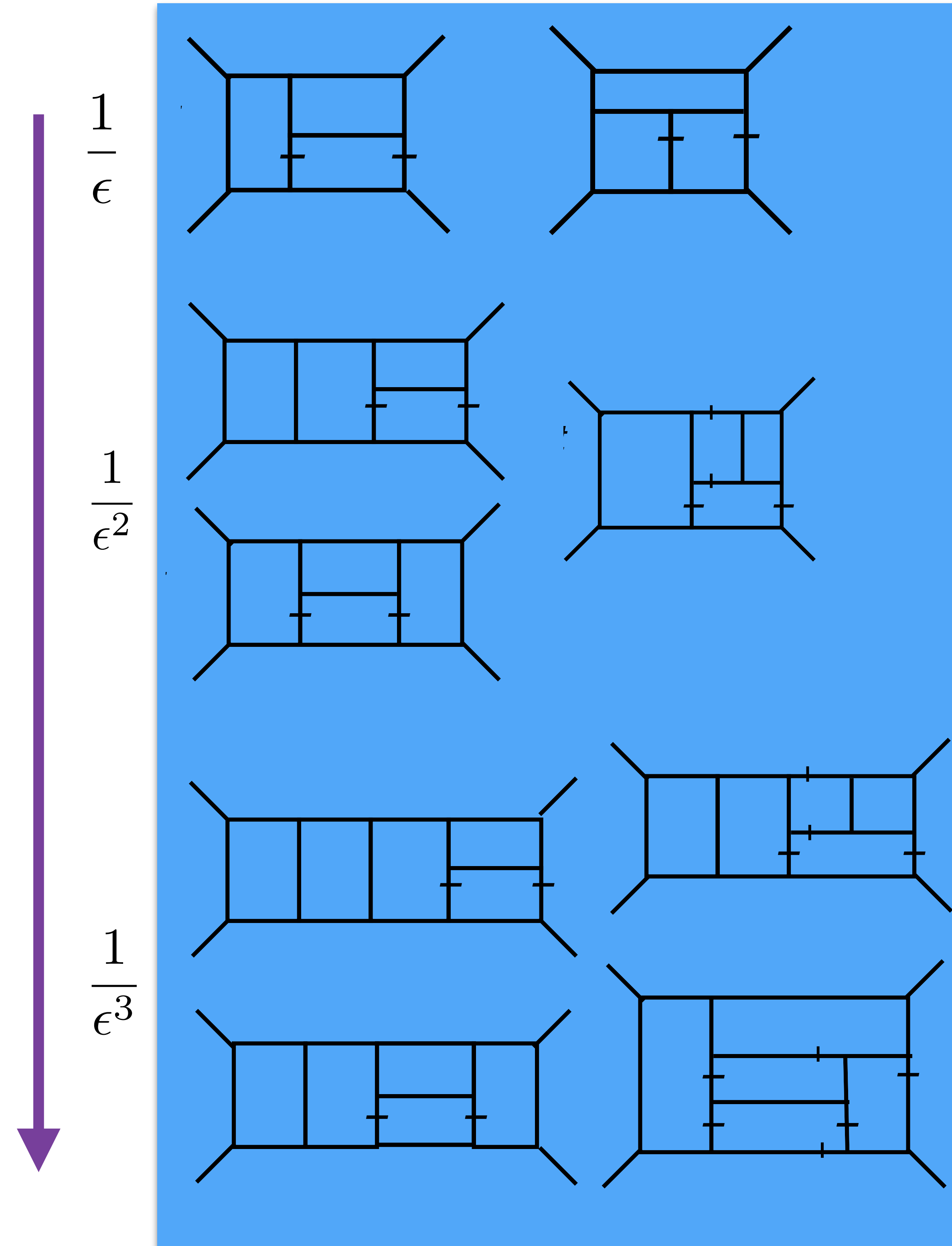


**In the limit ε->0 the full expression is FINITE !**

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- **The result of an alternative approach might be quite different from the PT one.**