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in quantum field theory

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RADIATIVE CORRECTIONS, DIVERGENCES, REGULARIZATION, RENORMALIZATION, RENORMALIZATION GROUP AND ALL THAT IN EXAMPLES IN QUANTUM FIELD THEORY

The present lectures are a practical guide to the calculation of radiative corrections to the Green functions in quantum field theory. The appearance of ultraviolet divergences is explained, their classification is given, the renormalization procedure which allows one to get the finite results is described, and the basis of the renormalization group in QFT is presented. Numerous examples of calculations in scalar and gauge theories are given. Quantum anomalies are discussed. In conclusion the procedure which allows one to get rid of infrared divergences in S-matrix elements is described. The lectures are based on the standard quantum field theory textbooks, the list of which is given at the end of the text.

These lectures were given to the 4-th year students of the Department of General and Applied Physics of the Moscow Institute of Physics and Technology (Technical University).

Figs.– 42, Refs.– 13.

Contents

0 Preface

Today there exist many excellent textbooks on quantum field theory. The most popular ones are listed in the bibliography to the present lectures. Nevertheless, everyone who gives lectures on quantum field theory faces the problem of selection of material and writing the lecture notes for students. The present text is just the lecture notes devoted to the radiative corrections in QFT. On this way, one encounters two problems, namely, the ultraviolet and the infrared divergences. Our task is to demonstrate how one can get rid of these divergences and obtain finite corrections to the cross-sections of elementary processes. During the course we describe the methods of Feynman diagram evaluation and regularization of divergences. In more detail, we consider the renormalization theory and elimination of ultraviolet divergencies in the Green functions off mass shell, as exemplified by scalar and gauge theories. In connection with the renormalization procedure we describe also the renormalization group formalism in QFT. As for the infrared divergences, in the literature one can find mainly the discussion of the IR divergencies in quantum electrodynamics. In non-Abelian theories as well as in QED with massless particles the situation is much more involved as there arise collinear divergences as well. In the last lecture, we show how one can get rid of these divergences using the methods developed in quantum chromodynamics. One more topic also related to divergences is the so-called anomalies. They also lead to unwanted ultraviolet divergent contributions. Therefore, a separate lecture is dedicated to the axial and conformal anomalies.

The presented text overlaps with many textbooks and is partly borrowed from there. However, the composition of the material and most of the calculations belong to the author, so we omit the direct references to any textbooks. It should be admitted that the style of presentation in different textbooks varies very much and the reader can choose the book according to his preferences. We mostly used the classical monograph by N.Bogoliubov and D.Shirkov when describing the renormalization theory and more modern book by M.Peskin and D.Schreder which we followed when discussing the infrared divergences.

Our experience in giving lectures on quantum field theory, the renormalization theory and the renormalization group tells us that this material is still complicated for perception and is not always presented clearly enough. One often meets with the lack of understanding of the complicated structure of the field theory which manifests itself in renormalization theory. Sometimes the nonrenormalizable theories are simplistically treated as the field theories with a dimensional coupling constant which otherwise have no difference from the renormalizable ones. The collinear divergences arising in theories with massless particles, despite a long history, have not also become the well-known part of the QFT course. Here we make an attempt of a simplified presentation of this complicated material. Of course, this means that one has to sacrifice some rigorousness and completeness. We hope that together with the existing literature the present lectures will serve the goal of clarification and mastering of quantum field theory and its applications to particle physics.

1 Lecture I: Radiative corrections. General analysis of divergent integrals

1.1 Radiative corrections

The formalism of quantum field theory, being the generalization of quantum mechanics to the case of an infinite number of degrees of freedom with nonconservation of the number of particles, allows one to describe the processes of scattering, annihilation, creation and decay of particles with the help of the set of well-defined rules. As in quantum mechanics the cross-section of any process is given by the square of the modulus of the probability amplitude calculated according to the Feynman rules for the corresponding Lagrangian integrated over the phase space. Since the exact calculations of the probability amplitudes seem to be impossible, one is bound to use the perturbation theory with a small parameter - the coupling constant - and get the result in the form of a power series. The leading terms of this series can be presented by Feynman diagrams without loops, the so-called tree diagrams. The examples of such diagrams for some typical processes in QED are shown in Fig.1.

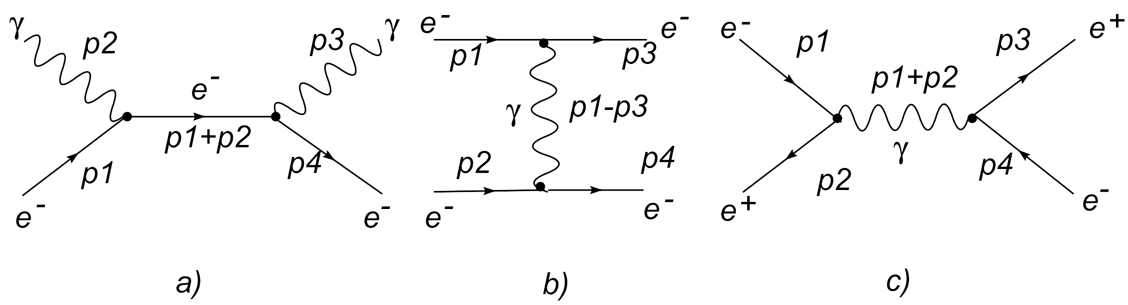


Figure 1: The examples of tree diagrams of different processes in QED:) the Compton scattering, b) the Mueller scattering, c) the annihilation of the particle-antiparticle pair. Shown are the momenta of external (real) and internal (virtual) particles

All the diagrams shown in Fig.1 are proportional to the square of the coupling constant e^2. They are constructed according to the well-known Feynman rules and do not contain any integration over momenta (when working in momentum representation) since due to the conservation of four-momentum all momenta are defined uniquely.

The situation changes when considering the next order of perturbation theory. As an example, in Fig.2 we show the corresponding diagrams for the Compton scattering.

They got the name of *radiative corrections* since in electrodynamics they correspond to the emission and absorption of photons. This name is also accepted in other theories for perturbative corrections. All these diagrams are proportional to the fourth power of the coupling constant e^4 and, hence, are the next order perturbations with respect to the tree diagrams. However, contrary to the tree diagrams, they contain a closed loop which requires the integration over the four-momenta running through the loop. Any loop corresponds to

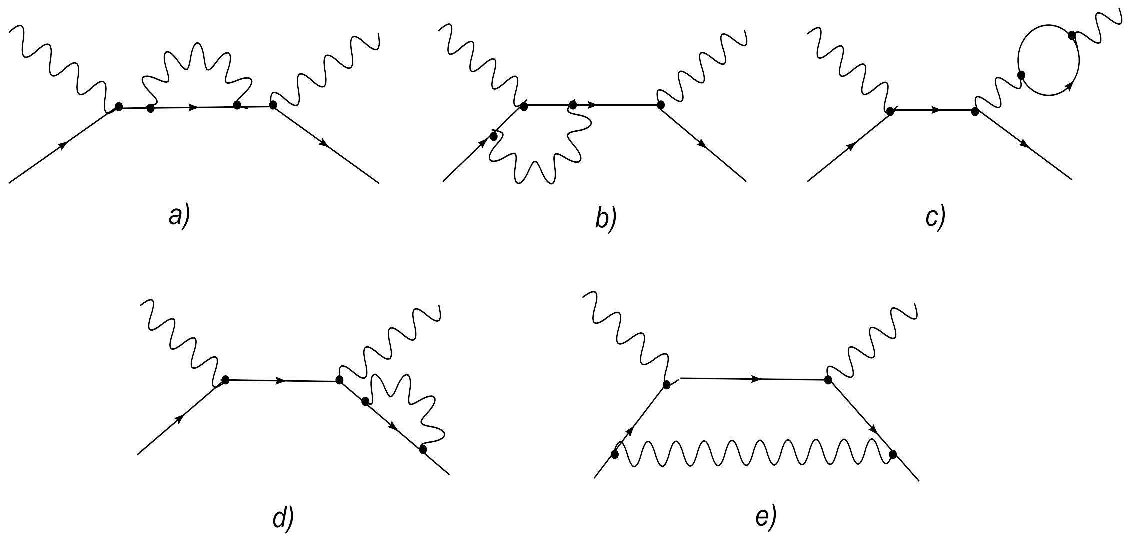


Figure 2: The one-loop diagrams for the process of the Compton scattering

the bifurcation of momenta similarly to the bifurcation of the electric current, according to the Kirchhoff rules, so that the total momentum is conserved but the momentum running along each line is arbitrary. Therefore, one has to integrate over it.

1.2 Divergence of integrals

Prior to calculating the radiative corrections let us consider the behaviour of the integrand and the integral as a whole. As an example we take the diagrams of the Compton scattering shown in Fig.2. The integral corresponding to the diagram shown in Fig.2.a) has the form

$$\int d^4k \frac{\gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\mu}{[k^2+i\varepsilon][(p-k)^2-m^2+i\varepsilon]}, \tag{1.1}$$

where the photon propagator is written in Feynman gauge and the integration takes place in Minkowskian space. We shall not calculate explicitly this integral now (we shall do it later) but consider the integrand from the point of view of the presence of singularities as well as the behaviour at small and large momenta.

The presence of poles in the propagators for momentum equal to the mass squared does not create any problem for the integration since according to the Feynman rules the denominator contains the infinitesimal imaginary term $\sim \varepsilon \to 0$, which defines the way to bypass the pole. The choice accepted in (1.1) corresponds to the causal Green function.

Consider now the case of $k_\mu \to 0$, the so-called *infrared* behaviour. Despite the presence of k^2 in the denominator, the singularity is absent due to the measure of the 4-dimensional integration which is also proportional to k^4. This is true for all such integrals. The singularities appear only for certain external momenta which are on mass shell and have a physical reason. Off shell the singularities are absent. For this reason we shall not discuss the infrared behaviour of the integrals so far.

Consider at last the case of $k_\mu \to \infty$, the so-called *ultraviolet* behaviour. Notice that in the denominator one has 4 powers of momenta, while in the numerator one has 1 plus 4 powers in the measure of integration. Hence one has 5-4=1, i.e. the integral is linearly divergent as $k_\mu \to \infty$. Is it the property of a particular integral or is it a general situation? What happens with the other diagrams?

Consider the integral corresponding to the diagram shown in Fig.2.). One has, according to the Feynman rules

$$\int d^4k \frac{\gamma^\mu(\hat{p_1}-\hat{k})+m)\gamma^\nu(\hat{p_2}-\hat{k})+m)\gamma^\mu}{k^2[(p_1-k)^2-m^2][(p_2-k)^2-m^2]}. \tag{1.2}$$

We are again interested in the behaviour for $k_\mu \to \infty$. The counting of the powers of momenta in the numerator and the denominator gives: 6 in the denominator and 2 in the numerator plus 4 in the integration measure. Altogether one has 6-6=0, i.e., the integral is logarithmically divergent as $k_\mu \to \infty$.

Here we met the difficulty called the ultraviolet divergence of the integrals for the radiative corrections. The examples considered above are not exceptional but the usual ones. The corrections are infinite, which makes perturbation theory over a small parameter meaningless. The way out of this trouble was found with the help of the renormalization theory which will be considered later and now we try to estimate the divergence of the integrals in a theory with an arbitrary Lagrangian.

1.3 General analysis of ultraviolet divergences

Consider an arbitrary Feynman diagram G shown in Fig.3. and try to find out whether it is

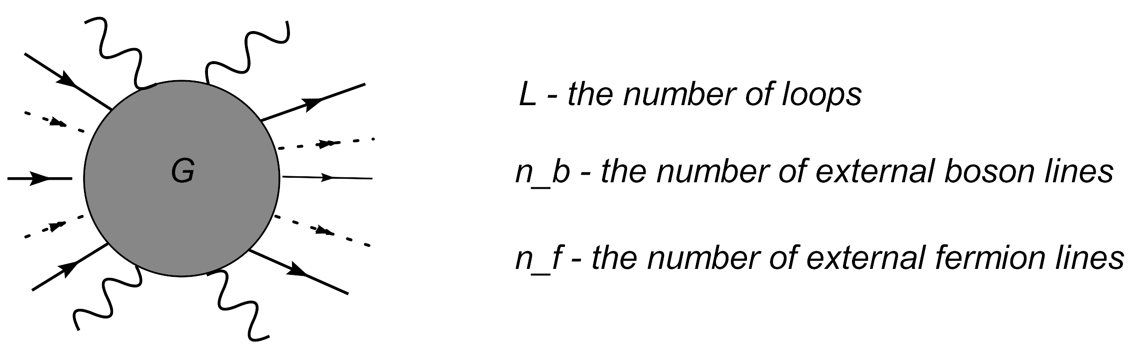


Figure 3: An arbitrary diagram containing L integrations

ultraviolet divergent or not. For this purpose we have to calculate the number of powers of momenta in the integrand: each internal loop leads to integration d^4p that gives 4 powers of momenta; each derivative in the vertex gives the momentum in p-space, i.e., 1; each internal line gives a propagator which behaves as p^{r_l}/p^2, i.e., $r_l - 2$ powers of momenta, where $r_l = 0, 1, 2$ for various fields. Combining all these powers together we get the quantity called *the index of divergence of the diagram* (UV)

$$\omega(G) = 4L + \sum_{vertices} \delta_v + \sum_{internal\ lines} (r_l - 2), \tag{1.3}$$

where L is the number of loops and δ_v is the number of derivatives in a vertex v.

The absence of the ultraviolet divergences means that $\omega(G) < 0$. However, one has to be careful, there might be subdivergences in some subgraphs. Hence, the necessary condition for finiteness is

$$\text{The finiteness condition (UV):} \quad \omega(\gamma_i) < 0, \quad \forall \gamma_i \subset G,$$

where γ_i are all possible subgraphs of the graph G including the graph G itself.

There exists, however, a simpler way to answer the same question which does not need to analyse all the diagrams. One can see it directly from the form of the Lagrangian. To see this, let us introduce the quantity called *the index of the vertex* (UV)

$$\omega_v = \delta_v + b_v + \frac{3}{2} f_v - 4, \tag{1.4}$$

where δ_v, b_v and f_v are the number of derivatives, internal boson and fermion lines, respectively. Then the index of a diagram (1.3) can be written as

$$\omega(G) = \sum_{vertices} \omega_v^{max} + 4 - n_b - \frac{3}{2} n_f, \tag{1.5}$$

where ω_v^{max} corresponds to the vertex where all the lines are internal, n_b and n_f are the number of external boson and fermion lines, and we have used the fact that usually $r_l(boson) = 0$ and $r_l(fermion) = 1$.

Equation (1.5) tells us that the finiteness ($\omega(G) < 0$) can take place if $\omega_v \leq 0$ and the number of external lines is big enough. Prior to the formulation of conditions when it happens, let us consider some examples.

Example 1: The scalar theory $\mathcal{L}_{int} = -\lambda\varphi^4$.

In this case $\delta_v = 0,\ f_v = 0,\ b_v = 4$ and, hence, $\omega_v^{max} = 0$. Thus, according to (1.5), $\omega(G) = 4 - n_b - \frac{3}{2} n_f$ and everything is defined by the number of external lines. The situation is illustrated in Fig.4.

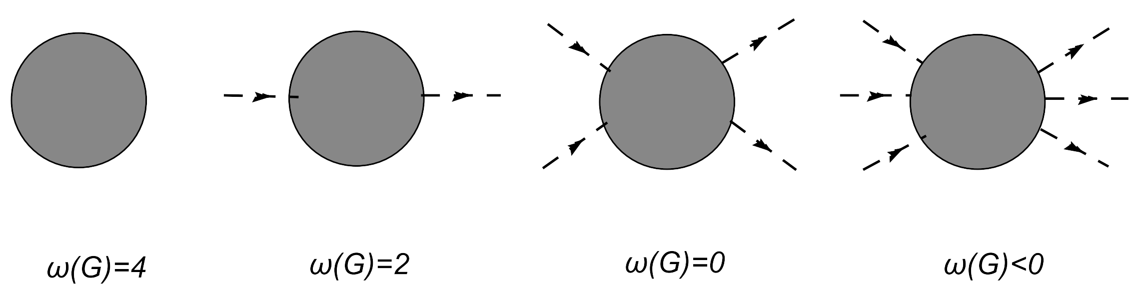


Figure 4: The indices of divergence of the diagrams in the scalar theory

We see that there exists a limited number of divergent structures in the φ^4 theory. These are the vacuum graphs, the two- and four-point functions. All the other diagrams having more than 4 external lines are convergent (though may have divergent subgraphs).

Example 2: Quantum Electrodynamics $\mathcal{L}_{int} = \bar{\psi}\hat{A}\psi$.

In this case $\delta_v = 0, f_v = 2, b_v = 1,\ \omega_v^{max} = 0$. Hence, $\omega(G) = 4 - n_b - \frac{3}{2} n_f$ and the situation is similar to the previous example, everything is defined by external lines. Divergent are the vacuum diagrams ($\omega(G) = 4$), the photon propagator ($\omega(G) = 2$), the electron propagator ($\omega(G) = 1$) and the triple vertex ($\omega(G) = 0$). All the other diagrams are convergent.

Example 3: Four-fermion interaction $\mathcal{L}_{int} = G\bar{\psi}\psi\bar{\psi}\psi$.

Here $\delta_v = 0, f_v = 4, b_v = 0$, $\omega_v^{max} = 2$ and, hence, $\omega(G) = 2N - \frac{3}{2}n_f$. Therefore, increasing the number of vertices we get new divergent diagrams independently of the number of external lines. The number of divergent structures happens to be infinite.

Thus, the key role is played by the maximal index of the vertex. All the theories may be classified according to the value of $\omega_v^{\max}$:

$$\omega_v^{\max} = \begin{cases} < 0 & Finite\ number\ of\ divergent\ diagrams, \\ \ \ 0 & Finite\ number\ of\ divergent\ structures, \\ > 0 & Infinite\ number\ of\ divergent\ structures. \end{cases} \tag{1.6}$$

Below we show that for the first two types of theories we can handle the ultraviolet divergences with the help of the renormalization procedure. The theories with $\omega_v^{\max} = 0$ are called *renormalizable*, the theories with $\omega_v^{\max} > 0$ are called *nonrenormalizable*, and the theories with $\omega_v^{\max} < 0$ are called *superrenormalizable*.

1.4 The analysis of dimensions

The property of a theory with respect to ultraviolet divergences can be reformulated in terms of dimensions. Consider for this purpose an arbitrary term of the interaction Lagrangian which is the product of the field operators and their derivatives

$$\mathcal{L}_I(x) = g \prod_{i,j} \varphi_i(x) \partial \varphi_j(x). \tag{1.7}$$

Consider the action which is the four-dimensional integral of the Lagrangian density

$$A = \int d^4x \mathcal{L}(x), \tag{1.8}$$

and find the dimensions of parameters in eq.(1.7). As a unit of measure we take the dimension of a mass equal to 1. Then the dimension of length $[L] = -1$, the dimension of time is also $[T] = -1$, the dimension of derivative $[\partial_\mu] = 1$, the dimension of momenta $[p_\mu] = 1$. Since the action is dimensionless (we use the natural units where $\hbar = c = 1$)

$$[A] = 0,$$

the dimension of the Lagrangian is

$$[\mathcal{L}] = 4, \quad (D - \text{in D dimensional space.})$$

This gives us the dimensions of the fields. Indeed, from the kinetic term for the scalar field one finds

$$[(\partial\phi)^2] = 4 \to [\phi] = 1, \quad (\frac{D-2}{2} \text{ in D dimensional space}),$$

for the spinor field

$$[\bar{\psi}\hat{\partial}\psi] = 4 \to [\psi] = \frac{3}{2}, \quad (\frac{D-1}{2} \text{ in D dimensional space}),$$

for the vector field

$$[(\partial_\mu A_\nu - \partial_\nu A_\mu)^2] = 4 \to [A_\mu] = 1, \quad (\frac{D-2}{2} \text{ in D dimensional space}).$$

This allows one to find the dimension of the coupling constant in (1.7)

$$[g] = 4 - \delta_v - b_v - \frac{3}{2} f_v = -\omega_v^{max}. \tag{1.9}$$

Then the classification of interactions (1.6) can be written as

$$[g] = \begin{cases} > 0 & \textit{Superrenormalizable theories}, \\ \ \ 0 & \textit{Renormalizable theories}, \\ < 0 & \textit{Nonrenormalizable thoeries}. \end{cases} \tag{1.10}$$

Consider which category various theories belong to. For this purpose we have to calculate the dimensions of the couplings.

<u>Illustration</u>

$$\begin{array}{lll}
\mathcal{L}_{\varphi^3} = -\lambda\varphi^3 & \Rightarrow & [\lambda] = 1, \quad \textit{SuperRen.} \\
\mathcal{L}_{\varphi^4} = -\lambda\varphi^4 & \Rightarrow & [\lambda] = 0, \quad \textit{Ren.} \\
\mathcal{L}_{QED} = e\bar{\psi}\gamma^\mu A_\mu \psi & \Rightarrow & [e] = 0, \quad \textit{Ren.} \\
\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{4}\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c\right]^2 & \Rightarrow & [g] = 0, \quad \textit{Ren.} \\
\mathcal{L}_{Yukawa} = y\bar{\psi}\varphi\psi & \Rightarrow & [y] = 0. \quad \textit{Ren.}
\end{array}$$

Thus, all these models are renormalizable.

$$\begin{array}{lll}
\mathcal{L} = -h\varphi^6 & \Rightarrow & [h] = -2, \quad \textit{Nonren.} \\
\mathcal{L} = G\bar{\psi}\psi\bar{\psi}\psi & \Rightarrow & [G] = -2 \quad \textit{Nonren.} \\
\mathcal{L} = \kappa\bar{\psi}\partial_\mu V_\mu \psi & \Rightarrow & [\kappa] = -1 \quad \textit{Nonren.} \\
\mathcal{L} = \gamma\bar{\psi}\partial_\mu\varphi\gamma^\mu\psi & \Rightarrow & [\gamma] = -1. \quad \textit{Nonren.}
\end{array}$$

All these models on the contrary are nonrenormalizable. Notice that they include the four-fermion or current-current interaction which was previously used in the theory of weak interactions.

Hence, we come to the following conclusion: the only renormalizable interactions in four dimensions are:

i) the φ^4 interaction;

ii) the Yukawa interaction;

iii) the gauge interaction;

iv) the theory φ^3 is superrenormalizable. It contains only two divergent diagrams shown in Fig.5.

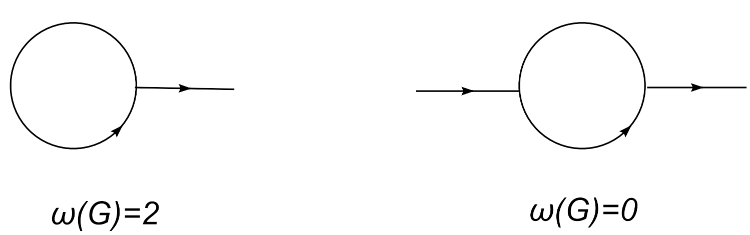


Figure 5: The only divergent diagrams in the ϕ^3 theory

If one looks at the spins of particles involved in the interactions, one finds out that they are strongly restricted. The renormalizable interactions contain only the fields with spins 0, 1/2 and 1. All the models with spins 3/2, 2, etc. are nonrenormalizable. The latter include also gravity. Indeed, the coupling constant in this case is the Newton constant which has dimension equal to $[G] = -2$, i.e., quantum gravity is nonrenormalizable.

Since we do not know how to handle the nonrenormalizable interactions because the ultraviolet divergences are out of control, there are only three types of interactions which are used in the construction of the Standard Model of fundamental interactions, namely the φ^4, the Yukawa and the gauge interactions with the scalar, spinor and vector particles.

Here one has to make a comment concerning the vector fields with $M \neq 0$. Remind the form of the propagator of the massive vector field

$$\overline{V_\mu V_\nu} = i\frac{g_{\mu\nu} - k_\mu k_\nu/M^2}{M^2 - k^2 - i\epsilon}.$$

It gives $r_l = 2$, which leads to some modification of the formulas used above and finally to the nonrenormalizability of the theory. The only known way to avoid this difficulty is the spontaneous breaking of symmetry. In this case,

$$\overline{V_\mu V_\nu} = i\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{M^2 - k^2 - i\epsilon},$$

that gives $r_l = 0$ and the theory happens to be renormalizable. This mechanism is used in the Standard Model to give masses to the intermediate weak bosons without breaking the renormalizability of the theory.

2 Lecture II: Regularization

The divergences which appear in radiative corrections are not yet a catastrophe for a theory (remind, for example, the infinite self-energy of an electric charge in its own Coulomb field) but require a quantitative description. To get a finite difference of the two infinite quantities, one has to give them some meaning. This can be achieved by introducing a kind of regularization of divergent integrals. The most natural way of regularization is to cut off the integral on the upper or lower bound of integration. There are also different ways of regularization based on a modification of the integrand or of the measure of integration. Below we consider three most popular kinds of regularization: the ultraviolet cutoff in Euclidean space (Λ-regularization), the Pauli-Villars regularization, and the dimensional regularization.

2.1 Euclidean integral and the ultraviolet cutoff

All the integrals in quantum field theory are written in Minkowski space; however, the ultraviolet divergence appears for large values of modulus of momentum and it is useful to regularize it in Euclidean space. Transition to Euclidean space can be achieved by replacing the zeroth component of momentum $k_0 \to ik_4$, so that the squares of all momenta and the scalar products change the sign $k^2 = k_0^2 - \vec{k}^2 \to -k_4^2 - \vec{k}^2 = -k_E^2$ and the measure of integration becomes equal to $d^4k \to id^4k_E$, where the integration over the fourth component of momenta goes along the imaginary axis. To go to the integration along the real axis, one has to perform the (Wick) rotation of the integration contour by 90^o (see. Fig.6). This is possible since the integral over the big circle vanishes and during the transformation of the contour it does not cross the poles.

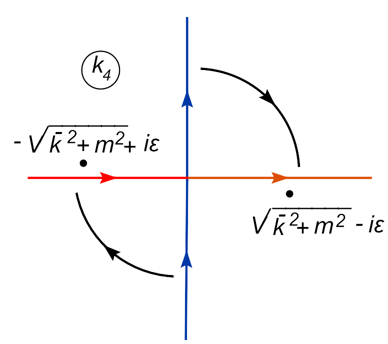


Figure 6: The Wick rotation of the integration contour

When transferring to Euclidean space the poles in all the propagators disappear. Now the integral in 4-dimensional Euclidean space can be evaluated in spherical coordinates and the integral over the modulus can be cut on the upper limit. Let us demonstrate how this method works in the case of the simplest scalar diagram shown in Fig.7. The corresponding

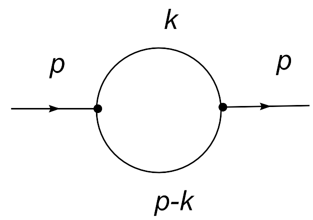


Figure 7: The simplest divergent diagram in a scalar theory

pseudo-Euclidean integral has the form

$$I(p^2) = \frac{1}{(2\pi)^4} \int \frac{d^4k}{[k^2 - m^2][(p-k)^2 - m^2]}. \tag{2.1}$$

Transforming it to Euclidean space one gets

$$I(p_E^2) = \frac{i}{(2\pi)^4} \int \frac{d^4k}{[k_E^2 + m^2][(p-k)_E^2 + m^2]}. \tag{2.2}$$

(in what follows the index will be omitted.)

For calculation of this kind of integrals we use the following approach. First, we transform the product of several brackets in the denominator into the single bracket with the help of the so-called Feynman parametrization. The following general formula is valid:

$$\begin{aligned} \frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}} &= \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} \int_0^1 dx_1 dx_2 \cdots dx_n \\ &\cdot \frac{\delta(1 - x_1 - x_2 - \cdots - x_n) x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1}}{[A_1 x_1 + A_2 x_2 + \cdots A_n x_n]^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}}. \end{aligned} \tag{2.3}$$

Here $\Gamma(\alpha)$ is the Euler Γ-function which has the following properties:

$$\Gamma(1) = 1,\ \Gamma(n+1) = n!,\ x\Gamma(x) = \Gamma(x+1),\ \Gamma(1+x) = e^{\left[-x\gamma_E + \sum_{n=2}^{\infty} \frac{(-x)^n}{n} \zeta(n)\right]},$$

where γ_E is the Euler constant and $\zeta(n)$ is the Riemann zeta-function. The Γ-function is finite for positive values of the argument and has simple poles at negative integer values and at zero.

In our case, $(n = 2, \alpha_1 = \alpha_2 = 1)$ and eq.(2.3) has the form:

$$\begin{aligned} \frac{1}{[k^2 + m^2][(p-k)^2 + m^2]} &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \int_0^1 \frac{dx_1 x_2 \delta(1 - x_1 - x_2)}{\left[[k^2 + m^2]x_1 + [(p-k)^2 + m^2]x_2\right]^2} \\ &= \int_0^1 \frac{dx}{[k^2 - 2pkx + p^2x + m^2]^2}. \end{aligned} \tag{2.4}$$

Thus, integral (2.2) can be written as

$$I(p^2) = \frac{i}{(2\pi)^4} \int_0^1 dx \int \frac{d^4k}{[k^2 - 2kpx + p^2x + m^2]^2} \overset{k \to k - px}{=} \frac{i}{(2\pi)^4} \int_0^1 dx \int \frac{d^4k}{[k^2 + p^2x(1-x) + m^2]^2} \tag{2.5}$$

Now the integral depends only on the modulus of k and one can use the spherical coordinates:

$$I(p^2) = \frac{i}{(2\pi)^4} \int_0^1 dx \; \Omega_4 \int_0^\Lambda \frac{k^3 dk}{[k^2 + p^2 x(1-x) + m^2]^2}, \tag{2.6}$$

where the volume of the 4-dimensional sphere equals $\Omega_4 = 2\pi^2$ (in general $\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$). The integral over the modulus can be easily calculated

$$\frac{1}{2} \int_0^{\Lambda^2} \frac{k^2 dk^2}{[k^2 + p^2 x(1-x) + m^2]^2} = \frac{1}{2} \log\left(\frac{\Lambda^2}{p^2 x(1-x) + m^2}\right) + 1, \tag{2.7}$$

and, as one can see, is logarithmically divergent at the upper limit. The full answer has the form

$$I(p^2) = \frac{i}{16\pi^2} \int_0^1 dx \left(\log\left(\frac{\Lambda^2}{p^2 x(1-x) + m^2}\right) + 1 \right). \tag{2.8}$$

The last integral over x can also be evaluated and takes the simplest form in the limiting cases for $m = 0$ or $p = 0$. Now one can go back to Minkowski space $p_E^2 => -p^2$.

The regularization with the ultraviolet cut-off is quite natural and relatively simple. The drawback of this regularization is Euclidean rather than Lorentzian invariance and also the absence of the gauge invariance. Therefore, it is not useful in the gauge theories. However, one should notice that the noninvariance of a regularization is acceptable since the invariance is restored when removing the regularization . Still, this aspect complicates the calculation as one has to take care of the validity of all the identities.

2.2 Pauli-Villars Regularization

Another method of regularization which is called the Pauli-Villars regularization is based on the introduction of a set of additional heavy fields with a wrong sign of the kinetic term. These fields are not physical and are introduced essentially with the purpose of regularization of divergent integrals. The main trick is in the replacement

$$\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2}, \tag{2.9}$$

where $M \to \infty$ is the mass of the Pauli-Villars fields. As a result, the propagator for large momenta decreases faster, which ensures the convergence of the integrals. The divergences manifest themselves as logs and powers of M^2 instead of the cutoff parameter Λ^2.

One uses sometimes the modifications of the Pauli-Villars regularization when the replacement (2.9) is performed not for each propagator but for the loop as a whole. This method of regularization is called the regularization over circles. It is used in Abelian gauge theories for the loops made of the matter fields. This way one can preserve the gauge invariance. However, in non-Abelian theories we face some problems related to the loops of the gauge fields which cannot become massive without violating the gauge invariance. This problem is often solved by introducing an additional regularization for the vector fields, for example, with the help of higher derivatives. Here we will not consider this regularization.

The positive property of the Pauli-Villars regularization is the explicit Lorentz and gauge (in abelian case) invariance, but it requires complicated calculations since one has to calculate massive diagrams, while massless integrals are much simpler.

2.3 Dimensional Regularization

The most popular in gauge theories is the so-called dimensional regularization. In this case, one modifies the integration measure.

The technique of dimensional regularization consists of analytical continuation from an integer to a noninteger number of dimensions. Basically one goes from some D to $D-2\epsilon$, where $\epsilon \to 0$. In particular, we will be interested in going from 4 to $4-2\epsilon$ dimensions. In this case, all the ultraviolet and infrared singularities manifest themselves as pole terms in ϵ. To perform this continuation to non-integer number of dimensions, one has to define all the objects such as the metric, the measure of integration, the γ matrices, the propagators, etc. Though this continuation is not unique, one can define a self-consistent set of rules, which allows one to perform the calculations.

The metric: $g_4^{\mu\nu} \to g_{4-2\epsilon}^{\mu\nu}$. Though it is rather tricky to define the metric in non-integer dimensions, one usually needs only one relation, namely $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = D = 4-2\epsilon$.

The measure: $d^4q \to (\mu^2)^\epsilon d^{4-2\epsilon}q$, where μ is a parameter of dimensional regularization with dimension of a mass. The integration with this measure is defined by an analytical continuation from the integer dimensions.

The γ matrices : The usual anticommutation relation holds $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$; however, some relations involving the dimension are modified:

$$\gamma^\mu\gamma_\mu = D = 4-2\epsilon; \quad Tr\gamma^\mu\gamma^\nu = g^{\mu\nu}Tr1 = g^{\mu\nu}\begin{cases} 2^{[D/2]} \\ 4 \end{cases} .$$

Usually $Tr1 = 4$ is taken. Then the γ-algebra is straightforward:

$$Tr\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = Tr1[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\nu\rho}g^{\mu\sigma}],$$

$$\gamma^\mu\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\mu\gamma^\nu + 2g^{\mu\nu}\gamma^\mu = -(4-2\epsilon)\gamma^\nu + 2\gamma^\nu = -(2-2\epsilon)\gamma^\nu, \quad \text{etc.}$$

What is not well-defined is the γ^5 since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and cannot be continued to an arbitrary dimension. This creates a problem in dimensional regularization since there is no consistent way of definition of γ^5.

The propagator : In momentum space the continuation is simple

$$\frac{1}{p^2-m^2} \to \frac{1}{p^2-m^2}.$$

However, in coordinate space one has: (take $m=0$ for simplicity)

$$\int \frac{d^4p}{p^2}e^{ipx} \sim \frac{1}{x^2} \Rightarrow \int \frac{d^{4-2\epsilon}p}{p^2}e^{ipx} \sim \frac{1}{[x^2]^{1-\epsilon}}.$$

The basic integrals: The main idea is to calculate the integral in the space-time dimension where it is convergent and then analytically continue the answer to the needed dimension.

Consider the earlier discussed example (2.1) and use the Euclidean representation (2.5). Let us rewrite it formally in D-dimensional space

$$\int \frac{d^Dk}{[k^2+M^2]^2} = \frac{\Omega_D}{2}\int_0^\infty \frac{(k^2)^{D/2-1}dk^2}{[k^2+M^2]^2}, \qquad M^2 \equiv p^2x(1-x)+m^2. \tag{2.10}$$

The integral over k^2 is now the table one

$$\int_0^\infty \frac{(k^2)^{D/2-1}dk^2}{[k^2+M^2]^2} \overset{k^2\to k^2M^2}{=} (M^2)^{\frac{D}{2}-2}\int_0^\infty \frac{x^{D/2-1}dx}{(x+1)^2} = (M^2)^{\frac{D}{2}-2}\,\frac{\Gamma(\frac{D}{2})\Gamma(2-\frac{D}{2})}{\Gamma(2)}, \qquad (2.11)$$

where we assume that the dimension D is such that the integral exists. In this case this is 2 and 3. The main formula (2.11) allows one to perform the analytical continuation over D into the region $D = 4-2\varepsilon$. For $\varepsilon = 0$, i.e., in 4 dimensions, the integral does not exist since the Γ-function has a pole at zero argument. However, in the vicinity of zero we get a regularized expression.

Collecting all together we get

$$I(p^2) = \frac{i}{(2\pi)^D}\frac{\Omega_D}{2}\int_0^1 dx\frac{\Gamma(D/2)\Gamma(2-D/2)}{[p^2x(1-x)+m^2]^{2-D/2}}. \qquad (2.12)$$

Substituting now $D = 4-2\varepsilon$ and transforming back into the pseudo-Euclidean space one finds

$$I(p^2) = \frac{i(-\pi)^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}}\Gamma(\varepsilon)\int_0^1 \frac{dx(\mu^2)^\varepsilon}{[p^2x(1-x)-m^2]^\varepsilon} \qquad (2.13)$$

Expanding the denominator into the series over ε, we finally arrive at

$$I(p^2) = \frac{i}{16\pi^2}\Gamma(1+\varepsilon)\left(\frac{1}{\varepsilon} - \int_0^1 dx\log[\frac{p^2x(1-x)-m^2}{-\mu^2}] + \log(4\pi)\right). \qquad (2.14)$$

Comparing it with eq.(2.8) we see that the ultraviolet divergence now takes the form of the pole over ε instead of the logarithm of the cutoff. This is less visual but much simpler in the calculations and also is automatically gauge invariant.

We present below the main integrals needed for the one-loop calculations. They can be obtained via the analytical continuation from the integer values of D. We will write them down directly in the pseudo-Euclidean space.

$$\int \frac{d^Dp}{[p^2-2kp+m^2]^\alpha} = i\frac{\Gamma(\alpha-D/2)}{\Gamma(\alpha)}\frac{(-\pi)^{D/2}}{[m^2-k^2]^{\alpha-D/2}}, \qquad (2.15)$$

$$\begin{aligned}
\int \frac{d^{4-2\epsilon}p}{[p^2-2kp+m^2]^2} &= i\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{(-\pi)^{2-\epsilon}}{[m^2-k^2]^\epsilon}, \qquad \Gamma(\epsilon) \sim \frac{1}{\epsilon} \to \infty,\\
\int \frac{d^{4-2\epsilon}p\;\; p_\mu}{[p^2-2kp+m^2]^2} &= i\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{(-\pi)^{2-\epsilon}k_\mu}{[m^2-k^2]^\epsilon}, \qquad (2.16)\\
\int \frac{d^{4-2\epsilon}p\;\; p_\mu p_\nu}{[p^2-2kp+m^2]^2} &= i(-\pi)^{2-\epsilon}\left[\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{k_\mu k_\nu}{[m^2-k^2]^\epsilon} + \frac{g^{\mu\nu}}{2}\frac{\Gamma(\epsilon-1)}{\Gamma(2)}\frac{1}{[m^2-k^2]^{\epsilon-1}}\right]
\end{aligned}$$

The key formula is (2.15). All the rest can be obtained from it by the differentiation. Notice the singularity in the r.h.s. of (2.15) for $\alpha = D/2-n$, $n = 0,1,\ldots$ These integrals remain non-regularized. However, they usually do not appear in the real calculations.

Let us mention one important rule used in dimensional regularization and related to the massless theories. By definition it is accepted that zero to any power is zero. Thus, for example, the following integral is zero

$$\int \frac{d^Dk}{(k^2)^\alpha} = 0, \qquad \forall\ \alpha. \qquad (2.17)$$

In fact, here we have a cancellation of the ultraviolet and infrared divergences which both have the form of a pole over $1/\varepsilon$. There is no any inconsistency here and this way of doing is self-consistent in the calculations of dimensionally regularized integrals.

This rule leads, in particular, to the vanishing of all the diagrams of the tad-pole type in the massless case. However, in the massive case they survive and are important for the restoration of the gauge invariance. As it will be clear later, in the Standard Model the tad-poles give their contribution to the renormalization of the quark masses and provide the transversality of the vector propagator in a theory with spontaneous symmetry breaking.

3 Lecture III: Examples of Calculations. One-loop Integrals

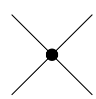
All further calculations will be performed using dimensional regularization. Below we show how the rules described above can be applied to calculate in various models of quantum field theory.

3.1 The scalar theory

We start with the simplest scalar case and consider the theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4. \tag{3.1}$$

The Feynman rules in this case are:

 $= \dfrac{i}{p^2 - m^2}$,  $= -i\lambda$

First, we find the one-loop divergent diagrams. As it follows from Fig.4, they are the propagator of the scalar field and the quartic vertex.

The propagator: In the first order there is only one diagram of the tad-pole type shown in Fig.8.

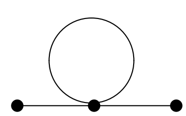


Figure 8: The one-loop propagator diagram

The corresponding integral is

$$J_1(p^2) = \frac{-i\lambda}{(2\pi)^{4-2\varepsilon}} \frac{i}{2} \int \frac{d^{4-2\varepsilon}k(\mu^2)^\varepsilon}{k^2 - m^2}, \tag{3.2}$$

where 1/2 is the combinatoric factor. Calculating the integral (3.2), according to (2.16), we find

$$J_1(p^2) = \frac{-i\lambda}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(-1+\varepsilon)}{2\Gamma(1)} m^2 (\frac{\mu^2}{m^2})^\varepsilon = \frac{i\lambda}{32\pi^2} m^2 \left[\frac{1}{\varepsilon} + 1 - \gamma_E + \log(4\pi) - \log\frac{m^2}{\mu^2}\right] \tag{3.3}$$

The fact that the integral diverges quadratically manifests itself in the structure of the multiplier $\Gamma(-1+\varepsilon)$ which has a pole at $\varepsilon = 0$ as well as at $\varepsilon = 1$. However, since we are interested in the limit $\varepsilon \to 0$, we expand the answer in the Loran series in ε. As one can see, even in the case of quadratically divergent integrals the divergence takes the form of a simple pole over ε, but the integral has the dimension equal to two. Notice, however, that for $m = 0$ the integral equals zero in accordance with the properties of dimensional regularization mentioned above.

The vertex: Here one also has only one diagram but the external momenta can be adjusted in several ways (see Fig.9). As a result the total contribution to the vertex function consists

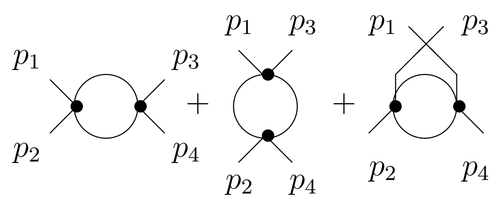


Figure 9: The one-loop vertex diagram

of three parts

$$I_1 = I_1(s) + I_1(t) + I_1(u),$$

where we introduced the commonly accepted notation for the Mandelstam variables (we assume here that the momenta p_1 and p_2 are incoming and the momenta p_3 and p_4 are outgoing)

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2,\ t = (p_1 - p_3)^2 = (p_2 - p_4)^2,\ u = (p_1 - p_4)^2 = (p_2 - p_3)^2,$$

and the integral equals

$$I_1(s) = \frac{(-i\lambda)^2}{48} \frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}} i^2 \int \frac{d^{4-2\varepsilon}k}{[k^2 - m^2][(p-k)^2 - m^2]} \tag{3.4}$$

(1/48 is the combinatoric coefficient). We have already calculated this integral and the answer has the form (2.14). Now we perform the calculation in a different and simpler way applicable to the massless integrals.

Two comments are in order. The first one concerns the evaluation of the combinatoric coefficient. It comes from the expansion of the S-matrix within the Wick theorem. In the case when all the particles are different like, for example, in QED, the combinatoric coefficient is usually 1. For identical particles their permutations are taken into account already in the Lagrangian (the factors 1/2 and /4! in (3.1)) and lead to nontrivial coefficients. There exists a simple method to calculate the combinatoric coefficient in these cases. The coefficient equals 1/Sym, where Sym is the symmetry factor of a diagram. Consider the diagram shown in Fig.9. If one does not distinguish the arrangement of momenta, then the diagram has the following symmetries: the permutation of external lines entering into the left vertex, the permutation of external lines entering into the right vertex, the permutation of the vertices, the permutation of internal lines. Altogether one has: $2 \times 2 \times 2 \times 2 = 16$. Hence, the combinatoric coefficient equals 1/16 but, since we distinguish three different momentum arrangements, one has 1/48. The same rule is valid for the multiloop diagrams and we will use it in the next section.

The second comment is related to the calculation of the massless integrals which are much simpler, and in some cases one can get the answer without any explicit integration. The method, which we will describe below, is applicable to a certain type of massless integrals and is based on conformal properties of the massless integrals depending on one external argument and uses the symmetry between the coordinate and momentum representations.

The key formula is the Fourier-transformation of the propagator of a massless particle

$$\int \frac{d^4p\ e^{ipx}}{p^2} = \frac{i\pi^2}{x^2}, \tag{3.5}$$

which can be generalized to an arbitrary dimension and any power of the propagator as follows:

$$\int \frac{d^Dp\ e^{ipx}}{(p^2)^\alpha} = i(-\pi)^{D/2}\frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}\frac{1}{(x^2)^{D/2-\alpha}}. \tag{3.6}$$

Obviously, this formula is also valid for the coordinate integration instead of momentum. This way the transition from momentum representation to the coordinate one and vice versa is performed with the help of (3.6) and is accompanied by the factor $\frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}$.

Let us go back to the diagram Fig.9. In momentum space it corresponds to the integral over the momenta running along the loop. However, in coordinate space it is just the product of the two propagators and does not contain any integration. Therefore, the integral in momentum space can be replaced by the Fourier-transform of the square of the propagator. Since in the massless case all the propagators in both momentum and coordinate representation are just the powers of p^2 or x^2, all of them are easily calculated with the help of relation (3.6).

In the case of the integral (3.4) for $m=0$ one first has to mentally transform both the propagators into coordinate space which, according to (3.6), gives the factor $(\frac{\Gamma(1-\varepsilon)}{\Gamma(1)})^2$, then multiply the obtained propagators (this gives $1/(x^2)^{2-2\varepsilon}$)) and transform the obtained result back into momentum space that gives the factor $\frac{\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}$ and the power of momenta $1/(p^2)^\varepsilon$ (the same as in the argument of the last Γ-function). Besides this, each loop contains the factor $i(-\pi)^{2-\varepsilon}$. Collecting all together one gets

$$I_1(s) = \frac{(-i\lambda)^2}{48}\frac{(\mu^2)^\varepsilon i^2}{(2\pi)^{4-2\varepsilon}}\int\frac{d^{4-2\varepsilon}k}{k^2(p-k)^2} = \frac{\lambda^2}{48}\frac{i\pi^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}}\left(\frac{\mu^2}{-s}\right)^\varepsilon\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(2-2\varepsilon)}$$

$$= \frac{i}{48}\frac{\lambda^2}{(4\pi)^{2-\varepsilon}}\left[\frac{\mu^2}{-s}\right]^\varepsilon\frac{1}{\varepsilon(1-2\varepsilon)}\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} = \frac{i}{48}\frac{\lambda^2}{16\pi^2}[\frac{1}{\varepsilon}+2-\gamma_E+\log 4\pi+\ln\frac{\mu^2}{-s}],$$

which coincides with (2.14) at $m=0$.

The described method for calculation of massless integrals is applicable to any integral depending on one external momentum (propagator type) and allows one to perform the calculations in any number of loops simply writing down the corresponding factors without explicit integration. In the case when the integral depends on more than one external momentum (like for a triangle or a box) and they cannot be put equal to zero the method is not directly applicable though some modifications are available. We do not consider them here.

The four-point vertex in the one-loop approximation thus equals (we take the common factor $1/4!\phi^4$ out of the brackets):

$$\Gamma_4 = -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left(\frac{3}{2\varepsilon}+3-\frac{3}{2}\gamma_E+\frac{3}{2}\log 4\pi+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right)\right\}. \tag{3.7}$$

As one can see, the Euler constant and the logarithm of 4π always accompany the pole term $1/\varepsilon$ and can be absorbed into the redefinition of μ^2.

3.2 Quantum electrodynamics

Consider now the calculation of the diagrams in the gauge theories. We start with quantum electrodynamics. The QED Lagrangian has the form

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2_{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu A_\mu\psi - \frac{1}{2\xi}(\partial_\mu A_\mu)^2, \tag{3.8}$$

where the electromagnetic stress tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the last term in (3.8) fixes the gauge. In what follows we choose the Feynman or the diagonal gauge ($\xi = 1$).

The Feynman rules corresponding to the Lagrangian (3.8) are shown in Fig.10.

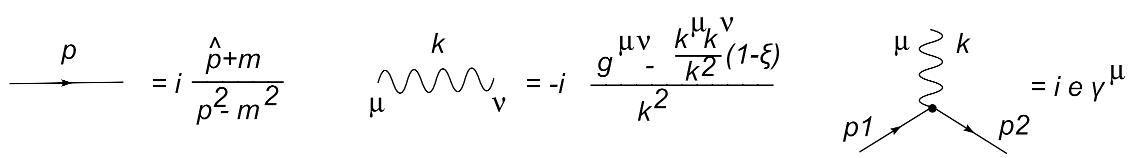


Figure 10: The Feynman rules for QED

In quantum electrodynamics the divergences appear only in the photon propagator, the electron propagator, and the triple vertex. The one-loop divergent diagrams are shown in Fig.11.

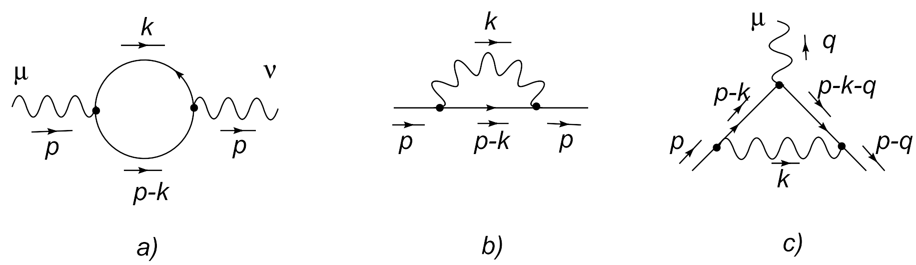


Figure 11: The one-loop divergent diagrams in QED

We begin with the vacuum polarization graph. It is given by the diagram shown in Fig. 11a). The corresponding expression looks like:

$$\Pi_{\mu\nu}(p) = (-)\frac{e^2}{(2\pi)^4}\int d^4k \frac{Tr[\gamma^\mu(m+\hat{k})\gamma^\nu(m+\hat{k}-\hat{p})]}{[m^2-k^2][m^2-(k-p)^2]}, \tag{3.9}$$

where the "-" sign comes from the fermion loop and $\hat{q} \equiv \gamma^\mu q_\mu$. We first go to dimension $4-2\epsilon$. Then the integral (3.9) becomes

$$\Pi^{Dim}_{\mu\nu}(p) = (-)\frac{e^2(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k \frac{Tr[\gamma^\mu(m+\hat{k})\gamma^\nu(m+\hat{k}-\hat{p})]}{[m^2-k^2][m^2-(k-p)^2]}, \tag{3.10}$$

Let us put $m = 0$ for simplicity. This will allow us to get a simple answer at the end. First, we calculate the trace of the γ-matrices:

$$Tr\gamma^\mu\hat{k}\gamma^\nu(\hat{k}-\hat{p}) = Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)k^\rho(k-p)^\sigma = 4k^\rho(k-p)^\sigma[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}].$$

So the integral now looks like

$$I^{Dim}_{\rho\sigma}(p) = (-)\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int\frac{d^{4-2\varepsilon}kk^\rho(k-p)^\sigma}{k^2(k-p)^2}.$$

Using the Feynman parametrization and performing the integration according to the formulae given above one finds

$$I^{Dim}_{\rho\sigma}(p) = (-)\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int_0^1 dx\int\frac{d^{4-2\varepsilon}kk^\rho(k-p)^\sigma}{[k^2-2pkx+p^2x]^2} \tag{3.11}$$

$$= (-)i\frac{(-\mu^2)^\varepsilon\pi^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}}\left\{-\Gamma(\varepsilon)\int_0^1\frac{dxp^\rho p^\sigma x(1-x)}{[p^2x(1-x)]^\varepsilon}+\Gamma(\varepsilon-1)\frac{g^{\rho\sigma}}{2}\int_0^1\frac{dx}{[p^2x(1-x)]^{\varepsilon-1}}\right\}.$$

To evaluate the remaining integrals, we use the standard integral for the Euler beta-function

$$\int_0^1 dxx^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

which gives in our case

$$\int_0^1 dxx^{1-\varepsilon}(1-x)^{1-\varepsilon} = \frac{\Gamma(2-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}.$$

Thus, the integral (3.11) becomes

$$I^{Dim}_{\rho\sigma}(p) = \frac{i}{16\pi^2}(4\pi)^\varepsilon\left(-\frac{\mu^2}{p^2}\right)^\varepsilon\frac{\Gamma^2(2-\varepsilon)\Gamma(\varepsilon)}{\Gamma(4-2\varepsilon)}\left[p^\rho p^\sigma+\frac{1}{2}\frac{g^{\rho\sigma}p^2}{1-\varepsilon}\right], \tag{3.12}$$

where we have used that $\Gamma(-1+\varepsilon) = -\frac{\Gamma(\varepsilon)}{1-\varepsilon}$. Multiplying eq.(3.12) by the trace

$$[g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma}]p^\rho p^\sigma = p^\mu p^\nu+p^\nu p^\mu-g^{\mu\nu}p^2 = 2p^\mu p^\nu-g^{\mu\nu}p^2,$$

$$[g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma}]g^{\rho\sigma}p^2 = g^{\mu\nu}p^2+g^{\mu\nu}p^2-g^{\mu\nu}(4-2\varepsilon)p^2 = -(2-2\varepsilon)p^2g^{\mu\nu},$$

we find

$$\begin{aligned}\Pi^{Dim}_{\mu\nu}(p) &= i\frac{4e^2}{16\pi^2}(4\pi)^\varepsilon\left(-\frac{\mu^2}{p^2}\right)^\varepsilon\frac{\Gamma^2(2-\varepsilon)\Gamma(\varepsilon)}{\Gamma(4-2\varepsilon)}\left[2p^\mu p^\nu-g^{\mu\nu}p^2-g^{\mu\nu}p^2\right]\\ &= -i\frac{8e^2}{16\pi^2}(4\pi)^\varepsilon\left(-\frac{\mu^2}{p^2}\right)^\varepsilon(g^{\mu\nu}p^2-p^\mu p^\nu)\frac{\Gamma^2(2-\varepsilon)\Gamma(\varepsilon)}{\Gamma(4-2\varepsilon)}.\end{aligned} \tag{3.13}$$

Expanding now over ε with the help of

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon),\ \Gamma(2-\varepsilon) = (1-\varepsilon)\Gamma(1-\varepsilon),\ \Gamma(4-2\varepsilon) = (3-2\varepsilon)(2-2\varepsilon)(1-2\varepsilon)\Gamma(1-2\varepsilon),$$

we finally get

$$\begin{aligned}\Pi^{Dim}_{\mu\nu}(p) &= -i\frac{e^2}{16\pi^2}(4\pi)^\varepsilon\left(-\frac{\mu^2}{p^2}\right)^\varepsilon(g^{\mu\nu}p^2-p^\mu p^\nu)\frac{4(1+5/3\varepsilon)}{3\varepsilon}e^{-\gamma\varepsilon}\\ &= -ie^2\frac{g^{\mu\nu}p^2-p^\mu p^\nu}{16\pi^2}\frac{4}{3}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+\frac{5}{3}\right],\\ &= i(g^{\mu\nu}p^2-p^\mu p^\nu)\Pi^{Dim}(p^2),\end{aligned} \tag{3.14}$$

where

$$\Pi^{Dim}(p^2) = -\frac{e^2}{16\pi^2}\frac{4}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log\frac{-\mu^2}{p^2} + \frac{5}{3}\right]. \tag{3.15}$$

Given the expression for the vacuum polarization one can construct the photon propagator as shown in Fig.12.

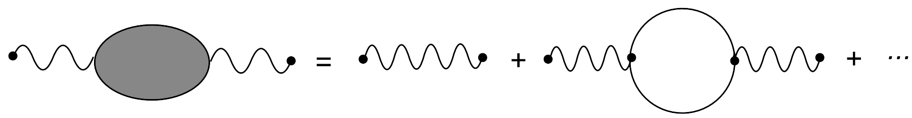


Figure 12: The photon propagator in QED

One has

$$\begin{aligned}
G_{\mu\nu}(p) &= \frac{-i}{p^2}g^{\mu\nu} + \frac{-i}{p^2}g^{\mu\rho}\Pi_{\rho\sigma}\frac{-i}{p^2}g^{\sigma\nu} + \cdots \\
&= \frac{-i}{p^2}g^{\mu\nu} - \frac{\Pi^{\mu\nu}}{p^4} + \cdots = \frac{-i}{p^2}g^{\mu\nu} - \frac{i(g^{\mu\nu} - p^\mu p^\nu/p^2)}{p^2}\Pi(p^2) + \cdots \\
&= \frac{-i}{p^2}(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})(1 + \Pi(p^2) + \cdots) - \frac{i}{p^2}\frac{p^\mu p^\nu}{p^2},
\end{aligned}$$

where $\Pi(p^2)$ is given by eq.(3.15). Notice that the radiative corrections are always proportional to the transverse tensor $P_{\mu\nu} = g_{\mu\nu} - p_\mu p_\nu/p^2$. This is a consequence of the gauge invariance and follows from the Ward identities.

Consider now the electron self-energy graph Fig.11b). The corresponding integral is

$$\Sigma(\hat{p}) = -\frac{e^2}{(2\pi)^4}\int d^4k\frac{\gamma^\mu(\hat{p} - \hat{k} + m)\gamma^\mu}{k^2[(p-k)^2 - m^2]}. \tag{3.16}$$

Acting in a usual way we go to dimension $4 - 2\varepsilon$, convert the indices of the γ-matrices and introduce the Feynman parametrization. The result is

$$\Sigma^{Dim}(\hat{p}) = -\frac{e^2(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int_0^1 dx\int\frac{d^{4-2\varepsilon}k[-2(1-\varepsilon)(\hat{p} - \hat{k}) + (4-2\varepsilon)m]}{[k^2 - 2kpx + p^2x - m^2x]^2}. \tag{3.17}$$

The integral over k can now be evaluated according to the standard formulas

$$\Sigma^{Dim}(\hat{p}) = -i\frac{e^2}{16\pi^2}\frac{(-\mu^2)^\varepsilon}{(4\pi)^{-\varepsilon}}\Gamma(\varepsilon)\int_0^1 dx\frac{-2(1-\varepsilon)\hat{p}(1-x) + (4-2\varepsilon)m}{[p^2x(1-x) - m^2x]^\varepsilon}. \tag{3.18}$$

This expression can be expanded in series in ε

$$\begin{aligned}
\Sigma^{Dim}(\hat{p}) &= -i\frac{e^2}{16\pi^2}\left[-\frac{\hat{p} - 4m}{\varepsilon} + \hat{p} - 2m - (\hat{p} - 4m)(-\gamma_E + \log(4\pi))\right. \\
&+ \left.\int_0^1 dx[2\hat{p}(1-x) - 4m]\log\frac{p^2x(1-x) - m^2x}{-\mu^2}\right].
\end{aligned} \tag{3.19}$$

Notice that the linear divergence of the integral manifests itself as a simple pole in ε, and the coefficient has the dimension equal to 1 and is Lorentz invariant (this is either $\hat{p}$ or m).

At last, consider the vertex function Fig.11c). The corresponding integral is

$$\Gamma_1(p,q) = \frac{e^3}{(2\pi)^4}\int d^4k \frac{\gamma^\nu(\hat{p}-\hat{k}-\hat{q}+m)\gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\nu}{[(p-k-q)^2-m^2][(p-k)^2-m^2]k^2}. \tag{3.20}$$

Transfer to dimension $4-2\varepsilon$ and introduce the Feynman parametrization. This gives

$$\begin{aligned}\Gamma_1^{Dim}(p,q) &= \frac{e^3(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\Gamma(3)\int_0^1 dx\int_0^x dy \\ &\times \int\frac{d^{4-2\varepsilon}k[\gamma^\nu(\hat{p}-\hat{k}-\hat{q}+m)\gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\nu]}{[((p-k-q)^2-m^2)y+((p-k)^2-m^2)(x-y)+k^2(1-x)]^3}.\end{aligned} \tag{3.21}$$

The integral over k is straightforward and gives

$$\begin{aligned}&\Gamma_1^{Dim}(p,q) = ie\frac{e^2}{16\pi^2}\frac{(-\mu^2)^\varepsilon}{(4\pi)^{-\varepsilon}}\int_0^1 dx\int_0^x dy \\ &\left\{\Gamma(1+\varepsilon)\frac{[\gamma^\nu(\hat{p}(1-x)-\hat{q}(1-y)+m)\gamma^\mu(\hat{p}(1-x)+\hat{q}y+m)\gamma^\nu]}{[(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x]^{1+\varepsilon}}\right. \\ &\left.+\frac{\Gamma(\varepsilon)}{2}\frac{\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\rho\gamma^\nu}{[(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x]^\varepsilon}\right\}.\end{aligned} \tag{3.22}$$

As one can see, the first integral is finite and the second one is logarithmically divergent. Expanding in series in ε we find

$$\begin{aligned}&\Gamma_1^{Dim}(p,q) = ie\frac{e^2}{16\pi^2}\left\{\frac{\gamma^\mu}{\varepsilon}-2\gamma^\mu-\gamma^\mu(\gamma_E-\log(4\pi))\right. \\ &- 2\gamma^\mu\int_0^1 dx\int_0^x dy\ \log\left[\frac{(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x}{-\mu^2}\right] \\ &\left.+ \int_0^1 dx\int_0^x dy\frac{\gamma^\nu(\hat{p}(1-x)-\hat{q}(1-y)+m)\gamma^\mu(\hat{p}(1-x)+\hat{q}y+m)\gamma^\nu}{(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x}\right\}.\end{aligned} \tag{3.23}$$

3.3 Quantum chromodynamics

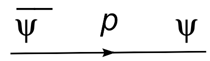
Consider now the non-Abelian gauge theories and, in particular, QCD. The Lagrangian of QCD has the form

$$\begin{aligned}\mathcal{L}_{QD} &= -\frac{1}{4}(F^a_{\mu\nu})^2+\bar\psi(i\gamma^\mu\partial_\mu-m)\psi+g\bar\psi\gamma^\mu A^a_\mu T^a\psi-\frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 \\ &+ \partial_\mu\bar{c}^a\partial_\mu c^2+gf^{abc}\partial_\mu\bar{c}^aA^b_\mu c^c,\end{aligned} \tag{3.24}$$

where the stress tensor of the gauge field is now $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$ and the last terms represent the Faddeev-Popov ghosts.

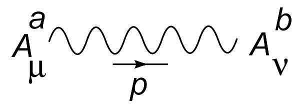
The Lagrangian (3.24) generates the following set of Feynman rules:

the spinor propagator



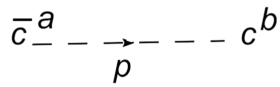
$= \frac{i}{\hat{p} - m}$

the vector propagator



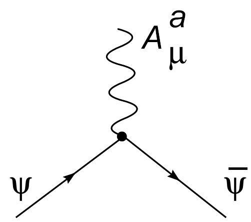
$\frac{-i\delta^{ab} g^{\mu\nu}}{p^2}$

the ghost propagator



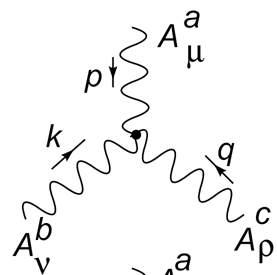
$= \frac{i\delta^{ab}}{p^2}$

the spino-gauge vertex



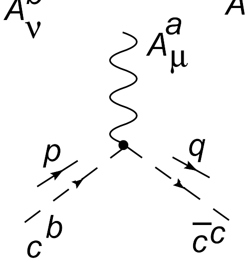
$-ig\gamma^{\mu} T^{a}$

the triple gauge vertex



$$= -gf^{abc}[(p-q)^{\rho} g^{\mu\nu} \\ +(q-k)^{\mu} g^{\rho\nu} \\ +(k-p)^{\nu} g^{\mu\rho}]$$

the ghost-gauge vertex



$= -gf^{abc} q^{\mu}$

Consider the one-loop divergent diagrams. We start with the gluon propagator. Besides the diagram shown in Fig.11), one has additional contributions to the vacuum polarization from the diagrams shown in Fig.13. The first diagram takes into account the gluon self-interaction and the second one the contribution of the Faddeev-Popov ghosts. (As has already been mentioned, the tad-pole diagrams should not be included since they are automatically zero.) These diagrams depend on the choice of the gauge, and to evaluate them we have to fix the gauge. In what follows we choose the Feynman gauge (or the diagonal gauge) for the gluon field.

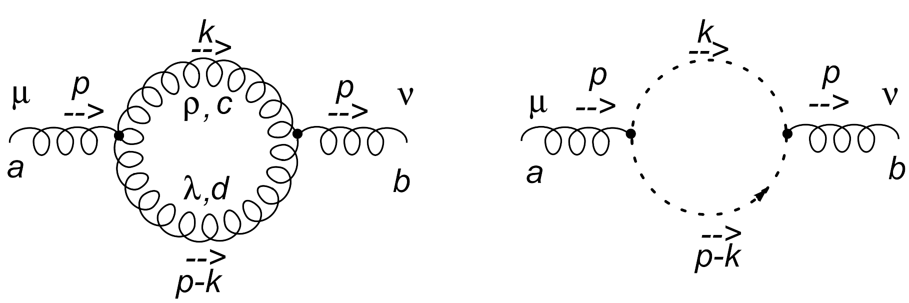


Figure 13: The vacuum polarization diagrams in the Yang-Mills theory

Then for the first diagram we have the expression

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(p) &= \frac{g^2C_A\delta^{ab}}{2(2\pi)^4}\int\frac{d^4k}{k^2(k-p)^2}[(2p-k)^\rho g^{\mu\lambda}+(2k-p)^\mu g^{\rho\lambda}-(k+p)^\lambda g^{\mu\rho}] \\ &\times\ [(2p-k)^\rho g^{\lambda\nu}-(k+p)^\lambda g^{\nu\rho}+(2k-p)^\nu g^{\rho\lambda}], \qquad (3.25)\end{aligned}$$

where 1/2 is a combinatorial factor and C_2 is the quadratic Casimir operator which for the SU(N) group equals N. It comes from the contraction of the gauge group structure constants f^{abc}

$$f^{abc}f^{dbc} = C_2\delta^{ad}.$$

Contracting the indices and going to $4-2\epsilon$ dimensions, one gets

$$\begin{aligned}&\Pi^{Dim\ (ab)}_{\mu\nu}(p) = \delta^{ab}\frac{g^2C_A}{2}\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int\frac{d^{4-2\varepsilon}k}{k^2(k-p)^2}\{g^{\mu\nu}[4p^2+k^2+(k-p)^2] \\ &+(3-2\varepsilon)(2k-p)^\mu(2k-p)^\nu-(2p-k)^\mu(2p-k)^\nu-(k+p)^\mu(k+p)^\nu\}. \qquad (3.26)\end{aligned}$$

To calculate the integrals, one can use the formulas given above. The first step is the Feynman parametrization, eq.(2.4), and then the momentum integration is performed according to eqs.(2.16). Applying these rules we get for the integral (3.26)

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = i\frac{g^2C_A\delta^{ab}}{(4\pi)^{2-\varepsilon}}\left[\frac{-\mu^2}{p^2}\right]^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}[g^{\mu\nu}p^2(\frac{19}{2}-6\varepsilon)-p^\mu p^\nu(11-7\varepsilon)]. \quad (3.27)$$

The second diagram corresponds to the integral

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = i\frac{g^2C_A\delta^{ab}}{(4\pi)^{2-\varepsilon}}\left[\frac{-\mu^2}{p^2}\right]^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}[g^{\mu\nu}p^2(\frac{19}{2}-6\varepsilon)-p^\mu p^\nu(11-7\varepsilon)]. \quad (3.28)$$

here the "-" sign comes from the Fermi statistics of the ghost fields.

Calculation is now straightforward and gives

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = i\frac{g^2C_A\delta^{ab}}{(4\pi)^{2-\varepsilon}}\left(-\frac{\mu^2}{p^2}\right)^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}[g^{\mu\nu}p^2/2+p^\mu p^\nu(1-\varepsilon)]. \quad (3.29)$$

Adding up the two contributions together, one finally has

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = iC_A\frac{2g^2\delta^{ab}}{16\pi^2}(4\pi)^\varepsilon\left[\frac{-\mu^2}{p^2}\right]^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}(5-3\varepsilon)[g^{\mu\nu}p^2-p^\mu p^\nu] \quad (3.30)$$

or expanding in ε

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = iC_A\delta^{ab}g^2\frac{g^{\mu\nu}p^2-p^\mu p^\nu}{16\pi^2}\frac{5}{3}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+\frac{31}{15}\right]. \quad (3.31)$$

Acting the same way as in QED one can calculate the contribution to the gluon propagator.

Notice that the final result for the sum of the two diagrams is again proportional to the transverse tensor $P_{\mu\nu} = g_{\mu\nu} - p_\mu p_\nu/p^2$. This is not true, however, for the diagram with the gauge fields and is valid only if one takes into account the ghost contribution. Notice also the opposite sign of the resulting expression compared to that of eq.(3.14). This is due to a

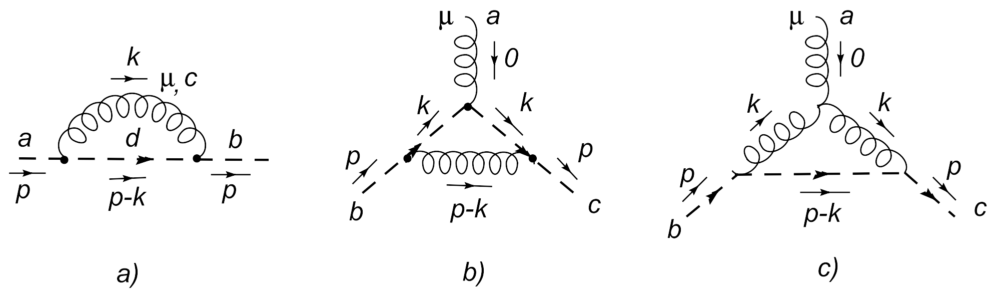


Figure 14: The ghost propagator and the ghost-gluon vertex diagrams in QCD

non-Abelian nature of the gauge fields and has very important consequences to be discussed later.

Consider also the ghost propagator. Here there is only one diagram shown in Fig.14a). It corresponds to the integral

$$\Pi^{Dim\ (ab)}(p) = -C_A\delta^{ab}\frac{g^2(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k\frac{k^\mu p^\mu}{k^2(k-p)^2}, \tag{3.32}$$

which equals

$$\begin{aligned}\Pi^{Dim\ (ab)}(p) &= -iC_A\delta^{ab}\frac{g^2}{2(4\pi)^{2-\varepsilon}}\left(\frac{-\mu^2}{p^2}\right)^\varepsilon p^2\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \\ &= -iC_A\delta^{ab}\frac{g^2}{32\pi^2}p^2\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+2\right]. \end{aligned} \tag{3.33}$$

Analogously one can calculate the vertex diagrams. We consider in more detail the calculation of the ghost-gluon vertex as a simpler one. The corresponding diagrams are shown in Fig.14. To simplify the evaluation, we put one of the momenta equal to zero. Then the first diagram gives the integral

$$V_{1\rho}^{Dim\ (abc)}(p) = i\frac{C_A}{2}f^{abc}\frac{g^3(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k\frac{k^\mu k^\rho p^\mu}{(k^2)^2(k-p)^2}. \tag{3.34}$$

Using the equality $kp = 1/2[k^2+p^2-(k-p)^2]$ and substituting it into (3.34) we find that the first two terms are reduced to the standard integrals and the last one leads to the tad-pole structure and is equal to zero. Adding up all together we get

$$\begin{aligned}V_{1\rho}^{Dim\ (abc)}(p) &= -C_A\frac{1}{4}f^{abc}\frac{g^3}{(4\pi)^{2-\varepsilon}}\left(-\frac{\mu^2}{p^2}\right)^\varepsilon p^\rho\frac{\Gamma(\varepsilon)\Gamma(2-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(3-2\varepsilon)}(1+2\varepsilon) \\ &= -C_A\frac{1}{8}f^{abc}\frac{g^3}{16\pi^2}p^\rho\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+4\right]. \end{aligned} \tag{3.35}$$

The second diagram gives

$$V_{2\rho}^{Dim\ (abc)}(p) = -i\frac{C_A}{2}f^{abc}\frac{g^3(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k\frac{(p-k)^\mu p^\nu[k^\nu g^{\mu\rho}+k^\mu g^{\nu\rho}-2k^\rho g^{\mu\nu}]}{(k^2)^2(k-p)^2}. \tag{3.36}$$

Contracting the indices in the numerator we have $(p-k)^\rho kp + p^\rho k(p-k) - 2k^\rho p(p-k)$, which after integration leads to

$$\begin{aligned} V_{2\rho}^{Dim\ (abc)}(p) &= -C_A \frac{3}{8} f^{abc} \frac{g^3}{(4\pi)^{2-\varepsilon}} \left(-\frac{\mu^2}{p^2}\right)^{\varepsilon} p^\rho \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} (1-\frac{2}{3}\varepsilon) \\ &= -C_A \frac{3}{8} f^{abc} \frac{g^3}{16\pi^2} p^\rho \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log \frac{-\mu^2}{p^2} + \frac{4}{3}\right]. \end{aligned} \tag{3.37}$$

Adding up the two contributions together we find

$$V_\rho^{Dim\ (abc)}(p) = -C_A \frac{1}{2} f^{abc} \frac{g^3}{16\pi^2} p^\rho \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log \frac{-\mu^2}{p^2} + 2\right]. \tag{3.38}$$

Having in mind that at the tree level the vertex has the form $V_\rho^{tree\ (abc)}(p) = -g f^{abc} p^\rho$ we get the vertex function in the one-loop approximation as

$$V_\rho^{(abc)}(p) = -g f^{abc} p^\rho \left\{1 + C_A \frac{1}{2} \frac{g^2}{16\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log \frac{-\mu^2}{p^2} + 2\right]\right\}. \tag{3.39}$$

4 Lecture IV: Renormalization. General Idea

Thus, we have convinced ourselves that the integrals for the radiative corrections are indeed ultraviolet divergent in accordance with the naive power counting. The question then is: how to get a sensible result for the cross-sections of the scattering processes, decay widths, etc? To answer this question let us see what is the reason for divergences at large values of momenta. In coordinate space the large values of momenta correspond to the small distances. Hence, the ultraviolet divergences allow for the singularities at small distances. Indeed, the simplest divergent loop diagram (Fig.7) in coordinate space is the product of two propagators. Each propagator is uniquely defined in momentum as well as in coordinate space, but the square of the propagator has already an ill-defined Fourier-transform, it is ultraviolet divergent. The reason is that the square of the propagator is singular as $x^2 \to 0$ and behaves like $1/(x^2)^2$. In fact, the causal Green function (the propagator) is the so-called distribution which is defined on smooth functions. It has the δ-function like singularities and needs an additional definition for the product of several such functions at a single point. The discussed diagram is precisely this product.

The general approach to the elimination of the ultraviolet divergences known as the $\mathcal{R}$-operation was developed in the 1950s. It consists in the introduction to the initial Lagrangian of additional local (or quasi-local) terms, called the counter-terms, which serve the task of the definition of the product of distributions at the coinciding points. The counter-terms lead to additional diagrams which cancel the ultraviolet divergences. The peculiarity of this procedure, being the subject of the Bogoliubov-Parasiuk theorem, is in that the singularities are local in coordinate space, i.e., are the functions of a single point and can contain only a finite number of derivatives. In the theories belonging to the renormalizable class, where the number of divergent structures is finite, the number of types of the counter-terms is also finite, they repeat the terms of the original Lagrangian. This means that the introduction of the counter-terms in this case is equivalent to the modification of the coefficients of various terms., i.e. to the modification of the normalization of these terms. That is why this procedure was called the *renormalization* procedure.

It should be stressed that the parameters of the original Lagrangian like the masses, the coupling constants and the fields themselves are not, strictly speaking, observable. They can be infinite. It is important that the renormalized parameters which enter the final answers are meaningful.

Below we show by several examples of renormalizable theories how one introduces the counter-terms into the Lagrangian, how they lead to the renormalization of the original parameters and how the renormalization procedure allows one to get finite results for the Green functions.

4.1 The scalar theory. The one-loop approximation

We start with the one-loop approximation and consider for simplicity the scalar theory (3.1). It belongs to the renormalizable type and has a finite number of ultraviolet divergent structures. The one-loop divergent diagrams in this theory were calculated in the third lecture. Here we are interested in the singular parts, i.e., the poles in ε. They are given by eqs. (3.3) and (3.7.

$$\text{The propagator}: \quad Sing\ J_1(p^2) = -im^2(\tfrac{\lambda}{16\pi^2})(-\tfrac{1}{2\varepsilon}),$$

$$\text{The vertex}: \quad Sing\ \Gamma_4(s,t,u) = -i\lambda(\tfrac{\lambda}{16\pi^2})(-\tfrac{3}{2\varepsilon}).$$

Note that the singular parts do not depend on momenta, i.e. their Fourier-transform has the form of the δ-function in coordinate space.

In order to remove the obtained singularities we add to the Lagrangian (3.1) extra terms, the counter-terms equal to the singular parts with the opposite sign (the factor i belongs to the S-matrix and does not enter into the Lagrangian), namely,

$$\Delta\mathcal{L} = \frac{1}{2\varepsilon}\frac{\lambda}{16\pi^2}\,(-\frac{m^2}{2}\phi^2) + \frac{\lambda}{16\pi^2}\frac{3}{2\varepsilon}\,(-\frac{\lambda}{4!}\phi^4). \tag{4.1}$$

These counter-terms correspond to additional vertices shown in Fig.15, where the cross

Figure 15: The one-loop counter-terms in the scalar theory

denotes the contribution corresponding to (4.1). With account taken of the new diagrams the expressions for the propagator (3.3) and the vertex (3.7) become

$$J_1(p^2) = \frac{i\lambda}{32\pi^2}m^2\left(1-\gamma_E+\log(4\pi)-\log(m^2/\mu^2)\right). \tag{4.2}$$

$$\Delta\Gamma_4 = i\lambda\left\{\frac{\lambda}{16\pi^2}\left(3-\frac{3}{2}\gamma_E+\frac{3}{2}\log(4\pi)+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right)\right\}. \tag{4.3}$$

Notice that the obtained expressions have no infinities but contain the dependence on the regularization parameter μ^2 which was absent in the initial theory. The appearance of this dependence on a dimensional parameter is inherent in any regularization and is called the dimensional transmutation, i.e., an appearance of a new scale in a theory.

What we have done is equivalent to *subtraction* of divergences from the diagrams. In doing this we have subtracted just the singular parts. This way of subtraction is called the *minimal subtraction scheme* or the MS-scheme. One can make the subtraction differently, for instance, subtract also the finite parts. It is useful to subtract the Euler constant and $\log 4\pi$ which accompany the pole terms. This subtraction scheme is called the *modified minimal subtraction scheme* or the $\overline{MS}$-scheme. It is equivalent to the redefinition of the parameter μ^2. Another popular scheme of subtraction is the so-called MOM-scheme when the subtractions are made for fixed values of momenta. For example, in the case of the vertex function one can make the subtraction at the point $s=t=u=l^2$. This subtraction is called the subtraction at a symmetric point.

The difference between various subtraction schemes is in the finite parts; in the one-loop approximation this is just the constant independent of momentum, however, in higher loops one already has momentum dependent terms. Therefore, the finite parts of the Green functions depend on a *subtraction scheme*. Note that this dependence in general is not reduced to the redefinition of the parameter μ, since there are usually a few divergent Green functions and all of them are independent.

Thus, in the three subtraction schemes discussed above we have three different values for the vertex function

$$\begin{aligned}
\Gamma_4^{MS} &= -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left[3-\frac{3}{2}\gamma_E+\frac{3}{2}\log 4\pi+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right]\right\},\\
\Gamma_4^{\overline{MS}} &= -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left[3+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right]\right\},\\
\Gamma_4^{MOM} &= -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left[\frac{1}{2}\ln\frac{l^2}{-s}+\frac{1}{2}\ln\frac{l^2}{-t}+\frac{1}{2}\ln\frac{l^2}{-u}\right]\right\}.
\end{aligned}$$

The counter-terms are also different. It is useful to write them in the following way

$$\Delta\mathcal{L} = -(Z-1)\frac{m^2}{2}\phi^2-(Z_4-1)\frac{\lambda}{4!}\phi^4, \tag{4.4}$$

where for different subtraction schemes one has

$$\begin{aligned}
Z^{MS} &= 1+\frac{1}{2\varepsilon}\frac{\lambda}{16\pi^2},\\
Z^{\overline{MS}} &= 1+[\frac{1}{2\varepsilon}+1-\gamma_E+\log(4\pi)]\frac{\lambda}{16\pi^2},\\
Z_4^{MS} &= 1+\frac{3}{2\varepsilon}\frac{\lambda}{16\pi^2},\\
Z_4^{\overline{MS}} &= 1+[\frac{3}{2\varepsilon}-3\gamma_E+3\log(4\pi)]\frac{\lambda}{16\pi^2},\\
Z_4^{MOM} &= 1+[\frac{3}{2\varepsilon}+3-3\gamma_E+3\log(4\pi)+\frac{3}{2}\ln\frac{\mu^2}{l^2}]\frac{\lambda}{16\pi^2}.
\end{aligned}\tag{4.5}$$

The Lagrangian (3.1) together with the counter-terms (4.4) can be written as

$$\mathcal{L}+\Delta\mathcal{L} = Z_2\frac{1}{2}(\partial_\mu\varphi)^2-Z\frac{m^2}{2}\varphi^2-Z_4\frac{\lambda}{4!}\varphi^4=\mathcal{L}_{Bare}, \tag{4.6}$$

where the renormalization constants Z and Z_4 are given by (4.5) and the renormalization constant Z_2 in the one-loop approximation equals 1.

Writing the "bare" Lagrangian in the same form as the initial one but in terms of the "bare" fields and couplings

$$\mathcal{L}_{Bare} = \frac{1}{2}(\partial_\mu\varphi_B)^2-\frac{m_B^2}{2}\varphi_B^2-\frac{\lambda_B}{4!}\varphi_B^4 \tag{4.7}$$

and comparing it with (4.6), we get the connection between the "bare" and renormalized quantities

$$\varphi_B=\sqrt{Z_2}\varphi,\quad m_B^2=ZZ_2^{-1}m^2,\quad \lambda_B=Z_4Z_2^{-2}\lambda. \tag{4.8}$$

Equations (4.7) and (4.8) imply that the one-loop radiative corrections calculated from the Lagrangian (4.7) with parameters chosen according to (4.8,4.5) are finite.

4.2 The scalar theory. The two-loop approximation

Consider now the two-loop diagrams. For simplicity and in order to complete all the integrations we restrict ourselves to the massless case. Since we are going to calculate the diagrams off mass shell, no infrared divergences may appear.

The propagator: In this order of PT there is only one diagram shown in Fig.16.

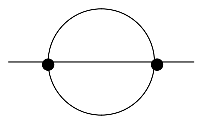


Figure 16: The two-loop propagator type diagram

The corresponding integral equals

$$J_2(p^2) = \frac{(-i\lambda)^2}{3!}\frac{i^3(\mu^2)^{2\varepsilon}}{(2\pi)^{8-4\varepsilon}}\int \frac{d^{4-2\varepsilon}k d^{4-2\varepsilon}q}{q^2(k-q)^2(p-k)^2},$$

(1/3! is a combinatorial coefficient). Let us use the method of evaluation of the massless diagrams described above. One has to transform each of the propagators into coordinate space, multiply them and transform back to momentum space. This reduces to writing down the corresponding transformation factors. One gets

$$J_2(p^2) = \frac{i\lambda^2}{6}\frac{(i\pi^2)^{2-\varepsilon}}{(2\pi)^{8-4\varepsilon}}p^2\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(-1+2\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(1)\Gamma(3-3\varepsilon)}$$

$$= \frac{i}{6}\frac{\lambda^2}{(16\pi^2)^2}\left[\frac{\mu^2}{-p^2}\right]^{2\varepsilon}\frac{p^2}{(2-3\varepsilon)(1-3\varepsilon)(1-2\varepsilon)2\varepsilon} = \frac{i}{24}\frac{\lambda^2}{(16\pi^2)^2}p^2\left[\frac{1}{\varepsilon}+\frac{13}{2}+2\ln\frac{\mu^2}{-p^2}\right],$$

where the Euler constant and $\log 4\pi$ are omitted.

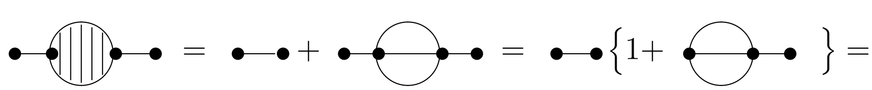
The appeared ultraviolet divergence, the pole in ε, can be removed via the introduction of the (quasi)local counter-term

$$\Delta\mathcal{L} = \frac{1}{2}(Z_2-1)(\partial\phi)^2, \tag{4.9}$$

where the wave function renormalization constant Z_2 in the $\overline{MS}$ scheme is obtained by taking the singular part of the integral with the opposite sign

$$Z_2 = 1 - \frac{1}{24\varepsilon}\left(\frac{\lambda}{16\pi^2}\right)^2. \tag{4.10}$$

After that the propagator in the massless case takes the form



$$= \frac{i}{p^2}\left\{1 - \frac{1}{24}\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{13}{2}+2\ln\frac{\mu^2}{-p^2}\right)\right\}. \tag{4.11}$$

Figure 17: The two-loop vertex diagrams

The vertex: In the given order there are two diagrams (remind that in the massless case the tad-poles equal to zero) shown in Fig.17.

The first diagram by analogy with the one-loop case equals the sum of s, t and u channels

$$I_{21} = I_{21}(s) + I_{21}(t) + I_{21}(u),$$

where each integral is nothing else but the square of the one-loop integral

$$I_{21}(s) = \frac{(-i\lambda)^3}{96}\left(\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}} i^2 \int \frac{d^{4-2\varepsilon}k}{k^2(p-k)^2}\right)^2 = -\frac{i}{96}\frac{\lambda^3}{(16\pi^2)^2}\left(\frac{1}{\varepsilon} + 2 + \ln\frac{\mu^2}{-s}\right)^2. \tag{4.12}$$

(1/96 is the combinatorial coefficient).

Opening the bracket we, for the first time here, come across the second order pole term $1/\varepsilon^2$ and the single pole $\log(-\mu^2/s)/\varepsilon$ accompanying it. This latter pole is not harmless since its Fourier-transform is not a local function of coordinates. This means that it can not be eliminated by a local counter-term. This would be an unremovable problem if it were not the one-loop counter-terms (4.1) which created the new vertices shown in Fig.15. In the same order of λ^3 one gets additional diagrams presented in Fig.18.

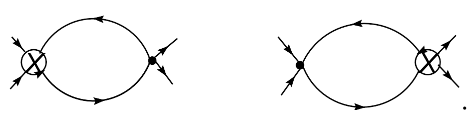
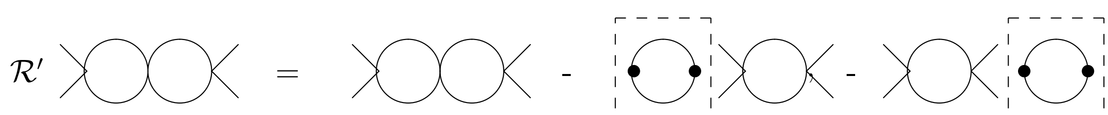


Figure 18: The diagrams with the counter-terms in the two-loop approximation

These diagrams lead to the subtraction of divergences in the subgraphs (left and right) in the first diagram of Fig.17. The subtraction of divergent subgraphs (the $\mathcal{R}$-operation without the last subtraction called the $\mathcal{R}'$-operation) looks like



where the subgraph surrounded with the dashed line means its singular part, and the rest of the graph is obtained by shrinking down the singular subgraph to a point. The result has the form

$$\mathcal{R}' I_{21}(s) = -\frac{i}{4}\frac{\lambda^3}{(16\pi^2)^2}\left\{\left(\frac{1}{\varepsilon} + 2 + \ln\frac{\mu^2}{-s}\right)^2 - \frac{2}{\varepsilon}\left(\frac{1}{\varepsilon} + 2 + \ln\frac{\mu^2}{-s}\right)\right\} =$$

$$= -\frac{i}{4}\frac{\lambda^3}{(16\pi^2)^2}\left(-\frac{1}{\varepsilon^2} + 4 + \ln^2\frac{\mu^2}{-s} + 4\ln\frac{\mu^2}{-s}\right).$$

Notice that after the subtractions of subgraphs the singular part is local, i.e. in momentum space does not contain $\ln p^2$. The terms with the single pole $1/\varepsilon$ are absent since the diagram can be factorized into two diagrams of the lower order.

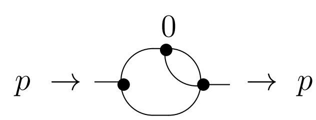
The contribution of a given diagram to the vertex function equals

$$\begin{aligned}\Delta\Gamma_4 &= -i\lambda\left\{\frac{1}{4}\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{3}{\varepsilon^2}+12\right.\right. \\ &\left.\left.+\ln^2\frac{\mu^2}{-s}+4\ln\frac{\mu^2}{-s}+\ln^2\frac{\mu^2}{-t}+4\ln\frac{\mu^2}{-t}+\ln^2\frac{\mu^2}{-u}+4\ln\frac{\mu^2}{-u}\right)\right\}\end{aligned} \tag{4.13}$$

The contribution to the renormalization constant of the four-point vertex in the $\overline{MS}$ scheme is equal to the singular part with the opposite sign

$$\Delta Z_4 = +\frac{3}{4\varepsilon^2}\left(\frac{\lambda}{16\pi^2}\right)^2. \tag{4.14}$$

The second diagram with the crossed terms contains 6 different cases. Consider one of them. Since we are interested here in the singular parts contributing to the renormalization constants, we perform some simplification of the original integral. We use a very important property of the minimal subtraction scheme that the renormalization constants depend only on dimensionless coupling constants and do not depend on the masses and the choice of external momenta. Therefore, we put all the masses equal to zero, and to avoid artificial infrared divergences, we also put equal to zero one of the external momenta. Then the diagram becomes the propagator type one:



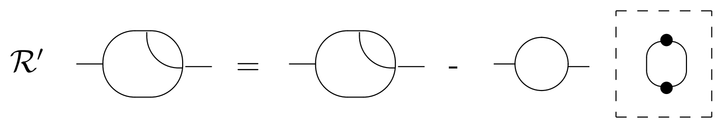
The corresponding integral is:

$$I_{22}(p^2) = \frac{(-i\lambda)^3}{48}\frac{(\mu^2)^{2\varepsilon}}{(2\pi)^{8-4\varepsilon}}i^4\int\frac{d^{4-2\varepsilon}q\,d^{4-2\varepsilon}k}{q^2(k-q)^2k^2(p-k)^2},$$

(1/48 is the combinatorial coefficient). Since putting one of the momenta equal to zero we reduced the diagram to the propagator type, we can again use the advocated method to calculate the massless integral. One has

$$\begin{aligned}I_{22}(p^2) &= \frac{i\lambda^3}{48}\frac{(\mu^2)^{2\varepsilon}}{(2\pi)^{8-4\varepsilon}}i\pi^2\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(2-2\varepsilon)}\int\frac{d^{4-2\varepsilon}k}{(k^2)^{1+\varepsilon}(p-k)^2} \\ &= -\frac{i}{48}\frac{\lambda^3}{(16\pi^2)^2}\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1-\varepsilon)\Gamma(2\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(2-2\varepsilon)\Gamma(1+\varepsilon)\Gamma(1)\Gamma(2-3\varepsilon)} \\ &= -\frac{i}{48}\frac{\lambda^3}{(16\pi^2)^2}\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{1}{2\varepsilon^2(1-2\varepsilon)(1-3\varepsilon)} \\ &= -\frac{i}{48}\frac{\lambda^3}{(16\pi^2)^2}\left\{\frac{1}{2\varepsilon^2}+\frac{5}{2\varepsilon}+2+\frac{\ln(-\mu^2/p^2)}{\varepsilon}+\ln^2\frac{\mu^2}{-p^2}+5\ln\frac{\mu^2}{-p^2}\right\}.\end{aligned}$$

As one can see, in this case we again have the second order pole in ε and, accordingly, the single pole with the logarithm of momentum. The reason of their appearance is the presence of the divergent subgraph. Here we again have to look at the counter-terms of the previous order which eliminate the divergence from the one-loop subgraph. The subtraction of divergent subgraphs (the $\mathcal{R}$-operation without the last subtraction) looks like



or

$$\mathcal{R}' I_2(s) = -\frac{i}{2}\frac{\lambda^3}{(16\pi^2)^2}\left\{\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{1}{2\varepsilon^2(1-2\varepsilon)(1-3\varepsilon)} - \left(\frac{\mu^2}{-p^2}\right)^{\varepsilon}\frac{1}{\varepsilon^2(1-2\varepsilon)}\right\}$$

$$= -\frac{i}{2}\frac{\lambda^3}{(16\pi^2)^2}\left\{\left(\frac{1}{2\varepsilon^2} + \frac{5}{2\varepsilon} + 2 + \frac{\ln(-\mu^2/p^2)}{\varepsilon} + \ln^2\frac{\mu^2}{-p^2} + 5\ln\frac{\mu^2}{-p^2}\right)\right.$$

$$\left. - \left(\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + 4 + \frac{\ln(-\mu^2/p^2)}{\varepsilon} + \frac{1}{2}\ln^2\frac{\mu^2}{-p^2} + 2\ln\frac{\mu^2}{-p^2}\right)\right\} =$$

$$= -\frac{i}{2}\frac{\lambda^3}{(16\pi^2)^2}\left\{-\frac{1}{2\varepsilon^2} + \frac{1}{2\varepsilon} - 2 + \frac{1}{2}\ln^2\frac{\mu^2}{-p^2} + 3\ln\frac{\mu^2}{-p^2}\right\}.$$

Once again, after the subtraction of the divergent subgraph the singular part is local, i.e. in momentum space does not depend on $\ln p^2$.

The contribution to the vertex function from this diagram is:

$$\Delta\Gamma_4 = -i\lambda\left\{\frac{1}{2}\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{3}{\varepsilon^2} + \frac{3}{\varepsilon} - 12 + \frac{1}{2}\ln^2\frac{\mu^2}{-p^2} + 3\ln\frac{\mu^2}{-p^2} + \dots\right)\right\} \tag{4.15}$$

and, accordingly,

$$\Delta Z_4 = \left(\frac{3}{2\varepsilon^2} - \frac{3}{2\varepsilon}\right)\left(\frac{\lambda}{16\pi^2}\right)^2. \tag{4.16}$$

Thus, due to (4.5) and (4.16) in the two-loop approximation the quartic vertex renormalization constant in the $\overline{MS}$ scheme looks like:

$$Z_4 = 1 + \frac{3}{2\varepsilon}\frac{\lambda}{16\pi^2} + \left(\frac{\lambda}{16\pi^2}\right)^2\left(\frac{9}{4\varepsilon^2} - \frac{3}{2\varepsilon}\right). \tag{4.17}$$

With taking account of the two-loop renormalization of the propagator (4.10) one has:

$$Z_\lambda = Z_4 Z_2^{-2} = 1 + \frac{3}{2\varepsilon}\frac{\lambda}{16\pi^2} + \left(\frac{\lambda}{16\pi^2}\right)^2\left(\frac{9}{4\varepsilon^2} - \frac{17}{12\varepsilon}\right). \tag{4.18}$$

The statement is that the counter-terms introduced this way eliminate all the ultraviolet divergences up to two-loop order and make the Green functions and hence the radiative corrections finite. In the case of nonzero mass, one should also add the mass counter-term.

4.3 The general structure of the R-operation

We are ready to formulate now the general procedure of getting finite expressions for the Green functions off mass shell in an arbitrary local quantum field theory. It consists of:

In any order of perturbation theory in the coupling constant one introduces to the Lagrangian the (quasi) local counter-terms. They perform the subtraction of divergences in the diagrams of a given order. The subtraction of divergences in the subgraphs is provided by the counter-terms of the lower order. After the subtraction of divergences in the subgraphs the rest of the divergences are always local. The Green functions of the given order calculated on the basis of the initial Lagrangian with account of the counter-terms are ultraviolet finite.

The structure of the counter-terms as functions of the field operators depends on the type of a theory. According to the classification discussed in the first lecture, the theories are divided into three classes: *superrenormalizable* (a finite number of divergent diagrams), *renormalizable* (a finite number of types of divergent diagrams) and *non-renormalizable* (a infinite number of types of divergent diagrams). Accordingly, in the first case one has a finite number of counter-terms; in the second case, a infinite number of counter-terms but they repeat the structure of the initial Lagrangian, and in the last case, one has an infinite number of structures with an increasing number of the fields and derivatives.

In the case of renormalizable and superrenormalizable theories, since the counter-terms repeat the structure of the initial Lagrangian, the result of the introduction of counter-terms can be represented as

$$\mathcal{L} + \Delta\mathcal{L} = \mathcal{L}_{Bare} = \mathcal{L}(\phi_B, \{g_B\}, \{m_B\}), \tag{4.19}$$

i.e., $\mathcal{L}_{Bare}$ is the same Lagrangian $\mathcal{L}$ but with the fields, masses and coupling constants being the "bare" ones related to the renormalized quantities by the multiplicative equalities

$$\phi_i^{Bare} = Z_i^{1/2}(\{g\}, 1/\varepsilon)\phi, \quad g_i^{Bare} = Z_g^i(\{g\}, 1/\varepsilon)g_i, \quad m_i^{Bare} = Z_m^i(\{g\}, 1/\varepsilon)m_i, \tag{4.20}$$

where the renormalization constants Z_i depend on the renormalized parameters and the parameter of regularization (for definiteness we have chosen $1/\varepsilon$). In some cases the renormalization can be nondiagonal and the renormalization constants become matrices.

The renormalization constants are not unique and depend on the renormalization scheme. This arbitrariness, however, does not influence the observables expressed through the renormalized quantities. We will come back to this problem later when discussing the group of renormalization. In the gauge theories Z_i may depend on the choice of the gauge though in the minimal subtraction scheme the renormalizations of the masses and the couplings are gauge invariant.

In the minimal schemes the renormalization constants do not depend on dimensional parameters like masses and do not depend on the arrangement of external momenta in the diagrams. This property allows one to simplify the calculation of the counter-terms putting the masses and some external momenta to zero, as it was exemplified above by calculation of the two-loop diagrams. In making this trick, however, one has to be careful not to create artificial infrared divergences. Since in dimensional regularization they also have the form of poles in ε, this may lead to the wrong answers.

In renormalizable theory the finite Green function is obtained from the "bare" one, i.e., is calculated from the "bare" Lagrangian by multiplication on the corresponding renormalization constant

$$\Gamma(\{p^2\}, \mu^2, g_\mu) = Z_\Gamma(1/\varepsilon, g_\mu)\Gamma_{Bare}(\{p^2\}, 1/\varepsilon, g_{Bare}), \tag{4.21}$$

where in the n-th order of perturbation theory the "bare" parameters in the r.h.s. have to be expressed in terms of the renormalized ones with the help of relations (4.20) taken in the (n-1)-th order. The remaining constant Z_Γ creates the counter-term of the n-th order of the form $\Delta\mathcal{L} = (Z_\Gamma - 1)O_\Gamma$, where the operator O_Γ reflects the corresponding Green function. If the Green function is finite by itself (for instance, has many legs), then one has to remove the divergences only in the subgraphs and the corresponding renormalization constant $Z_\Gamma = 1$.

Note that since the propagator is inverse to the operator quadratic in fields in the Lagrangian, the renormalization of the propagator is also inverse to the renormalization of the 1-particle irreducible two-point Green function

$$D(p^2, \mu^2, g_\mu) = Z_2^{-1}(1/\varepsilon, g_\mu) D_{Bare}(p^2, 1/\varepsilon, g_{Bare}). \tag{4.22}$$

The propagator renormalization constant is also the renormalization constant of the corresponding field, but the fields themselves, contrary to the masses and couplings, do not enter into the expressions for observables.

We would like to stress once more that the $\mathcal{R}$-operation works independently on the fact renormalizable or non-renormalizable the theory is. In local theory the counter-terms are local anyway. But only in renormalizable theory the counter-terms are reduced to the multiplicative renormalization of the finite number of fields and parameters.

One can perform the $\mathcal{R}$-operation for each diagram separately. For this purpose one has first of all to subtract the divergences in the subgraphs and then subtract the divergence in the diagram itself which has to be local. This serves as a good test that the divergences in the subgraphs are subtracted correctly. In this case the $\mathcal{R}$-operation can be symbolically written in a factorized form

$$\mathcal{R}G = \prod_{div.subgraphs} (1 - M_\gamma) G, \tag{4.23}$$

where G is the initial diagram, M is the subtraction operator (for instance, subtraction of the singular part of the regularized diagram) and the product goes over all divergent subgraphs including the diagram itself. By a subgraph we mean here the 1-particle irreducible diagram consisting of the vertices and lines of the diagram which is UV divergent. The 1-particle irreducible is called the diagram which can not be made disconnected by deleting of one line.

We have demonstrated above the application of the $\mathcal{R}$-operation to the two–loop diagrams in a scalar theory. Consider some other examples of diagrams with larger number of loops shown in Fig.19. They appear in the ϕ^4 theory in the three-loop approximation.

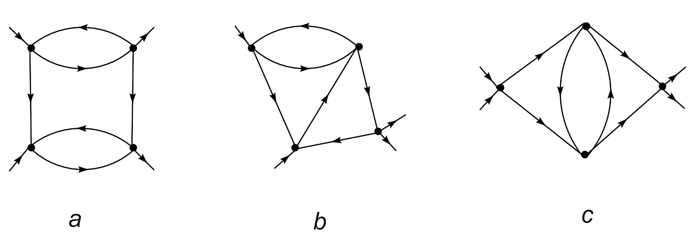


Figure 19: The multiloop diagrams in the ϕ^4 theory

In order to perform the $\mathcal{R}$-operation for these diagrams one first has to find out the divergent subgraphs. They are shown in Fig.20.

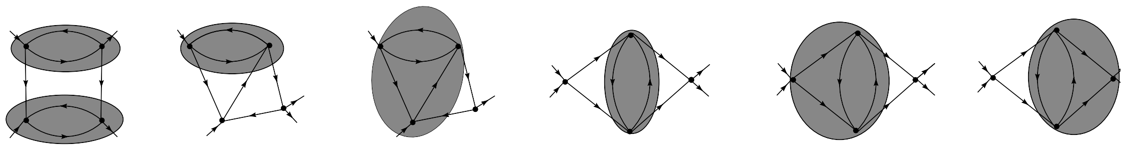


Figure 20: The divergent subgraphs in the diagrams of Fig.19

Let us use the factorized representation of the $\mathcal{R}$-operation in the form of (4.23). For the three chosen diagrams one has, respectively,

$$
\begin{aligned}
RG_a &= (1-M_G)(1-M_{\gamma_1})(1-M_{\gamma_1'})G_a, \\
RG_b &= (1-M_G)(1-M_{\gamma_2})(1-M_{\gamma_1})G, \\
RG_c &= (1-M_G)(1-M_{\gamma_2})(1-M_{\gamma_2'})(1-M_{\gamma_1})G,
\end{aligned}
$$

where γ_1 and γ_2 are the one- and two-loop divergent subgraphs shown in Fig.20.

The result of the application of the $\mathcal{R}$-operation without the last subtraction ($\mathcal{R}'$-operation) for the diagrams of interest graphically is as follows:

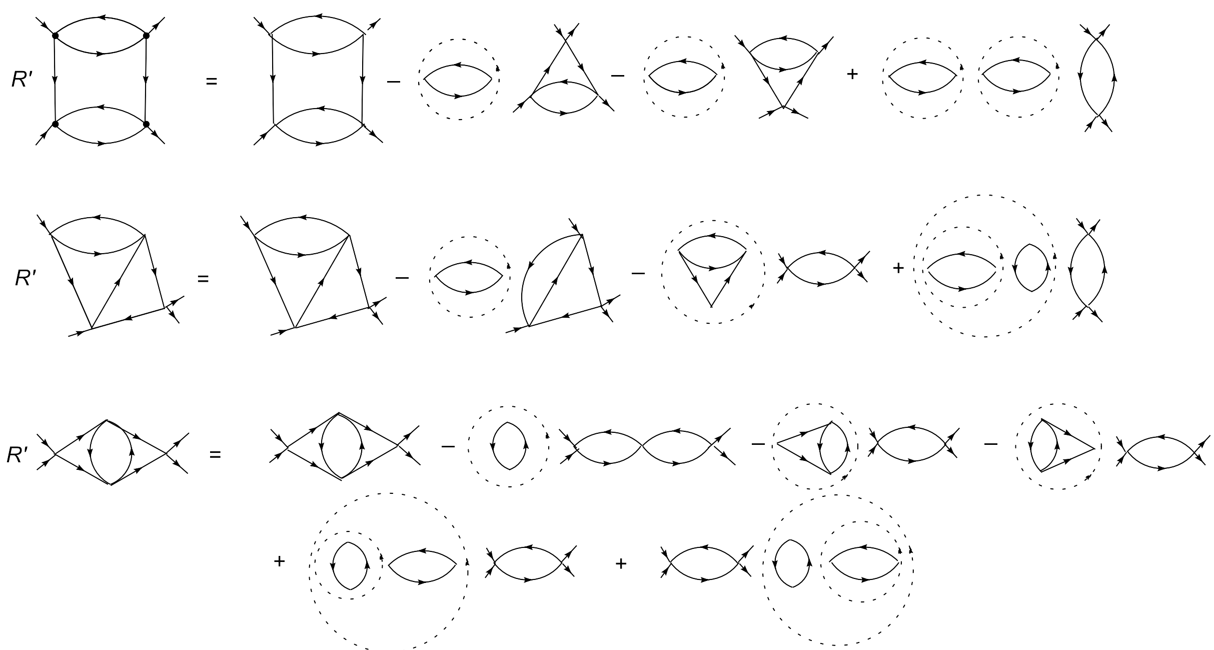
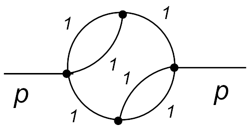


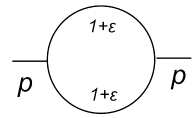
Figure 21: The $\mathcal{R}'$-operation for the multiloop diagrams

Here, as before, the graph surrounded with the dashed circle means its singular part and the remaining graph is obtained by shrinking the singular subgraph to a point.

Let us demonstrate how the $\mathcal{R}'$-operation works for the diagram Fig.19a). Since the result of the $\mathcal{R}'$-operation does not depend on external momenta, we put two momenta on the diagonal to be equal to zero so that the integral takes the propagator form. Then we can use the method based on Fourier-transform, as it was explained above. One has

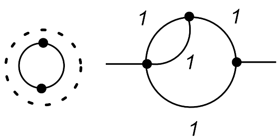


$$= \left(\Gamma(1-\varepsilon)\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\right)^2$$

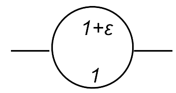


$$= \left(\Gamma(1-\varepsilon)\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\right)^2 \left(\Gamma(1-\varepsilon)\frac{\Gamma^2(1-2\varepsilon)\Gamma(3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(2-4\varepsilon)}\right)\left(\frac{\mu^2}{p^2}\right)^{3\varepsilon} \cong \frac{1}{\varepsilon^3(1-2\varepsilon)^2(1-4\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{3\varepsilon}.$$

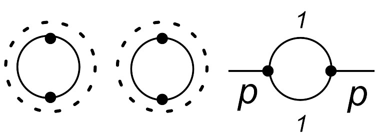
We use here the angular integration measure in the $4-2\varepsilon$ dimensional space accepted above, which results in the multiplication of the standard expression by $\Gamma(1-\varepsilon)$ in order to avoid the unwanted transcendental functions. Following the scheme shown in Fig.21 we get



$$= \frac{1}{\varepsilon}\Gamma(1-\varepsilon)\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}$$

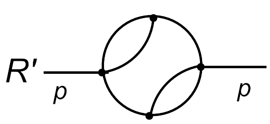


$$= \frac{1}{\varepsilon}\Gamma(1-\varepsilon)\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\Gamma(1-\varepsilon)\frac{\Gamma(1-\varepsilon)\Gamma(1-2\varepsilon)\Gamma(2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(2-3\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{2\varepsilon} \cong \frac{1}{\varepsilon^3(1-2\varepsilon)(1-3\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{2\varepsilon}.$$



$$= \frac{1}{\varepsilon^2}\Gamma(1-\varepsilon)\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{\varepsilon} \cong \frac{1}{\varepsilon^3(1-2\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{\varepsilon}.$$

Combining all together one finds



$$\cong \frac{1}{\varepsilon^3(1-2\varepsilon)^2(1-4\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{3\varepsilon} - 2\frac{1}{\varepsilon^3(1-2\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{\varepsilon} + \frac{1}{\varepsilon^3(1-2\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{\varepsilon}$$

$$= \frac{1-\varepsilon-\varepsilon^2}{\varepsilon^3}.$$

Note the cancellation of all nonlocal contributions. The singular part after the R'-operation is always local.

The realization of the $\mathcal{R}'$-operation for each diagram G allows one to find the contribution of a given diagram to the corresponding counter-term and, in the case of a renormalizable theory, to find the renormalization constant equal to

$$Z = 1 - \mathcal{K}\,\mathcal{R}'G, \tag{4.24}$$

where $\mathcal{K}$ means the extraction of the singular part. Adding the contribution of various diagrams we get the resulting counter-term of a given order and, accordingly, the renormalization constant.

5 Lecture V: Renormalization. Gauge Theories and the Standard Model

Consider now the gauge theories. The difference from the scalar case is in the relations between various renormalization constants which follow from the gauge invariance. If the regularization and the renormalization scheme do not break the symmetry these relations hold automatically. In the opposite case, this is an additional requirement imposed on the counter-terms.

5.1 Quantum electrodynamics

Quantum electrodynamics (3.8) is a renormalizable theory; hence, the counter-terms repeat the structure of the Lagrangian. They can be written as

$$\Delta\mathcal{L}_{QED} = -\frac{Z_3-1}{4}F_{\mu\nu}^2 + (Z_2-1)i\bar{\psi}\hat{\partial}\psi - m(Z-1)\bar{\psi}\psi + e(Z_1-1)\bar{\psi}\hat{A}\psi. \qquad (5.1)$$

The term that fixes the gauge is not renormalized. In the leading order of perturbation theory we calculated the corresponding diagrams with the help of dimensional regularization (see (3.15),(3.19),(3.23)). Their singular parts with the opposite sign give the proper renormalization constants. They are, respectively,

$$\begin{aligned} Z_1 &= 1-\frac{e^2}{16\pi^2}\frac{1}{\varepsilon}, \\ Z_2 &= 1-\frac{e^2}{16\pi^2}\frac{1}{\varepsilon}, \\ Z_3 &= 1-\frac{e^2}{16\pi^2}\frac{4}{3\varepsilon}, \\ Z &= 1-\frac{e^2}{16\pi^2}\frac{4}{\varepsilon}. \end{aligned} \qquad (5.2)$$

Adding (5.1) with (3.8) we get

$$\begin{aligned} &\mathcal{L}_{QED}+\Delta\mathcal{L}_{QED} = -\frac{Z_3}{4}F_{\mu\nu}^2 + Z_2 i\bar{\psi}\hat{\partial}\psi - mZ\bar{\psi}\psi + eZ_1\bar{\psi}\hat{A}\psi - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 \\ &= -\frac{1}{4}F_{\mu\nu B}^2 + i\bar{\psi}_B\hat{\partial}\psi_B - mZZ_2^{-1}\bar{\psi}_B\psi_B + eZ_1Z_2^{-1}Z_3^{-1/2}\bar{\psi}_B\hat{A}_B\psi_B \\ &\quad -\frac{Z_3^{-1}}{2\xi}(\partial_\mu A_{\mu B})^2, \end{aligned} \qquad (5.3)$$

that gives

$$\psi_B = Z_2^{1/2}\psi, \quad A_B = Z_3^{1/2}A, \quad m_B = ZZ_2^{-1}m, \quad e_B = Z_1Z_2^{-1}Z_3^{-1/2}e, \quad \xi_B = Z_3\xi. \qquad (5.4)$$

The gauge invariance here manifests itself in two places. First, the transversality of the radiative correction to the photon propagator means that the gauge fixing term is not renormalized and, hence, the gauge parameter ξ is renormalized as a gauge field. Second, the gauge invariance connects the vertex Green function and the fermion propagator (the Ward identity), which leads to the identity $Z_1 = Z_2$. Since the dimensional regularization which we use throughout the calculations does not break the gauge invariance, this identity is satisfied automatically (see (5.2)). This means that the renormalization of the coupling (5.4) is defined by the photon propagator only. Note, however, that this is not true in general in a non-Abelian theory.

5.2 Quantum chromodynamics

The complications which appear in non-Abelian theories are caused by the presence of many vertices with the same coupling as it follows from the gauge invariance. Hence, they have to renormalize the same way, i.e there appear new identities, called the Slavnov-Taylor identities. The full set of the counter-terms in QCD looks like

$$\begin{aligned}\Delta\mathcal{L}_{QD} &= -\frac{Z_3-1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g(Z_1-1)f^{abc}A^a_\mu A^b_\nu \partial_\mu A^c_\nu \\ &-(Z_4-1)\frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\mu A^e_\nu + (\tilde{Z}_3-1)\partial_\mu \bar{c}^a\partial_\mu c^a + g(\tilde{Z}_1-1)f^{abc}\partial_\mu\bar{c}^a A^b_\mu c^c \\ &+i(Z_2-1)\bar{\psi}\hat{\partial}\psi - m(Z-1)\bar{\psi}\psi + g(Z_{1\psi}-1)\bar{\psi}\hat{A}^a T^a\psi, \end{aligned} \tag{5.5}$$

that being added to the initial Lagrangian gives

$$\begin{aligned}\mathcal{L}_{QD} + \Delta\mathcal{L}_{QD} &= -\frac{Z_3}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - gZ_1 f^{abc}A^a_\mu A^b_\nu\partial_\mu A^c_\nu \\ &-Z_4\frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\mu A^e_\nu - \tilde{Z}_3\partial_\mu\bar{c}^a\partial c^a - g\tilde{Z}_1 f^{abc}\partial_\mu\bar{c}^a A^b_\mu c^c \\ &+iZ_2\bar{\psi}\hat{\partial}\psi - mZ\bar{\psi}\psi + gZ_{1\psi}\bar{\psi}\hat{A}^aT^a\psi - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 \\ &= -\frac{1}{4}(\partial_\mu A^a_{\nu B} - \partial_\nu A^a_{\mu B})^2 - gZ_1Z_3^{-3/2}f^{abc}A^a_{\mu B}A^b_{\nu B}\partial_\mu A^c_{\nu B} \\ &-Z_4Z_3^{-2}\frac{g^2}{4}f^{abc}f^{ade}A^b_{\mu B}A^c_{\nu B}A^d_{\mu B}A^e_{\nu B} + \partial_\mu\bar{c}^a_B\partial_\mu c^a_B + g\tilde{Z}_1\tilde{Z}_3^{-1}Z_3^{-1/2}f^{abc}\partial_\mu\bar{c}^a_B A^b_{\mu B}c^c_B \\ &+\frac{Z_3^{-1}}{2\xi}(\partial_\mu A^a_{\mu B})^2 + i\bar{\psi}_B\hat{\partial}\psi_B - mZZ_2^{-1}\bar{\psi}_B\psi_B + gZ_{1\psi}Z_2^{-1}Z_3^{-1/2}\bar{\psi}_B\hat{A}^a_B T^a\psi_B. \end{aligned} \tag{5.6}$$

This results in the relations between the renormalized and the "bare" fields and couplings

$$\begin{aligned} &\psi_B = Z_2^{1/2}\psi, \quad A_B = Z_3^{1/2}A, \quad c_B = \tilde{Z}_3^{1/2}c, \\ &m_B = ZZ_2^{-1}m, \quad g_B = Z_1Z_3^{-3/2}g, \quad \xi_B = Z_3\xi, \\ &Z_1Z_3^{-1} = \tilde{Z}_1\tilde{Z}_3^{-1}, \quad Z_4 = Z_1^2Z_3^{-1}, \quad Z_{1\psi}Z_2^{-1} = Z_1Z_3^{-1}. \end{aligned} \tag{5.7}$$

The last line of equalities follows from the requirement of identical renormalization of the coupling in various vertices and represents the Slavnov-Taylor identities for the singular parts.

The explicit form of the renormalization constants in the lowest approximation follows from the one-loop diagrams calculated earlier (see (3.14), (3.19), (3.23), (3.31), (3.33), (3.39). Aa usual, one has to take the singular part with the opposite sign. One has in the $\overline{MS}$ scheme

$$
\begin{aligned}
Z_2 &= 1-\frac{g^2}{16\pi^2}\frac{C_F}{\varepsilon}, \\
Z_3 &= 1+\frac{g^2}{16\pi^2}(\frac{5}{3\varepsilon}C_A-\frac{4}{3\varepsilon}T_f n_f), \\
Z &= 1-\frac{g^2}{16\pi^2}\frac{4C_F}{\varepsilon}, \\
\tilde{Z}_1 &= 1-\frac{g^2}{16\pi^2}\frac{C_A}{2\varepsilon}, \\
\tilde{Z}_2 &= 1+\frac{g^2}{16\pi^2}\frac{C_A}{2\varepsilon}, \\
Z_g &= \tilde{Z}_1\tilde{Z}_2^{-1}Z_3^{-1/2}=1-\frac{g^2}{16\pi^2}(\frac{11}{6\varepsilon}C_A-\frac{4}{3\varepsilon}T_f n_f),
\end{aligned}
\tag{5.8}
$$

where the following notation for the Casimir operators of the gauge group is used

$$f^{abc}f^{dbc}=C_A\delta^{ad},\quad (T^aT^a)_{ij}=C_F\delta_{ij},\quad Tr(T^aT^b)=T_F\delta^{ab}.$$

For the $SU(N)$ group and the fundamental representation of the fermion fields they are equal to

$$C_A=N,\quad C_F=\frac{N^2-1}{2N},\quad T_F=\frac{1}{2}.$$

5.3 The Standard Model of fundamental interactions

In the Standard Model of fundamental interactions besides the gauge interactions and the quartic interaction of the Higgs fields there are also Yukawa type interactions of the fermion fields with the Higgs field. These interactions are also renormalizable and is characterized by the Yukawa coupling constants, one for each fermion field. The peculiarity of the SM is that the masses of the fields appear as a result of spontaneous symmetry breaking when the Higgs field develops a vacuum expectation value. As a result the masses are not independent but are expressed via the coupling constant multiplied by the vacuum expectation value. Here there are two possibilities: to treat the Yukawa couplings as independent quantities and to renormalize them in a usual way and then express the renormalized masses via the renormalized couplings or to start with the masses of particles and to treat the Yukawa couplings as secondary quantities. The first approach is usually used within the minimal subtraction scheme where the renormalizations do not depend on masses. On the contrary, in the scheme when the subtraction is carried out on mass shell (the so-called "on-shell" scheme), one usually takes masses of particles as the basis. Under this way of subtraction the pole of the propagator is not shifted and the renormalized mass coincides with the mass of a physical particle. Below we consider the renormalizations in the SM in the $\overline{MS}$ scheme and concentrate on the renormalization of the fields and the couplings.

Another property of the Standard Model is that it has the gauge group $SU_c(3)\times SU_L(2)\times U_Y(1)$ which is spontaneously broken to $SU_c(3)\times U_{EM}(1)$. In the theories with spontaneously broken symmetry, according to the Goldstone theorem there are massless particles, the goldstone bosons. These particles indeed are present in the SM but they are not the physical degrees of freedom and due to the Higgs effect are absorbed by vector bosons turning into longitudinal degrees of freedom of massive vector particles.

Thus, there are two possibilities to formulate the SM as a theory with spontaneous symmetry breaking: the *unitary* formulation in which nonphysical degrees of freedom are absent and vector bosons have three degrees of freedom, and the so-called *renormalizable* formulation in which goldstone bosons are present in the spectrum and vector fields have two degrees of freedom. These two formulations correspond to two different choices of the gauge in spontaneously broken theory.

In unitary gauge we have only physical degrees of freedom, i.e., the theory is automatically unitary, hence the name of this gauge. However, the propagator of the massive vector fields in this case has the form

$$G_{\mu\nu}(k) = -i\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}}{k^2 - M^2},$$

i.e., does nor decrease when momentum goes to infinity. This leads to the increase in the power of divergences and the theory happens to be formally nonrenormalizable despite the coupling constant being dimensionless. We have mentioned this fact in the first lecture.

On the other hand, in renormalizable gauge, where the vector fields have two degrees of freedom, the propagator behaves as

$$G_{\mu\nu}(k) = -i\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2},$$

which obviously leads to a renormalizable theory which explains the name of this gauge. However, the presence of the goldstone bosons calls into question the unitarity of the theory since transitions between the physical and unphysical states become possible.

Since all the gauges are equivalent, one can work in any of them but in the unitary gauge one has to prove the renormalizability while in the renormalizable gauge one has to prove unitarity. The gauge invariance of observables preserved in a spontaneously broken theory should guarantee the fulfilment of both the requirements simultaneously. Note that in spontaneous symmetry breaking the symmetry of the Lagrangian is preserved, it is the boundary condition that breaks the symmetry.

The rigorous proof of that the theory is simultaneously renormalizable and unitary is not so obvious and eventually was awarded the Nobel prize, but can be seen by using some intermediate gauge called the R_ξ-gauge. The gauge fixing term in this case is chosen in the form

$$-\frac{1}{2\xi}(\partial_\mu A^a_\mu - \xi g F^a_i \chi_i)^2, \quad gF^a_i = \frac{v}{2}\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \\ 0 & 0 & g' \end{pmatrix},$$

where v is the vacuum expectation value of the Higs field, and χ_i are the goldstone bosons. In this gauge the vector propagator has the form

$$G_{\mu\nu}(k) = -i\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi M^2}(1-\xi)}{k^2 - M^2},$$

and at $\xi = 0$ corresponds to the renormalizable gauge while as $\xi \to \infty$ it corresponds to the unitary one. Since all the observables do not depend on ξ, we can choose $\xi = 0$ when investigating the renormalizability properties and choose $\xi = \infty$ in examining the unitarity.

Since we are interested here in the renormalizability of the SM, in what follows we will work in a renormalizable gauge.

The Lagrangian of the Standard Model consists of the following three parts:

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs}, \tag{5.9}$$

The gauge part is totally fixed by the requirement of the gauge invariance leaving only the values of the couplings as free parameters

$$\begin{aligned}\mathcal{L}_{gauge} &= -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^i_{\mu\nu} - \frac{1}{4}B_{\mu\nu}B_{\mu\nu} \\ &+ i\overline{L}_\alpha\gamma^\mu D_\mu L_\alpha + i\overline{Q}_\alpha\gamma^\mu D_\mu Q_\alpha + i\overline{E}_\alpha\gamma^\mu D_\mu E_\alpha \\ &+ i\overline{U}_\alpha\gamma^\mu D_\mu U_\alpha + i\overline{D}_\alpha\gamma^\mu D_\mu D_\alpha + (D_\mu H)^\dagger(D_\mu H),\end{aligned} \tag{5.10}$$

where the following notation for the covariant derivatives is used

$$\begin{aligned}G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc}G^b_\mu G^c_\nu, \\ W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk}W^j_\mu W^k_\nu, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ D_\mu L_\alpha &= (\partial_\mu - i\frac{g}{2}\tau^i W^i_\mu + i\frac{g'}{2}B_\mu)L_\alpha, \\ D_\mu E_\alpha &= (\partial_\mu + ig' B_\mu)E_\alpha, \\ D_\mu Q_\alpha &= (\partial_\mu - i\frac{g}{2}\tau^i W^i_\mu - i\frac{g'}{6}B_\mu - i\frac{g_s}{2}\lambda^a G^a_\mu)Q_\alpha, \\ D_\mu U_\alpha &= (\partial_\mu - i\frac{2}{3}g' B_\mu - i\frac{g_s}{2}\lambda^a G^a_\mu)U_\alpha, \\ D_\mu D_\alpha &= (\partial_\mu + i\frac{1}{3}g' B_\mu - i\frac{g_s}{2}\lambda^a G^a_\mu)D_\alpha.\end{aligned}$$

The Yukawa part of the Lagrangian which is needed for the generation of the quark and lepton masses is also chosen in the gauge invariant form and contains arbitrary Yukawa couplings (we ignore the neutrino masses, for simplicity)

$$\mathcal{L}_{Yukawa} = y^L_{\alpha\beta}\overline{L}_\alpha E_\beta H + y^D_{\alpha\beta}\overline{Q}_\alpha D_\beta H + y^U_{\alpha\beta}\overline{Q}_\alpha U_\beta \tilde{H} + h.c., \tag{5.11}$$

where $\tilde{H} = i\tau_2 H^\dagger$.

At last the Higgs part of the Lagrangian contains the Higgs potential which is chosen in such a way that the Higgs field acquires the vacuum expectation value and the potential itself is stable

$$\mathcal{L}_{Higgs} = -V = m^2 H^\dagger H - \frac{\lambda}{2}(H^\dagger H)^2. \tag{5.12}$$

Here there are two arbitrary parameters: m^2 λ. The ghost fields and the gauge fixing terms are omitted.

The Lagrangian of the SM contains the following set of free parameters:

- 3 gauge couplings g_s, g, g';
- 3 Yukawa matrices $y^L_{\alpha\beta}, y^D_{\alpha\beta}, y^U_{\alpha\beta}$;
- Higgs coupling constant λ;

- Higgs mass parameter m^2;
- the number of the matter fields (generations).

All particles obtain their masses due to spontaneous breaking of the $SU_{left}(2)$ symmetry group via a nonzero vacuum expectation value (v.e.v.) of the Higgs field

$$< H >= \begin{pmatrix} v \\ 0 \end{pmatrix}, \qquad v = m/\sqrt{\lambda}. \tag{5.13}$$

As a result, the gauge group of the SM is spontaneously broken down to

$$SU_c(3) \otimes SU_L(2) \otimes U_Y(1) \Rightarrow SU_c(3) \otimes U_{EM}(1).$$

The physical weak intermediate bosons are linear combinations of the gauge ones

$$W^{\pm}_\mu = \frac{W^1_\mu \mp iW^2_\mu}{\sqrt{2}}, \qquad Z_\mu = -\sin\theta_W B_\mu + \cos\theta_W W^3_\mu \tag{5.14}$$

with masses

$$m_W = \frac{1}{\sqrt{2}} gv, \qquad m_Z = m_W/\cos\theta_W, \qquad \tan\theta_W = g'/g, \tag{5.15}$$

while the photon field

$$\gamma_\mu = \cos\theta_W B_\mu + \sin\theta_W W^3_\mu \tag{5.16}$$

remains massless.

The matter fields acquire masses proportional to the corresponding Yukawa couplings:

$$M^u_{\alpha\beta} = y^u_{\alpha\beta} v, \;\; M^d_{\alpha\beta} = y^d_{\alpha\beta} v, \;\; M^l_{\alpha\beta} = y^l_{\alpha\beta} v, \;\; m_H = \sqrt{2} m. \tag{5.17}$$

The mass matrices have to be diagonalized to get the quark and lepton masses.

The explicit mass terms in the Lagrangian are forbidden because they are not $SU_{left}(2)$ symmetric. They would destroy the gauge invariance and, hence, the renormalizability of the Standard Model. To preserve the gauge invariance we use the mechanism of spontaneous symmetry breaking which, as was explained above, allows one to get the renormalizable theory with massive fields.

The Feynman rules in the SM include the ones for QED and QCD with additional new vertices corresponding to the $SU(2)$ group and the Yukawa interaction, as well as the vertices with goldstone particles if one works in the renormalizable gauge. We will not write them down due to their complexity, though the general form is obvious.

Consider the one-loop divergent diagrams in the SM. Besides the familiar diagrams in QED and QCD discussed above one has the diagrams presented in Fig.22. The diagrams containing the goldstone bosons are omitted. The calculation of these diagrams is similar to what we have done above. Therefore, we show only the results for the renormalization constants of the fields and the coupling constants. They have the form (for the gauge fields we use the Feynman gauge)

$$\begin{aligned}
Z_{2Q_L} &= 1 - \frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{36}g'^2 + \frac{3}{4}g^2 + \frac{4}{3}g_s^2 + \frac{1}{2}y_U^2 + \frac{1}{2}y_D^2], \\
Z_{2u_R} &= 1 - \frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{4}{9}g'^2 + \frac{4}{3}g_s^2 + y_U^2],
\end{aligned}$$

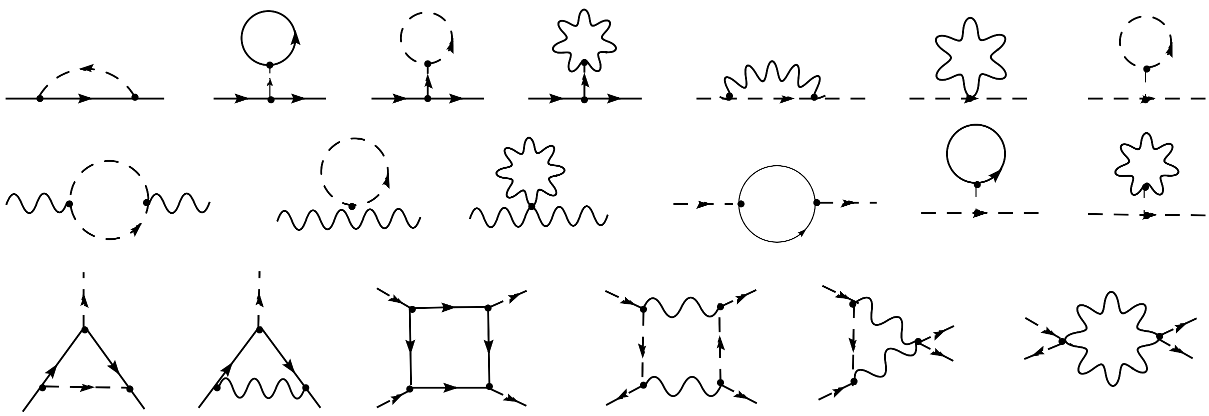


Figure 22: Some divergent one-loop diagrams in the SM. The dotted line denotes the Higgs field, the solid line - the quark and lepton fields, and the wavy line - the gauge fields

$$
\begin{aligned}
Z_{2d_R} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{9}g'^2+\frac{4}{3}g_s^2+y_D^2],\\
Z_{2L_L} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{4}g'^2+\frac{3}{4}g^2+\frac{1}{2}y_L^2],\\
Z_{2e_R} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[g'^2+y_L^2],\\
Z_{2H} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{2}g'^2+\frac{3}{2}g^2-3y_U^2-3y_D^2-y_L^2],\\
Z_{3B} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{20}{9}N_F+\frac{1}{6}N_H]g'^2 \qquad \mathrm{U(1)_Y}\ \text{boson}\\
Z_{3A} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[3-\frac{32}{9}N_F]e^2 \qquad \text{photon}\\
Z_{3W} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{10}{3}-\frac{1}{3}(N_F+3N_F)-\frac{1}{6}N_H]g^2,\\
Z_{3G} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[5-\frac{4}{3}N_F]g_s^2,\\
Z_{g_3^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-11+\frac{4}{3}N_F]g_s^2,\\
Z_{g_2^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{22}{3}+\frac{4}{3}N_F+\frac{1}{6}N_H]g^2,\\
Z_{g'^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{20}{9}N_F+\frac{1}{6}N_H]g'^2,\\
Z_{y_U^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{17}{12}g'^2-\frac{9}{4}g^2-8g_s^2+\frac{9}{2}y_U^2+\frac{3}{2}y_D^2+y_L^2],\\
Z_{y_D^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{5}{12}g'^2-\frac{9}{4}g^2-8g_s^2+\frac{3}{2}y_U^2+\frac{9}{2}y_D^2+y_L^2],\\
Z_{y_L^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{15}{4}g'^2-\frac{9}{4}g^2+\frac{9}{4}y_L^2+3y_U^2+3y_D^2],\\
Z_\lambda &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{3}{2}g'^2-\frac{9}{2}g^2+2(3y_U^2+3y_D^2+y_L^2)+6\lambda\\
&\quad -2(3y_U^4+3y_D^4+y_L^4)/\lambda+(\frac{3}{8}g'^4+\frac{9}{8}g^4+\frac{3}{4}g^2g'^2)/\lambda],
\end{aligned}
$$

where, for simplicity, we ignored the mixing between the generations and assumed the

Yukawa matrices to be diagonal.

The difference from the expressions considered above is that the renormalization constant of the scalar coupling contains the terms of the type g^4/λ and y^4/λ. This is because writing the counter-term for the quartic vertex we factorized λ. The counter-terms themselves are proportional to g^4 and y^4 and are not equal to zero. Thus, the quantum corrections generate a new interaction even if it is absent initially. Since the gauge and Yukawa interactions belong to the renormalizable type, the number of types of the counter-terms is finite and the only new interaction which is generated this way, if it was absent, is the quartic scalar one. With allowance for this interaction the model is renormalizable.

Since the masses of all the particles are equal to the product of the gauge or Yukawa couplings and the vacuum expectation value of the Higgs field, in the minimal subtraction scheme the mass ratios are renormalized the same way as the ratio of couplings. To find the renormalization of the mass itself, one should know how the v.e.v. is renormalized or find explicitly the mass counter-term from Feynman diagrams. In this case, one has also to take into account the tad-pole diagrams shown in Fig.22, including the diagrams with goldstone bosons.

For illustration we present the renormalization constant of the b-quark mass in the SM

$$\begin{aligned} Z_{m_b} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\sum_l \frac{y_l^4}{\lambda}+3\sum_q \frac{y_q^4}{\lambda}-\frac{3}{2}\lambda+\frac{3}{4}(y_b^2-y_t^2) \\ &- \frac{3}{16}\frac{(g^2+g'^2)^2}{\lambda}-\frac{3}{8}\frac{g^4}{\lambda}-3Q_b(Q_b-T_b^3)g'^2-4g_s^2]. \end{aligned} \tag{5.18}$$

The result for the t-quark can be obtained by replacing b by t. For the light quarks the Yukawa constants are very small and can be ignored in eq.(5.18).

Note that here we again have the Higgs self-interaction coupling λ in the denominator. It appears from the tad-pole diagrams but, contrary to the previous case, the renormalization constant Z_{m_q} is not multiplied by λ and the denominator is not cancelled. This does not lead to any problems in perturbation theory since by order of magnitude $\lambda \sim g^2 \sim y^2$ and the loop expansion is still valid.

6 Lecture VI: Renormalization Group

The procedure formulated above allows one to eliminate the ultraviolet divergences and get the finite expression for any Green function in any local quantum field theory. In renormalizable theories this procedure is reduced to the multiplicative renormalization of parameters (masses and couplings) and multiplication of the Green function by its own renormalization constant. This is true for any regularization and subtraction scheme. Thus, for example, in the cutoff regularization and dimensional regularization the relation between the "bare" and renormalized Green functions looks like

$$\Gamma(\{p^2\},\mu^2,\{g_\mu\}) = Z_\Gamma(\Lambda^2/\mu^2,\{g_\mu\})\Gamma_{Bare}(\{p^2\},\Lambda,\{g_{Bare}\}) \tag{6.1}$$

$$\Gamma(\{p^2\},\mu^2,\{g_\mu\}) = Z_\Gamma(1/\varepsilon,\{g_\mu\})\Gamma_{Bare}(\{p^2\},1/\varepsilon,\{g_{Bare}\}), \tag{6.2}$$

where $\{p^2\}$ is the set of external momenta, $\{g\}$ is the set of masses and couplings, and

$$g_{Bare} = Z_g((\Lambda^2/\mu^2,\{g_\mu\})g \quad \text{or} \quad g_{Bare} = Z_g((1/\varepsilon,\{g_\mu\})g.$$

It is obvious that the operation of multiplication by the constant Z obeys the group property. Indeed, after the elimination of divergences one can multiply the couplings, masses and the Green functions by finite constants and this will be equivalent to the choice of another renormalization scheme. Since these finite constants can be changed continuously, we have a continuous Lie group which got the name of renormalization group. The group transformations of multiplication of the couplings and the Green functions are called the Dyson transformations.

6.1 The group equations and solutions via the method of characteristics

In what follows we stick to dimensional regularization and rewrite relation (6.2) in the form

$$\Gamma_{Bare}(\{p^2\},1/\varepsilon,\{g_{Bare}\}) = Z_\Gamma^{-1}(1/\varepsilon,\{g_\mu\})\Gamma(\{p^2\},\mu^2,\{g_\mu\}). \tag{6.3}$$

It is obvious that the l.h.s. of this equation does not depend on the parameter of dimensional transmutation μ and, hence, the r.h.s. should not also depend on it. This allows us to write the functional equation for the renormalized Green function. Differentiating it with respect to the continuous parameter μ one can get the differential equation which has a practical value: solving this equation one can get the improved expression for the Green function which corresponds to summation of an infinite series of Feynman diagrams.

Consider an arbitrary Green function Γ obeying equation (6.2) with the normalization condition

$$\Gamma(\{p^2\},\mu^2,0) = 1.$$

Differentiating (6.2) with respect to μ^2 one gets:

$$\mu^2\frac{d}{d\mu^2}\Gamma = \left(\mu^2\frac{\partial}{\partial\mu^2} + \mu^2\frac{\partial g}{\partial\mu^2}\frac{\partial}{\partial g}\right)\Gamma = \mu^2\frac{d\ln Z_\Gamma}{d\mu^2}Z_\Gamma\Gamma_{Bare},$$

or

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta(g)\frac{\partial}{\partial g}+\gamma_\Gamma\right)\Gamma(\{p^2\},\mu^2,g_\mu)=0, \tag{6.4}$$

where we have introduced the so-called beta function $\beta(g)$ and the anomaly dimension of the Green function $\gamma_\Gamma(g)$ defined as

$$\begin{aligned}\beta &= \mu^2\frac{dg}{d\mu^2}|_{g_{bare}}, &(6.5)\\ \gamma_\Gamma &= -\mu^2\frac{d\ln Z_\Gamma}{d\mu^2}|_{g_{bare}}. &(6.6)\end{aligned}$$

Equation (6.4) is called the *renormalization group equation* in partial derivatives (in Ovsyannikov form). In the western literature it is also called the Callan-Simanzik equation.

The solution of the renormalization group equation can be written in terms of characteristics:

$$\Gamma\left(e^t\frac{\{p^2\}}{\mu^2},g\right)=\Gamma\left(\frac{\{p^2\}}{\mu^2},\bar g(t,g)\right)\mathrm{e}^{\int\limits_0^t\gamma_\Gamma(\bar g(t,g))dt}, \tag{6.7}$$

where the characteristic equation is (for definiteness we restrict ourselves to a single coupling)

$$\frac{d}{dt}\bar g(t,g)=\beta(\bar g),\quad \bar g(0,g)=g. \tag{6.8}$$

The quantity $\bar g(t,g)$ is called the *effective charge* or *effective coupling*.

We will consider the useful properties of this solution (6.7) later and we first derive several other similar equations. Since the vertex function usually comes with the coupling, one can consider the product

$$g\Gamma\left(\frac{\{p^2\}}{\mu^2},g\right). \tag{6.9}$$

If Γ is the n-point function, then the renormalization of the coupling g is given by

$$g_{Bare}=Z_\Gamma Z_2^{-n/2}g,$$

and the product (6.9) is renormalized as

$$g\Gamma=Z_2^{n/2}g_{Bare}\Gamma_{Bare}.$$

Hence, one has the same equation as (6.2) with solution (6.7) but with $Z_\Gamma=Z_2^{n/2}$ and $\gamma_\Gamma=-n/2\gamma_2$. (Recall that the anomalous dimension γ_2 is defined with respect to the renormalization constant Z_2^{-1}.)

Furthermore, one can construct the so-called *invariant charge* by multiplying the product (6.9) by the corresponding propagators

$$\xi=g\Gamma\left(\frac{\{p^2\}}{\mu^2},g\right)\prod_i^n D^{1/2}\left(\frac{p_i^2}{\mu^2},g\right). \tag{6.10}$$

The invariant charge ξ, being RG-invariant, obeys the RG equation without the anomalous dimension and plays an important role in the formulation of the renormalization group together with the effective charge. In some cases, for instance in the MOM subtraction scheme, the effective and invariant charges coincide.

The usefulness of solution (6.7) is that it allows one to sum up an infinite series of logs coming from the Feynman diagrams in the infrared ($t \to -\infty$) or ultraviolet ($t \to \infty$) regime and improve the usual perturbation theory expansions. This in its turn extends the applicability of perturbation theory and allows one to study the infrared or the ultraviolet asymptotics of the Green functions.

To demonstrate the power of the RG, let us consider the invariant charge in a theory with a single coupling and restrict ourselves to the massless case. Let the perturbative expansion be

$$\xi(\frac{p^2}{\mu^2}, g) = g(1 + bg \ln \frac{p^2}{\mu^2} + ...). \tag{6.11}$$

The β function in the one-loop approximation is given by

$$\beta(g) = bg^2. \tag{6.12}$$

Notice that the coefficient b of the logarithm in eq.(6.11) coincides with that of the β function. Alternatively the β function can be defined as the derivative of the invariant charge with respect to logarithm of momentum

$$\beta(g) = p^2 \frac{d}{dp^2} \xi(\frac{p^2}{\mu^2}, g)|_{p^2=\mu^2}. \tag{6.13}$$

This definition is useful in the MOM scheme where the mass is not considered as a coupling but as a parameter and the renormalization constants depend on it. We will come back to the discussion of this question below when considering different definitions of the mass.

According to eq.(6.7) (with vanishing anomalous dimension) the RG-improved expression for the invariant charge corresponding to the perturbative expression (6.11) is:

$$\xi_{RG}(\frac{p^2}{\mu^2}, g) = \xi_{PT}(1, \bar{g}(\frac{p^2}{\mu^2}, g)) = \bar{g}(\frac{p^2}{\mu^2}, g), \tag{6.14}$$

where we have put in eq.(6.7) $p^2 = \mu^2$ and then replaced t by $t = \ln p^2/\mu^2$. The effective coupling is a solution of the characteristic equation

$$\frac{d}{dt}\bar{g}(t, g) = b\bar{g}^2, \quad \bar{g}(0, g) = g, \quad t \equiv \ln \frac{p^2}{\mu^2}. \tag{6.15}$$

The solution of this equation is

$$\bar{g}(t, g) = \frac{g}{1 - bgt}. \tag{6.16}$$

Being expanded over t, the geometrical progression (6.16) reproduces the expansion (6.11); however, it sums the infinite series of terms of the form $g^n t^n$. This is called the leading log approximation (LLA) in QFT. To get the correction to the LLA, one has to consider the next term in the expansion of the β function. Then one can sum up the next series of terms of the form $g^n t^{n-1}$ which is called the next to leading log approximation (NLLA), etc. This procedure allows one to describe the leading asymptotics of the Green functions for $t \to \pm\infty$.

Consider now the Green function with non-zero anomalous dimension. Let its perturbative expansion be

$$\Gamma(\frac{p^2}{\mu^2}, g) = 1 + cg \ln \frac{p^2}{\mu^2} + ... \tag{6.17}$$

Then in the one-loop approximation the anomalous dimension is

$$\gamma(g) = cg. \tag{6.18}$$

Again the coefficient of the logarithm coincides with that of the anomalous dimension. In analogy with eq.(6.13) the anomalous dimension can be defined as a derivative with respect to the logarithm of momentum

$$\gamma(g) = p^2 \frac{d}{dp^2} \ln \Gamma(\frac{p^2}{\mu^2}, g)|_{p^2=\mu^2}. \tag{6.19}$$

Substituting (6.18) into eq.(6.7), one has in the exponent

$$\int_0^t \gamma(\bar{g}(t,g))dt = \int_g^{\bar{g}} \frac{\gamma(g)}{\beta(g)} dg = \int_g^{\bar{g}} \frac{cg}{bg^2} dg = \frac{c}{b} \ln \frac{\bar{g}}{g}.$$

This gives for the Green function the improved expression

$$\Gamma_{RG} = \left(\frac{\bar{g}}{g}\right)^{-c/b} = \left(\frac{1}{1-bgt}\right)^{c/b} \approx 1 + ct + ... \tag{6.20}$$

Thus, one again reproduces the perturbative expansion, but expression (6.20) again contains the whole infinite sum of the leading logs. To get the NLLA, one has to take into account the next term in eq.(6.18) together with the next term of expansion of the β function.

All the formulas can be easily generalized to the case of multiple couplings and masses.

6.2 The effective coupling

By virtue of the central role played by the effective coupling in RG formulas, consider it in more detail. The behaviour of the effective coupling is determined by the β function. Qualitatively, the β function can exhibit the behaviour shown in Fig.23. We restrict ourselves to the region of small couplings.

In the first case, the β-function is positive. Hence, with increasing momentum the effective coupling unboundedly increases. This situation is typical of most of the models of QFT in the one-loop approximation when $\beta(g) = bg^2$ and $b > 0$. The solution of the RG equation for the effective coupling in this case has the form of a geometric progression (6.16). It is characterized by the presence of a pole at high energies, called the Landau pole. We will consider this pole in detail later.

In the second case, the β-function is negative and, hence, the effective coupling decreases with increasing momentum. This situation appears in the one-loop approximation when $b < 0$, which takes place in the gauge theories. Here we also have a pole but in the infrared region.

In the third case, the β-function has zero: at first, it is positive and then is negative. This means that for small initial values the effective coupling increases; and for large ones,

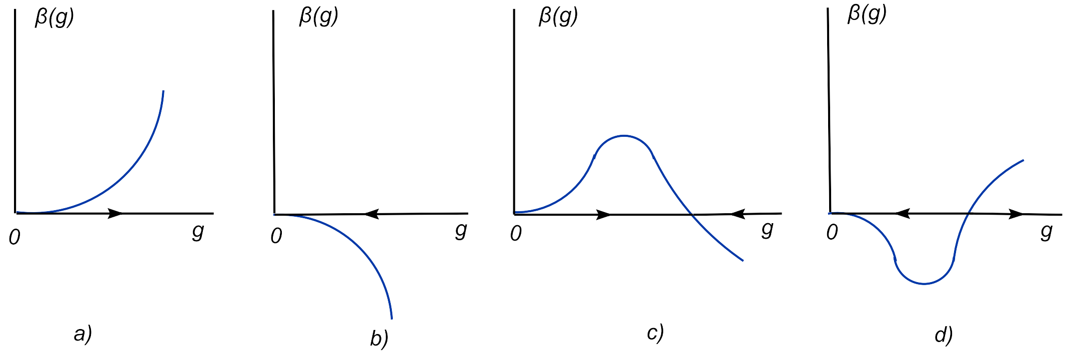


Figure 23: The possible form of the β-function. The arrows show the behaviour of the effective coupling in the ultraviolet regime ($t \to \infty$)

decreases. In both the cases, with increasing momentum it tends to the fixed value defined by the zero of the β-function. This is the so-called ultraviolet stable *fixed point*. It appears in some models in higher orders of perturbation theory.

Eventually, in the last case one also has the fixed point but now for the small initial coupling it decreases and for the large one it increases, i.e., with increasing momentum the effective coupling moves away from the fixed point, it is ultraviolet unstable. On the contrary, with decreasing momentum it tends to the fixed point, i.e., it is infrared stable. It appears in some models in lower dimensions, for instance, in statistical physics.

6.3 Dimensional regularization and the $\overline{MS}$ scheme

Consider now the calculation of the β function and the anomalous dimensions in some particular models within the dimensional regularization and the minimal subtraction scheme. Note that in transition from dimension 4 to $4-2\varepsilon$ the dimension of the coupling is changed and the "bare" coupling acquires the dimension $[g_B] = 2\varepsilon$. That is why the relation between the "bare" and renormalized coupling contains the factor $(\mu^2)^\varepsilon$

$$g_B = (\mu^2)^\varepsilon Z_g g. \tag{6.21}$$

Hence, even before the renormalization when $Z_g = 1$, in order to compensate this factor the dimensionless coupling g should depend on μ. Differentiating (6.21) with respect to μ^2 one gets

$$0 = \varepsilon Z_g g + \frac{d\log Z_g}{d\log\mu^2} Z_g g + Z_g \frac{dg}{d\log\mu^2},$$

i.e.,

$$\beta_{4-2\varepsilon}(g) \equiv \frac{dg}{d\log\mu^2} = -\varepsilon g + g\frac{d\log Z_g}{d\log\mu^2} = -\varepsilon g + \beta_4(g). \tag{6.22}$$

In the $\overline{MS}$ scheme the renormalization constants are given by the pole terms in $1/\varepsilon$ expansion and so does the bare coupling. They can be written as

$$Z_\Gamma = 1 + \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n} = 1 + \sum_{n=1}^{\infty}\sum_{m=n}^{\infty} \frac{c_{nm} g^m}{\varepsilon^n}. \tag{6.23}$$

And similarly

$$g_{Bare} = (\mu^2)^\varepsilon \left[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] = (\mu^2)^\varepsilon \left[g + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_{nm} g^{m+1}}{\varepsilon^n} \right]. \tag{6.24}$$

Differentiating eq.(6.23) with respect to $\ln \mu^2$ and having in mind the definitions (6.5) and (6.6), one has:

$$-[1 + \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n}] \gamma_\Gamma(g) = [-\varepsilon g + \beta(g)] \frac{d}{dg} \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n}.$$

Equalizing the coefficients of equal powers of ε, one finds

$$\gamma_\Gamma(g) = g \frac{d}{dg} c_1(g), \tag{6.25}$$

$$g \frac{d}{dg} c_n(g) = [\gamma_\Gamma(g) + \beta(g) \frac{d}{dg}] c_{n-1}(g), \quad n \geq 2. \tag{6.26}$$

One sees that the coefficients of higher poles c_n, $n \geq 2$ are completely defined by that of the lowest pole c_1 and the β function. In its turn the β-function is also defined by the lowest pole. To see this, consider eq.(6.24). Differentiating it with respect to $\ln \mu^2$ one has

$$\varepsilon \left[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] + [-\varepsilon g + \beta(g)] \left[1 + \frac{d}{dg} \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] = 0. \tag{6.27}$$

Equalizing the coefficients of equal powers of ε, one finds

$$\beta(g) = (g \frac{d}{dg} - 1) a_1(g), \tag{6.28}$$

$$(g \frac{d}{dg} - 1) a_n(g) = \beta(g) \frac{d}{dg} a_{n-1}(g), \quad n \geq 2. \tag{6.29}$$

Thus, knowing the coefficients of the lower poles one can reproduce all the higher order divergences. This means that they are not independent, all the information about them is connected in the lowest pole. In particular, substituting in (6.29) the perturbative expansion (6.24) one can solve the recurrent equation and find for the highest pole term

$$a_{nn} = a_{11}^n, \tag{6.30}$$

i.e. in the leading order one has the geometric progression

$$g_B = \mu^{2\varepsilon} \frac{g}{1 - g a_{11}/\varepsilon}, \tag{6.31}$$

which reflects the fact that the effective coupling in the LLA is also given by a geometric progression (6.16).

The pole equations are easily generalized for the multiple couplings case, the higher poles are also expressed through the lower ones though the solutions of the RG equations are more complicated.

Consider now some particular models and calculate the corresponding β-functions and the anomalous dimensions.

The ϕ^4 theory

The renormalization constants in the $\overline{MS}$ scheme up to two loops are given by eqs. (4.10,4.14,4.18). ($g \equiv \lambda/16\pi^2$)

$$
\begin{aligned}
Z_4 &= 1+\frac{3}{2\varepsilon}g+g^2\left(\frac{9}{4\varepsilon^2}-\frac{3}{2\varepsilon}\right), &(6.32)\\
Z_2^{-1} &= 1+\frac{g^2}{24\varepsilon}, &(6.33)\\
Z_g &= 1+\frac{3}{2\varepsilon}g+g^2\left(\frac{9}{4\varepsilon^2}-\frac{17}{12\varepsilon}\right). &(6.34)
\end{aligned}
$$

Notice that the higher pole coefficient $a_{22} = 9/4$ in the last expression is the square of the lowest pole one $a_{11} = 3/2$ in accordance with eq.(6.30).

Applying now eqs.(6.25) and (6.28) one gets

$$
\begin{aligned}
\gamma_4(g) &= \frac{3}{2}g-3g^2, &(6.35)\\
\gamma_2(g) &= \frac{1}{12}g^2, &(6.36)\\
\beta(g) &= g(\gamma_4+2\gamma_2)=\frac{3}{2}g^2-\frac{17}{6}g^2. &(6.37)
\end{aligned}
$$

One can see from eq.(6.37) that the first coefficient of the β-function is 3/2, i.e., the ϕ^4 theory belongs to the type of theories shown in Fig.23a). In the leading log approximation (LLA) one has a Landau pole behaviour. In the two-loop approximation (NLLA) the β-function gets a non-trivial zero and the effective coupling possesses an UV fixed point like the one shown in Fig.23). However, this fixed point is unstable with respect to higher orders and is not reliable. Here we encounter the problem of divergence of perturbation series in quantum field theory, they are the so-called asymptotic series which have a zero radius of convergence.

QED

In QED in the one-loop approximation the renormalization constants in the Feynman gauge are given by eq.(5.2). Due to the Ward identities the renormalization of the coupling is defined by the photon wave function renormalization constant Z_3 and is gauge invariant. Equation (5.2) allows one to determine the anomalous dimensions and the β-function

$$
\begin{aligned}
\gamma_1(\alpha) &= -\alpha, &(6.38)\\
\gamma_2(\alpha) &= \alpha, &(6.39)\\
\gamma_3(\alpha) &= \frac{4}{3}\alpha, &(6.40)\\
\gamma_m(\alpha) &= -4\alpha, &(6.41)\\
\beta_\alpha(\alpha) &= \frac{4}{3}\alpha^2, &(6.42)
\end{aligned}
$$

where we use the notation $\alpha \equiv e^2/16\pi^2$.

Thus, in QED in the one-loop approximation the effective coupling behaves the same way a in the ϕ^4 theory and has a Landau pole in the LLA. In this theory, the next term of expansion of the β-function is also calculated. It has the same sign.

QCD

In QCD the calculation of the β function can be based on various vertices. The result should be the same due to the gauge invariance. To simplify the calculations, we choose the ghost-ghost-vector vertex. The renormalization constants in the one-loop approximation in the Feynman gauge are given by (5.8) and lead to the following anomalous dimensions and the β-function:

$$\tilde{\gamma}_1(\alpha) = -\frac{C_2}{2}\alpha, \tag{6.43}$$

$$\tilde{\gamma}_2(\alpha) = -\frac{C_2}{2}\alpha, \tag{6.44}$$

$$\gamma_3(\alpha) = -(\frac{5}{3}C_2 - \frac{2}{3}n_f)\alpha, \tag{6.45}$$

$$\beta_\alpha(\alpha) = \alpha(2\tilde{\gamma}_1 + 2\tilde{\gamma}_2 + \gamma_3) = -(\frac{11}{3}C - \frac{2}{3}n_f)\alpha^2, \tag{6.46}$$

where like in QED we take $\alpha \equiv g^2/16\pi^2$, the Casimir operator C in the case of SU(3) groups is equal to 3, and n_f is the number of quark flavours.

One can see from eq.(6.46) that if the number of flavours is less than $\frac{11}{2}C_2 = \frac{33}{2}$, the β-function is negative and the effective coupling decreases and tends to zero with increasing momentum. This type of behaviour of the effective coupling is called the *asymptotic freedom*. It takes place only in gauge theories.

6.4 Λ_{QCD}

The solution of the characteristic equation for the effective coupling, which is a differential equation of the first order, depends on initial conditions. Therefore, the solution (6.16) depends on the choice of the initial point and the value of the coupling at this point. However, this choice is not unique and one can choose another initial point and another value of the coupling and still get the same solution, as it is shown in Fig.24.

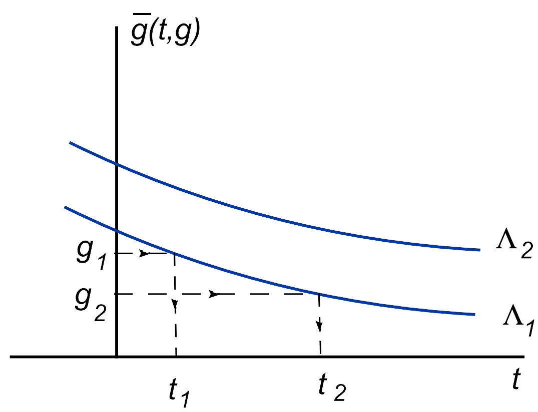


Figure 24: Different parametrizations of the effective coupling. Each curve is characterized by a single parameter Λ

In fact, every curve is not characterized by two numbers (the initial point and the coupling), but by one number and the transition from one curve to another is defined by the change of this number. To see this, consider the one-loop expression for the effective coupling in a gauge theory and rewrite it in equivalent form

$$\bar{g}(\frac{Q^2}{\mu^2}, g_\mu) = \frac{g_\mu}{1 - \beta_0 g_\mu \ln \frac{Q^2}{\mu^2}} = \frac{1}{\frac{1}{g_\mu} - \beta_0 \ln \frac{Q^2}{\mu^2}} \equiv -\frac{1}{\beta_0 \ln \frac{Q^2}{\Lambda^2}} = \bar{g}(\frac{Q^2}{\Lambda^2}), \tag{6.47}$$

where we have introduced the notation

$$\Lambda^2 = \mu^2 e^{-\frac{1}{\beta_0 \alpha_\mu}}. \tag{6.48}$$

This quantity is called Λ_{QCD} in quantum chromodynamics and can be introduced in any model. The numerical value of Λ is defined from experiment.

Equation (6.48) can be generalized to any number of loops. For this purpose, let us rewrite the RG equation for the effective coupling in the Gell-Mann – Low form. One has

$$\ln \frac{Q^2}{\mu^2} = \int_{g_\mu}^{g_Q} \frac{dg}{\beta_g(g)}. \tag{6.49}$$

Combining the lower limit with $\ln \mu^2$ one gets

$$\ln \frac{Q^2}{\Lambda^2} = \int^{g_Q} \frac{dg}{\beta_g(g)}, \tag{6.50}$$

where

$$\Lambda^2 = \mu^2 exp \left(\int^{g_\mu} \frac{dg}{\beta_g(g)} \right), \tag{6.51}$$

which is the generalization of eq.(6.48) for an arbitrary number of loops.

The quantity Λ, introduced this way, is μ-independent but depends on the renormalization scheme due to the scheme dependence of the β-function. However, the scheme dependence of Λ is given exactly (!) in one-loop order. Indeed, since Λ does not depend on μ, let us choose μ in such a way that $g_\mu \to 0$. Then for the β-function one can use the perturbative expansion

$$\beta_\alpha(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3 + ...$$

or

$$\int \frac{d\alpha}{\beta(\alpha)} = -\frac{1}{\beta_0 \alpha} + \ln \alpha + O(\alpha).$$

In this limit the ratio of two parameters Λ belonging to two different schemes is

$$\ln \frac{\Lambda_1^2}{\Lambda_2^2} = -\frac{1}{\beta_0} \left[\frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right] = -\frac{1}{\beta_0} \left[c_1 - c_2 \right], \tag{6.52}$$

where the coefficients c_1 and c_2 are calculated in the one-loop order. They can be found from perturbative expansion of any physical quantity in two different schemes

$$\begin{aligned} R &= g_1(1 + c_1 g_1 + ...) \\ &= g_2(1 + c_2 g_2 + ...). \end{aligned}$$

Since Λ does not depend on g, one can take any value of g, and eq.(6.52) is always valid. The difference $c_1 - c_2$ does not depend on a particular choice of R (though each of them depends) and is universal.

It should be noted that the quantities like the invariant or effective coupling, the β-function, etc. are not directly observable. Therefore, their dependence on the subtraction scheme does not contradict the independence of predictions of the method of calculations. We perform the perturbative expansion over the coupling which is scheme dependent, but the coefficients are also scheme dependent. As a result, within the given accuracy defined by the order of perturbation theory the answer is universal.

In the minimal subtraction schemes when the renormalizations depend only on dimensionless couplings, the one-loop renormalization constants and hence the anomalous dimensions and the β-function are the same in all schemes; the difference starts from two loops. The exception is the β-function in a theory with a single coupling like QED, QCD or the ϕ^4 theory, where the difference starts from three loops. Indeed, if one has two subtraction schemes M_1 and M_2 so that the couplings in two schemes are related by

$$g_2 = q(g_1) = g_1 + c g_1^2 + O(g_1^3),$$

then the β-functions $\beta_1(g_1)$ and $\beta_2(g_2)$ are connected by the relation

$$\beta_2(g_2) = \frac{dq(g_1)}{dg_1}\beta_1(g_1)$$

and their perturbative expansions are

$$\begin{aligned}
\beta_1(g_1) &= \beta_0 g_1^2 + \beta_1 g_1^3 + \beta_2 g_1^4 + ...,\\
\beta_2(g_2) &= \beta_0 g_2^2 + \beta_1 g_2^3 + \beta_2' g_2^4 + ...
\end{aligned}$$

so that the first two terms of the β-function are universal.

As for the further terms of expansion, they depend on the renormalization scheme and one can use this dependence as discretion, for instance, one can put all of them equal to zero. Then we would have an exact β-function. However, one should have in mind that it is not valuable by itself but rather in the aggregate with the PT expansion for the Green functions for which we construct the solution of the RG equation. This expansion in our "exact" scheme is unknown.

6.5 The running masses

In the minimal subtraction scheme the renormalization of the mass is performed the same way as the renormalization of the couplings, i.e., the mass is treated as an additional coupling and is renormalized multiplicatively, namely,

$$m_{Bare} = Z_m m,$$

where the mass renormalization constant Z_m is independent of the mass parameters and depends only on dimensionless couplings. Then, in full analogy with the effective coupling one can introduce the *effective* or the "running" mass

$$\frac{d}{dt}\bar{m}(t,g) = \bar{m}\gamma_m(\bar{g}), \quad \bar{m}(0,g) = m_0. \tag{6.53}$$

Solving this equation together with the equation for the effective coupling (6.8) one has

$$\bar{m}(t,g) = m_0 e^{\int_0^t \gamma_m(\bar{g}(t,g))dt} = m_0 e^{\int_g^{\bar{g}} \frac{\gamma(g)}{\beta(g)}dg}. \qquad (6.54)$$

In the one-loop order

$$\beta(\alpha) = b\alpha^2, \quad \gamma_m(\alpha) = c\alpha$$

and the solution is

$$m(t) = m_0 \left(\frac{\alpha(t)}{\alpha_0}\right)^{c/b}.$$

This is the running mass!

The natural question arises: what is the physical mass measured in experiment and how is it related to the running mass and at what scale?

To answer this question, consider why the mass is running. This is due to the radiative corrections. If one considers the value of momentum which is bigger than the mass, i.e. $p^2 > m^2$, then the particles are created, they are running inside the loops and give the contribution to the running. On the contrary, if $p^2 < m^2$, particles are not created, they "decouple" and do not contribute to the running. In the MOM scheme this takes place automatically because for the momentum smaller than the mass the diagram simply disappears. In the minimal scheme, on the contrary, this does not happen. Hence, it is quite natural in this case to stop the running at the value of $p^2 = m^2$ and to identify the physical mass with the running mass at the scale of the mass, i.e

$$m^2 = \bar{m}^2(m^2).$$

However, this is true only up to finite corrections. Let us come back to the definition of the mass term in the Lagrangian. It is chosen in such a way that the propagator of a particle, which is the inverse to the quadratic form, has the pole at $p^2 = m^2$. Therefore, a more appropriate definition of the physical mass is the position of the pole of the propagator with allowance for the radiative corrections, .i.e.,

physical mass $\equiv$ pole mass

This definition of a mass does not depend on a scale and it is also scheme independent and may have physical meaning. The pole mass can be expressed through the running mass at the scale of a mass with finite and calculable corrections.

Consider as an example the quark mass in QCD. The quark propagator is graphically presented in Fig.25.

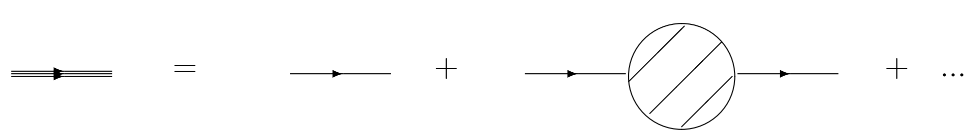


Figure 25: The quark propagator

The corresponding expression is

$$G(\hat{p},m) = \frac{i}{\hat{p}-m} + \frac{i}{\hat{p}-m}(iA\hat{p}+iBm)\frac{i}{\hat{p}-m} + ...$$
$$= \frac{i}{\hat{p}-m}\left[1 - \frac{A\hat{p}+Bm}{\hat{p}-m} + ...\right] = \frac{i}{\hat{p}-m}\;\frac{1}{1+\frac{A\hat{p}+Bm}{\hat{p}-m}} = \frac{i}{\hat{p}-m+A\hat{p}+Bm}.$$

The pole mass is now defined as a root of the equation

$$\hat{p}(1+A(p^2)) - m(1-B(p^2)) = 0, \tag{6.55}$$

which gives in the lowest order

$$m_{pole} = m\frac{1-B(m^2)}{1+A(m^2)} = m[1-A(m^2)-B(m^2)].$$

To calculate the functions A and B, consider the one-loop diagram shown in Fig.26.

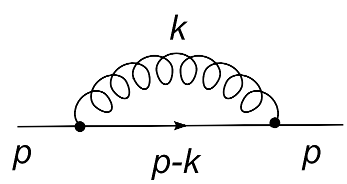


Figure 26: The quark propagator in one loop in QCD

The corresponding expression is

$$\Sigma = -\frac{g_s^2}{(2\pi)^4}C_F\int\frac{dk\;\gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\nu}{[(p-k)^2-m^2]}\frac{g^{\mu\nu}}{k^2} \tag{6.56}$$

and was calculated earlier. The result has the form (3.19)

$$A(p^2,m^2) = \frac{g_s^2}{16\pi^2}C_F\left[\frac{1}{\varepsilon}-1-2\int_0^1 dx(1-x)\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right], \tag{6.57}$$

$$B(p^2,m^2) = \frac{g_s^2}{16\pi^2}C_F\left[-\frac{4}{\varepsilon}+2+4\int_0^1 dx\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right]. \tag{6.58}$$

After subtraction of divergences in the $\overline{MS}$-scheme one has

$$A^{\overline{MS}}(p^2,m^2) = -\frac{g_s^2}{16\pi^2}C_F\left[1+2\int_0^1 dx(1-x)\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right], \tag{6.59}$$

$$B^{\overline{MS}}(p^2,m^2) = \frac{g_s^2}{16\pi^2}C_F\left[2+4\int_0^1 dx\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right]. \tag{6.60}$$

Substituting $p^2=m^2$, one finds

$$A^{\overline{MS}}(m^2,m^2) = 2+\ln\frac{\mu^2}{m^2},\quad B^{\overline{MS}}(m^2,m^2) = -6-4\ln\frac{\mu^2}{m^2}. \tag{6.61}$$

Thus, for the radiative correction to the pole mass we have

$$m_{pole} = m(\mu)\left[1 + \frac{\alpha_s C_F}{4\pi}(4 + 3\ln\frac{\mu^2}{m^2})\right]. \tag{6.62}$$

Substituting $C_F = 4/3$ and $\mu^2 = m^2$ one obtains the desired relation between the pole mass and the running mass at the mass scale

$$m_{pole} = m(m)\left[1 + \frac{4}{3}\frac{\alpha_s}{\pi}\right]. \tag{6.63}$$

7 Lecture VII: Zero Charge and Asymptotic Freedom

Since the behaviour of the effective coupling has so essential consequences we consider two typical examples which are realized in quantum field theory in the one-loop approximation and presumably take place in a full theory. Usually, one speaks about the zero charge behaviour or the asymptotic freedom. We explain below what it means.

7.1 The zero charge

The notion of the zero charge appeared in QED in the leading log approximation. This is what takes place within the renormalization group method in the one-loop approximation. If one writes down the expression for the renormalized coupling as a function of the "bare" coupling, i.e. inverts eq.(6.31), one gets

$$g = \frac{g_B}{1+\beta_0 g_B/\varepsilon} = \frac{g_B}{1+\beta_0 g_B \log\Lambda^2}, \tag{7.1}$$

where the first coefficient of the β-function $\beta_0 > 0$. Then, removing the regularization, i.e., for $\varepsilon \to 0$ or $\Lambda \to \infty$, the renormalized coupling tends to zero independently of the value of the "bare" coupling. This is what is called the *zero charge*. For the effective coupling considered above the zero charge corresponds to the behaviour shown on the left panel of Fig.27 which is characterized by the Landau pole at high energies.

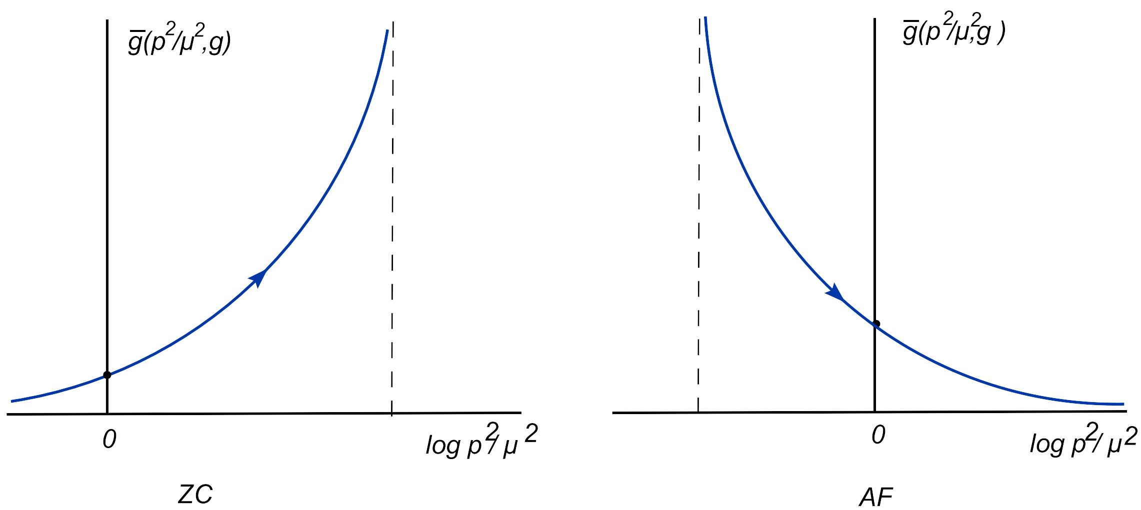


Figure 27: The behaviour of the effective coupling: the zero charge (left) and the asymptotic freedom (right)

The zero charge behaviour is typical of QED, the ϕ^4 theory for positive quartic coupling and also the Yukawa type interactions, i.e., in those theories where the β-function is positive.

It is obvious that in the vicinity of the pole the perturbation theory does not work and, hence, the one-loop formula is not applicable. However, for small momenta transfer the one-loop approximation is reliable. For instance, in QED the effective expansion parameter is $e^2/16\pi^2 = \alpha/4\pi \approx 1/137/4\pi \approx 5.8 \cdot 10^{-4}$ and the next loop corrections (which have the same sign) do not play any essential role. The behaviour of the effective coupling in QED in the region up to 100 GeV has got the experimental confirmation in measuring the fine structure constant at the LEP accelerator. At the scale equal to the mass of the Z-boson M_Z the fine structure constant is not 1/137 but $\alpha(M_Z) \approx 1/128$, which is in a good agreement with the one-loop formula.

The large momenta transfer in this case are limited by the pole provided the pole does not disappear in a full theory. It is still unclear how higher orders of perturbation theory influence this behaviour since the perturbation series is divergent and it is impossible to make definite conclusions without additional nonperturbative information.

The presence of the Landau pole indicates the presence of unphysical ghost states. To see this, consider the photon propagator in QED which due to the Ward identities coincides with the invariant charge and in the leading log approximation has the form of a geometric progression

$$G(p^2) = -i\frac{g^{\mu\nu} - p^\mu p^\nu/p^2}{p^2}\frac{1}{1 - \frac{4}{3}\sum Q^2 \frac{\alpha_0}{4\pi}\log(-p^2/m^2)}, \tag{7.2}$$

where Q is the electric charge of a particle (in the units of electron charge) running round the loop.

This expression has a pole in the Euclidean region at $p^2 = -m^2 exp(\frac{3\pi}{\alpha_0 Q^2 n_f})$. Substituting $m = m_e = 0.5$ MeV, $\alpha_0 \simeq 1/137$ and $\sum Q^2 = [(4/9 + 1/9)3 + 1)3] = 8$, one gets $p^2 \simeq -(5 \cdot 10^{31})^2$ GeV^2. That is the pole is very far off, even beyond the Planck scale, and at low energies one can ignore it. However, the presence of the pole indicates the presence of a new asymptotic state and the residue at the pole defines the norm of this state. In the case of the Landau pole the residue is negative, i.e., the new state is a ghost, it has the wrong sign of the kinetic term in the Lagrangian. This fact, in its turn, leads to negative probabilities, which indicates internal inconsistency of the theory.

Usually, it is assumed that there are two ways out of this trouble: either the higher order corrections improve the behaviour of the theory at high momenta so that the Landau pole disappears, or that the zero charge theory is contradictory by itself, but at high energies it is part of a more general theory where the behaviour of the coupling is improved. The example of such a behaviour is given by the Grand Unified Theories where QED is one of the branches of a non-Abelian gauge theory with the asymptotically free behaviour. In both the cases the theory at high energies is modified. At the same time, the zero charge theory is infrared free, i.e. for small momenta transfer the coupling goes to zero.

7.2 The asymptotic freedom

The name *asymptotic freedom* originates from the non-Abelian gauge theories where it was found that the sign of the first coefficient of the β-function is negative. The effective coupling in this case behaves as is shown in the right panel of Fig.27 and tends to zero at high momenta transfer. This means that quarks in QCD are quasi-free particles, i.e., practically do not interact. This way one explains the success of the so-called parton model of the strong interactions at high energies, according to which the proton behaves as a set of free partons, and at high energies the interaction takes place with the individual partons and their interaction does not play any role.

The behaviour of the effective coupling in QCD at high energies was tested at various accelerators and in various experiments and the validity of the renormalization group formula was confirmed. The accuracy of modern measurements assumes the inclusion of the next terms of perturbative expansion. In QCD in the $\overline{MS}$ scheme the four terms of the β-function are known. Below we present the two-loop expression

$$\beta_\alpha(\alpha_s) = -\frac{1}{4\pi}[11 - \frac{2}{3}n_f]\alpha_s^2 - \frac{1}{(4\pi)^2}[102 - \frac{38}{3}n_f]\alpha_s^3 + O(\alpha_s^4). \tag{7.3}$$

As one can see, if the number of quarks in not too big, both the coefficients of the β-function are negative. All the experimental data fit a single curve for the effective coupling with the parameter $\Lambda_{QCD} \simeq 200$ MeV (see Fig.28)

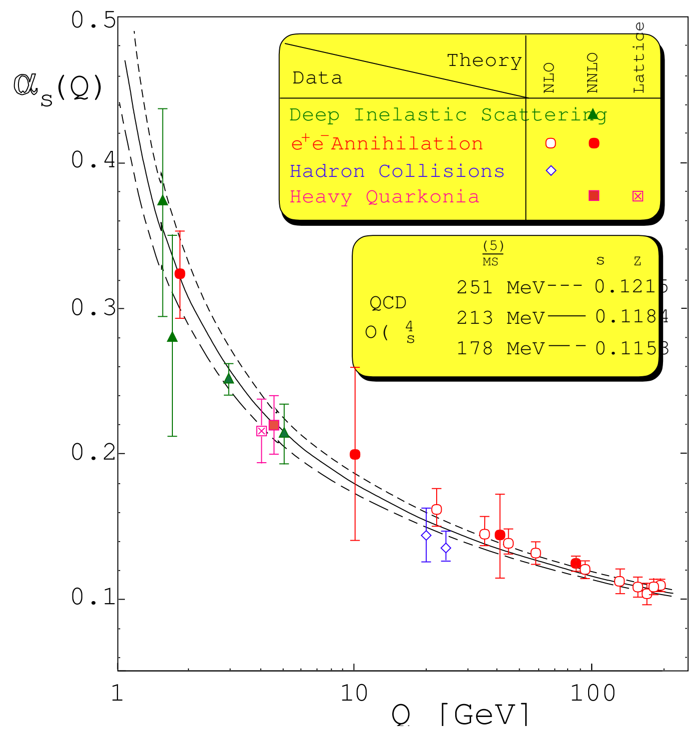


Figure 28: The variation of the effective coupling of the strong interactions α_s with energy

In four-dimensional space the asymptotic freedom occurs only in non-Abelian gauge theories. But in the case when one has several interactions, like in the Standard Model, the non-Abelian coupling may draw other couplings into the asymptotically free region. Consider, for instance, the behaviour of the Yukawa couplings in the SM. For simplicity, let us take a single Yukawa coupling for the t-quark and a single gauge coupling. Then in the one-loop approximation the equations for the effective couplings look like

$$\begin{aligned} \frac{dg}{dt} &= -bg^2, \quad g \equiv \frac{g_s^2}{16\pi^2}, \\ \frac{dy}{dt} &= y(ay - cg), \quad y \equiv \frac{y_t^2}{16\pi^2}, \quad t \equiv \log\frac{q^2}{q_0^2}, \end{aligned} \tag{7.4}$$

where the coefficients b, a and c are always positive and for the SM are equal to $7, 9/2$ and 8, respectively. The solutions to these equations are

$$\begin{aligned} g &= \frac{g_0}{1+bg_0t}, \quad y=\frac{y_0E}{1-ay_0F}, \\ E(t) &= (g/g_0)^{c/b}, \quad F(t)=\int_0^t E(t')dt'. \end{aligned} \tag{7.5}$$

In the case of a single Yukawa coupling it can be written in an explicit form

$$y=\frac{y_0(\frac{g}{g_0})^{c/b}}{1+\frac{y_0}{g_0}\frac{a}{c-b}[(\frac{g}{g_0})^{c/b-1}-1]}. \tag{7.6}$$

Graphically, it can be presented in a phase diagram shown in Fig.29. For the initial condition

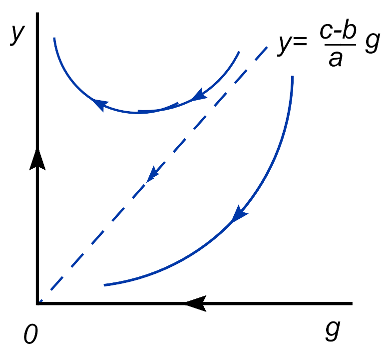


Figure 29: The behaviour of the Yukawa and gauge couplings for various initial conditions

such that $y_0 > (c-b)/a\ g_0$ the Yukawa coupling increases with momenta and has the Landau pole, while for $y_0 \leq (c-b)/a\ g_0$ it demonstrates the asymptotically free behaviour. In a similar way in the Grand Unified Theories one can reach the asymptotic freedom for all the couplings.

The back side of the asymptotic freedom at high energies is the presence of a pole at low energies or the infrared pole. In this region, we also go beyond the validity of perturbation theory since the coupling increases. To find the true behaviour of the coupling one has to attract independent nonperturbative information. However, in QCD the region near the infrared pole $p \sim \Lambda_{QCD}$ is in the phase of hadronization, i.e., in this region the quark-gluon description is no more adequate. Therefore, the behaviour of the effective coupling in this region is not described by perturbative QCD.

7.3 The screening and anti-screening of the charge

The variation of the coupling with momenta transfer or with the scale, which is the characteristic feature of quantum field theory, has its analog in a classical theory. This analogy allows one to understand the qualitative reason for the variation of the coupling.

Indeed, let us consider the electromagnetic phenomena. Consider the dielectric medium and put the test electric charge in it. The medium will be polarized. The electric dipoles present in the medium will be rearranged in such a way as to screen the charge (see Fig.30). This is a consequence of the Coulomb law: the opposite charges are attracted and the same charges are repulsed. This is the essence of the electric screening phenomena.

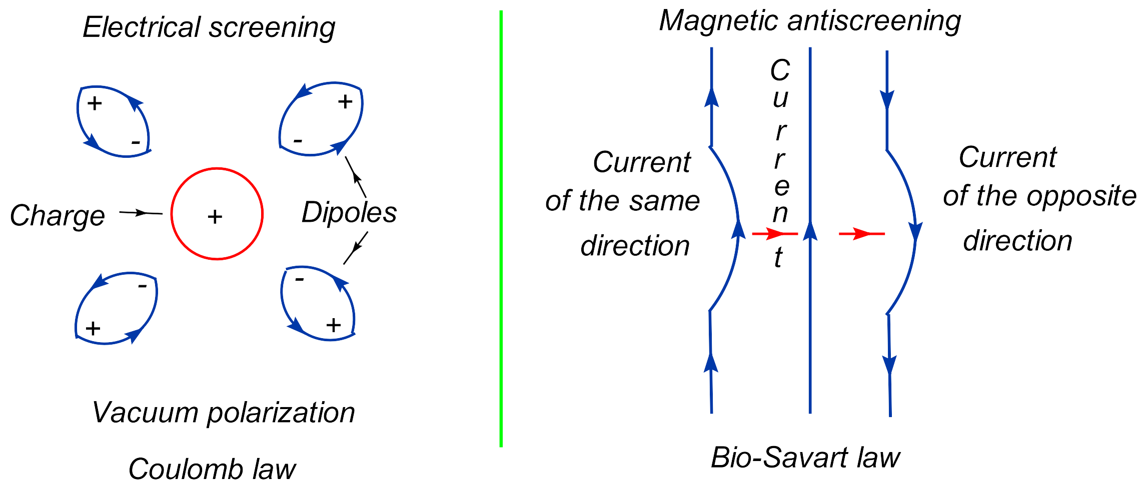


Figure 30: The electric screening and magnetic anti-screening

The opposite situation occurs in magnetic medium. According to the Bio-Savart law, the electric currents of the same direction are attracted and the opposite direction are repulsed (see Fig.30). This leads to the anti-screening in magnetic medium.

In quantum field theory the role of the medium is played by the vacuum. The vacuum is polarized in the presence of created virtual pairs. The matter particles as well as transversely polarized quanta of the gauge fields act like the electric dipoles in the dielectric and cause the screening of the charge. At the same time, the longitudinal quanta of the gauge fields behave like currents and cause the anti-screening. These two effects are in competition (see eq.(3.31) above) and, for instance, in QCD with a small number of quarks the effect of anti-screening prevails.

Thus, the couplings become the functions of the distance or momentum transfer described by the renormalization group equations.

8 Lecture VIII: Anomalies

The gauge invariance leads to numerous relations between various operators and their vacuum averages, i.e., the Green functions. We have already come across such relations called the Ward or the Slavnov-Taylor identities. They are the consequences of the gauge symmetry of the classical theory. In case when one has divergences in a theory and is bound to use some regularization, the validity of these identities depends on invariance of the regularization. However, one can always perform the subtraction of divergences in such a way that the finite parts obey these relations.

The exception from this rule is the so-called *anomalies.* By anomalies one usually means the violation in quantum theory of some relation, for instance, the conservation of the current or the Ward identity following from the symmetry properties of a classical theory. The well-known examples of quantum anomalies is the anomaly of the trace of the energy-momentum tensor or the axial anomaly. The characteristic feature of the anomaly is the impossibility of its removing by the redefinition of any quantities or parameters.

8.1 The axial anomaly

Consider quantum electrodynamics. Let us define the vector and the axial vector currents

$$j_\mu = \bar{\psi}\gamma^\mu\psi, \quad j_\mu^5 = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{8.1}$$

In classical theory the equations of motion lead to the conservation or partial conservation of the current

$$\partial_\mu j_\mu = 0, \quad \partial_\mu j_\mu^5 = 2imj^5, \tag{8.2}$$

where $j^5 = \bar{\psi}\gamma^5\psi$.

On the other hand, as a consequence of the gauge invariance, the vector and the axial vertices obey the Ward identities

$$\begin{aligned}(p-p')^\mu\Gamma_\mu(p,p') &= S^{-1}(p) - S^{-1}(p'), &(8.3)\\ (p-p')^\mu\Gamma_\mu^5(p,p') &= S^{-1}(p)\gamma^5 + \gamma^5 S^{-1}(p') + 2m\Gamma^5(p,p'), &(8.4)\end{aligned}$$

where Γ_μ, Γ_μ^5 and Γ^5 are the vector, axial and pseudoscalar vertices, respectively, and S is the fermion propagator.

If one looks how the identities (8.3,8.4) are fulfilled in perturbation theory, one first of all has to introduce some regularization due to the presence of the ultraviolet divergences. If the regularization is gauge invariant, then the vector Ward identity is satisfied in any order of PT. For the axial identity there are two types of diagrams: in the first one the axial current is in the outgoing fermion line, and in the second one the axial current is in the internal loop (see Fig.31). For the first type of a diagram the identity (8.4) is satisfied, and for the second type there exists one famous triangle diagram (see Fig.32) where it is violated due to the ultraviolet divergence of the integral.

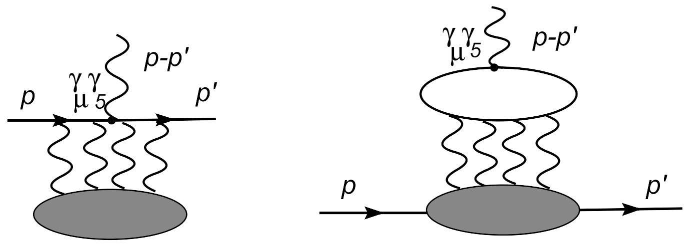


Figure 31: The diagrams with the axial current in external and internal fermion lines

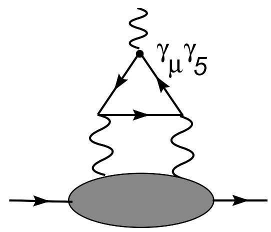
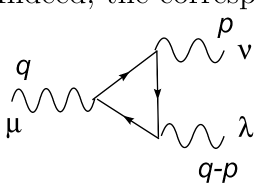


Figure 32: The anomalous triangle diagram for the axial current

Indeed, the corresponding integral in momentum space looks like



$$= (-)(-ie)^2 \int \frac{d^4k}{(2\pi)^4} Tr\left[\gamma^\mu\gamma^5 \frac{i\hat{k}}{k^2}\gamma^\nu \frac{i(\hat{k}+\hat{p})}{(p+k)^2}\gamma^\lambda \frac{i(\hat{k}+\hat{q})}{(q+k)^2}\right] \tag{8.5}$$

and is formally divergent requiring the regularization.

To preserve the conservation of the gauge invariance, it is useful to introduce the dimensional regularization; however, here we for the first time face a problem since the γ^5 matrix has no natural and consistent continuation to non-integer dimension. Two properties of the γ^5 matrix, namely, the anticommutation with all γ^μ, $\mu = 0,1,2,3$ and the property of the trace $Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$ are in contradiction if the dimension is noninteger. To calculate the axial anomaly, we use the following trick: we use the formula for the trace but reject the property of anticommutativity of γ^5. This allows one to perform al the calculations in a consistent and unambiguous way.

The divergence of the axial current can be obtained by multiplication of (8.5) by iq^μ which gives

$$e^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr\left[\hat{q}\gamma^5\hat{k}\gamma^\nu(\hat{k}+\hat{p})\gamma^\lambda(\hat{k}+\hat{q})\right]}{k^2(k+p)^2(k+q)^2} \tag{8.6}$$

Using the cyclic property of the trace we move $\hat{q}$ to the right and write it as $\hat{q} = (\hat{q}+\hat{k}) - \hat{k}$. Then the first term multiplied by $\hat{k}+\hat{q}$ gives $(k+q)^2$ and cancels with the denominator. As a result, one gets the integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{Tr\left[\hat{q}\gamma^5\hat{k}\gamma^\nu(\hat{k}+\hat{p})\gamma^\lambda\right]}{k^2(k+p)^2},$$

which depends only on p and after the integration turns to zero due to the antisymmetry of the trace with the γ^5 matrix.

In the second term we will drag $\hat{k}$ to the left until it is multiplied by $\hat{k}$ giving k^2. As a result, at each step we always get the trace of four γ-matrices with γ^5 for which we have the formula with the ϵ-tensor. We obtain in the numerator

$$-4i\epsilon^{\alpha\nu\beta\lambda}k^\alpha(k+p)^\beta[(k+q)^2-q^2]+8i\epsilon^{\alpha\nu\beta\rho}k^\alpha(k+p)^\beta q^\rho k^\lambda-4i\epsilon^{\alpha\nu\lambda\rho}k^\alpha q^\rho[(k+p)^2-p^2]$$
$$-4i\epsilon^{\nu\alpha\lambda\rho}p^\alpha q^\rho k^2+8i\epsilon^{\alpha\beta\lambda\rho}k^\alpha p^\beta q^\rho k^\nu.$$

Despite the fact that the integral is formally divergent, using a dimensional regularization and collecting all terms together we finally get the finite answer equal to

$$-\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\lambda}p^\mu q^\rho=-\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\lambda}p^\mu(q-p)^\rho, \tag{8.7}$$

One has to add to this expression the same diagram but with the replacement $p\leftrightarrow q-p,\nu\leftrightarrow\lambda$ and take the sum, but the answer is already invariant with respect to this replacement. Multiplying (8.7) by $A_\nu(p)A_\lambda((q-p)$ and transforming to the coordinate representation, one gets

$$\partial_\mu j^5_\mu=\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\lambda}\partial_\mu A_\nu\partial_\rho A_\lambda=\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}. \tag{8.8}$$

As a result one has the following modification of equations for the divergence of the axial current and the axial vertex

$$\partial_\mu j^5_\mu=2imj^5+\frac{\alpha}{4\pi}F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}, \tag{8.9}$$

$$(p-p')^\mu\Gamma^5_\mu(p,p')=S^{-1}(p)\gamma^5+\gamma^5S^{-1}(p')+2m\Gamma^5(p,p')-i\frac{\alpha}{4\pi}F(p,p'), \tag{8.10}$$

where $F(p,p')$ is the vertex with insertion of the operator $F\tilde{F}$. The appearance of the r.h.s in these equations is called *anomaly* known as the Adler-Bell-Jackiw or triangle anomaly.

The most essential here is not the violation of the Ward identity but the fact that subtracting the anomaly and restoring the "normal" Ward identity for the axial vertex we violate the conservation of the vector current. In other words, it is impossible to satisfy the conservation of axial and vector currents *simultaneously*.

Notice that the violation of the conservation of the axial current preserving the conservation of the vector current (8.9) can be obtained by accurately calculating the matrix element for the divergence of the axial current in x-space splitting the arguments of the field operators. Consider the vacuum average of the divergence of the axial current, and to avoid the singularity for the product of two operators at coinciding points, split the arguments. Then to preserve the gauge invariance, we have to insert between the operators the exponent of the Wilson line. The axial current then takes the form

$$j^5_\mu(x)=\lim_{\varepsilon\to0}\{\bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5\exp[-ie\int\limits_{x-\varepsilon/2}^{x+\varepsilon/2}dz^\nu A_\nu(z)]\psi(x-\varepsilon/2)\}, \tag{8.11}$$

and for the divergence we get

$$\begin{aligned}\partial_\mu j^5_\mu(x) &= \lim_{\varepsilon\to0}\{\partial_\mu\bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5\exp[-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2}dz^\nu A_\nu(z)]\psi(x-\varepsilon/2)\\ &+ \bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5\exp[-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2}dz^\nu A_\nu(z)]\partial_\mu\psi(x-\varepsilon/2)\\ &+ \bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5[-ie\varepsilon^\nu\partial_\mu A_\nu(x)]\exp[-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2}dz^\nu A_\nu(z)]\psi(x-\varepsilon/2)\}.\end{aligned} \tag{8.12}$$

Using the equations of motion

$$\gamma^\mu \partial_\mu \psi = -ie\hat{A}\psi, \quad \partial_\mu \bar{\psi}\gamma^\mu = ie\bar{\psi}\hat{A}$$

and keeping the terms of the order of ε we find

$$\begin{aligned}\partial_\mu j^5_\mu(x) &= \lim_{\varepsilon\to 0}\{\partial_\mu \bar{\psi}(x+\varepsilon/2)[-ie\hat{A}(x+\varepsilon/2) - ie\hat{A}(x-\varepsilon/2) \\ &\qquad -ie\varepsilon^\nu \gamma^\mu \partial_\mu A_\nu(x)]\gamma^5 \psi(x-\varepsilon/2)\} \\ &= \lim_{\varepsilon\to 0}\{\bar{\psi}(x+\varepsilon/2)[-ie\varepsilon^\nu \gamma^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu)]\gamma^5\psi(x-\varepsilon/2)\} \qquad (8.13)\end{aligned}$$

Now we have to calculate the vacuum average over the fermion vacuum (the photon field is assumed to be external) which means that we have to permute the fermion operators. The permutation function of the fermion operators is singular and this is the reason for appearance of a nonzero term similarly to the appearance of triangle anomaly due to divergency of the integral. Indeed, calculating the propagator of the fermion in external field and keeping the terms linear in the photon field, we get

$$S(y-z) = \int \frac{d^4k}{(2\pi)^4} e^{ik(y-z)} \frac{i\hat{k}}{k^2} + \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} e^{i(k+p)y} e^{-ikz} \frac{i(\hat{k}+\hat{p})}{(k+p)^2}(-ie\hat{A}(p))\frac{i\hat{k}}{k^2} + ... \quad (8.14)$$

The propagator (8.14) is singular as $y \to z$; however, the first term does not give a contribution to the divergence, while the second one leads to

$$\begin{aligned}&\langle \bar{\psi}(x+\varepsilon/2)\gamma^\mu\gamma^5\psi(x-\varepsilon/2)\rangle = \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} e^{ipx} e^{-ik\varepsilon} Tr[\frac{i(\hat{k}+\hat{p})}{(k+p)^2}(-ie\hat{A}(p))\frac{i\hat{k}}{k^2}\gamma^\mu\gamma^5] \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} e^{ipx} e^{-ik\varepsilon} \frac{4e\epsilon^{\mu\nu\rho\sigma}(k+p)_\nu A_\rho(p) k_\sigma}{(k+p)^2 k^2}. \qquad (8.15)\end{aligned}$$

To find the limit as $\varepsilon \to 0$, one can expand the integrand for large k, which gives

$$\begin{aligned}\langle \bar{\psi}(x+\varepsilon/2)\gamma^\mu\gamma^5\psi(x-\varepsilon/2)\rangle &= 4e\epsilon^{\mu\nu\rho\sigma}\int \frac{d^4p}{(2\pi)^4} e^{ipx} p_\nu A_\rho(p) \int \frac{d^4k}{(2\pi)^4} e^{-ik\varepsilon}\frac{k_\sigma}{k^4} \\ = -4e\epsilon^{\mu\nu\rho\sigma} i\partial_\nu A_\rho(x)\frac{2\varepsilon_\sigma}{16\pi^2\varepsilon^2} &= -e\epsilon^{\mu\nu\rho\sigma} iF_{\nu\rho}(x)\frac{\varepsilon_\sigma}{4\pi^2\varepsilon^2}, \qquad (8.16)\end{aligned}$$

Substituting this expression into (8.13) we find

$$\partial_\mu j^5_\mu = \lim_{\varepsilon\to 0}\{-e\epsilon^{\mu\nu\rho\sigma} iF_{\nu\rho}(x)\frac{\varepsilon_\sigma}{4\pi^2\varepsilon^2}(-ie\varepsilon^\tau F_{\mu\tau})\} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho}F_{\sigma\mu}, \qquad (8.17)$$

that coincides with (8.9).

The axial anomaly has one very important property: the obtained formulas (8.9) and (8.10) are *exact* in all orders of perturbation theory, i.e., have no radiative corrections. More rigorous statement is: there exists such a renormalization scheme (and it was constructed explicitly) that the radiative corrections to the axial anomaly are absent. This statement is the subject of the Adler-Bardeen theorem. Graphically, this means the cancellation of the contributions of the diagrams shown in Fig.33, which was checked by explicit calculation.

The Adler-Bardeen theorem is valid also in non-Abelian theories. It has important consequences: if the anomaly is compensated in the lowest order, it will not appear further.

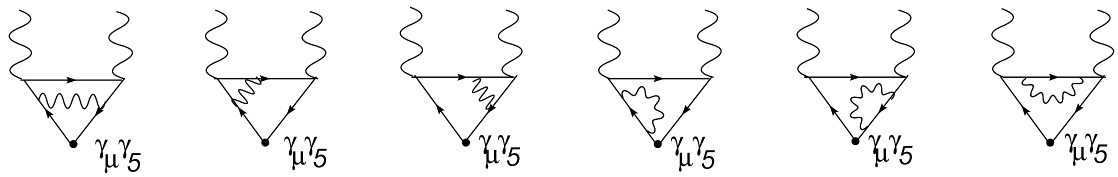


Figure 33: Cancellation of radiative corrections to the axial anomaly

8.2 Consequences of the axial anomaly

Let us ask the question what are the consequences of the axial anomaly? Here one has to distinguish two cases: when the operator of the axial current is an external operator with respect to the Lagrangian and when it is present in the interaction Lagrangian.

In the first case, the presence of anomaly does not lead to any troubles and even may be useful. Thus, for instance, in the current algebra which describes the low energy hadron interactions, the axial anomaly is responsible for the neutral pion decay $\pi^0 \to 2\gamma$ and is in agreement with the experiment.

In the second case, the triangle anomaly leads to that the ultraviolet renormalizations of the vector vertex do not remove all divergences from the axial vertex. This has destructive consequences for the renormalizability of the whole theory. To see this, compare the two processes of the elastic scattering of leptons: $\nu_e + e \to \nu_e + e$ and $\nu_\mu + e \to \nu_e + \mu$ in the Standard Model. Graphically, in the lowest order they differ by one diagram containing the triangle anomaly (See Fig.34).

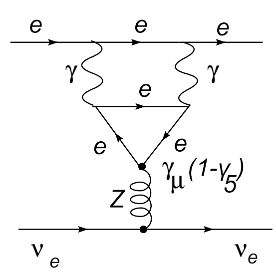


Figure 34: The anomaly in the process of lepton scattering in the Standard Model

As a result, after the renormalization the amplitude of $\nu_\mu e$-scattering has finite radiative corrections, while that of νe-scattering is divergent. This led to nonrenormalizability of the theory and was a serious problem for the left-right nonsymmetric model with $SU_L(2) \times U(1)$ symmetry before the introduction of the -quark. Remarkably, the -quark introduced by Glashow, Iliopoulos and Maiani for suppression of the neutral current changing strangeness leads to the compensation of the contributions of quarks and leptons to triangle anomaly and restores the renormalizability of the theory.

In the Standard Model due to its left-right asymmetry the presence of the axial currents for quarks and leptons leads to several kinds of triangle anomalies where all three gauge fields may be in the vertices of the triangle. However, not all of them lead to anomalies. In general, the anomaly is proportional to the trace

$$Tr\ T^a\{T^b, T^c\},$$

where the matrix T^a is the generator of the corresponding gauge group in the representation corresponding to the fields that run inside the triangle. The necessary condition of the existence of anomaly is the presence of the complex representations and the nontrivial anticommutator of the generators of the group. Among the simple Lie groups which satisfy this requirement, only the groups $SU(n)$, $SO(4n+2)$ and E_6 have complex representations and out of them only the $SU(n), n > 2$ and $SO(6)$ groups have a symmetric invariant needed for the construction of the anomaly. The gauge theories built on other groups are free from anomalies.

The non-vanishing anomalies corresponding to the symmetry group of the Standard Model $SU_c(3) \times SU_L(2) \times U_Y(1)$ are presented in Fig.35 where the gauge fields adjusted to the groups $U(1)$ and $SU(2)$ are shown prior to mixing. The particles that run over the triangle can be either left or right quarks and leptons. Particles of different helicity give the opposite sign contribution to the axial anomaly.

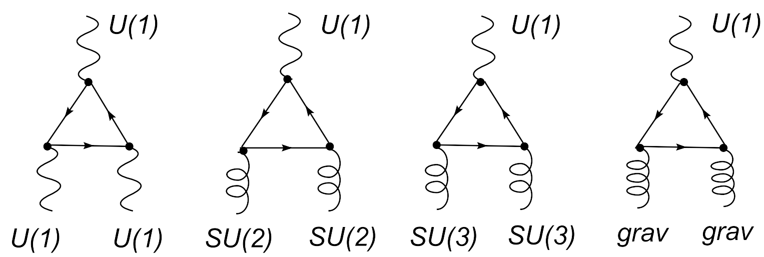


Figure 35: The triangle anomaly in the Standard Model

In the first case, the anomaly is proportional to the trace of the cube of hypercharge $TrY^3 = TrY_L^3 - TrY_R^3$ and its absence is achieved by the cancellation of the contributions of quarks and leptons in each generation

$$\begin{array}{rcl} TrY^3 = & = & 3\left[(\frac{1}{3})^3 + (\frac{1}{3})^3 - (\frac{4}{3})^3 - (-\frac{2}{3})^3\right] + (-1)^3 + (-1)^3 - (-2)^3 = 0. \\ & & \uparrow \ \ \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \\ & & colour \ \ u_L \quad d_L \quad u_R \quad d_R \quad \nu_L \quad e_L \quad e_R. \end{array} \tag{8.18}$$

In further diagrams the anomaly is proportional to, respectively,

$$\begin{aligned} TrY_L &= 3\left(\frac{1}{3} + \frac{1}{3}\right) - 1 - 1 = 0, \\ TrY_q &= 3\left(\frac{1}{3} + \frac{1}{3} - \frac{4}{3} - (-\frac{2}{3})\right) = 0, \\ TrY &= 3\left(\frac{1}{3} + \frac{1}{3} - \frac{4}{3} - (-\frac{2}{3})\right) - 1 - 1 - (-2) = 0. \end{aligned} \tag{8.19}$$

This way the anomaly is miraculously canceled in all the cases and does not break the renormalizability of the SM.

8.3 The conformal anomaly

Another example of quantum anomaly is the conformal anomaly or the anomaly of the trace of the energy-momentum tensor. The requirement of conformal (scale) invariance means the invariance of the action with respect to the transformation

$$x_\mu \to x_\mu e^{-\sigma}, \quad \phi(xe^{-\sigma}) \to e^{\Delta\sigma}\phi(x), \tag{8.20}$$

where Δ is the dimension of a field. This condition is fulfilled in the classical Lagrangian if it has no dimensional parameters. In this case, according to the Noether theorem, there exists a conserved current called the dilatation current $D^\mu = \Theta^{\mu\nu}x_\nu$, so that

$$\partial_\mu D^\mu = \Theta^\mu_\mu,$$

where Θ^μ_ν is the symmetric energy-momentum tensor.

The easiest way to see it is to define the energy-momentum tensor as a variation of the action of the matter fields with respect to the space-time metric in the external gravitational filed

$$\Theta^{\mu\nu} = 2\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\ \mathcal{L}(x). \tag{8.21}$$

The scale transformation can be realized as a variation of the metric

$$g_{\mu\nu}(x) \to e^{2\sigma}g_{\mu\nu}(x). \tag{8.22}$$

This means that the variation of the Lagrangian under this transformation is the trace of $\Theta^{\mu\nu}$. The deviation of the trace of the energy-momentum tensor from zero indicates the violation of the scale (and hence conformal) invariance.

In the quantum case, due the presence of the ultraviolet divergences the new scale appears. This is the same phenomenon of dimensional transmutation discussed above. Therefore, the scale invariance of the action is violated.

Since the coupling constant becomes scale dependent, its variation with the scale (8.20) takes the form

$$\delta g = \sigma\mu\frac{dg}{d\mu} = \sigma\beta(g). \tag{8.23}$$

Hence, for the variation of the Lagrangian we get

$$\delta\ \mathcal{L} = \sigma\frac{\delta\mathcal{L}}{\delta g_i}\beta_i(\{g\}), \tag{8.24}$$

i.e.,

$$\partial_\mu D^\mu = \Theta^\mu_\mu = \frac{\delta\mathcal{L}}{\delta g_i}\beta_i(\{g\}). \tag{8.25}$$

This relation is known as *the trace anomaly* of the energy-momentum tensor.

Similarly to the axial anomaly, relation (8.25) can be checked by perturbation theory. However, in this case the result is defined by the full β-function calculated in all orders of PT.

9 Lecture IX: Infrared Divergences

One more problem that we encounter on the way of calculating the finite expressions for the probabilities of physical processes is the presence of the so-called *infrared divergences*. They appear when calculating the matrix elements of the scattering matrix on shell, i.e., when the squares of external momenta are equal to the corresponding masses squared and the theory contains massless particles like photons or gluons. The infrared divergences can be of two types: the divergences for small values of momenta (the genuine infrared divergences) and the divergences at parallel momenta (the collinear divergences). Contrary to the ultraviolet divergences, the infrared divergences have a clear physical meaning: a massless particle with a very small momentum can not be registered and with momentum parallel to another particle cannot be distinguished. For this reason in the theories with massless particles one has to define the physical process to be evaluated in a proper way.

9.1 The double logarithmic asymptotics

For illustration consider the process of creation of a muon pair in the e^+e^- annihilation. The leading diagrams for this process are shown in Fig.36.

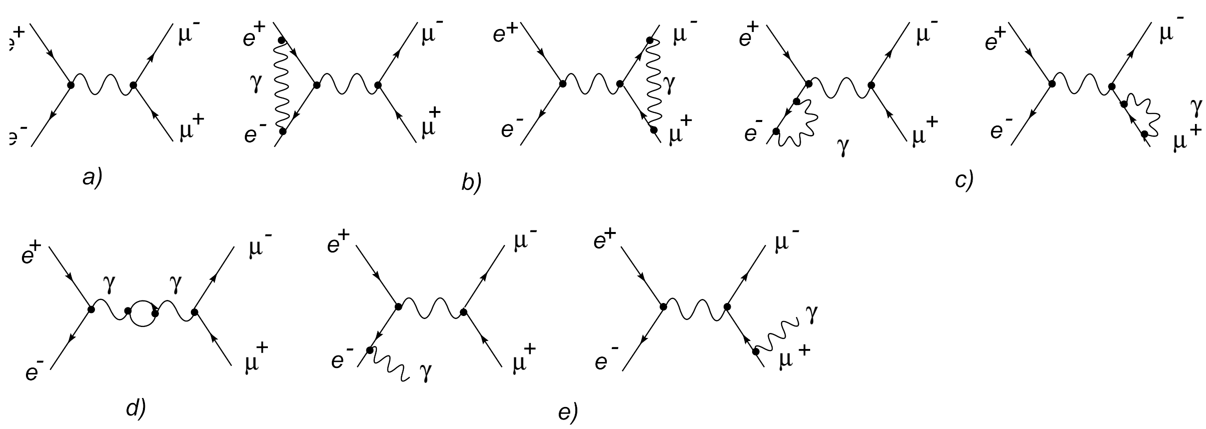


Figure 36: The diagrams contributing to the process $e^+e^- \to \mu^+\mu^-$ in QED: a) the leading order, b)- d) the virtual corrections of the order of α, e) the real corrections of the order of α.

The first diagram is the tree amplitude, it gives the contribution in the leading order. The radiative corrections due to emission of virtual photons (Fig.36 b)) are the corrections to the vertex function considered above (see (3.23)). It is easy to see that if one puts in this formula all fermion momenta on mass shell, i.e. $p^2 = (p-q)^2 = m^2$, then in the second integral in the denominator one gets $[-m^2x^2 + q^2y(x-y)]$. Performing the change of variables $y \to yx$ so that all the integrations are performed within the limits [0,1], we get $[-m^2x^2 + q^2x^2y(1-y)]$, and the integral (with account of the Jacobian = x) is logarithmically divergent as $x \to 0$.

The appeared divergence has the infrared nature. Like the ultraviolet one it can be regularized, for instance, by introducing the nonzero photon mass or cutting the integral over momenta at the lower limit, or with the help of dimensional regularization but it cannot be removed by any renormalization.

Let us calculate this diagram on mass shell introducing the nonzero photon mass m_{ph} into the virtual photon line. This will not break the gauge invariance since, as it will be clear later, after the cancellation of the IR divergences one can put the mass of a photon equal to zero.

Let us go back to eq.(3.23), remove the UV divergence by the minimal subtraction and go to the mass shell for the fermion fields taking into account that the external fermion operators obey the Dirac equation $(\hat{p}-m)u(p)=0$ and $\bar{u}(p-q)(\hat{p}-\hat{q}-m)=0$. Then after some exercise we obtain for the vertex function the following expression:

$$\Gamma_1^R(p,q) = ie\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q^\nu}{2m}\right], \qquad \sigma^{\mu\nu} \equiv i\frac{\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu}{2}, \tag{9.1}$$

where the form-factors $F_i(q^2)$ have the form

$$\begin{aligned} F_1(q^2) &= \frac{e^2}{16\pi^2}\left[-2-2\int_0^1 dx\int_0^1 dy\ x\ \log\left(\frac{-m^2x^2+q^2x^2y(1-y)}{-\mu^2}\right)\right. \\ &+ \left.\int_0^1 dx\int_0^1 dy\ x\ \frac{2m^2(2-2x-x^2)-2q^2(1-xy)(1-x+xy)}{-m^2x^2+q^2x^2y(1-y)-m_{ph}^2(1-x)}\right], \qquad (9.2) \\ F_2(q^2) &= \frac{e^2}{16\pi^2}\left[\int_0^1 dx\int_0^1 dy\ x\ \frac{-4m^2x(1-x)}{-m^2x^2+q^2x^2y(1-y)-m_{ph}^2(1-x)}\right]. \qquad (9.3) \end{aligned}$$

The form factor F_2 is IR convergent and does not need any regularization. Substituting $m_{ph}=0$, we get

$$F_2(q^2) = \frac{\alpha}{4\pi}\int_0^1 dy\frac{2m^2}{m^2-q^2y(1-y)}. \tag{9.4}$$

For $q^2=0$ it can be easily calculated and equals

$$F_2(q^2=0) = \frac{\alpha}{2\pi}, \tag{9.5}$$

which is nothing else but the first correction to the g-factor, which is called the anomalous magnetic moment of electron (muon).

As for the form factor F_1, it is IR divergent. We calculate its divergent part in the limit $m_{ph}\to 0$. It comes only from the second integral in (9.2). To simplify the integration, we notice that the divergence is defined by the region of the parameter $x\sim 0$. Therefore, we put $x=0$ everywhere in the numerator and in the coefficient of m_{ph} in the denominator. Then one gets

$$F_1(q^2) \simeq \frac{e^2}{16\pi^2}\int_0^1 dy\int_0^1 xdx\frac{2(2m^2-q^2)}{[-m^2+q^2y(1-y)]x^2-m_{ph}^2}. \tag{9.6}$$

The integral over x is now easily evaluated

$$F_1(q^2) \simeq \frac{\alpha}{4\pi}\int_0^1 dy\frac{2m^2-q^2}{[-m^2+q^2y(1-y)]}\log\left(\frac{-m^2+q^2y(1-y)-m_{ph}^2}{-m_{ph}^2}\right). \tag{9.7}$$

The remaining integral over y is also simple. We calculate it in the limit $-q^2 \to \infty$. Then it takes the form

$$F_1(q^2) \simeq -\frac{\alpha}{4\pi}\int_0^1 dy \frac{q^2}{[-m^2+q^2y(1-y)]}\log\left(\frac{-q^2}{m_{ph}^2}\right) \simeq -\frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{m_{ph}^2}\right). \tag{9.8}$$

The obtained double logarithmic behaviour of the form-factor is called *the Sudakov* double logarithm. It contains the infrared cutoff in the form of the photon mass. In the amplitude of creation of the muon pair there are two of such form factors for the electron and the muon vertices, respectively. The corrections to the fermion and the photon propagators do not contain the IR divergences. Thus, the cross-section of the process $e^+e^- \to \mu^+\mu^-$ is logarithmically divergent. In order to understand the reason of appearance of the IR divergence and to find the method of its elimination, consider the process of creation of the muon pair from the point of view of an observer.

9.2 The soft photon emission

During the process of electron-positron annihilation the muon pair is created with momenta that satisfy the conservation law and can be measured. However, they are registered with some accuracy, and momentum smaller than some value which depends on a particular detector is not registered. Therefore, if besides the muon pair the photon with momentum smaller than this value is created, then this process with emission of the "soft" γ-quantum $e^+e^- \to \mu^+\mu^-\gamma$ is experimentally indistinguishable from the initial process $e^+e^- \to \mu^+\mu^-$. The diagrams corresponding to the process $e^+e^- \to \mu^+\mu^-\gamma$ are shown in Fig.36 e). They contain an additional vertex and hence additional coupling, but being squared give a correction to the main process of the order of α, exactly as the radiative corrections due to the virtual photon.

Let us compare the differential cross-sections of the precess $e^+e^- \to \mu^+\mu^-$ in the one-loop approximation and $e^+e^- \to \mu^+\mu^-\gamma$ in the tree approximation. We have, respectively,

$$\frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) = \left(\frac{d\sigma}{d\Omega}\right)_0\left[1-\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m_{e,\mu}^2}\right)\log\left(\frac{-q^2}{m_{ph}^2}\right)+...+\mathcal{O}(\alpha^2)\right] \tag{9.9}$$

$$\frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-\gamma) = \left(\frac{d\sigma}{d\Omega}\right)_0\left[+\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m_{e,\mu}^2}\right)\log\left(\frac{-q^2}{m_{ph}^2}\right)+...+\mathcal{O}(\alpha^2)\right] \tag{9.10}$$

where the second cross-section is written down without derivation which we will perform later. As follows from eqs.(9.9,9.10), each of these cross-sections is IR divergent, but in the sum the divergences cancel and one gets the finite answer.

What is observable after all? In fact, neither the first nor the second process is observable separately. In a real detector with limited sensitivity one observes the process of creation of the muon pair plus an arbitrary number of soft photons with the total energy below the sensitivity threshold. In a given order of perturbation theory we have to sum the cross-sections of the two processes in order to get the observed cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{observable} = \left(\frac{d\sigma}{d\Omega}\right)(e^+e^- \to \mu^+\mu^-) + \left(\frac{d\sigma}{d\Omega}\right)(e^+e^- \to \mu^+\mu^-\gamma, E < E_{min}). \tag{9.11}$$

The latter cross-section is given by the same formula (9.10) with the replacement in the second logarithm of the photon energy by E_{min}. Thus, we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{observable} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2_{e,\mu}}\right)\log\left(\frac{-q^2}{E^2_{min}}\right) + ... + \mathcal{O}(\alpha^2)\right]. \qquad (9.12)$$

As one can see, for the proper statement of the problem the cross-section of the observable process is finite and does not depend on the IR regulator. At the same time, it depends on the sensitivity of the detector E_{min} and for improved sensitivity tends to infinity. However, this infinity also is not physical and is the artefact of perturbation theory: when the logarithm becomes large we go beyond the scope of applicability of perturbation theory and it is necessary to perform the summation of these corrections by analogy with what happens with the ultraviolet logarithms within the renormalization group method.

Thus, the IR divergences appear due to the contributions of the photons with "soft" momenta: real with the energy smaller than E_{min} and virtual with momenta $k^2 < E^2_{min}$. What is important is that the momenta of fermions are on mass shell, otherwise the singularities in the propagator do not arise. The typical diagram of higher order contains a big amount of real and virtual photon lines (see Fig.37).

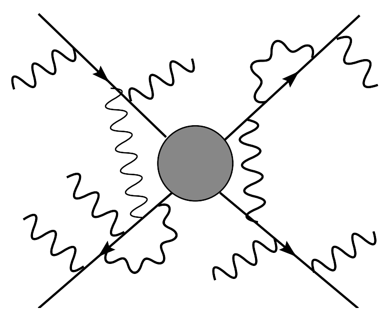


Figure 37: The hard process with creation of the soft photons

Let us try to sum up the contributions of these soft photons. Consider first the external fermion line with the outgoing photons (real and virtual).

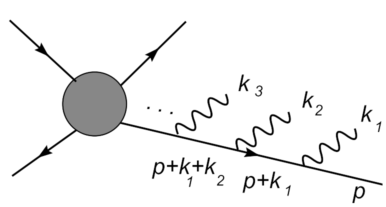


Figure 38: The emission of the soft photons from the fermion line

It corresponds to the following expression:

$$\bar{u}(p)\,(-ie\gamma^{\mu_1})\,\frac{i(\hat{p}+\hat{k}_1+m)}{2pk_1}\,(-ie\gamma^{\mu_2})\,\frac{i(\hat{p}+\hat{k}_1+\hat{k}_2+m)}{2p(k_1+k_2)+O(k^2)}\,\cdots \qquad (9.13)$$
$$\cdots\,(-ie\gamma^{\mu_n})\,\frac{i(\hat{p}+\hat{k}_1+\cdots+\hat{k}_n+m)}{2p(k_1+\cdots k_n)+O(k^2)}\;\;iM_{hard}.$$

We use now the fact that the operator $\bar{u}(p)$ obeys the Dirac equation $\bar{u}(p)(\hat{p}-m)=0$ and omit the momenta $k_i \ll p$ in the numerator. Then we get

$$\bar{u}(p)\gamma^{\mu_1}(\hat{p}+m)\gamma^{\mu_2}(\hat{p}+m)\cdots = \bar{u}(p)2p^{\mu_1}\gamma^{\mu_2}(\hat{p}+m)\cdots = \bar{u}(p)2p^{\mu_1}2p^{\mu_2}\cdots . \tag{9.14}$$

Hence, eq.(9.13) takes the form

$$\bar{u}(p)\ (e\frac{p^{\mu_1}}{pk_1})(e\frac{p^{\mu_2}}{p(k_1+k_2)})\cdots(e\frac{p^{\mu_n}}{p(k_1+\cdots+k_n)}). \tag{9.15}$$

The next step is the summation over all the permutations of the photon lines and the permutations of momenta k_i. (So far we have not distinguished between the real and virtual photons, we will do it later.) This operation is non-trivial but leads to the simple result. One has

$$\sum_{permutations} \frac{1}{pk_1}\frac{1}{p(k_1+k_2)}\cdots\frac{1}{p(k_1+k_2+\cdots+k_n)} = \frac{1}{pk_1}\frac{1}{pk_2}\cdots\frac{1}{pk_n}. \tag{9.16}$$

The same procedure can be applied to the incoming fermion line. The difference is that the fermion momentum has the opposite direction which leads to the replacement of $(p+k_i)^2$ to $(p-k_i)^2$ in the propagator, i.e., the change of the sign $p \to -p$ in the denominator. Collecting both factors together we get the following expression for the amplitude of emission of soft photons from arbitrary points of the incoming and the outgoing line (Fig.39):

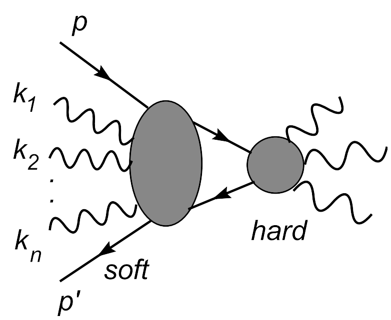


Figure 39: The emission of soft photons from arbitrary points of the incoming and the outgoing lines

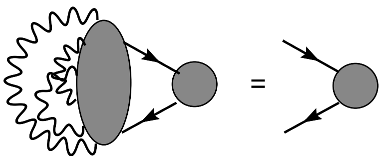
$$\mathcal{M} = \bar{u}(p')\ i\mathcal{M}_{hard}\ u(p)\ e\left(\frac{p'^{\mu_1}}{p'k_1}-\frac{p^{\mu_1}}{pk_1}\right) e\left(\frac{p'^{\mu_2}}{p'k_2}-\frac{p^{\mu_2}}{pk_2}\right)\cdots e\left(\frac{p'^{\mu_n}}{p'k_n}-\frac{p^{\mu_n}}{pk_n}\right). \tag{9.17}$$

Now we have to decide which photons are real and which are virtual. The virtual photon can be obtained by joining the two photon momenta k_i and k_j, taking $k_i=-k_j=k$, multiplying by the photon propagator and integrating over k. In this way for any virtual photon we get the expression:

$$\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\left(\frac{p'}{p'k}-\frac{p}{pk}\right)\left(\frac{p'}{-p'k}-\frac{p}{-pk}\right), \tag{9.18}$$

where the factor $1/2$ compensates the double counting due to permutation of k_i and k_j. The obtained integral is nothing else but the vertex function in the one-loop approximation, i.e., the form factor $F_1(q^2)$.

If the number of virtual photons equals n, one gets the product of n expressions like (9.18) and the factor $1/n!$ taking into account the permutations which do not change the result. The full answer is obtained with the help of summation over the soft virtual photons, which gives



$$= \times \sum_{n=0}^{\infty} \frac{F_1^n}{n!} = \bar{u}(p')\; i\mathcal{M}_{hard}\; u(p) \exp(F_1). \tag{9.19}$$

At the same time, if the real photon is emitted, then instead of the propagator one has to multiply the amplitude by the polarization operator, sum up over all polarizations and integrate the square of the matrix element over the photon phase space. In this case, one gets the following expression:

$$I(q^2) = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{-g^{\mu\nu}}{2|k|} \left(\frac{p'^\mu}{p'k} - \frac{p^\mu}{pk}\right)\left(\frac{p'^\nu}{p'k} - \frac{p^\nu}{pk}\right), \tag{9.20}$$

which is the element of the cross-section of the process $e^+e^- \to \mu^+\mu^-\gamma$. The integration over the modulus of the three-vector $\vec{k}$ has to be performed within the limits (m_{ph}, E_{min}). Contracting the indices one gets

$$I(q^2) = -\frac{e^2}{(2\pi)^3} \int \frac{d^3k}{2|k|} \left(\frac{p'^2}{(p'k)^2} - 2\frac{p'p}{(pk)(p'k)} + \frac{p^2}{(pk)^2}\right). \tag{9.21}$$

The first and the last integrals are equal to each other. Let us consider the last one and choose the frame where $\vec{p} = 0$. This gives

$$I_1 = -\frac{e^2}{(2\pi)^3} 4\pi \int_{m_{ph}}^{E_{min}} \frac{k^2dk}{2k} \frac{m^2}{(mk)^2} = -\frac{\alpha}{2\pi} \log(\frac{E^2_{min}}{m^2_{ph}}). \tag{9.22}$$

As for the second integral, we proceed in the following way: first we also choose the frame $\vec{p} = 0$, and then we covariantize the answer. One has

$$\begin{aligned} I_2 &= \frac{e^2}{(2\pi)^3} 2\pi \int_{m_{ph}}^{E_{min}} \frac{k^2dk}{k} \int_{-1}^{1} d\cos\theta \frac{m\sqrt{\vec{p'}^2 + m^2}}{(mk)(\sqrt{\vec{p'}^2 + m^2}k - |\vec{p'}|k\cos\theta)} \\ &= \frac{\alpha}{2\pi} \log(\frac{E^2_{min}}{m^2_{ph}}) \frac{\sqrt{\vec{p'}^2 + m^2}}{|\vec{p'}|} \log\left(\frac{\sqrt{\vec{p'}^2 + m^2} - |\vec{p'}|}{\sqrt{\vec{p'}^2 + m^2} + |\vec{p'}|}\right). \end{aligned} \tag{9.23}$$

Covariantizing this answer and having in mind that $q = p - p'$, $p^2 = p'^2 = m^2$ and, hence, $q^2 = 2m^2 - 2m\sqrt{\vec{p'}^2 + m^2}$ one gets

$$I_2(q^2) = \frac{\alpha}{2\pi} \log(\frac{E^2_{min}}{m^2_{ph}}) \frac{2m^2 - q^2}{\sqrt{-q^2(4m^2 - q^2)}} \log\left(\frac{2m^2 - q^2 - \sqrt{-q^2(4m^2 - q^2}}{2m^2 - q^2 + \sqrt{-q^2(4m^2 - q^2}}\right). \tag{9.24}$$

Thus,

$$I(q^2) = \frac{\alpha}{2\pi} \log(\frac{E^2_{min}}{m^2_{ph}}) \left[\frac{2m^2 - q^2}{\sqrt{-q^2(4m^2 - q^2)}} \log\left(\frac{2m^2 - q^2 - \sqrt{-q^2(4m^2 - q^2}}{2m^2 - q^2 + \sqrt{-q^2(4m^2 - q^2}}\right) - 2\right]. \tag{9.25}$$

In the limit $-q^2 \to \infty$ we get the desired answer

$$I(q^2) \to \frac{\alpha}{\pi} \log(\frac{E^2_{min}}{m^2_{ph}}) \log(\frac{-q^2}{m^2}), \tag{9.26}$$

coinciding with (9.10).

If there are n real photons, there are n such contributions and the symmetry factor $1/n!$ taking into account the identity of the final particles. The cross-section of the process with emission of an arbitrary number of photons with the energy smaller than E_{min} hence equals

$$\sum_{n=0}^{\infty} \frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^- + n\gamma) = \frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) \times \sum_{n=0}^{\infty} \frac{I^n}{n!} = \frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-)e^I. \tag{9.27}$$

Combining the results for the real and virtual photons one gets the final expression for the observable cross-section with emission of an arbitrary number of photons with the energy smaller than E_{min}

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{observable} &= \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp{(2F_1)} \times \exp{(I)} \\ = \left(\frac{d\sigma}{d\Omega}\right)_0 &\exp\left[-\frac{\alpha}{\pi}\log(\frac{-q^2}{m^2_{e,\mu}})\log(\frac{-q^2}{m^2_{ph}})\right] \exp\left[\frac{\alpha}{\pi}\log(\frac{-q^2}{m^2_{e,\mu}})\log(\frac{E^2_{min}}{m^2_{ph}})\right] \\ = \left(\frac{d\sigma}{d\Omega}\right)_0 &\exp\left[-\frac{\alpha}{\pi}\log(\frac{-q^2}{m^2_{e,\mu}})\log(\frac{-q^2}{E^2_{min}})\right]. \end{aligned} \tag{9.28}$$

The obtained expression is valid in all orders of perturbation theory. The exponential factor does not depend on the IR cutoff but on the sensitivity of the detector. It is called the *Sudakov form factor*. When E_{min} tends to zero, the form factor decreases and in the limit $E_{min} \to 0$ vanishes. This is the manifestation of the statement that he amplitude of creation of the fermion pair without accompanying soft photons indeed vanishes: the charged particle inevitably emits the low frequency electromagnetic waves. This means that the cross-section of elastic electron scattering without inclusion of emission of bremsstrahlung quanta should vanish, precisely as it follows from eq.(9.28).

Let us estimate the value of the Sudakov form factor for some real process. A good example is the cross-section of e^+e^- annihilation into hadrons which in the leading order in the fine structure constant is described by one diagram with Z-boson exchange in the s-channel. The cross-section has a maximum in the Z-boson peak where it is described by the Breit-Wigner resonance formula. The energy is equal to the Z-boson mass M_Z and the energy resolution is defined by the Z-boson width Γ_Z. Substituting the values $M_Z = 91.187$ GeV, $\Gamma_Z = 2.496$ GeV, $m_e = 0.5$ MeV, $\alpha = 1/128$ into the form factor (9.28) we get

$$\exp\left[-\frac{\alpha}{\pi}\log(\frac{M^2_Z}{m^2_e})\log(\frac{M^2_Z}{\Gamma^2_Z})\right] \approx 0.648.$$

As one can see, the form factor, despite the smallness of the fine structure constant, considerably departs from unity and has to be taken into account when analysing the experimental data.

9.3 The cancellation of the infrared divergences

The considered example is typical of the QED and one can make the general statement concerning the infrared divergences for the elements of the S-matrix.

The infrared divergences in radiative corrections to the cross-section of any physical process in QED are cancelled in every order of perturbation theory if to the cross-section of the elastic process one adds the inelastic cross-section of the process with emission of an arbitrary number of additional photons integrated over the phase space with the requirement that the total photon energy does not exceed some value E_{min}.

This statement is also valid for the cross-sections of the processes in non-Abelian gauge theories like the electroweak theory and some processes in QCD, though in this case, due to the self-interaction of the non-Abelian gauge fields, there is no full factorization with the exponentiation, and the proof of this statement presents some problem. Nevertheless, for many processes the result has the same form. Thus, for example, the electromagnetic form-factor in QCD has the same Sudakov form (9.28) but with the replacement $\alpha \to C_F \alpha_s$.

Thus, one can say that the problem of obtaining the ultraviolet and the infrared finite radiative corrections to the cross-sections of the physical processes is solved in two steps: first, with the help of the renormalization procedure one gets rid of the ultraviolet divergences, which is under full control in renormalizable theories; second, defining the correct physical process including the emission of the soft quanta, the cancellation of the infrared divergences takes place.

As we will see below, this is not sufficient in non-Abelian gauge theories with massless gauge fields. They contain additional divergences which require some ads-inn to the described procedure. We will consider this question in the last lecture.

10 Lecture X: Collinear Divergences

10.1 The collinear divergences in massless theory

The obtained result (9.28) for the cross-section of creation of the muon pair in the process of e^+e^--annihilation with emission of additional soft photons is typical of the theories with a massive fermion and massless photons. It can be generalized to non-Abelian theories with massless gluon, though the gluon interactions cause some problems in proving the cancellation of the IR divergences. Note, however, that eq. (9.28) contains the logarithmic singularity with respect to the fermion mass, and if the latter tends to zero, one has the new divergence. This would not cause any problem since all the fermions are massive but the masses of the electron and the light quarks are so small compared to the characteristic energies of the scattering process that with good precision it is reasonable to neglect them. As for the QCD, considering the processes with gluons in initial states due to the self-interaction of the gluons we face this problem for the gluon amplitudes.

Let us analyse what is the reason for the appearance of the new divergence after the IR divergence at small photon momenta if regularized by introducing the photon mass. Consider for this purpose eq. (9.17) for the contribution of the real or virtual photons. The difference is that in one case the integration goes over the four-momentum of the virtual photon; and in the other case, over the three-momentum of the real photon, but what is essential that for the massless electron its propagator takes the form

$$\frac{1}{2pk} = \frac{1}{2(p^0k^0 - \vec{p}\vec{k})} \simeq \frac{1}{2(|\vec{p}||\vec{k}| - |\vec{p}||\vec{k}|\cos\theta)} = \frac{1}{2|\vec{p}||\vec{k}|(1-\cos\theta)}, \tag{10.1}$$

where θ is the angle between the electron and photon momenta. (In the case of a virtual photon we use the fact that the contribution to the singularity comes from the region of photon momentum close to the mass shell.)

Thus, the divergence appearing in the massless case comes from the integration over the angles and not over the modulus, as in the case of the IR divergence, and is related to the collinearity of momenta of two particles. For this reason it is called *the collinear divergence*. To get rid of these divergences, one can introduce the angular sensitivity of the detector analogously to the IR divergence. This would reflect the fact that two massless particles having almost parallel momenta are not distinguishable from a single particle with the same total momentum. Hence, the observed cross-section should include besides the main process the process of emission of the soft photons and the process of emission of the collinear photons with the kinematically allowed absolute values of momenta.

However, in real life the quarks and leptons are massive though their masses are very small; therefore, the problem of collinear divergences occurs for the processes with the gluon fields. Since the gluons are not free particles but exist inside hadrons, any process with the gluons has a similar process with quarks and it is reasonable to consider them together. For this reason, one usually speaks about *the inclusive* processes where besides the particles of the main process one includes the creation of all kinematically allowed particles, in particular the gluons. In this case, we do not impose any restriction on the gluon energy, we do not introduce any detector sensitivity to the energy or the angle, but sum over all the possibilities. It happens, however, that this is not sufficient to get the finite answer. It is necessary to take into account the possibility of existence of collinear gluons in the initial state, and only after this one can get the finite answer for the cross-section of the observable process.

The multiloop analysis in this case is much more complicated and is the subject of the Kinoshita-Lee-Nauenberg theorem which states:

The infrared and collinear divergences in a massless theory are cancelled in the cross-section of any process if one takes into account the existence in the initial and final states of an arbitrary number of the soft quanta as well as the particles having the parallel momenta with the same total momentum. The probabilities of these processes integrated over the phase space of these additional soft (collinear) quanta in the initial and final states should be added to the probability of the initial process.

As an illustration we consider the model example of the electron-proton (quark) scattering and put all the masses equal to zero. We will be interested in the radiative corrections in the first order with respect to the strong coupling α_s. The corresponding diagrams are shown in Fig.40.

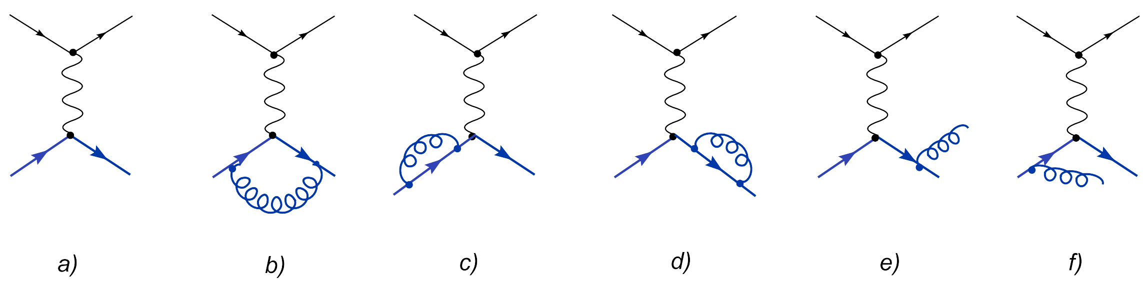


Figure 40: The process of electron-quark scattering in the first order in α_s:) the Born diagram, b)-d) the corrections due to the virtual gluons, e)-f) the corrections due to the real gluons

We have already calculated the matrix elements corresponding to these diagrams, but now we proceed in a different way. Since the ultraviolet divergences which appear in the diagrams b)-d) are compensated due to the Ward identity in QED ($Z_1 = Z_2$), all the arising divergences are solely infrared and collinear. To extract them we will use the dimensional regularization. Then both the divergences are manifested in the form of the poles over ε and, since we have both of them, there will be poles of the first and the second order.

We start with the virtual corrections. The diagrams of self-energy c) and d) in the massless case are identically zero due to the above-mentioned property of a massless integral depending on one argument equal to zero ($p^2 = 0$ on the mass shell). As we explained, here one has the cancellation of the UV and the IR divergences. Therefore, all divergences in the vertex diagram b) may be considered as infrared. (The UV divergences should cancel with the UV ones from the self-energy diagrams and the latter in their turn cancel with the IR). The integral for the vertex part is defined by two form factors $F_1(q^2)$ and $F_2(q^2)$ (9.1). Taking the expression for the vertex function (3.22) as the starting point, we put $m = 0$ and go to the mass shell. The result is

$$F_1(q^2) = -C_F\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-q^2}\right)^{\varepsilon}(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+8), \tag{10.2}$$

$$F_2(q^2) = 0, \tag{10.3}$$

where instead of the logarithm of the photon mass as the IR regulator we have the pole over ε. In order to avoid the transcendental numbers, we used the helpful definition of the angular measure in the space of $4-2\varepsilon$ dimensions and multiplied the standard expression by $\Gamma(1-\varepsilon)/(4\pi)^{\varepsilon}$. Then the constants like γ_E, $log(4\pi)$ and $\zeta(2)$ disappear from the intermediate expressions. Due to the cancellation of divergences in the final expressions, this redefinition does not influence the answer.

Thus, the cross-section for the diagrams with virtual gluon has the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{virt} = \left(\frac{d\sigma}{d\Omega}\right)_0\left[1-2C_F\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\varepsilon}(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+8)\right], \tag{10.4}$$

where the differential cross-section in the Born approximation is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{2E^2}\left(\frac{s^2+u^2-\varepsilon t^2}{t^2}\right)\left(\frac{\mu^2}{s}\right)^{\varepsilon}. \tag{10.5}$$

In the c.m. frame $s=E^2, t=-E^2/2(1-\cos\theta), u=-E^2/2(1+\cos\theta)$, where the angle θ is the electron scattering angle.

Consider now the diagrams with the emission of the real gluons e) and f). Besides the squares of each of the diagrams one should also take into account the interference term. The calculation in fact repeats that in QED but instead of the photon mass we again use the dimensional regularization and do not restrict the integration region over the momentum of additional gluon. The calculation is a bit tedious, after contracting all the indices the phase integral takes the form

$$d\sigma_{2\to 3} = \frac{1}{2\pi E^2}\int d^Dp_3\delta^+(p_3^2)\int\frac{d^Dk}{(2\pi)^D}\delta^+(k^2)\delta^+((p_4-k)^2)|M|^2_{p_4=p_1+p_2-p_3} \tag{10.6}$$

$$|M|^2 = \frac{e^4g^2}{4}8\frac{M_0+\epsilon M_1+\epsilon^2M_2}{t(s+t+u)},$$

$$M_0 = 4s-8p_1k-4p_2k+\frac{-8(p_1k)^2+4(2s+t)p_1k-(3s^2+t^2+u^2+2st)}{p_2k},$$

$$M_1 = -4(s+u)+8p_1k+8p_2k+\frac{8(p_1k)^2-4(s+t+u)p_1k+2(s+t+u)^2-2(u+s)t}{p_2k}$$

$$M_2 = 4(s+t+u)-4p_2k-\frac{(s+t+u)^2}{p_2k} = -\frac{(s+t+u+2p_2k)^2}{p_2k}.$$

It is useful to pass to the spherical coordinates and use the c.m. frame. After the integration over the phase volume the result can be represented in the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{real} = \left(\frac{d\sigma}{d\Omega}\right)_0\left[2C_F\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\varepsilon}(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+8)\right]+C_F\frac{\alpha^2}{E^2}\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{s}\right)^{\varepsilon}\left(\frac{\mu^2}{-t}\right)^{\varepsilon}(\frac{f_1}{\varepsilon}+f_2), \tag{10.7}$$

where the functions f_1 and f_2 in the c.m. frame are ($x = \cos\theta$)

$$\begin{aligned} f_1 &= -2\frac{(1-x)(x^3+5x^2-3x+5)\log(\frac{1-x}{2})-(x-1)^2(x+1)(x-11)/4}{(1-x)^2(1+x)^2}, \qquad (10.8)\\ f_2 &= -\frac{1}{(1-x)^2(1+x)^2}\Big[(1-x)(x^3+5x^2-3x+5)\log^2(\frac{1-x}{2})\\ &+\frac{1}{2}(1-x)(3x^3+15x^2+77x-31)\log(\frac{1-x}{2})+(1+x)^2(x^2+5x+3)\pi^2\\ &-12(9x^2+2x+5)Li_2(\frac{1+x}{2})+\frac{1}{2}(1-x)(1+x)(5x^2-42x-23)\Big]. \qquad (10.9)\end{aligned}$$

As one can see from the comparison of the cross-sections of the processes with the virtual (10.4) and the real gluons (10.7), in the sum the second order poles cancel. However, the total cancellation of divergences does not happen. The remaining divergences in the form of a single pole have a collinear nature. As was already mentioned, for their cancellation one has to define properly the initial states. The point is that the massless quark can emit the collinear gluon which will carry part of the initial momentum and in this case, it is impossible to distinguish one particle propagating with the speed of light from the two flying parallel.

10.2 The quark distributions and the splitting functions

To take into account this possibility, let us come back to the scattering process and assume that the initial quark has emitted the parallel gluon (see Fig.41). The two particles can be almost parallel with small relative transverse momentum. The three four-momenta can be

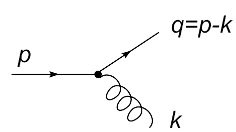


Figure 41: The diagram corresponding to the splitting of the quark into the quark and the gluon

chosen in the form:

$$p = (p;0,0,p), \quad q \approx (zp;p_\perp,0,zp), \quad k \approx ((1-z)p;-p_\perp,0,(1-z)p),$$

so that all of them obey the condition $p^2 = q^2 = k^2 = 0$ with the accuracy up to $p_\perp^2$. It is helpful, however, to use another method, namely to choose the momenta in such a way that they obey the mass shell condition with the accuracy up to $p_\perp^4$, but to give up the energy conservation in the order of $p_\perp^2$. The advantage of this approach consists in the use of formulas for the spinors and the polarization vectors on mass shell. Therefore, we choose the momenta as follows:

$$p = (p;0,0,p), \; q \approx (zp+\frac{p_\perp^2}{2zp};p_\perp,0,zp), \; k \approx ((1-z)p+\frac{p_\perp^2}{2(1-z)p};-p_\perp,0,(1-z)p).$$

The square of the matrix element corresponding to the process of splitting on mass shell in this case can be written in the standard form

$$|M(q\to qG)|^2 = \frac{g^2}{2}{}_F Tr(\gamma^\mu \hat{p}\gamma^\nu \hat{q})\sum_{pol}\epsilon^{*\mu}\epsilon^\nu, \qquad (10.10)$$

where the factor 1/2 comes from the averaging over the spin states. Here we must take into account the physical polarizations of the gluon only, i.e.

$$\sum_{pol} \epsilon^{*\mu}\epsilon^{\nu} \to \delta^{ij} - \frac{k^i k^j}{(\vec{k})^2},$$

which gives

$$|M(q \to qG)|^2 = 4g^2{}_F \left[p^0 q^0 - \frac{(\vec{p}\vec{k})(\vec{q}\vec{k})}{(\vec{k})^2} \right], \tag{10.11}$$

or, substituting the values of momenta,

$$|M(q \to qG)|^2 = C_F \frac{2g^2 p_\perp^2}{z(1-z)} \frac{1+z^2}{1-z}, \qquad z < 1. \tag{10.12}$$

The obtained expression does not depend on the choice of momenta and has a universal character.

Now one can calculate the cross-section of the process of interest. Graphically, it will be the same diagram Fig.40); however, the additional gluon will be referred not to the final state but to the initial one. Here we use the standard Feynman rules when the energy conservation law is not violated, but the massless particle is slightly off shell. Since in the case of interest the quark with momentum q is virtual, it is useful to choose the momenta like

$$p = (p; 0, 0, p), q \approx (zp - \frac{p_\perp^2}{2(1-z)p}; p_\perp, 0, zp), k \approx ((1-z)p + \frac{p_\perp^2}{2(1-z)p}; -p_\perp, 0, (1-z)p).$$

In this case,

$$q^2 = -\frac{p_\perp^2}{1-z}. \tag{10.13}$$

Then the cross-section of the process can be written in the factorized form

$$d\sigma(p) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} |M_{q\to qG}|^2 (\frac{1}{q^2})^2 (\frac{p^0 z}{p^0}) d\sigma(pz), \tag{10.14}$$

where the factor $(\frac{p^0 z}{p^0})$ is due to fact that the cross-section is normalized to the energy of initial particles, and we have replaced the quark with the energy p^0 by the quark with the energy zp^0.

Rewriting the differential d^3k in terms of the new variables

$$d^3k = pdzd^2p_\perp = pdz\pi dp_\perp^2,$$

and substituting the value of the matrix element (10.12) and q^2 from (10.13), we get

$$\begin{aligned} d\sigma(p) &= C_F \frac{2}{16\pi^2} \int \frac{pdzdp_\perp^2}{(1-z)p} \frac{(1-z)^2}{p_\perp^4} \frac{2p_\perp^2}{z(1-z)} \frac{1+z^2}{1-z} zd\sigma(pz) \\ &= C_F \frac{\alpha_s}{2\pi} \int \frac{dzdp_\perp^2}{p_\perp^2} \frac{1+z^2}{1-z} d\sigma(pz). \end{aligned} \tag{10.15}$$

The integral over the transverse momentum is divergent at zero and this is nothing else but the manifestation of the collinear divergence. The upper limit is not of great importance, it is restricted by kinematic considerations. We assume that the integration over $p_\perp^2$ goes from zero to some scale Q^2. Later, we will see that one can change this scale analogously to the change of the ultraviolet scale μ^2.

To extract the divergence we use the dimensional regularization. Changing the dimension of transverse integration from 2 to $2-2\varepsilon$ one gets

$$\begin{aligned} d\sigma(p) &= C_F\frac{\alpha_s}{2\pi}\int_0^1 dz\frac{1+z^2}{1-z}\int_0^{Q^2}\frac{(p_\perp^2)^{-\varepsilon}(-\mu^2)^\varepsilon dp_\perp^2}{p_\perp^2}d\sigma(pz) \\ &= C_F\frac{\alpha_s}{2\pi}\int_0^1 dz\frac{1+z^2}{1-z}\ \frac{1}{\varepsilon}\left(-\frac{\mu^2}{Q^2}\right)^\varepsilon d\sigma(pz). \end{aligned} \tag{10.16}$$

At first sight the obtained expression still contains the pole in the integrand as $z\to 1$. However, it only looks like a singularity. It came from the matrix element (10.12), which we have calculated only for $z<1$ and it needs to be redefined for $z\to 1$. We will come back to this question below and, at first, discuss the interpretation of relation (10.16).

Let us introduce the notion of distribution of the initial quark with respect to the fraction of the carried momentum z: $q(z)$. Then the initial distribution corresponds to $q(z)=\delta(1-z)$, and the emission of a gluon leads to the splitting: the quark carries the fraction of momentum equal z, while the gluon - $(1-z)$. The probability of this event is given by the so-called *splitting functions* $P_{qq}(z)$ and $P_{qG}(1-z)$. In the lowest order of perturbation theory in α_s the quark and gluon distributions can be written in the form

$$q(z,Q^2) = \delta(1-z)+\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}\left(\frac{\mu^2}{Q^2}\right)^\varepsilon P_{qq}(z), \tag{10.17}$$

$$G(z,Q^2) = \frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}\left(\frac{\mu^2}{Q^2}\right)^\varepsilon P_{qG}(1-z), \tag{10.18}$$

where the splitting functions are defined by the corresponding matrix elements one of which for $P_{qq}(z)$ has been calculated in the leading order in α_s earlier (see (10.12)). The result has the following form:

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right), \tag{10.19}$$

$$P_{qG}(z) = \frac{z^2+(1-z)^2}{2}. \tag{10.20}$$

Note that eq. (10.19) contains the redefinition of the function $P_{qq}(z)$ at the point $z=1$ mentioned above, namely the sign " + " should be understood as the following integration rule:

$$\int_0^1 dz\frac{f(z)}{(1-z)_+}\equiv\int_0^1 dz\frac{f(z)-f(1)}{(1-z)},$$

and the coefficient of the δ-function is defined from the requirement of conservation of the number of quarks

$$\int_0^1 q(x,Q^2)dz=1 \quad\Rightarrow\quad \int_0^1 P_{qq}(z)dz=0.$$

Thus, eq. (10.16) together with the Born diagram can be written as

$$d\sigma(p)=\int_0^1 dz\ q(z,Q^2)\ d\sigma(pz), \tag{10.21}$$

where the quark distribution $q(z, Q^2)$ is given by (10.17).

It seems strange at first sight that the answer depends on the scale Q^2 which defines the quark distribution. However, it has the physical interpretation. This is the measure of collinearity of the emitted gluons that can be distinguished, i.e., it refers to the definition of the initial state. In fact, in the massless case one cannot define the initial state that contains just the quark, it exists together with the set of collinear gluons. (The same is true for the massless electron with collinear photons.) This scale is sometimes called the factorization scale, at this scale the scattering cross-section (10.21) takes the factorized form. The factorization scale can be varied. The dependence of the quark and the gluon distributions on the scale is governed by the so-called DGLAP equations well known in QCD.

10.3 The finite answers

Thus, besides the two contributions to the cross-section from the virtual and the real gluons there is one more contribution related to the splitted initial state (10.16). In the lowest order of perturbation theory in α_s it can be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = \frac{1}{\varepsilon}\frac{\alpha_s}{2\pi}\int_0^1 dz \left(\frac{\mu^2}{Q_f^2}\right)^{\varepsilon} P_{qq}(z)\frac{d\sigma_0}{d\Omega}(pz), \tag{10.22}$$

where the Born cross-section is given by (10.5) with the replacement of the initial quark momentum p by pz, and the factorization scale Q_f^2 is an arbitrary quantity associated with the quark distribution function. Note that the scale Q_f^2 may depend on z. It is quite natural to choose the factorization scale equal to the characteristic scale of the process of interest. Thus, in our case this choice corresponds to $Q_f^2 = -\hat{t}$, where $\hat{t}$ is the Mandelstam parameter t for the process where p is replaced by pz. One has $\hat{t} = t\frac{2z}{(z+1)+(z-1)x}$. This leads to the following result:

$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = C_F\frac{\alpha^2}{2E^2}\frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{s}\right)^{\varepsilon}\left(\frac{\mu^2}{-t}\right)^{\varepsilon}(-\frac{f_1}{\varepsilon} + f_3), \tag{10.23}$$

where f_1 is given by (10.8) and

$$\begin{aligned} f_3 \quad = \quad & -\frac{1}{(1-x)^2(1+x)^2}\Big[2(1-x)(x^3+x^2-33x+7)\log(\frac{1-x}{2}) \\ & +12(9x^2+2x+5)Li_2(\frac{1+x}{2}) - (1+x)^2(x^2+5x+3)\pi^2 \\ & -\frac{1}{2}(1-x)(1+x)(11x^2-19)\Big]. \end{aligned} \tag{10.24}$$

Comparing the obtained expression with (10.4) and (10.7) we see that the last divergence cancels and the final expression for the cross-section of the electron-quark scattering with account of possible creation of the gluon in the initial and final states takes the form ($x = \cos\theta$)

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right) \quad &= \quad \left(\frac{d\sigma}{d\Omega}\right)_{virt} + \left(\frac{d\sigma}{d\Omega}\right)_{real} + \left(\frac{d\sigma}{d\Omega}\right)_{split} \qquad (10.25) \\ = \frac{\alpha^2}{2E^2}&\left\{\frac{x^2+2x+5}{(1-x)^2} - \frac{\alpha_s}{2\pi}\frac{C_F}{(1-x)(1+x)^2}\Big[(x^3+5x^2-3x+5)\log^2\frac{1-x}{2}\right. \\ &\left. +\frac{1}{2}(7x^3+19x^2-55x-3)\log\frac{1-x}{2} - (1+x)(3x^2+21x+2)\Big]\right\}. \end{aligned}$$

This expression is our final answer for the cross-section of the physical process of electron-quark scattering where the initial and the final state include the soft and collinear gluons. It includes also the definition of the initial state and can be recalculated for the alternative choice of the factorization scale similar to what happens to the ultraviolet scale which defines the coupling constant. Thus, we practically deal with the scattering not of individual particles but rather with coherent states with a fixed total momentum. Only this process has a physical meaning.

In Fig.42, we show the differential cross-section of this process as a function of the electron scattering angle: $\frac{E^2}{\alpha^2}\frac{d\sigma}{d\Omega}(\cos\theta)$. We have chosen here the strong coupling $\alpha_s = 0.2$, and $C_F = 4/3$. As one can see, the inclusion of the radiative correction $\sim \alpha_s$ practically

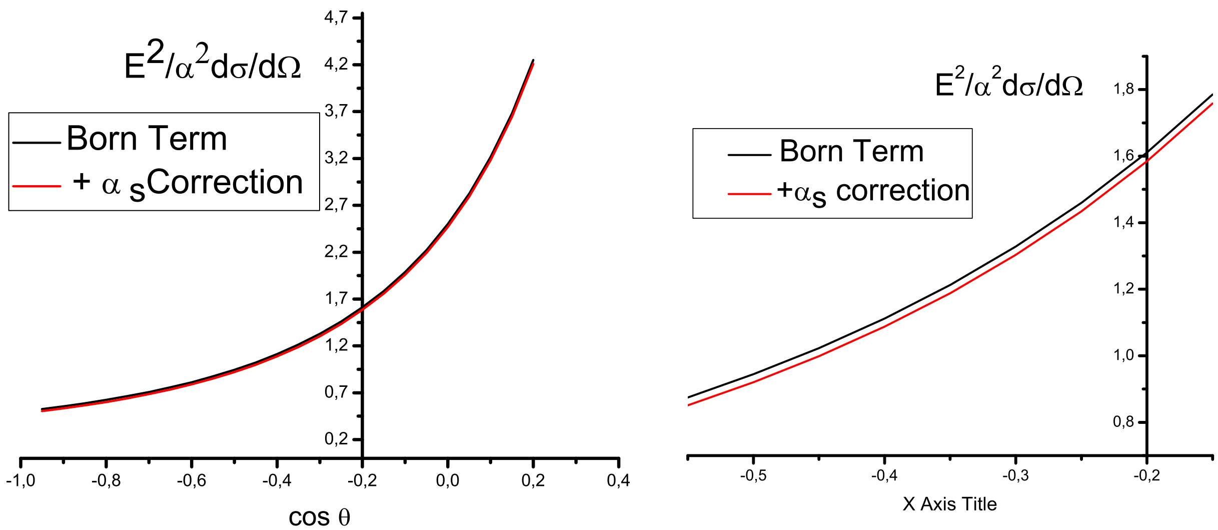


Figure 42: The differential cross-section of eq scattering in the Born approximation and with allowance for the α_s correction. On the right plane the same plot is shown in the bigger scale

does not change the result, the difference from the Born approximation is less than a per cent, that justifies the use of perturbation theory.

Let us stress once more that the obtained answer for the cross-section of the observable process depends on: a) the ultraviolet subtraction scheme that manifests itself, in particular, in the appearance of the ultraviolet scale μ^2 (canceled in our case in the lowest order of perturbation theory) and b) the definition of the initial coherent state, which manifests itself in the appearance of the factorization scale Q_f^2. The universality in the description of the physical processes is based on the fact that choosing the UV and the IR scale one way or another and fitting the experimental data of some process, one can then recalculate the obtained values of the running coupling and of the quark (lepton) distribution for any other choice of the scales. This way the result for the observable quantities does not depend on a particular choice of these scales and is universal.

11 Afterword

Local quantum field theory, being the mathematical basis of elementary particle physics, is the logical continuation of quantum mechanics. It exploits the same basic ideas, but describing the system with an infinite number of degrees of freedom permits the creation and annihilation of particles in the course of the interaction. The modern formulation is based on the interaction representation which assumes the existence of the asymptotic states of the free fields. In the S-matric approach we presume that these fields interact in a local way in the space-time, and calculating the S-matrix elements one can find the probabilities of various processes. The most developed and reliable method of these calculations is the perturbation theory in the coupling constant which is similar to the one in quantum mechanics. However, due to a much more complicated structure of the field theory, the methods of perturbation theory encounter problems which have no analogy in quantum mechanics, namely the divergence of the appearing integrals for the radiative corrections. We have shown in these lectures how one can deal with these divergences which have the ultraviolet and the infrared nature and how to get the finite answers for the probabilities of the physical processes. We did not aim to prove the main theorems like the Bogoliubov-Parasiuk or the Kinoshita-Lee-Nauenberg theorem, but have exemplified how they work. The explicit calculations allow one to convince himself in the validity of the final conclusions.

It should be noted that the formalism of quantum field theory contains the physical principles which we have to follow sometimes not realizing it. Thus, for example, the ultraviolet divergences restrict the type of the interaction and, contrary to quantum mechanics, there are only a few types of allowed Lagrangians. Not without reason the renormalizability played such an important role in the formation of the Standard Model. The other example is the notion of the asymptotic states. Even starting with the free fields within the perturbation theory, from the requirement of the cancellation of the infrared divergences we come to the definition of the physical initial and final states which are essentially the coherent states.

The very fact that the gravitational interaction does not fit to the general scheme probably means that local quantum field theory has a limited applicability and should be replaced by a more general construction. It might be nonlocal like in the string theory, or multidimensional one like in the brane-world theory. However, in any case, in the low energy limit one has the local quantum field theory though possibly going beyond the Standard Model that we considered here.

Acknowledgments

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