From Amplitudes to Form Factors in N=4 SYM Theory

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2. Gluon scattering amplitudes
   - PT Weak coupling case: All loop result
3. Formfactors of Gauge Invariant Operators
4. Cancellation of IR Divergences
   - Virtual Corrections
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N=4 Super Yang-Mills Theory

- $\mathcal{N} = 4$ Super Yang-Mills theory is the most supersymmetric theory without gravity
- Field content: 1 massless gauge boson, 4 massless (Majorana) spin 1/2 fermions, 6 real (or 3 complex) massless spin 0 bosons All fields are in adjoint representation of the gauge group (Take $SU(N_c)$)
- There is only one coupling for gauge, Yukawa and scalar interactions.
- The theory is exactly scale invariant, conformal field theory at quantum level, i.e. the β function identically vanishes at all orders of PT

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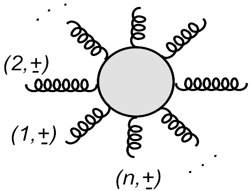

All outgoing gluons with helicity + or -
on mass shell
In the leading N_c order (planar limit)

- Colour decomposition of amplitudes in N=4 SYM theory for $N_c \to \infty$

$$\mathcal{A}_n^{(l)} = g^{n-2}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \sum_{perm} Tr(T^{a_{\sigma(1)}}, ..., T^{a_{\sigma(n)}}) A_n^{(l)}(a_{\sigma(1)}, ..., a_{\sigma(n)}),$$

where $\mathcal{A}_n$ - physical amplitude, A_n - partial amplitude, a_i - is color index of $i-th$ external "gluon"

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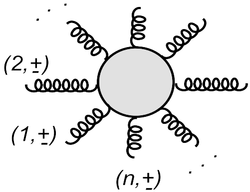

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Perturbation theory

- Bern, Dixon & Smirnov's conjecture: $M_n^{(L)}(\varepsilon) \equiv A_n^{(L)}/A_n^{(0)}$

$$\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^L M_n^{(L)}(\varepsilon) = exp\left[\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(f^{(l)}(\varepsilon) M_n^{(1)}(l\varepsilon) + C^{(l)} + E_n^{(l)}(\varepsilon)\right)\right]$$

$$f^{(l)}(\varepsilon) = f_0^{(l)}(\varepsilon) + \varepsilon f_1^{(l)}(\varepsilon) + \varepsilon^2 f_2^{(l)}(\varepsilon)$$

- $$\mathcal{M}_n(\varepsilon) = exp\left[-\frac{1}{4}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(\frac{\gamma_K^{(l)}}{(l\varepsilon)^2} + \frac{2G_0^{(l)}}{l\varepsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\varepsilon} + \frac{1}{4}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \gamma_K^{(l)} F_n^{(1)}(0)\right]$$

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- Cusp anomalous dimension appears in RG eq. for the expectation value of a Wilson line with a cusp

  Loop expansion $$\gamma_K = \sum_{l=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^l \gamma_K^{(l)}$$

  $$\gamma_K^{(1)} = 4, \quad \gamma_K^{(2)} = -8\zeta_2, \quad \gamma_K^{(3)} = 88\zeta_4, \ldots$$

- Classical solution (Alday & Maldacena) for the scattering amplitude

  $$\gamma_K(g^2) \sim \frac{\sqrt{g^2 N_c}}{\pi}, \quad G_0(g^2) \sim \sqrt{g^2 N_c}\frac{1-\log 2}{2\pi}, \quad for\ g^2 N_c \to \infty$$

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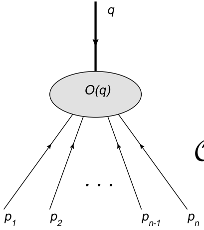
Formfactors of gauge invariant operators

Local operator

$$F(p_1^{\lambda_1}, p_2^{\lambda_2}, ..., p_n^{\lambda_n}) = < 0|\mathcal{O}|p_1^{\lambda_1} p_2^{\lambda_2} ... p_n^{\lambda_n} >$$

Using $\mathcal{N} = 1$ superfield notation for chiral and vector fields

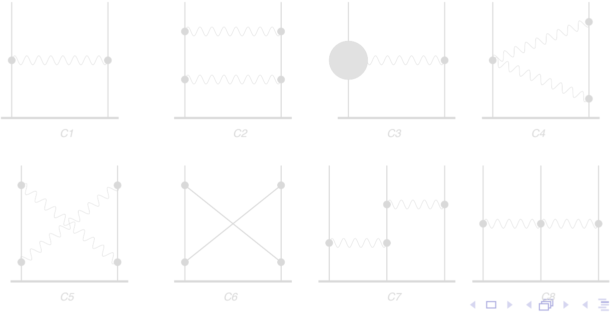


$$\begin{aligned}
\mathcal{C}_{IJ} &= Tr(\Phi_I \Phi_J), I \neq J \\
\mathcal{V}_I^J &= Tr(e^{-gV} \bar{\Phi}^J e^{gV} \Phi_I), I \neq J \\
\mathcal{O}_I^{(n)} &= Tr((\Phi_I)^n), \\
\mathcal{K} &= \sum_I Tr(e^{-gV} \bar{\Phi}^I e^{gV} \Phi_I).
\end{aligned}$$

Formfactor diagrams

- One-loop diagrams in $\mathcal{N}=1$ superspace

- Two-loop diagrams in $\mathcal{N}=1$ superspace



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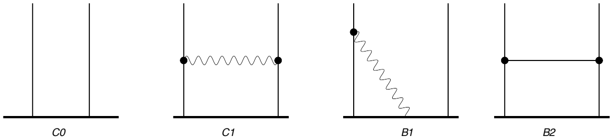

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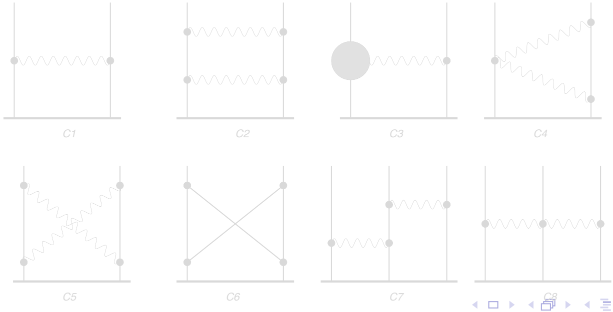

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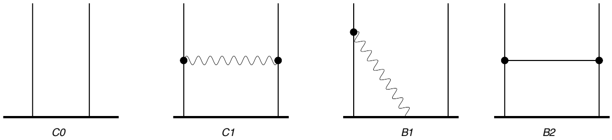

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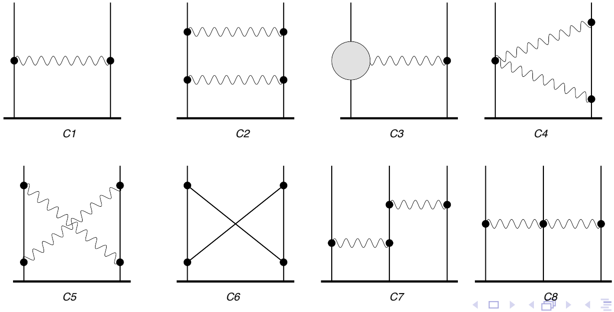

Exponentiation of IR divergencies in 1 and 2 loops

$$\mathcal{F}(p_1 \dots p_n) = \mathcal{F}_{tree}(p_1 \dots p_n)(1 + \text{loops})$$

$$\mathcal{M} = \frac{\mathcal{F}}{\mathcal{F}_{tree}} = (1 + loops) = \sum_{l=0} (g^2 N_c)^l \mathcal{M}^{(l)}.$$

$$\mathcal{M} = Exp\left[\sum_{i=1}^{2} \frac{1}{2}\left(\hat{M}(s_{i,i+1}/\mu^2)\right) + O(\epsilon)\right](1 + finite).$$

$$\hat{M}(s_{i,i+1}/\mu^2) = -\frac{1}{2}\sum_{l}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(\frac{\gamma_{cusp}^{(l)}}{(l\epsilon)^2} + \frac{G^{(l)}}{l\epsilon} + C^{(l)}\right)\left(\frac{s_{i,i+1}}{\mu^2}\right)^{l\epsilon}$$

$$s_{i,i+1} = (p_i + p_{i+1})^2$$

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  Collinear anomalous dimension $G^{(1)} = 0, G^{(2)} = -\zeta(3)$
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- $\mathcal{O}^{(n)}, n \geq 3$
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  finite$\neq 0$ and in general DOES NOT exponentiate
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- In collinear limit ($s_{23} \to 0$)

$$\log(\mathcal{M}) = \left\{ \sum_{i=1}^{2} (\frac{g^2 N_c}{16\pi^2}) \left(\frac{\mu^2}{s_{i\ i+1}}\right)^{\epsilon} \left(-\frac{1}{\epsilon^2} + \frac{\zeta_2}{2}\right) + \sum_{i=1}^{n} (\frac{g^2 N_c}{16\pi^2})^2 (\frac{\mu^2}{s_{i\ i+1}})^{2\epsilon} \left(\frac{42\zeta_2 + 3\log^2\frac{s_{12}}{s_{13}}}{96\epsilon^2} + \frac{23\zeta_3}{8\epsilon} - \frac{1}{2880}\left(75\log^4\frac{s_{12}}{s_{13}} + 120\pi^2\log^2\frac{s_{12}}{s_{13}} - 317\pi^4\right)\right)\right\}.$$

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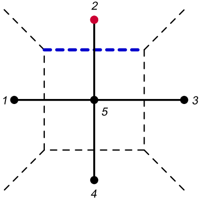
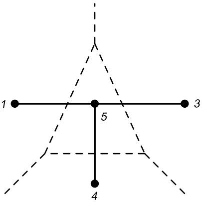
Dual conformal invariance

Drummond, Henn, Smirnov and Sokatchev

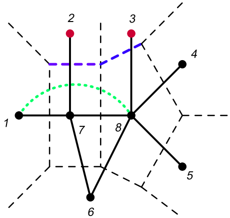


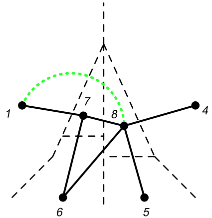
b)



c)

$$C^{1-loop} = \lim_{x_2 \to \infty} x_{12}^2 x_{34}^2 \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \int \frac{d^4 x_5 \; x_{34}^2}{x_{15}^2 x_{35}^2 x_{45}^2} = \Phi\left(\frac{x_{34}^2}{x_{13}^2}, \frac{x_{14}^2}{x_{13}^2}\right)$$





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From partial amplitudes to cross-sections

To obtain the cross sections from partial amplitudes one has to compute the square of them. In the the planar limit it is just:

$$\Phi_n(p_1^\pm, ..., p_n^\pm) = g^{2n-4}(\frac{g^2 N_c}{16\pi^2})^{2l} \sum_{colors} \mathcal{A}_n^{(l)} \mathcal{A}_n^{(l)*} =$$

$$2g^{2n-4} N_c^{n-2}(N_c^2 - 1)(\frac{g^2 N_c}{16\pi^2})^{2l} \sum_{perm} |A_n^{(l)}(a_{\sigma(1)}, ..., a_{\sigma(n-1)}, a_n)|^2$$

Then the cross-section is

$$d\sigma_n(p_{in}) = \Phi_n(p_1^\pm, ..., p_n^\pm) d\phi_{n-2},$$

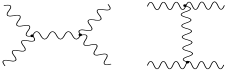
where $d\phi_{n-2}$ is the phase space of the outgoing particles:

$$d\phi_{n-2} \sim \delta^D(p_{in} - p_{fin}) \mathcal{S}_n \prod_{k=1}^{n-2} \delta^+(p_k^2) d^D p_k,$$

$\mathcal{S}_n$ - is the measurement function and integration goes over $D = 4 - 2\varepsilon$ dimensions.

2×2 gluon scattering. Feynman Diagrams

- Tree level



- 1loop

$+$ *permutations*

2×2 gluon scattering. Feynman Diagrams

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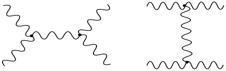
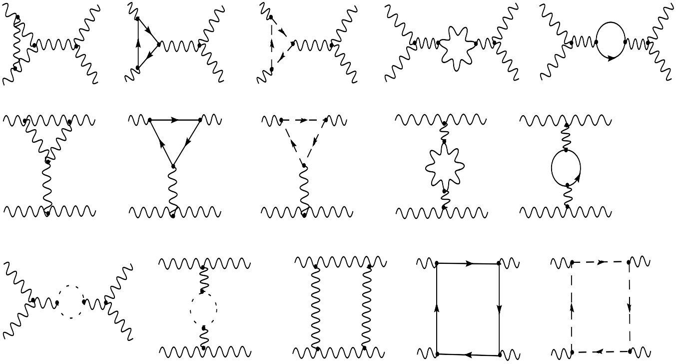


- 1loop



$+$ *permutations*

Virtual Correction (MHV)

- Born Term $\qquad c \equiv \cos\theta$

$$\left(\frac{d\sigma}{d\Omega}\right)_0^{--++} = \frac{\alpha^2 N_c^2}{8E^2}\frac{s^4(s^2+t^2+u^2)}{s^2t^2u^2}\left(\frac{\mu^2}{s}\right)^\varepsilon = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^\varepsilon \frac{3+c^2}{(1-c^2)^2}$$

- Virtual Correction

$$\left(\frac{d\sigma}{d\Omega}\right)_{virt}^{--++} = \frac{\alpha^2 N_c^2}{8E^2}\left(\frac{\mu^2}{s}\right)^\varepsilon \left\{\frac{\alpha N_c}{2\pi}\frac{s^4}{s^2t^2u^2}\left[-\frac{8}{\varepsilon^2}\left(((\frac{\mu^2}{-t})^\varepsilon + (\frac{\mu^2}{-u})^\varepsilon)s^2 \right.\right.\right.$$

$$\left. +((\frac{\mu^2}{s})^\varepsilon + (\frac{\mu^2}{-t})^\varepsilon)u^2 + ((\frac{\mu^2}{s})^\varepsilon + (\frac{\mu^2}{-u})^\varepsilon)t^2\right)$$

$$\left.\left. +\frac{16}{3}\pi^2(s^2+t^2+u^2) + 4(u^2\log^2(\frac{-s}{t}) + t^2\log^2(\frac{-s}{u}) + s^2\log^2(\frac{t}{u}))\right]\right\}$$

$$= \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\varepsilon}\left\{\frac{\alpha N_c}{2\pi}\left[-\frac{16}{\varepsilon^2}\frac{3+c^2}{(1-c^2)^2} + \frac{4}{\varepsilon}\left(\frac{5+2c+c^2}{(1-c^2)^2}\log(\frac{1-c}{2})\right.\right.\right.$$

$$\left.\left.\left. +(c\leftrightarrow -c)\right) + \frac{16(3+c^2)\pi^2}{3(1-c^2)^2} - \frac{16}{(1-c^2)^2}\log(\frac{1-c}{2})\log(\frac{1+c}{2})\right]\right\}$$

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- Virtual Correction

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$$= \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\varepsilon}\left\{\frac{\alpha N_c}{2\pi}\left[-\frac{16}{\varepsilon^2}\frac{3+c^2}{(1-c^2)^2} + \frac{4}{\varepsilon}\left(\frac{5+2c+c^2}{(1-c^2)^2}\log\left(\frac{1-c}{2}\right) + (c\leftrightarrow -c)\right) + \frac{16(3+c^2)\pi^2}{3(1-c^2)^2} - \frac{16}{(1-c^2)^2}\log\left(\frac{1-c}{2}\right)\log\left(\frac{1+c}{2}\right)\right]\right\}$$

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$$\left.+((\frac{\mu^2}{s})^{\varepsilon}+(\frac{\mu^2}{-t})^{\varepsilon})u^2+((\frac{\mu^2}{s})^{\varepsilon}+(\frac{\mu^2}{-u})^{\varepsilon})t^2\right)$$
$$\left.\left.+\frac{16}{3}\pi^2(s^2+t^2+u^2)+4(u^2\log^2(\frac{-s}{t})+t^2\log^2(\frac{-s}{u})+s^2\log^2(\frac{t}{u}))\right]\right\}$$

$$= \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\varepsilon}\left\{\frac{\alpha N_c}{2\pi}\left[-\frac{16}{\varepsilon^2}\frac{3+c^2}{(1-c^2)^2}+\frac{4}{\varepsilon}\left(\frac{5+2c+c^2}{(1-c^2)^2}\log(\frac{1-c}{2})\right.\right.\right.$$
$$\left.\left.\left.+(c\leftrightarrow -c)\right)+\frac{16(3+c^2)\pi^2}{3(1-c^2)^2}-\frac{16}{(1-c^2)^2}\log(\frac{1-c}{2})\log(\frac{1+c}{2})\right]\right\}$$

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$$\left.+((\frac{\mu^2}{s})^{\varepsilon}+(\frac{\mu^2}{-t})^{\varepsilon})u^2+((\frac{\mu^2}{s})^{\varepsilon}+(\frac{\mu^2}{-u})^{\varepsilon})t^2\right)$$
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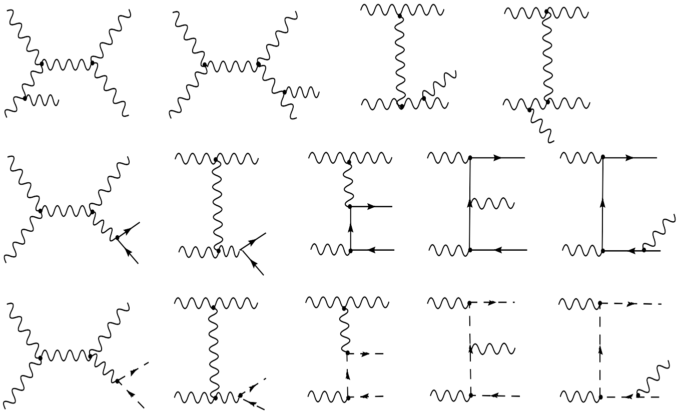
$$\left.\left.\left.+(c\leftrightarrow -c)\right)+\frac{16(3+c^2)\pi^2}{3(1-c^2)^2}-\frac{16}{(1-c^2)^2}\log(\frac{1-c}{2})\log(\frac{1+c}{2})\right]\right\}$$

2 × 3 gluon scattering. Feynman Diagrams

- Tree level (components)



Real Emission

- MHV

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+++)}_{Born} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\varepsilon}\frac{\alpha N_c}{2\pi}\left\{\frac{1}{\varepsilon^2}\frac{8(3+c^2)}{(1-c^2)^2}\right.$$
$$+\frac{1}{\varepsilon}\left[\frac{2}{(1+c)^2}\log(\frac{1-c}{2})+\frac{2}{(1-c)^2}\log(\frac{1+c}{2})+\frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2}\right.$$
$$\left.\left.+\frac{12(3+c^2)}{(1-c^2)^2}\log\frac{(1-\delta)}{\delta}\right]+\text{Finite part}\right\}$$

- Fermions

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+\bar{q}q)}_{Real} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\varepsilon}\frac{\alpha N_c}{2\pi}\left\{\frac{16}{\varepsilon}\left[\frac{(79-25c^2)}{3(1-c^2)^2)}\right.\right.$$
$$\left.\left.+\frac{2(3-c)^2}{(1-c)(1+c)^3})\log(\frac{1-c}{2})+\frac{2(3+c)^2}{(1-c)^3(1+c)}\log(\frac{1+c}{2})\right]+\text{Finite part}\right\}$$

- Sfermions

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+\tilde{\bar{q}}\tilde{q})}_{Real} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\varepsilon}\frac{\alpha N_c}{2\pi}\left\{\frac{8}{\varepsilon}\left[-\frac{2(10+7c^2)}{(1-c^2)^2}\right.\right.$$
$$\left.\left.-\frac{3(5-c)}{(1+c)^3}\log(\frac{1-c}{2})-\frac{3(5+c)}{(1-c)^3}\log(\frac{1+c}{2})\right]+\text{Finite part}\right\}$$

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$$+\frac{1}{\varepsilon}\left[\frac{2}{(1+c)^2}\log(\frac{1-c}{2})+\frac{2}{(1-c)^2}\log(\frac{1+c}{2})+\frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2}\right.$$
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$$+\frac{1}{\varepsilon}\left[\frac{2}{(1+c)^2}\log(\frac{1-c}{2}) + \frac{2}{(1-c)^2}\log(\frac{1+c}{2}) + \frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2}\right.$$
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Splitting of Massless states

- There are NO isolated massless asymptotic states!
- Massless particle can split into two (or more) indistinguishable collinear particles.
- One has to consider coherent states of parallel massless particles.
- Distribution function

$$g_i(z) = \delta(1-z) + \frac{\alpha}{2\pi\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\varepsilon}\sum_j P_{ij}(z)$$

$P_{ij}(z)$ - Splitting function, Q_f^2 - transverse momentum cutoff

- Initial splitting

$$d\sigma_{2\to2}^{spl,init} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dz \sum_{l=g,q,\Lambda} P_{gl}(z) \sum_{i,j=1,2;\ i\neq j} d\sigma_{2\to2}(zp_i, p_j, p_3, p_4)\mathcal{S}_2^{spl,init}(z)$$

- Final splitting

$$d\sigma_{2\to2}^{spl,fin} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon} d\sigma_{2\to2}(p_1,p_2,p_3,p_4)\int_0^1 dz \sum_{l=g,q,\Lambda} P_{gl}(z)\mathcal{S}_2^{spl,fin}(z)$$

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Initial and final state splitting (MHV)

- Initial

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+++)}_{InSplit} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\varepsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\varepsilon}$$

$$\frac{\alpha N_c}{2\pi}\left\{\frac{1}{\varepsilon}\left[-\frac{4(c^2+3)}{(1-c^2)^2}\left(\log\frac{1-c}{2}+\log\frac{1+c}{2}\right)-\frac{8(c^2+3)}{(1-c^2)^2}\log\frac{1-\delta}{\delta}\right.\right.$$
$$\left.\left.-\frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2}\right]+\text{Finite part}\right\}$$

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- Final

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+++)}_{FnSplit} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\varepsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\varepsilon}$$

$$\frac{\alpha N_c}{2\pi}\left\{-\frac{1}{\varepsilon}\frac{4(c^2+3)}{(1-c^2)^2}\log\frac{1-\delta}{\delta}+\text{Finite part}\right\}$$

Infrared-free sets (for any arbitrary δ)

- $$A^{MHV} = \frac{1}{2}\left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--++)}_{Virt} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+++)}_{Real} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+++)}_{InSplit} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+++)}_{FnSplit}$$

- $$B^{AntiMHV} = \frac{1}{2}\left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--++)}_{Virt} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--++-)}_{Real} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--++-)}_{InSplit} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--++-)}_{FnSplit}$$

- $$C^{Matter} = \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+\bar{q}q,\tilde{\bar{q}}\tilde{q})}_{Real} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+\bar{q}q,\tilde{\bar{q}}\tilde{q})}_{InSplit} + \left(\frac{d\sigma}{d\Omega_{14}}\right)^{(--+\bar{q}q,\tilde{\bar{q}}\tilde{q})}_{FnSplit}$$

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Infrared-free observables

- Registration of two fastest gluons of positive chirality

$$A^{MHV}\Big|_{\delta=1/3}+B^{AntiMHV}\Big|_{\delta=1}$$

- Registration of one fastest gluon of positive chirality

$$A^{MHV}\Big|_{\delta=1/3}+B^{AntiMHV}\Big|_{\delta=1/3}+C^{Matter}\Big|_{\delta=1}$$

- Anti MHV cross-section

$$B^{AntiMHV}\Big|_{\delta=1}+C^{Matter}\Big|_{\delta=1}\Rightarrow \text{Finite Part}$$

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The simplest IR finite answer so far ($Q_f = E$): N=4 SYM Anti MHV

$$\left(\frac{d\sigma}{d\Omega_{14}}\right)_{AntiMHV} = \frac{\alpha^2 N_c^2}{E^2}\left\{\frac{3+c^2}{(1-c^2)^2}\right.$$
$$-\frac{\alpha N_c}{2\pi}\left[2\frac{(c^4+2c^3+4c^2+6c+19)\log^2(\frac{1-c}{2})}{(1-c)^2(1+c)^4}+2\frac{(c^4-2c^3+4c^2-6c+19)\log^2(\frac{1+c}{2})}{(1-c)^4(1+c)^2}\right.$$
$$-8\frac{(c^2+1)\log(\frac{1+c}{2})\log(\frac{1-c}{2})}{(1-c^2)^2}+\frac{6\pi^2(3c^2+13)-5(61c^2+99)}{9(1-c^2)^2}$$
$$\left.\left.+2\frac{(11c^3+31c^2-47c+133)\log(\frac{1+c}{2})}{3(1-c)^3(1+c)^2}-2\frac{(11c^3-31c^2-47c-133)\log(\frac{1-c}{2})}{3(1+c)^3(1-c)^2}\right]\right\}$$

Summary

- Factorization (exponentiation) of IR divergences takes place with universal second order pole (cusp anomalous dimension) and non-universal first order pole (collinear dim).
- The finite part factorizes only in simple cases both for the gluon amplitudes and for the formfactors
- In observable cross-sections the IR divergences do cancel in accordance with Kinoshita-Lee-Nauenberg theorem

$$d\sigma_{obs}^{incl} = \sum_{n=2}^{\infty}\int_0^1 dz_1 q_1(z_1,\frac{Q_f^2}{\mu^2})\int_0^1 dz_2 q_2(z_2,\frac{Q_f^2}{\mu^2})\prod_{i=1}^{n}\int_0^1 dx_i q_i(x_i,\frac{Q_f^2}{\mu^2})\times$$

$$\times\, d\sigma^{2\to n}(z_1p_1,z_2p_2,...)S_n(\{z\},\{x\}) = g^4\sum_{L=0}^{\infty}\left(\frac{g^2}{16\pi^2}\right)^L d\sigma_L^{Finite}(s,t,u,Q_f^2)$$

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