

INTRODUCTION TO QUANTUM FIELD THEORY

Lecture Notes

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Abstract

These lectures contain an elementary introduction to the principles of quantum field theory. Our aim is, starting from the very beginning and discussing the basic concepts to build up the formalism needed to construct the Standard Model of particle interactions.

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1 Lecture I

Fields, Lagrangians, Equations of Motion, Conservation Laws, Quantization

The modern theory of fundamental interactions is the theory of **Quantised Fields**. The main object of quantum field theory is a quantum field.

1.1 Fields

The **Field** is the most fundamental and universal form of matter known so far which describes a system with an infinite number of degrees of freedom. It has already appeared in classical physics while describing the action of two sources at a distance (Fig.1). Because of a finite velocity of spread of interaction, limited by the velocity of light, the source does not feel the other one immediately. The substance carrying the interaction is called the field as it was introduced by Michael Faraday in the 19-th century.

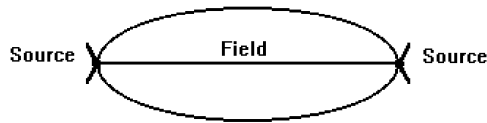


Figure 1:

The **Quantum Field** is the synthesis of the classical Faraday-Maxwell electromagnetic field and the field of probabilities in non-relativistic quantum mechanics. The quantum field is a single object replacing the fields and particles in classical physics. One field of that sort being a function of a space-time point x_μ describes all the particles of the same kind in the Universe.

The concept of the quantum field enables us to describe systems with an unconserved number of particles and also transformations of one particle into another. Elementary act of any interaction becomes the interaction of several quantum fields at a space-time point while the usual classical "forces" appear to be secondary effects resulting from the exchange of intermediate fields.

Mathematically, the field is a straightforward generalization of a point. In case of a field we have an infinite and continuous system of points with an infinite number of degrees of freedom.

1.2 Lagrangians

In what follows we consider the so-called **Lagrangian** approach to Quantum Field Theory. It is constructed in full analogy with an ordinary Lagrangian mechanics of a point. The main role is played by the **Lagrangian function** $L(t)$ that actually is a

function of dynamical variables and is a sum over all points of a system. In case of a field theory the sum is replaced by the integral over the space. The correspondence with classical mechanics is shown in Table 1.

Table 1.

<i>ThePoint</i>	<i>TheField</i>
$x_\mu^i \quad i = 1, 2, \dots, N$	$\varphi(x_\mu)$
$L(t) = \sum_{i=1}^N \mathcal{L}(x^i)$	$L(t) = \int_{space} d^3x \mathcal{L}(x)$
$A = \int_{t_1}^{t_2} L(t) dt$	$A = \int_{time} L(t) dt = \int_{space-time} d^4x \mathcal{L}(x)$

The action A then becomes an integral of the Lagrangian density function over the space-time.

The Lagrangian density, usually called simply the **Lagrangian**, obeys the following requirements:

1) **Lorentz (Poincare) Invariance.** This is a physical requirement of invariance of equations of motion obtained from the Lagrangian (see below eq.(1.1)) with respect to the choice of a frame. This means that the Lagrangian being a function of fields and their derivatives has no explicit dependence on x :

$$\mathcal{L}(x) = \mathcal{L}[\varphi(x), \partial\varphi(x), \dots].$$

The Lorentz invariance means that

$$\mathcal{L}'(x') = \mathcal{L}[\varphi'(x'), \partial\varphi'(x'), \dots] = \mathcal{L}[\varphi(x), \partial\varphi(x), \dots] = \mathcal{L}(x),$$

i.e. $\mathcal{L}(x)$ is a scalar, while the fields may belong to various representations of the Lorentz group like spinor, vector, etc.

2) **Locality.** This is a postulate which is experimentally verified. So far we have no evidence of deviation from local QFT. Locality of $\mathcal{L}(x)$ means that it contains a limited number of partial derivatives of the fields.

3) **Correspondence with classical mechanics.** Equations of motion in classical mechanics contain second-order derivatives with respect to time. This means that the Lagrangian should depend only on first-order derivatives of the fields in order to have the proper equations of motion., i.e.

$$\mathcal{L}(x) = \mathcal{L}[\varphi(x), \partial\varphi(x)].$$

4) **Unitarity.** This is the property of the S-matrix following from the requirement of conservation of the norm of a state vector. Unitarity in turn means that the Lagrangian should be a real (hermitean) function which provides real (hermitean) dynamical invariants like energy, momentum, etc.

It should be noted that the Lagrangian is not uniquely defined. One can add to it a total derivative of an arbitrary function

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial_\mu F^\mu$$

without changing the action, i.e. $A' = A$, and hence the equations of motion. Strictly speaking this is true if one ignores the topologically nontrivial contributions coming from the boundary conditions. This is sometimes used to choose the Lagrangian in the most useful form.

1.3 Equations of Motion

Equations of motion can be obtained from the Lagrangian with the help of the **principle of the least action**, which states that during the motion of a classical system the action along the path takes the minimal value, i.e. $\delta A = 0$. To find out the corresponding equation, consider the variation of the field $\delta u_i(x)$ such that $\delta u_i(x) = 0$ at the boundary. Then the equation $\delta A / \delta u_i = 0$ leads to the *Euler-Lagrange* equation of motion

$$\frac{\delta \mathcal{L}}{\delta u_i(x)} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu u_i(x)} = 0, \quad (1.1)$$

where $\partial_\mu \equiv \partial / \partial x_\mu$.

1.4 Dynamical Invariants

The equations of motion enable us to construct the dynamical invariants, i.e. time-independent quantities such as the energy, momentum, angular momentum, charge, etc.

Consider the variation $\delta \mathcal{L}$ corresponding to the transformation of the field $u_i \rightarrow u_i + \delta u_i$ with the argument x_μ being unchanged for the time being. One has

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta u_i} \delta u_i + \frac{\delta \mathcal{L}}{\delta \partial_\mu u_i} \delta(\partial_\mu u_i).$$

Using the equation of motion for the first term and the fact that in our case $\delta \partial_\mu u_i = \partial_\mu \delta u_i$, we find that

$$\delta \mathcal{L} = \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu u_i} \delta u_i + \frac{\delta \mathcal{L}}{\delta \partial_\mu u_i} \partial_\mu \delta u_i.$$

So, finally we have

$$\delta \mathcal{L} = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu u_i} \delta u_i \right].$$

Consider now a global (rigid) transformation $\delta u_i = i \alpha^a T_{ij}^a u_j$, where parameters $\alpha^a = \text{const}$ and T^a are the generators of the transformation. Then

$$\delta \mathcal{L} = -\alpha^a \partial_\mu j^{\mu a},$$

where the current $j^{\mu a}$ is

$$j^{\mu a} = -i \frac{\delta \mathcal{L}}{\delta \partial_\mu u_i} T_{ij}^a u_j. \quad (1.2)$$

If \mathcal{L} is invariant, i.e. $\delta \mathcal{L} = 0$, there will be conserved currents

$$\partial_\mu j^{\mu a} = 0.$$

Taking the integral over the whole space we have

$$\int d^3x \left(\frac{\partial}{\partial t} j^{0a} - \vec{\nabla} \cdot \vec{j}^a \right) = 0.$$

If \vec{j}^a vanishes at spatial infinity, then the term with the total derivative vanishes and we come to

$$\frac{d}{dt} \int d^3x j^{0a}(\vec{x}, t) = 0,$$

i.e. the charge

$$Q^a = \int d^3x j^{0a}(\vec{x}, t) \quad (1.3)$$

is conserved

$$\frac{dQ^a}{dt} = 0. \quad (1.4)$$

This is Neuther's theorem.

1.5 Noether's theorem

Let the action be invariant under some s -parameter transformation

$$\begin{cases} x_\mu \rightarrow x'_\mu = f_\mu(x, \omega), & \omega_k, \quad k = 1, 2, \dots, s \\ u_i(x) \rightarrow u'_i(x') = U_i(x, \omega), & \text{are the parameters} \end{cases}$$

so that $\delta A = 0$.

Then there exist s dynamical invariants

$$C_k = \int d^3x \Theta_k^0(x), \quad \frac{dC_k}{dt} = 0, \quad k = 1, 2, \dots, s,$$

where

$$\Theta_k^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu u_i} (\partial_\nu u_i X_k^\nu - \Psi_{i,k}) - X_k^\mu \mathcal{L}, \quad (1.5)$$

$$X_k^\mu = \frac{\partial f^\mu(x, \omega)}{\partial \omega_k} \Big|_{\omega=0}, \quad \Psi_{i,k} = \frac{\partial U_i(x, \omega)}{\partial \omega_k} \Big|_{\omega=0}. \quad (1.6)$$

The previous case (eq.(1.2)) corresponds to the internal symmetry transformation when

$$X_k^\mu = 0, \quad \Theta_k^\mu = j_k^\mu, \quad C_k = Q_k.$$

1.5.1 Example 1: Translation

$$\begin{cases} x'_\mu = x_\mu + \omega_\mu, & X_k^\mu = \delta_k^\mu, & \Theta_\nu^\mu = T_\nu^\mu, \\ u'_i(x') = u_i(x), & \Psi_{i,k} = 0, & C_\nu = P_\nu. \end{cases}$$

Here T_ν^μ and P_ν are the energy-momentum tensor and four-momentum, respectively

$$\begin{aligned} T^{\mu\nu}(\mathcal{L}) &= \frac{\delta \mathcal{L}}{\delta \partial_\mu u_i} \partial_\nu u_i - g^{\mu\nu} \mathcal{L}, \\ P^\nu &= \int d^3x T^{\nu 0}(\vec{x}, t). \end{aligned} \quad (1.7)$$

The conservation of the momentum $P^\nu(t_1) = P^\nu(t_2)$ results from the invariance of the Lagrangian with respect to translational invariance $\mathcal{L}(x + \omega) = \mathcal{L}(x)$.

1.5.2 Example 2: Lorentz rotation

$$\begin{cases} x'_\mu = x_\mu + \delta L_{\mu\nu} x_\nu, & X_\nu^{(\rho\sigma)} = x^\sigma \delta_\nu^\rho - x^\rho \delta_\nu^\sigma, \\ u'_i(x') = A_i^{j(\mu\nu)} u_j(x) \delta L_{\mu\nu}, & \Psi_{i,k}^{(\rho\sigma)} = A_i^{j(\rho\sigma)} u_j. \end{cases}$$

Here $A_i^{j(\rho\sigma)}$ depends on the spin of a field. The conserved quantity now is the total angular momentum

$$M^{\tau(\rho\sigma)} = \frac{\delta \mathcal{L}}{\delta \partial_\tau u_i} (\partial^\rho u_i x^\sigma - \partial^\sigma u_i x^\rho) \quad (1.8)$$

$$+ \mathcal{L} (x^\rho g^{\sigma\tau} - x^\sigma g^{\rho\tau}) - \frac{\delta \mathcal{L}}{\delta \partial_\tau u_i} A_i^{j(\rho\sigma)} u_j. \quad (1.9)$$

The last term is the spin-tensor $S^{\tau(\rho\sigma)}$.

We now illustrate these general expressions by some simple examples.

1.6 Illustration

1.6.1 Example 1: Scalar Fields

The Lagrangian with the above mentioned properties for a real scalar field is

$$\mathcal{L}_s = \frac{1}{2} \partial_\mu \varphi(x) \partial^\mu \varphi(x) - \frac{m^2}{2} \varphi^2(x). \quad (1.10)$$

The first term here is the kinetic energy and the last one is the mass term. Equation of motion following from (1.9) is the well known **Klein-Gordon** equation

$$(\square - m^2) \varphi(x) = 0, \quad (1.11)$$

where we use the notation $\square = -\partial^2 = -\partial_0^2 + \vec{\partial}_x^2$.

We construct now the dynamical invariants (1.7):

$$\begin{aligned} T^{\mu\nu} &= \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}, \\ T^{00} &= \mathcal{H} = \frac{1}{2} [\dot{\varphi}^2 + (\vec{\nabla} \varphi)^2 + m^2 \varphi^2], \\ P^k &= \int T^{0k} d\vec{x} = - \int \varphi \partial^k \varphi d\vec{x}, \end{aligned}$$

where the dot means the derivative with respect to t .

To solve the K-G equation, it is useful to go to the momentum representation. Performing the Fourier transform we have

$$\varphi(x) = \frac{1}{(2\pi)^2} \int d^4 k e^{ikx} \tilde{\varphi}(k),$$

where $\tilde{\varphi}(k)$ obeys the equation

$$(k^2 - m^2) \tilde{\varphi}(k) = 0. \quad (1.12)$$

The solution of eq.(1.11) is

$$\tilde{\varphi}(k) = \sqrt{2\pi}\delta(k^2 - m^2)\varphi(k),$$

where $\varphi(k)$ is an arbitrary function of k_μ with $k_0 = \pm\sqrt{\vec{k}^2 + m^2}$. Two possible signs of the square root correspond to two linearly independent solutions

$$\varphi(x) = \varphi^+(x) + \varphi^-(x) \quad (1.13)$$

such that

$$\varphi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int d^4k e^{\pm ikx} \delta(k^2 - m^2) \tilde{\varphi}^\pm(k), \quad k_0 = \pm\sqrt{\vec{k}^2 + m^2}.$$

Taking the integral over k_0 we get

$$\varphi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2k_0}} e^{\pm ikx} a^\pm(\vec{k}), \quad (1.14)$$

with the redefinition $a^\pm(\vec{k}) \equiv \tilde{\varphi}^\pm(k)/\sqrt{2k_0}$. Now the four-momentum takes the form

$$P^\mu = \int T^{0\mu} d\vec{x} = \int d\vec{k} k^\mu a^+(\vec{k}) a^-(\vec{k}) \quad (1.15)$$

or if one restores the initial order of multipliers

$$P^\mu = \frac{1}{2} \int d\vec{k} k^\mu [a^+(\vec{k}) a^-(\vec{k}) + a^-(\vec{k}) a^+(\vec{k})]. \quad (1.16)$$

One can give now the interpretation to these coefficients $a^\pm(\vec{k})$. As follows from eq.(1.15) the product $a^+(\vec{k}) a^-(\vec{k})$ represents the particle density with momentum \vec{k} and energy k_0 . We shall see after quantization that $a^\pm(\vec{k})$ will become the creation (annihilation) operators and the product will be the operator of the number of particles with a given momentum.

1.6.2 Example 2: Vector fields

The Lagrangian with due regard to the note on p.12 can be chosen to be

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{m^2}{2} U_\mu(x) U^\mu(x), \quad (1.17)$$

where $F_{\mu\nu}(x) = \partial_\mu U_\nu(x) - \partial_\nu U_\mu(x)$.

The corresponding equation of motion

$$\partial_\nu F^{\nu\mu} - m^2 U^\mu = (\square - m^2) U^\mu + \partial^\mu \partial_\nu U^\nu = 0 \quad (1.18)$$

is called the **Proca** equation. Taking the derivative of eq.(1.17) with respect to ∂_μ we get the constraint

$$\partial_\mu U^\mu = 0 \quad (1.19)$$

which is called the **Lorentz** condition. Thus, the Proca equation is equivalent to the K-G equation with the Lorentz condition. Therefore, the solutions of eq.(1.17) are the same as eqs.(1.12), (1.18).

$$U_\mu(x) = U_\mu^+(x) + U_\mu^-(x), \quad (1.20)$$

$$U_\mu^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2k_0}} e^{\pm i k x} U_\mu^\pm(\vec{k})$$

plus the constraint

$$k_0 U_0^\pm(\vec{k}) = k_n U^{\pm, n}(\vec{k}), \quad n = 1, 2, 3. \quad (1.21)$$

The four-momentum and spin-vector for a real vector field are (cf.eq.(1.14))

$$P^\nu = - \int d\vec{k} k^\nu [U_\mu^+(\vec{k}) U^{-, \mu}(\vec{k})],$$

$$\vec{S} = i \int d\vec{k} k^\nu [\vec{U}^+(\vec{k}) \times \vec{U}^-(\vec{k})], \quad (1.22)$$

where the arrow mean the three-vectors and the spin vector is defined by $S^a = \epsilon^{abc} \int d\vec{x} S_{bc}^0$. One can see that P^ν in eq.(1.21) is not positive definite due to the Minkowskian signature, while the positive definiteness of the Hamiltonian (P^0) is a necessary requirement for stability of a classical system. However, we have not yet used the constraint (1.20). Substituting eq.(1.20) into eq.(1.21) we have

$$-U_\mu U^\mu = \vec{U} \vec{U} - \frac{1}{k_0^2} (\vec{k} \vec{U}) (\vec{k} \vec{U}).$$

This form should be diagonalized, which can be done by introducing the local frame

$$\vec{U}(\vec{k}) = \vec{e}_1 a_1(\vec{k}) + \vec{e}_2 a_2(\vec{k}) + \frac{\vec{k}}{|\vec{k}|} \frac{k_0}{m} a_3(\vec{k}), \quad (1.23)$$

where the polarization vectors \vec{e}_i obey the normalization properties

$$(\vec{e}_i \cdot \vec{e}_j) = \delta_{ij}, \quad \vec{e}_3 = \frac{\vec{k}}{|\vec{k}|}.$$

Eq.(1.22) leads to $-U_\mu U^\mu = a_n a_n$, $n = 1, 2, 3$, so that eq.(1.21) becomes

$$P^\nu = \int d\vec{k} k^\nu \sum [a_n^+(\vec{k}) a_n^-(\vec{k})] =$$

$$= \frac{1}{2} \int d\vec{k} k^\nu \sum_n [a_n^+(\vec{k}) a_n^-(\vec{k}) + a_n^-(\vec{k}) a_n^+(\vec{k})],$$

which is now obviously positive definite. However, if we consider the projection of the spin-vector along the momentum, we find out that it is not diagonal

$$S_3 \sim i [a_1^+(\vec{k}) a_2^-(\vec{k}) - a_2^+(\vec{k}) a_1^-(\vec{k})] \quad (1.24)$$

making the interpretation less clear. To diagonalize eq.(1.23), we make the linear substitution

$$a_1^\pm = \frac{b_1^\pm + b_2^\pm}{\sqrt{2}}, \quad a_2^\pm = \frac{b_1^\pm - b_2^\pm}{i\sqrt{2}}, \quad a_3^\pm = b_3^\pm$$

and finally get

$$\begin{aligned} P^\nu &= \frac{1}{2} \int d\vec{k} k^\nu \sum_n [b_n^+(\vec{k}) b_n^-(\vec{k}) + b_n^-(\vec{k}) b_n^+(\vec{k})], \\ S_3 &\sim i[b_1^+(\vec{k}) b_1^-(\vec{k}) - b_2^+(\vec{k}) b_2^-(\vec{k})]. \end{aligned} \quad (1.25)$$

Now the interpretation is straightforward. For instance, $b_1^+ b_1^-$ is the density of particles with momentum \vec{k} , energy k_0 and projection of the spin equal to +1. Actually, $b_{1,2}^\pm$ correspond to circular polarization, while $a_{1,2}^\pm$ are linear polarization vectors. Due to the absence of $b_3^+ b_3^-$ in eq.(1.25) it corresponds to a zero spin projection.

Thus, we conclude that a vector field with non-zero mass describes particles with three possible spin projections equal to +1, 0, -1.

1.6.3 Example 3: Spinor fields

The Lagrangian now is

$$\mathcal{L} = \bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x) \quad (1.26)$$

and leads to the equation of motion which is the **Dirac** equation

$$(i\hat{\partial} - m)\psi(x) = 0. \quad (1.27)$$

Here $\hat{\partial} \equiv \gamma^\mu \partial_\mu$, γ^μ being the Dirac 4x4 γ -matrices. The spinor field $\psi_\alpha(x)$ is a column vector

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

and the conjugated one is a row vector $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$, where ψ^\dagger means hermitean conjugation.

Dynamical invariants here are also given by Noether's theorem. The energy-momentum tensor, charged current and spin-tensor are, respectively

$$\begin{aligned} T^{\mu\nu}(x) &= \frac{i}{2} [\bar{\psi} \gamma^\mu \partial^\nu \psi - \partial^\nu \bar{\psi} \gamma^\mu \psi], \\ J^\mu(x) &= \bar{\psi} \gamma^\mu \psi, \\ S^{\tau(\mu\nu)}(x) &= \frac{1}{4} \bar{\psi} (\gamma^\tau \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\tau) \psi, \quad \sigma^{\mu\nu} = \frac{1}{2i} [\gamma^\mu, \gamma^\nu]. \end{aligned} \quad (1.28)$$

To solve the Dirac equation we go to the momentum representation. Just like in the bosonic case, there are two solutions

$$\psi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{\sqrt{2p_0}} e^{\pm ipx} \psi^\pm(\vec{p}),$$

where $\psi^\pm(\vec{p})$ obey the equation

$$(m \pm \hat{p})\psi^\pm(\vec{p}) = 0. \quad (1.29)$$

In the $\vec{p} = 0$ frame this leads to

$$(\gamma^0 \pm I)\psi^\pm(0) = 0.$$

Taking the particular representation for γ^0 matrix

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

one gets the following two-parameter solutions for both the cases

$$\begin{aligned} \psi_\alpha^-(0) &= c_1\delta_{1\alpha} + c_2\delta_{2\alpha}, \\ \psi_\alpha^+(0) &= c_3\delta_{3\alpha} + c_4\delta_{4\alpha}. \end{aligned} \quad (1.30)$$

Here α is a spinor index running from 1 to 4 and $\delta_{i\alpha}$ is the Kronecker symbol.

Hence, there are two independent solutions of eq.(1.29) which are actually two spin eigenstates for both ψ^+ and ψ^- . To go to an arbitrary momentum one should perform the Lorentz rotation of eq.(1.30).

Let us introduce a pair of such normalized solutions to the Dirac equation (1.29):

$$v_\alpha^{s,\pm}(\vec{p}), \quad s = 1, 2; \quad \alpha = 1, 2, 3, 4$$

and the conjugated ones $v_\alpha^{*s,\pm}(\vec{p})$ defined by

$$v_\alpha^{*s,\pm}(\vec{p}) = (v_\alpha^{s,\pm}(\vec{p}))^*$$

with the normalization property

$$v_\alpha^{*s,\pm}(\vec{p})v_\alpha^{\tau,\pm}(\vec{p}) = \delta^{s\tau}.$$

Then for an arbitrary solution we have an expansion

$$\psi_\alpha^\pm(\vec{p}) = \sum_{s=1,2} v_\alpha^{s,\pm}(\vec{p})a_s^\pm(\vec{p}). \quad (1.31)$$

Substituting eq.(1.31) into eq.(1.28) we get

$$P^\nu = \frac{1}{2} \int d\vec{p} p^\nu \sum_s [a_s^{*+}(\vec{p})a_s^-(\vec{p}) - a_s^{*-}(\vec{p})a_s^+(\vec{p})]. \quad (1.32)$$

Note the very important difference in sign for the second term as compared to the bosonic case, eq.(1.15). The Hamiltonian happens not to be positive definite. To avoid this disaster, one needs the proper quantization procedure (Fermi-Dirac) to be discussed below.

For the interpretation of the coefficients $a_s^\pm(\vec{p})$ it is also useful to consider the spin projection and the charge operators:

$$\begin{aligned} S_3 &\sim \frac{1}{2}[a_1^{*+}a_1^- - a_2^{*+}a_2^- + a_1^{*-}a_1^+ - a_2^{*-}a_2^+], \\ Q &= \int \psi^* \psi d\vec{x} = \int d\vec{p} \sum_s [a_s^{*+}a_s^- + a_s^{*-}a_s^+]. \end{aligned} \quad (1.33)$$

Now the interpretation becomes clear as in the previous cases. We see that the Dirac field describes charged particles with two possible spin projections equal to $\pm 1/2$.

1.7 Quantization

The next step of our program is the quantization procedure as far as we know that at small distances Nature maintains the quantum properties. Quantization in field theory is a generalization of that one in Quantum mechanics with N particles when $N \rightarrow \infty$.

We start with the consideration of the most popular model which is directly connected with the free field theory, namely, with the harmonic oscillator.

1.7.1 Occupation number ("second" quantization) representation.

The Hamiltonian of the harmonic oscillator which enters into the **Schrödinger** equation is

$$H = \frac{1}{2}(p^2 + \omega^2 q^2), \quad (1.34)$$

where p and q are the momentum and coordinate, respectively. Solving the eigenvalue equation

$$H\psi_n = E_n\psi_n,$$

we find the spectrum $E_n = \hbar\omega(n + 1/2)$. The eigenfunctions happen to be connected with each other by acting of some operators a^+ and a

$$a^+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad a\psi_n = \sqrt{n}\psi_{n-1},$$

where

$$a^+ = \sqrt{\frac{\omega}{2}}\left(\hat{q} - \frac{i\hat{p}}{\omega}\right), \quad a = \sqrt{\frac{\omega}{2}}\left(\hat{q} + \frac{i\hat{p}}{\omega}\right)$$

are usually called the rising and lowering operators, respectively, and \hat{q} and \hat{p} are the operators of the coordinate and momentum in some representation. One can also define the ground state ψ_0 by $a\psi_0 = 0$.

The operators a^+ and a obey the following algebra

$$\begin{aligned} aa^+ &= \hat{n} + 1, \quad a^+a = \hat{n}, \quad [a, a^+] = 1, \\ [a, a] &= 0, \quad [a^+, a^+] = 0, \end{aligned} \quad (1.35)$$

where $\hat{n}\psi_n = n\psi_n$. The Hamiltonian now can be expressed in terms of these operators

$$H = \frac{\omega}{2}(aa^+ + a^+a) = \omega(\hat{n} + 1/2). \quad (1.36)$$

Because of the equidistant spectrum we can now interpret the operators to be

- a^+ - the creation operator of a quantum,
- a - the annihilation operator of a quantum,
- \hat{n} - the operator of a number of quanta.

Hence any n -quantum state can be obtained by acting of creation operators on the vacuum state

$$\psi_n = \frac{(a^+)^n}{\sqrt{n!}}\psi_0.$$

In case of N oscillators the Hamiltonian is simply a sum of those for the individual oscillators

$$H = \sum_{k=1}^N H_k = \sum_{k=1}^N \omega_k (a_k^+ a_k + 1/2), \quad (1.37)$$

while the algebra of the operators becomes

$$[a_k, a_l^+] = \delta_{kl}, \quad [a_k, a_l] = 0, \quad [a_k^+, a_l^+] = 0. \quad (1.38)$$

Thus, an arbitrary quantum state can be described by the so-called occupation numbers, i.e. the numbers telling you how many creation operators of a given type are there

$$\psi(n_1, \dots, n_N) = \prod_{1 \leq k \leq N} \left[\frac{(a_k^+)^{n_k}}{\sqrt{n_k!}} \right] \psi_0.$$

This is called the **occupation number** representation (or the **Fock** representation).

We will show now that the scalar field obeying the Klein-Gordon equation is equivalent to a set of oscillators (1.37). Consider the 3-dimensional Fourier transform

$$u(t, \vec{x}) = \int d\vec{k} [e^{i\vec{k}\vec{x}} a(t, \vec{k}) + e^{-i\vec{k}\vec{x}} a^*(t, \vec{k})].$$

Then, the K-G equation gives the equation of motion for $a(t, \vec{k})$

$$\ddot{a}(t, \vec{k}) + \omega_k^2 a(t, \vec{k}) = 0,$$

where the frequency $\omega_k^2 = \vec{k}^2 + m^2$. This is nothing else than the oscillator-type equation. If we now put the system into a "box" of volume $V = L^3$ with periodic boundary conditions, we will have a discrete set of possible values of momenta

$$\vec{k}(n_1, n_2, n_3) = \left\{ \frac{2\pi}{L}n_1, \frac{2\pi}{L}n_2, \frac{2\pi}{L}n_3 \right\}.$$

Each value of \vec{k}_n corresponds to an oscillator with the frequency

$$\omega_n^2 = m^2 + \frac{4\pi^2}{L^2}(n_1^2 + n_2^2 + n_3^2).$$

Thus, the field obeying the K-G equation corresponds to a set of oscillators labelled by three integers n_1, n_2, n_3 . Transition to an infinite volume $L \rightarrow \infty$ means that all the

sums like eq.(1.37) become integrals while the operators are renormalized according to the rules

$$\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{k}} \rightarrow \int d\vec{k}, \quad \left(\frac{L}{2\pi}\right)^3 \delta_{\vec{k}\vec{k}'} \rightarrow \delta(\vec{k} - \vec{k}'), \quad a_n \rightarrow \left(\frac{2\pi}{L}\right)^3 a(\vec{k}).$$

The commutation relations, eq.(1.38) become

$$[a(\vec{k}), a^+(\vec{l})] = \delta(\vec{k} - \vec{l}), \quad [a(\vec{k}), a(\vec{l})] = 0, \quad [a^+(\vec{k}), a^+(\vec{l})] = 0. \quad (1.39)$$

Therefore the **quantization** means the transformation from the field $u(x)$ to the operator $u(x)$ acting on a state vector ψ . Commutation relations follow from those for a^+ and a (eq.(1.39)) and will be obtained below.

1.7.2 Canonical quantization

This scheme of quantization is very natural from the point of view of classical mechanics in the framework of **Hamiltonian** equations of motion. In the canonical formalism the basic variables are the generalized coordinate q and generalized momentum p

$$p = \frac{\delta \mathcal{L}}{\delta \dot{q}}, \quad \dot{q} = \frac{\partial q}{\partial t}.$$

The Hamiltonian equations of motion for some dynamical variable $A(p, q)$ are of the form

$$\frac{dA(p, q)}{dt} = \{A, H\}, \quad (1.40)$$

where the **Poisson** bracket is

$$\{a, b\} \equiv \frac{\partial a}{\partial q} \frac{\partial b}{\partial p} - \frac{\partial a}{\partial p} \frac{\partial b}{\partial q}. \quad (1.41)$$

For an oscillator

$$H = \frac{1}{2}(p^2 + \omega^2 q^2), \quad \{q, p\} = 1, \quad (1.42)$$

so the equations of motion are

$$\dot{q} = \{q, H\} = p, \quad \dot{p} = \{p, H\} = -\omega^2 q.$$

Their solution can be expressed in terms of two functions $a^{(\pm)}(t)$:

$$q(t) = \frac{a^{(+)}(t) + a^{(-)}(t)}{\sqrt{2\omega}}, \quad p(t) = i\sqrt{\frac{\omega}{2}}(a^{(+)}(t) - a^{(-)}(t)),$$

where $a^{(\pm)}(t)$ obey the diagonal equations

$$\dot{a}^{(\pm)}(t) = \pm i\omega a^{(\pm)}(t)$$

with the solution

$$a^{(\pm)}(t) = a^{(\pm)}(0)e^{\pm i\omega t}.$$

After identification $a^{(+)} = a^+$, $a^{(-)} = a$, eqs.(1.42) take the form

$$H = \frac{\omega}{2}(aa^+ + a^+a), \quad \{a, a^+\} = -i. \quad (1.43)$$

Canonical quantization can now be performed according to the following **postulate of canonical quantization**: All dynamical variables (q, p, a, a^+, H , etc.) are the operators acting on the wave function ψ with the commutation relations obtained by the substitution

$$\{a, b\}_{classical} \rightarrow \{a, b\}_{quantum} = \frac{1}{i}[a, b].$$

Then the equation of motion for the operator A (1.40) becomes

$$i\frac{dA}{dt} = [A, H], \quad (1.44)$$

while the commutators are

$$[q, p] = i, \quad [a, a^+] = 1.$$

1.7.3 Relativistic scheme of quantization

Within the Lagrangian approach the quantization can be performed in a totally Lorentz covariant way. The **postulate of quantization** then is the following: all dynamical variables are expressed through the field operators $u_i(x)$ in the same way as in classical field theory with inclusion of commutation relations (to be specified).

For example, for the scalar field the four-momentum is given by eq.(1.15) where $a^\pm(\vec{k})$ are the operators.

To find out the physical meaning of the field operators $u^\pm(x)$ or $a^\pm(\vec{k})$, we consider the transformation properties of these operators under the group of translations

$$u'(x) = u(x - \omega) = U^{-1}(\omega)u(x)U(\omega). \quad (1.45)$$

Here $U(\omega) = e^{iP_\mu\omega^\mu}$ is the operator of translation, P_μ being the generator. Infinitesimal form of eq.(1.45) is

$$i\frac{\partial u(x)}{\partial x_\mu} = [u(x), P_\mu], \quad (1.46)$$

which is a generalization of eq.(1.44). For the Fourier components this leads to

$$k_\mu a^\pm(\vec{k}) = \mp [a^\pm(\vec{k}), P_\mu]. \quad (1.47)$$

Consider the eigenstate vector Φ_p

$$P_\mu \Phi_p = p_\mu \Phi_p.$$

Then we have from eq.(1.47)

$$\begin{aligned} P_\mu a^+(\vec{k})\Phi_p &= (p+k)_\mu a^+(\vec{k})\Phi_p, \\ P_\mu a^-(\vec{k})\Phi_p &= (p-k)_\mu a^-(\vec{k})\Phi_p. \end{aligned}$$

Hence

$$a^\pm(\vec{k})\Phi_p = \Phi_{p\pm k}, \quad k^2 = m^2.$$

Therefore, the interpretation of $a^\pm(\vec{k})$ is obvious:

$a^+(\vec{k})$ is the creation operator for a particle with momentum \vec{k} and mass m .

$a^-(\vec{k})$ is the annihilation operator for a particle with momentum \vec{k} and mass m .

Analogously,

$u^+(x)$ is the creation operator for a particle of mass m at a space-time point x_μ ,

$u^-(x)$ is the annihilation operator for a particle of mass m at a space-time point x_μ .

Due to the uncertainty principle we know either the position or the momentum of a particle.

The vacuum state is defined by

$$a^-(\vec{k})\Phi_0 = 0, \quad \Phi_0^* a^+(\vec{k}) = 0, \quad \Phi_0^* \Phi_0 = 1.$$

Then an arbitrary s -particle state is

$$\Phi_s = \int d\vec{k}_1 \dots d\vec{k}_s F_s^{(i_1, \dots, i_s)}(\vec{k}_1, \dots, \vec{k}_s) a_{i_1}^+(\vec{k}_1) \dots a_{i_s}^+(\vec{k}_s) \Phi_0, \quad (1.48)$$

where F_s is called the wave function of a system of s particles in momentum representation. For any state we have the **Fock** representation

$$\Phi = \sum_s \Phi_s$$

with Φ_s given by eq.(1.48).

1.7.4 Types of commutation relations

We will derive now the commutation relations between field operators in coordinate representation. according to the general rules they are

$$\{u_i(x), u_j(y)\}_{\mp} \rightarrow [u_i(x), u_j(y)]_{\mp} = \Delta_{ij}(x-y), \quad (1.49)$$

where the " \mp " signs correspond to the Bose and Fermi cases, respectively.

For the scalar fields we have eq.(1.39)

$$[a^-(\vec{k}), a^+(\vec{q})] = \delta(\vec{k} - \vec{q}).$$

Taking the integral over $d\vec{k} d\vec{q}$ we get

$$\begin{aligned} & \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{\sqrt{2k_0}} \frac{d\vec{q}}{\sqrt{2q_0}} e^{i(qy-kx)} \{[,] = \delta\} = \\ &= \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{\sqrt{2k_0}} \frac{d\vec{q}}{\sqrt{2q_0}} \delta(\vec{k} - \vec{q}) e^{i(qy-kx)} = \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2k_0} e^{ik(y-x)} \\ & \equiv \frac{1}{i} D^-(x-y), \end{aligned}$$

where we have introduced the notation

$$\frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2k_0} e^{ik(x-y)} = \frac{1}{i} D^-(y-x) = iD^+(x-y). \quad (1.50)$$

Now

$$\begin{aligned} [u^-(x), u^+(y)] &= -iD^-(x-y), \\ [u^+(x), u^-(y)] &= -iD^+(x-y), \\ [u(x), u(y)] &= -iD(x-y), \end{aligned} \quad (1.51)$$

where

$$D(x-y) = D^+(x-y) + D^-(x-y) = \frac{1}{(2\pi)^3} \int d^4k e^{-ik(x-y)} \epsilon(k_0) \delta(k^2 - m^2),$$

$$\epsilon(k_0) = \begin{cases} 1 & k_0 > 0, \\ -1 & k_0 < 0. \end{cases}$$

The obtained commutators obey the very important property of **local commutativity**, namely

$$[u(x), u(y)] = 0, \quad (x-y)^2 < 0,$$

which reflects the causal property of independence of two events connected by a space-like interval.

In case of charged massive vector fields the commutator is

$$[a_n^-(\vec{k}), a_m^+(\vec{q})] = \delta_{nm} \delta(\vec{k} - \vec{q}). \quad (1.52)$$

This leads to

$$\begin{aligned} [U_n^{*-}(\vec{k}), U_m^+(\vec{q})] &= \delta(\vec{k} - \vec{q}) [\delta_{nm} + \frac{k_n k_m}{m^2}], \\ [U_0^{*-}(\vec{k}), U_n^+(\vec{q})] &= -\frac{k_0 k_n}{m^2} \delta(\vec{k} - \vec{q}). \end{aligned}$$

In coordinate space we have

$$[U_\nu^*(x), U_\mu(y)] = (g_{\mu\nu} + \frac{1}{m^2} \frac{\partial^2}{\partial x_\nu \partial x_\mu}) iD(x-y) \quad (1.53)$$

with the samew function $D(x-y)$.

For the spinor field one should remind the problem with the positive definiteness of the energy. The four-momentum in eq.(1.32) is

$$P^\nu = \frac{1}{2} \int d\vec{p} p^\nu \sum_s [a_s^{*+}(\vec{p}) a_s^-(\vec{p}) - a_s^{*-}(\vec{p}) a_s^+(\vec{p})]$$

so we are forced to choose the anticommutator relation instead of a commutator one

$$[a_s^{*-}(\vec{k}), a_r^+(\vec{q})]_+ = \delta_{rs} \delta(\vec{k} - \vec{q}), \quad (1.54)$$

which leads to

$$P^\nu = \frac{1}{2} \int d\vec{p} p^\nu \sum_s [a_s^{*+}(\vec{p}) a_s^-(\vec{p}) + a_s^+(\vec{p}) a_s^{*-}(\vec{p})], \quad (1.55)$$

that is now positive definite. The anticommutator, eq.(1.54) is of a fermi type and leads to the following anticommutator for spinor operators in coordinate space

$$[\psi(x), \bar{\psi}(y)] = \frac{1}{i} S(x - y), \quad (1.56)$$

where

$$S(x - y) = (i\hat{\partial}_x + m)D(x - y).$$

1.7.5 Normal Ordering

Knowledge of a commutation relation helps us to define the so-called normal form of an operator which is useful when calculating the matrix elements.

Normal form of an operator is the one where all the creation operators a^+ stand to the left of all annihilation operators a^- , denoted by double dots $: \dots :$. For example, the product of two operators is

$$u^*(x)u(y) = u^{*+}(x)u^+(y) + u^{*+}(x)u^-(y) + \underline{u^{*-}(x)u^+(y)} + u^{*-}(x)u^-(y).$$

Then the normal product is defined by

$$: u^*(x)u(y) := u^{*+}(x)u^+(y) + u^{*+}(x)u^-(y) + \underline{u^+(y)u^{*-}(x)} + u^{*-}(x)u^-(y).$$

Thus,

$$u^*(x)u(y) =: u^*(x)u(y) : -iD^-(x - y).$$

When calculating the matrix elements one has

$$\Phi_0^* : \dots : \Phi_0 = 0,$$

so the only contribution comes from the commutators.

The following postulate is usually applied: All the dynamical variables are written in the normal form. For example,

$$\mathcal{L} = \frac{1}{2} : \partial_\mu \varphi(x) \partial^\mu \varphi(x) : - \frac{m^2}{2} : \varphi^2(x) :$$

2 Lecture II

Continuous Symmetries, Global and Local Symmetries, Gauge Fields

We now proceed to the construction of interacting fields. The **Guiding idea**: forms of interactions are dictated by **symmetry principles**. The Lagrangians are constructed to be invariant under transformations of the fields with respect to definite symmetry groups.

2.1 Continuous Symmetries

Transformations can be discrete or continuous. Examples of **discrete** transformations playing an important role in physics are

$$P - \text{reflection of space } \vec{x} \rightarrow -\vec{x} : P\varphi(\vec{x}, t) = \varphi(-\vec{x}, t),$$

$$T - \text{reflection of time } t \rightarrow -t : T\varphi(\vec{x}, t) = \varphi(\vec{x}, -t),$$

$$C - \text{charge conjugation: } C\varphi(x) = \varphi^*(x).$$

The fundamental property of local QFT is the *CPT* - invariance.

The idea of symmetry is very attractive and quite natural. We meet it everywhere in Nature. A human body, at least from outside, is left-right symmetric. Well known snowflakes (Fig.2) are invariant under i) reflection with respect to cc' plane, ii) rotation through $\frac{\pi}{3}k$, where k is an integer.

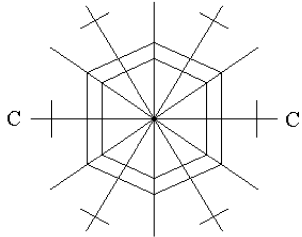


Figure 2:

If we consider a polygon (n -angular), it is invariant under rotations through $\frac{2\pi}{n}k$. When n tends to infinity, we get the invariance under an arbitrary rotation, i.e. we come to a **continuous** symmetry called the $U(1)$ invariance.

2.2 Global Symmetries

2.2.1 Global $U(1)$ symmetry

$U(1)$ is the group of unitary 1×1 matrices with a unit determinant, i.e. just the phase factors of the form $e^{i\alpha}$. Clearly

$$e^{i\alpha} e^{i\beta} = e^{i\beta} e^{i\alpha},$$

so the group is Abelian. The transformation looks as follows

$$U(\alpha)\varphi(x) = e^{i\alpha Q}\varphi(x), \quad (2.1)$$

where α is a real parameter independent of x and Q is the charge of φ , the generator of the transformation.

The Lagrangian

$$\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2 \quad (2.2)$$

is clearly invariant under $U(1)$. It can only involve pairs $\varphi^*\varphi$. The Noether's current (1.2) is ($\delta\varphi = i\alpha\varphi$)

$$j^\mu = -i\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\varphi + i\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi^*}\varphi^* = -i(\partial^\mu\varphi^*\varphi - \partial^\mu\varphi\varphi^*).$$

The complex fields φ^*, φ can be written in terms of their real components

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}, \quad \varphi^* = \frac{\varphi_1 - i\varphi_2}{\sqrt{2}}.$$

The Lagrangian (2.2) becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 - \frac{m^2}{2}(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2. \quad (2.3)$$

It is invariant under rotation

$$\begin{aligned} \varphi_1 &\rightarrow \varphi'_1 = \cos\alpha\varphi_1 - \sin\alpha\varphi_2, \\ \varphi_2 &\rightarrow \varphi'_2 = \sin\alpha\varphi_1 + \cos\alpha\varphi_2. \end{aligned}$$

For $\alpha \ll 1$

$$\varphi'_1 = \varphi_1 - \alpha\varphi_2, \quad \varphi'_2 = \varphi_2 + \alpha\varphi_1.$$

Hence

$$\delta\varphi_i = i\alpha t^j_i\varphi_j, \quad t = \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}.$$

The Noether's current then is

$$j^\mu = -i\partial^\mu\varphi_1(i\varphi_2) - i\partial^\mu\varphi_2(-i\varphi_1) = \partial^\mu\varphi_1\varphi_2 - \partial^\mu\varphi_2\varphi_1. \quad (2.4)$$

Note that φ_1 and φ_2 are degenerate: $m_1^2 = m_2^2 = m^2$. This is a typical consequence of a symmetry, but not necessary, as will be seen later.

Another example of invariant Lagrangian is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi, \quad (2.5)$$

where the current takes the form

$$j^\mu = i\bar{\psi}(x)\gamma^\mu\psi.$$

2.2.2 Global SU(2) Symmetry

SU(2) is the group of unitary 2x2 matrices with a unit determinant

$$U(\alpha) = e^{i\alpha^a T^a}, \quad (2.6)$$

where the generators T^a ($a = 1, 2, 3$) are $T^a = \tau^a/2$, τ^a being the Pauli matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Due to nonzero commutators

$$[T^a, T^b] = i\epsilon^{abc}T^c,$$

where ϵ^{abc} is a totally antisymmetric tensor, $\epsilon^{123} = 1$, SU(2) is a non-Abelian group.

Consider a doublet $\underline{2}$ of scalars $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$, where $\varphi_{1,2}$ are the complex fields. The conjugated field is $\varphi^+ = (\varphi_1^*, \varphi_2^*)$. Then the transformation of the fields under (2.6) is

$$\begin{aligned} \varphi' &= U\varphi = e^{i\alpha^a T^a} \varphi, \\ \varphi^{+'} &= \varphi^+ U^+ = \varphi^+ e^{-i\alpha^a T^a}. \end{aligned}$$

Note that $U^+U = 1$, i.e. $T^+ = T$.

The Lagrangian invariant under SU(2) looks like (2.2)

$$\mathcal{L} = \partial_\mu \varphi^+ \partial^\mu \varphi - m^2 \varphi^+ \varphi - \lambda (\varphi^+ \varphi)^2. \quad (2.7)$$

For $\alpha \ll 1$ $\delta\varphi_i = i\alpha^a (T^a)_{ij} \varphi_j$, so that the Noether's current (1.2) is

$$j^{\mu a} = i \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi_i} T_i^{aj} \varphi_j - i \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi_i^+} (-\varphi^{+j} T_j^{ai}) = -\frac{i}{2} [\partial^\mu \varphi^{+i} T_i^{aj} \varphi_j - \partial^\mu \varphi_i T_j^{ai} \varphi^{+j}].$$

Again the symmetry has forced φ_1 and φ_2 to be degenerate. This only happens if there is a unique nondegenerate lowest energy state- vacuum. In our case $\varphi_1 = \varphi_2 = 0$.

2.2.3 SU(N) Group

A natural generalization of SU(2) group is SU(N). It is often used in particle physics and we consider some general properties of this group.

SU(N) is the group of NxN unitary matrices with unit determinant:

$$U \in SU(N), \quad U^+U = 1, \quad \det U = 1.$$

The number of generators of SU(N) is the number of real parameters characterizing the general SU(N) matrix. In general, an NxN matrix has $2N^2$ real parameters. The condition $U^+U = 1$ imposes N^2 constraints and that of $\det U = 1$ adds one more constraint. Hence, the total number left is $N^2 - 1$, which gives the number of generators of SU(N). The usual parametrization is like eq.(2.6)

$$U = \exp\left[i \sum_{a=1}^{N^2-1} \alpha^a T^a\right] \equiv \exp(i\alpha^a T^a), \quad (2.8)$$

where T^a are the generators. They obey the following properties:

$$\begin{aligned} U^{-1} = U^+ : \exp(-i\alpha^a T^a) = \exp(-i\alpha^a T^{+a}) &\Rightarrow T^a = T^{+a}, \\ \det U = 1 : \det U = \exp \text{Tr} \ln U = \exp \text{Tr}(i\alpha^a T^a) &\Rightarrow \text{Tr} T^a = 0. \end{aligned}$$

The matrices T^a belong to the Lie algebra of SU(N) group with the commutator

$$[T^a, T^b] = i f^{abc} T^c, \quad (2.9)$$

where f^{abc} (numbers) are called the structure constants of the group. Clearly, f^{abc} is antisymmetric in its indices: the f^{abc} is a set of numbers which completely characterize the group.

Note that up to now we have considered the set of $N^2 - 1$, NxN matrices T^a : these are known as the generators of SU(N) in the **fundamental representation**. But any set of $N^2 - 1$ traceless hermitean matrices which satisfies the Lie algebra (2.9) can also be a group.

Consider N complex fields φ_i ($i = 1, 2, \dots, N$) forming a vector

$$\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}.$$

The SU(N) transformation is

$$\varphi \rightarrow \varphi' = U\varphi$$

or in the component notation

$$\varphi_i \rightarrow \varphi'_i = U_i^j \varphi_j; \quad \delta\varphi_i = i(\alpha^a T^a)_i^j \varphi_j.$$

N fields transformed in this way are said to transform according to the **fundamental representation** of SU(N), denoted by \underline{N} . A complex conjugate field φ_i^* is transformed as

$$\varphi_i^* \rightarrow \varphi_i'^* = U_i^{*j} \varphi_j^*.$$

If we take transpose, i.e. write φ_j^* as a row vector we will get the hermitean conjugate field

$$(\varphi_i^*)^T = \varphi^{+i}$$

transformed as

$$\varphi^{+i} \rightarrow \varphi^{+i'} = \varphi^{+j} (U^+)^j_i.$$

N fields transformed in this way are said to transform according to the **conjugate fundamental representation** of SU(N), denoted by \overline{N} .

Combining the fundamental representation with its conjugate we produce the direct (tensor) product of representations being the NxN matrix

$$(\varphi \otimes \chi^+)_i^j = \varphi_i \chi^{+j} = (\varphi_i \chi^{+j} - \frac{1}{N} \delta_i^j \varphi_k \chi^{+k}) + \frac{1}{N} \delta_i^j \varphi_k \chi^{+k}. \quad (2.10)$$

If we take the trace of eq.(2.10)

$$\text{Tr}(\varphi \otimes \chi^+) = \varphi_k \chi^{+k}$$

we find that the first term is a traceless matrix with $N^2 - 1$ independent components:

$$M_i^j = \varphi_i \chi^{+j} - \frac{1}{N} \delta_i^j \varphi_k \chi^{+k},$$

while the remaining matrix is diagonal with one component

$$S_i^j = \frac{1}{N} \delta_i^j \varphi_k \chi^{+k}.$$

It is useful to see how M and S are transformed under $\text{SU}(N)$. One has

$$M_i^j \rightarrow (M')_i^j = U_i^l \varphi_l \chi^{+m} U_m^{+j} - \frac{1}{N} \delta_i^j U_k^l \varphi_l \chi^{+m} U_m^{+k}.$$

Using the unitarity properties this leads to

$$(M')_i^j = U_i^l \varphi_l \chi^{+m} U_m^{+j} - \frac{1}{N} \delta_i^j \varphi_l \chi^{+l}.$$

or in matrix notation

$$\begin{aligned} M &\rightarrow M' = U M U^+, \\ S &\rightarrow S' = S. \end{aligned} \tag{2.11}$$

The $N^2 - 1$ independent components of the matrix M_i^j are said to transform according to the **adjoint representation** of $\text{SU}(N)$ denoted by $\underline{N^2 - 1}$. The remaining component S_i^j is invariant under $\text{SU}(N)$, it transforms as a **singlet** denoted by $\underline{1}$.

So, we have shown the decomposition law to be valid for any $\text{SU}(N)$ group

$$\underline{N} \otimes \overline{\underline{N}} = \underline{N^2 - 1} \oplus \underline{1}.$$

Note that if we identify the structure constants f^{abc} with the matrix elements of $N^2 - 1 \times N^2 - 1$ matrix T^b

$$f^{abc} = i(T^b)_{ac}$$

the matrices T^b would be the generators of $\text{SU}(N)$ in the adjoint representation.

2.3 Local Symmetries and Gauge Fields

2.3.1 Local Symmetries

Local symmetries in field theory were introduced by Yang and Mills in 1954. They wondered what happens if one allows local symmetry transformations, i.e. the group parameters α^a be functions of space-time

$$\alpha^a = \alpha^a(x).$$

The transformations

$$\varphi(x) \rightarrow \varphi'(x) = U(x)\varphi(x) \quad (2.12)$$

are called the **gauge transformations**.

This step happens to be nontrivial one. The kinetic terms in the Lagrangians are no longer invariant. To see this, consider an N-component Dirac field ψ in the fundamental representation of $SU(N)$. The kinetic term behaves like

$$\begin{aligned} \bar{\psi}i\gamma^\mu\partial_\mu\psi &\rightarrow \bar{\psi}U^+(x)i\gamma^\mu\partial_\mu(U(x)\psi) = \\ &= \bar{\psi}i\underbrace{(U^+\gamma^\mu U)}_{\gamma^\mu}\partial_\mu\psi + \bar{\psi}U^+i\gamma^\mu(\partial_\mu U)\psi \end{aligned}$$

The last term is a new one which makes the Lagrangian non-invariant. To remedy this we introduce the so-called **covariant** derivative D_μ defined in such a way as to give the invariant kinetic term

$$\bar{\psi}'i\gamma^\mu D_\mu\psi \rightarrow \bar{\psi}'i\gamma^\mu D'_\mu\psi' = \bar{\psi}i\gamma^\mu D_\mu\psi. \quad (2.13)$$

Eq.(2.13) gives us the transformation of D_μ :

$$\bar{\psi}'i\gamma^\mu D'_\mu\psi' = \bar{\psi}U^+i\gamma^\mu D'_\mu U\psi \stackrel{def}{=} \bar{\psi}i\gamma^\mu D_\mu\psi.$$

Hence

$$D'_\mu = U D_\mu U^+ \quad (2.14)$$

and

$$(D_\mu\psi)' = U D_\mu\psi. \quad (2.15)$$

2.3.2 The Gauge Fields

Introducing the difference of two derivatives

$$V_\mu = \partial_\mu - D_\mu$$

we get with the help of eqs.(2.14), (2.15)

$$\begin{aligned} [(\partial_\mu - D_\mu)\psi]' &= \partial_\mu\psi' - U D_\mu\psi = \partial_\mu U\psi + U\partial_\mu\psi - U D_\mu\psi \\ &= U(\partial_\mu - D_\mu)\psi + \partial_\mu U\psi. \end{aligned}$$

Thus

$$(\partial_\mu - D_\mu)'\psi' = [U(\partial_\mu - D_\mu)U^+ + (\partial_\mu U)U^+]\psi.$$

This gives the transformation law for V_μ

$$V'_\mu = UV_\mu U^+ + (\partial_\mu U)U^+. \quad (2.16)$$

The first term of eq.(2.16) is the well known adjoint representation (2.11). It can be decomposed in the generators $V_\mu = V_\mu^a T^a$. The second term can also be expressed in the same way as $C_\mu^a T^a$.

Thus, we end up with $N^2 - 1$ **gauge fields** V_μ^a transformed according to the adjoint representation of $SU(N)$.

Now the locally invariant Lagrangian for fermions in the fundamental representation takes the form

$$\begin{aligned}\mathcal{L}_F &= \bar{\psi}i\gamma^\mu D_\mu\psi - m\bar{\psi}\psi \\ &= \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - i\bar{\psi}\gamma^\mu V_\mu^a T^a\psi\end{aligned}\quad (2.17)$$

and contains the **interaction**. We need a kinetic term for fields V_μ as well and it must be also gauge-invariant. To find it, we consider the commutator of two covariant derivatives

$$[D_\mu, D_\nu] \rightarrow [D'_\mu, D'_\nu] = U[D_\mu D_\nu - D_\nu D_\mu]U^+.$$

Acting on some field ψ we have

$$\begin{aligned}[D_\mu, D_\nu]\psi &= D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi) = \\ &= -(\partial_\mu V_\nu - \partial_\nu V_\mu - [V_\mu, V_\nu])\psi = -G_{\mu\nu}\psi.\end{aligned}$$

According to eqs. (2.14) and (2.15) the **strength tensor** $G_{\mu\nu}$ has the following transformation properties

$$G_{\mu\nu} \rightarrow G'_{\mu\nu} = UG_{\mu\nu}U^+.$$

Thus, the kinetic term which is invariant under $SU(N)$ is

$$\mathcal{L}_{gauge} = \frac{1}{2g^2}Tr(G_{\mu\nu}G^{\mu\nu}),\quad (2.18)$$

where g is a number introduced for a proper normalization (see below). The invariance of \mathcal{L}_{gauge} follows from the properties of the matrices:

$$Tr(G'_{\mu\nu}G'^{\mu\nu}) = Tr(UG_{\mu\nu}U^+UG^{\mu\nu}U^+) = Tr(U^+UG_{\mu\nu}G^{\mu\nu}U) = Tr(G_{\mu\nu}G^{\mu\nu}).$$

Consider now the coupling to scalar fields. The recipe is the following: substitute the covariant derivative D_μ instead of ∂_μ into the globally invariant Lagrangian, e.g. (2.7)

$$\mathcal{L}_s = (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi).\quad (2.19)$$

Eq.(2.19) is obviously gauge-invariant.

Note that $V_\mu = V_\mu^a T^a$. The same is true for $G_{\mu\nu}$:

$$\begin{aligned}G_{\mu\nu} &= \partial_\mu V_\nu^a T^a - \partial_\nu V_\mu^a T^a - V_\mu^b V_\nu^c [T^b, T^c] = \\ &= (\partial_\mu V_\nu^a - \partial_\nu V_\mu^a - if^{abc}V_\mu^b V_\nu^c)T^a = G_{\mu\nu}^a T^a.\end{aligned}$$

Redefining the fields $V_\mu^a = igA_\mu^a$ we have

$$\frac{1}{2g^2}Tr(G_{\mu\nu}G^{\mu\nu}) = \frac{1}{2g^2}G_{\mu\nu}^a G^{\mu\nu a} Tr(T^a T^a) = \frac{1}{4g^2}G_{\mu\nu}^a G^{\mu\nu a} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a},$$

where

$$G_{\mu\nu}^a = igF_{\mu\nu}^a, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

and we accept the usual normalization of the generators in the fundamental representation

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}.$$

Finally, the Lagrangian eqs. (2.17), (2.18) and (2.19) take the form

$$\begin{aligned} \mathcal{L}_{gauge} &= -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}, \\ \mathcal{L}_{fermion} &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + g\bar{\psi}\gamma^\mu A_\mu^a T_F^a \psi, \\ \mathcal{L}_{scalar} &= [(\partial_\mu - igA_\mu^a T_S^a)\Phi]^\dagger [(\partial^\mu + igA^{a\mu} T_S^a)\Phi] - V(\Phi^\dagger \Phi). \end{aligned} \quad (2.20)$$

They have the following remarkable properties:

- i) the universal coupling constant g ;
- ii) non-Abelian gauge theories are non-linear, i.e. contain self-interactions in a natural way;
- iii) the gauge fields are massless, an explicit mass term violates the gauge invariance.

2.3.3 Note on Quantization of the Gauge Fields

having the Lagrangian for the gauge fields we can quantize them according to the general rules. However, the straightforward quantization meets with some obstacles. To see this we consider the canonical quantization. The generalized canonical momenta corresponding to eqs. (2.18) are

$$p_\mu^a = \frac{\delta \mathcal{L}}{\delta \partial_0 A_\mu^a} = F_{\mu 0}^a.$$

Hence $p_0^a = F_{00}^a = 0$. At the same time we would like to preserve the usual commutation relation $[q, p] = 1$. Clearly, we have a contradiction. The reason is that the Lagrangian (2.18) is singular. A_0^a has no canonically conjugated momenta, it is not a dynamical variable, but a constraint, and can be eliminated from the equations.

Consider the gauge equation of motion (the Abelian case)

$$\square A_\nu + \partial_\nu(\partial_\mu A_\mu) = 0. \quad (2.21)$$

Applying the gauge transformation $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu f$ we see that the longitudinal part of the vector field $\partial_\mu A_\mu \equiv \chi(x)$ is not invariant, but is transformed like

$$\chi \rightarrow \chi' = \chi + \partial^2 f$$

and can be eliminated by choosing $\partial^2 f = -\chi$. This choice of a gauge $\partial_\mu A_\mu = \chi(x) = 0$ is called the Lorentz gauge.

Thus, the gauge field can be divided into two parts

$$A_\mu = A_\mu^{tr} + A_\mu^{long},$$

where the physical degrees of freedom are associated only with the transverse part obeying the equation

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A_\mu^{tr} = 0. \quad (2.22)$$

The differential operator of eq.(2.22) $K_{\mu\nu}^{tr} = g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$ has no inverse one needed for a solution. This is another aspect of the same problem. Note that this problem was absent in case of massive vector fields due to the mass term. Eqs. (1.22) are singular in the $m \rightarrow 0$ limit.

The solution of the problem can be achieved via the following recipe: introduce a symmetry breaking term to remove the degeneracy of the Lagrangian

$$\mathcal{L}^{tr} \rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A_\mu)^2, \quad (2.23)$$

where α is a number. Then eq. (2.21) becomes

$$\partial^\nu F_{\mu\nu} - \frac{1}{\alpha}\partial_\mu(\partial A) = \square A_\mu + (1 - \frac{1}{\alpha})\partial_\mu(\partial A) = 0 \quad (2.24)$$

and leads to

$$\square\partial^\mu A_\mu = \square\chi = 0. \quad (2.25)$$

The differential operator in eq.(2.24) now is

$$K_{\mu\nu}^{tr} \rightarrow K_{\mu\nu} = g_{\mu\nu}\square + (1 - \frac{1}{\alpha})\partial_\mu\partial_\nu.$$

Its Fourier transform is of the form

$$g_{\mu\nu}k^2 - (1 - \frac{1}{\alpha})k_\mu k_\nu = k^2 P_{\mu\nu}^{tr} - \frac{1}{\alpha}k^2 P_{\mu\nu}^l,$$

where

$$P_{\mu\nu}^{tr} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad P_{\mu\nu}^l = \frac{k_\mu k_\nu}{k^2}$$

are the corresponding projectors. The inverse operator now exists:

$$K_{\mu\nu}^{-1} = \frac{1}{k^2}P_{\mu\nu}^{tr} + \frac{d_l}{k^2}P_{\mu\nu}^l, \quad d_l = \alpha.$$

This enables us to solve eq.(2.24) making the Fourier transform. The diagonalization can now be achieved by introducing the local frame

$$A_\mu^\pm(\vec{k}) = e_\mu^1 a_1^\pm(\vec{k}) + e_\mu^2 a_2^\pm(\vec{k}) + e_\mu^3 a_3^\pm(\vec{k}) + e_\mu^0 a_0^\pm(\vec{k}), \quad (2.26)$$

with the commutation relation

$$[a_\mu^-(\vec{k}), a_\nu^+(\vec{q})] = g_{\mu\nu}\delta(\vec{k} - \vec{q}). \quad (2.27)$$

Note the wrong sign for a_0 (indefinite metric). However, due to the Lorentz condition

$$|\vec{k}| a_3^\pm(\vec{k}) - k_0 a_0^\pm(\vec{k}) = 0, \quad |\vec{k}| = k_0,$$

hence

$$a_3^+ a_3^- = a_0^+ a_0^-$$

and the four-momentum contains only the contribution of the transverse quanta

$$P^\nu = \int d\vec{k} k^\nu \sum_{s=1,2} a_s^+(\vec{k}) a_s^-(\vec{k}). \quad (2.28)$$

Eq.(2.28) is clearly positive definite. Eq.(2.27) leads to the following commutator of the fields in coordinate space

$$[A_\mu(x), A_\nu(y)] = i g_{\mu\nu} D(x-y), \quad (d_l = 1). \quad (2.29)$$

2.3.4 The Ghost Fields

The situation is more complicated in the non-Abelian case. Eq.(2.25) here becomes

$$D^\mu \partial_\mu \chi^a = 0 \quad (2.30)$$

or

$$\partial_\mu^2 \chi^a + g f^{abc} A_\mu^b \partial_\mu \chi^c = 0.$$

So, one can not put $\chi^a = 0$ any longer. It will be created due to the interaction with the other fields. Of course, one may ignore the χ -fields in external states. But how to eliminate unphysical contribution of the χ -fields from the loops? An elegant way to solve this problem has been proposed by Faddeev and Popov. They introduced the so-called ghost fields ξ^a , obeying the same equation (2.30) but with the Fermi statistics

$$\mathcal{L}_{ghost} = D_\mu \xi^{+a} \partial^\mu \xi^a. \quad (2.31)$$

These fields appear only in closed loops. Due to the Fermi statistics the contribution of the ghost fields has additional minus sign for each loop and cancels that of the χ -fields restoring unitarity in a physical subspace.

2.3.5 BRST Invariance

One may ask now what happens to the gauge invariance after all these perturbations? Hopefully, the answer is the following: the amplitudes of all the physical processes are invariant. The shortest way to show this is the so-called BRST-invariance. It turns out that the total Lagrangian

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{gauge-fixing} + \mathcal{L}_{ghost} \quad (2.32)$$

is invariant under some BRST transformation

$$\delta A_\mu^a = \lambda^+ D_\mu \xi^a, \quad \delta \xi^{+a} = -\frac{1}{\alpha} \lambda^+ \partial_\mu A_\mu^a, \quad \delta \xi^a = -\frac{1}{2} g f^{abc} \lambda^+ \xi^b \xi^c, \quad (2.33)$$

where λ is a constant anticommuting parameter and α is a gauge-fixing parameter.

BRST invariance is equivalent to the usual gauge invariance in the physical sector with the gauge transformation parameter being $\omega^a(x) = \lambda^+ \xi^a(x)$. This enables us to prove the gauge invariance of physical amplitudes.

3 Lecture III

Spontaneous Symmetry Breaking, Goldstone Particles, Higgs Effect

3.1 Spontaneous Symmetry Breaking

The breaking of symmetry sometimes plays a role as important as the symmetry itself. Solving equations of motion that are differential ones, we have to impose some boundary or initial conditions. Let the Lagrangian be invariant under some symmetry group. What about the initial state? There may be different possibilities. The potential may have symmetrical or asymmetrical minima, they may be stable or unstable (see Fig.3).

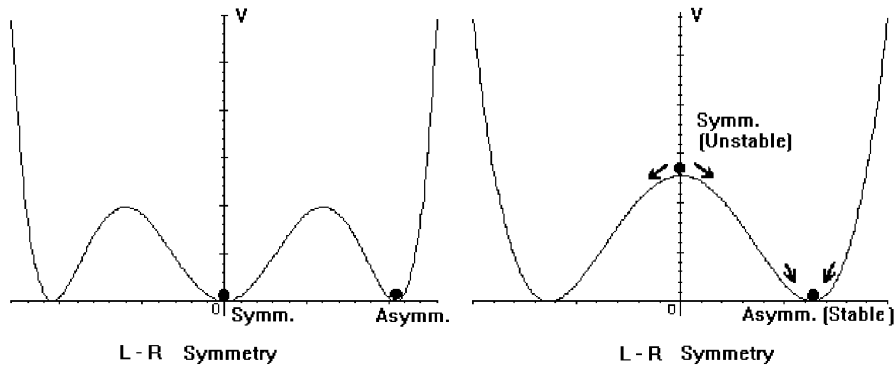


Figure 3:

Spontaneous symmetry breaking is the asymmetric ground (initial) state for a symmetric Lagrangian. The necessary condition for it is the degeneracy of the vacuum. Why the vacuum is so important? This is because in QFT particle excitations of a field φ are quantized fluctuations of φ around the lowest energy state (vacuum). The constant value of the field corresponding to the vacuum is called the vacuum expectation value (vev). Both the vev and the kind of fluctuations about it are determined by \mathcal{L} .

To determine the spectrum of particles we expand the potential about the minimum

$$\begin{aligned}
 V(\varphi_i) = & V(\varphi_{i0}) + \sum_i \frac{\partial V}{\partial \varphi_i} \Big|_{\varphi_i = \varphi_{i0}} (\varphi_i - \varphi_{i0}) + \\
 & + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \Big|_{\varphi_i = \varphi_{i0}} (\varphi_i - \varphi_{i0}) (\varphi_j - \varphi_{j0}) + \dots
 \end{aligned}$$

The mass matrix

$$M_{ij} = \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j} \Big|_{\varphi_i = \varphi_{i0}} \quad (3.1)$$

should be diagonalized to determine the particle spectrum.

3.1.1 Spontaneous Breaking of Discrete Symmetry

Consider the Lagrangian for a real scalar field obeying the discrete symmetry $\varphi \rightarrow -\varphi$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4. \quad (3.2)$$

When $m^2 > 0$ and $\lambda > 0$ the potential has the only minimum $\varphi_0 = 0$ (Fig.4a) and the mass matrix is

$$\frac{\partial^2 V}{\partial \varphi^2} \Big|_{\varphi=0} = m^2.$$

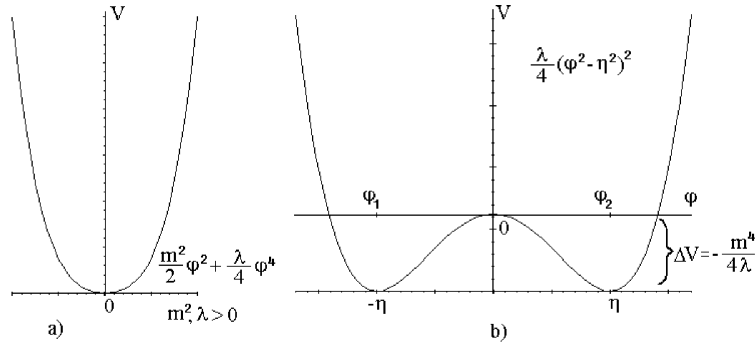


Figure 4:

The situation changes when $m^2 < 0$. Substituting in this case $m^2 \rightarrow -m^2$, we have

$$E^2 = \vec{p}^2 - m^2, \quad v = \frac{p}{m} = \frac{\sqrt{E^2 + m^2}}{E} > 1,$$

that means that the spectrum contains a tachyon state. However, this conclusion is based on the unstable vacuum $\varphi_0 = 0$. There are the other solutions

$$V' = 0 \Rightarrow \varphi_{1,2} = \pm \frac{m}{\lambda}, \quad \frac{\partial^2 V}{\partial \varphi^2} \Big|_{\varphi = \varphi_{1,2}} = 2m^2.$$

The potential now can be rewritten as

$$V(\varphi) = \frac{\lambda}{4}(\varphi^2 - \eta^2)^2, \quad \eta = \frac{m}{\sqrt{\lambda}}$$

and has two stable minima $\varphi_{1,2} = \pm \eta$ (Fig.4b).

The spontaneous symmetry breaking now corresponds to $\mathcal{L}(\varphi) = \mathcal{L}(-\varphi)$, but $\text{vac}(\varphi) \neq \text{vac}(-\varphi)$. Taking the asymmetrical vacuum state and shifting the field $\varphi = \eta + \chi$ we get the Lagrangian

$$\mathcal{L}(\chi) = \frac{1}{2}(\partial_\mu \chi)^2 - \lambda \eta^2 \chi^2 - \lambda \eta \chi^3 - \frac{\lambda}{4} \chi^4, \quad (3.3)$$

which does not manifest any obvious symmetry, describing a particle with the mass $\sqrt{2}m$.

The question naturally arises whether it is possible to restore the symmetry due to the tunneling process from one well to another. To answer this question, one has to calculate the tunneling amplitude proportional to $\exp(iS)$, where S is the action along the path. This action is purely imaginary and does go to infinity when the volume $V \rightarrow \infty$, so the probability of tunneling goes to zero. Hence, the restoration of symmetry is possible only in small domains.

3.1.2 Spontaneous Breaking of Continuous Symmetry

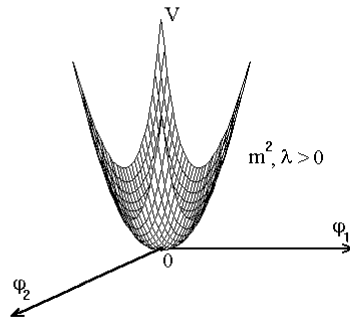


Figure 5:

Here we meet a new phenomenon. Consider for simplicity the U(1) invariant Lagrangian

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi^* \varphi) \quad (3.4)$$

or in terms of real fields

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - V(\varphi_1^2 + \varphi_2^2). \quad (3.5)$$

The potential V must be bounded from below so as to have a ground state. We choose

$$V = \frac{m^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2. \quad (3.6)$$

Now there are again two possibilities:

i) $m^2, \lambda > 0$. The potential is shown in Fig.5. The minimum is derived from the equations

$$\begin{aligned}\frac{\partial V}{\partial \varphi_1} &= m^2 \varphi_1 + \lambda \varphi_1 (\varphi_1^2 + \varphi_2^2) = 0, \\ \frac{\partial V}{\partial \varphi_2} &= m^2 \varphi_2 + \lambda \varphi_2 (\varphi_1^2 + \varphi_2^2) = 0.\end{aligned}\quad (3.7)$$

The only solution is $\varphi_{10} = \varphi_{20} = 0$. The symmetry is not broken. As before, we have the degenerated spectrum

$$m^2(\varphi_1) = m^2(\varphi_2) = m^2.$$

ii) $m^2 = -\mu^2 < 0, \lambda > 0$. The potential now looks like the bottom of the bottle shown

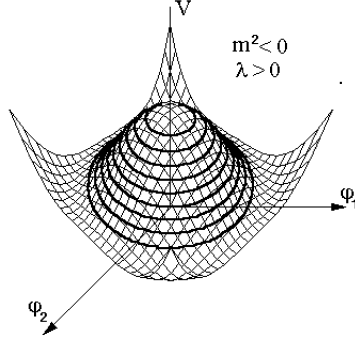


Figure 6:

in Fig.6. Equations (3.7) here have another solution

$$\varphi_{10}^2 + \varphi_{20}^2 = v^2 = \frac{\mu^2}{\lambda}.$$

Geometrically they are the points on a circle with the radius v in the (φ_1, φ_2) plane (see Fig.7). Physically, a non-zero value of the field corresponds to a condensation of pairs so that the value of the condensate $\langle \varphi^* \varphi \rangle = v^2 \neq 0$.

We see that the vacuum is no longer unique! The $O(2)$ symmetry transfers any point on this circle into another. One can choose any point, say $\varphi_{10} = v, \varphi_{20} = 0$ or $\varphi_{10} = \varphi_{20} = v/\sqrt{2}$, etc.

The mass matrix (3.1) is given by

$$\begin{aligned}\frac{\partial^2 V}{\partial \varphi_1 \partial \varphi_2} &= 2\lambda \varphi_1 \varphi_2, \quad \frac{\partial^2 V}{\partial \varphi_1^2} = -\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2) + 2\lambda \varphi_1^2, \\ \frac{\partial^2 V}{\partial \varphi_2^2} &= -\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2) + 2\lambda \varphi_2^2.\end{aligned}$$

To diagonalize it we choose φ_{10} or φ_{20} to vanish, say $\varphi_{10} = v, \varphi_{20} = 0$, then

$$\mathcal{M} = \begin{pmatrix} 2\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix}.\quad (3.8)$$

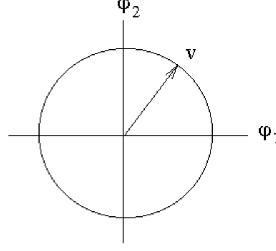


Figure 7:

Thus $\varphi'_1 = \varphi_1 - v$ corresponds to a massive particle of mass $m^2 = 2\lambda v^2 = 2\mu^2$, while $\varphi'_2 = \varphi_2$ is massless. This result is independent of the choice of the vev. For any point on a circle after diagonalization the conclusion is the same.

The Lagrangian with the new fields becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi'_1)^2 + \frac{1}{2}(\partial_\mu \varphi'_2)^2 - \lambda v^2 \varphi_1'^2 - \lambda v \varphi'_1 (\varphi_1'^2 + \varphi_2'^2) - \frac{\lambda}{4}(\varphi_1'^2 + \varphi_2'^2)^2. \quad (3.9)$$

The $O(2)$ symmetry is no longer manifest. It is spontaneously broken.

Consider another parametrization of the $U(1)$ model. If we write down the complex field in polar coordinates

$$\varphi(x) = \frac{1}{\sqrt{2}}\rho(x)e^{i\theta(x)}. \quad (3.10)$$

The derivative becomes

$$\partial_\mu \varphi(x) = \frac{1}{\sqrt{2}}e^{i\theta(x)}[\partial_\mu \rho(x) + i\partial_\mu \theta(x)\rho(x)],$$

so the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2(\partial_\mu \theta)^2 - V(\rho^2), \quad (3.11)$$

where

$$V(\rho^2) = -\mu^2 \rho^2 + \frac{\lambda}{4}\rho^4 = \frac{\lambda}{4}(\rho^2 - v^2)^2 - \frac{\lambda}{4}v^4. \quad (3.12)$$

We see that the field θ disappears out of the potential. This is a manifestation of the $U(1)$ symmetry. To find the mass of the particle corresponding to the "radial" field $\rho(x)$, we expand the potential around the minimum value $\rho_0 = v$. One finds

$$\rho(x) = v + \eta(x), \quad m_\rho^2 = 2\mu^2,$$

while $m_\theta = 0$ due to the absence of the quadratic θ -term in the Lagrangian (3.12).

So the final conclusion is the following: the spectrum of the model (3.5) when $m^2 = -\mu^2 < 0$ consists of two particles, one massive and one massless.

3.2 Goldstone Particles

The presence of massless particles is not the matter of chance, but is a general rule known as the **Goldstone theorem** that states that when a global continuous symmetry is spontaneously broken, there exists a number of massless particles (**Goldstone bosons**) equal to the number of broken symmetries.

Consider now the generalization of this result to a non-Abelian global symmetry.

3.2.1 SU(2) symmetry

Let $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ be a complex doublet of SU(2). Then an SU(2) invariant Lagrangian is

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - V(\varphi^\dagger \varphi). \quad (3.13)$$

If we take as before

$$V = -\mu^2(\varphi^\dagger \varphi) + \lambda(\varphi^\dagger \varphi)^2$$

the minimum will be achieved for

$$\varphi^\dagger \varphi = \mu^2/2\lambda \equiv v^2.$$

Choosing the parametrization like in eq.(3.10)

$$\varphi(x) = \frac{1}{\sqrt{2}} e^{i\frac{\tau^a \xi^a}{v}} \begin{pmatrix} 0 \\ H(x) + v \end{pmatrix} \quad (3.14)$$

we have

$$\partial_\mu \varphi(x) = \frac{1}{\sqrt{2}} e^{i\frac{\tau^a \xi^a}{v}} \left\{ \begin{pmatrix} 0 \\ \partial_\mu H(x) \end{pmatrix} + i\frac{\tau^a \partial_\mu \xi^a}{v} \begin{pmatrix} 0 \\ H(x) + v \end{pmatrix} \right\}.$$

Note that in eq.(3.14) we still have four real fields: ξ^a , $a = 1, 2, 3$ and H .

The Lagrangian (3.13) now becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \left\{ (0, \partial_\mu H) - i\frac{\partial_\mu \xi^a}{v} (0, H + v) \tau^a \right\} \\ &\times \left\{ \begin{pmatrix} 0 \\ \partial_\mu H(x) \end{pmatrix} + i\frac{\tau^a \partial_\mu \xi^a}{v} \begin{pmatrix} 0 \\ H(x) + v \end{pmatrix} \right\} - V(H + v) \\ &= \frac{1}{2} (\partial_\mu H)^2 + \frac{1}{2} (\partial_\mu \xi^a)^2 \frac{(H + v)^2}{v^2} - V(H + v). \end{aligned} \quad (3.15)$$

This looks just like the U(1) case, but the three fields ξ^a , ($a = 1, 2, 3$ in SU(2)) are found to be massless. $H(x)$ is massive and the original manifest SU(2) symmetry is no longer manifest, but becomes a "secret" symmetry. The number of massless states is in general given by the

Goldstone Theorem: *There are as many massless Goldstone bosons as there are directions in the space of the fields φ_i (internal symmetry space) along which the vacuum is degenerate, i.e. not unique.*

Consider a transformation of the vacuum generated by some charges $Q^a, a = 1, 2, \dots, N$. Let

$$\begin{aligned} Q^k|0\rangle &\neq 0, \quad k = 1, 2, \dots, M \quad (\text{broken generators}), \\ Q^n|0\rangle &= 0, \quad n = M + 1, \dots, N \quad (\text{unbroken generators}). \end{aligned}$$

According to the Noether theorem (eq.(1.4) and eq.(1.44))

$$\dot{Q}^a = 0 = i[H, Q^a].$$

Hence

$$[H, Q^a]|0\rangle = 0$$

or

$$H(Q^a|0\rangle) = Q^a(H|0\rangle). \quad (3.16)$$

Equation (3.16) means that for any $k = 1, 2, \dots, M$ the state $Q^a|0\rangle$ is degenerated with the vacuum state $|0\rangle$, i.e. corresponds to the massless excitations called the Goldstone bosons. Therefore the number of Goldstone bosons equals M , the number of broken generators.

Two comments are in order: i) the number of broken generators depends on the choice of representation of the scalar field; ii) in case of fermion generators Q^a the corresponding particles will be fermions. This possibility is realized in supersymmetric models.

To realize the necessity of massless particles in the case of spontaneous symmetry breaking, we concentrate on some properties of Goldstone bosons. Consider the transformation of some state vector under the group of symmetry

$$e^{i\alpha Q}|a(\vec{k})\rangle = |a(\vec{k})\rangle + i\alpha Q|a(\vec{k})\rangle + \dots,$$

where Q is the generator. Let

$$Q|a(\vec{k})\rangle \sim v, \quad (3.17)$$

where $v = 0$ corresponds to a symmetry state while $v \neq 0$ means that the state is not invariant under the symmetry transformation. As far as

$$Q = \int d^3x j^0(x),$$

we can find the matrix element of a current

$$\langle 0|j^\mu(x)|a(\vec{k})\rangle \sim v \cdot k^\mu.$$

Taking the derivative we have

$$\langle 0|\partial_\mu j^\mu(x)|a(\vec{k})\rangle \sim v \cdot k^2. \quad (3.18)$$

However, according to the Noether theorem the current is conserved, i.e. $\partial_\mu j^\mu(x) = 0$. To avoid the discrepancy with eq.(3.18) we must put either $v = 0$ (unbroken symmetry),

or $v \neq 0$, but $k^2 = m^2 = 0$ (Goldstone boson). Therefore, if a symmetry is broken, the massless particles inevitably appear.

Example: U(1) or SO(2) symmetry.

According to eq.(2.5) the current is

$$j^\mu = \partial^\mu \varphi_1 \varphi_2 - \partial^\mu \varphi_2 \varphi_1.$$

If $\langle \varphi_1 \rangle = v$, we shift the field $\varphi_1 = \varphi'_1 + v$ so that $\langle \varphi'_1 \rangle = 0$. Then the current becomes

$$j^\mu = -v \partial^\mu \varphi_2 - \varphi'_1 \partial^\mu \varphi_2 + \partial^\mu \varphi'_1 \varphi_2. \quad (3.19)$$

Taking the matrix element between the vacuum and one-particle states we see that the only contribution comes from the first term of eq.(3.19)

$$\langle 0 | j^\mu(x) | a_2(\vec{k}) \rangle = -v \langle 0 | \partial_\mu \varphi_2 | a_2(\vec{k}) \rangle = -v i k^\mu e^{i k x}.$$

So

$$\langle 0 | \partial_\mu j^\mu(x) | a_2(\vec{k}) \rangle = v k^2 e^{i k x}.$$

Thus, either $v = 0$ and $m_1 = m_2 = m_\lambda$ or $m_1 \neq 0$, and $m_2 = 0$, which corresponds to spontaneous symmetry breaking, $a_2^+(\vec{k})|0\rangle = |a_2(\vec{k})\rangle$ being the Goldstone boson.

3.3 The Higgs Effect

3.3.1 Spontaneous Breaking of Local Gauge Invariance

We now come to the phenomenon known as the **Higgs effect** - the appearance of the mass of the gauge boson due to spontaneous symmetry breaking and disappearance of Goldstone bosons.

Consider first the Abelian case, namely, local U(1) invariance. The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \varphi^* D^\mu \varphi + \mu^2 \varphi^* \varphi - \lambda(\varphi^* \varphi)^2 \quad (3.20)$$

is invariant under the local phase transformation

$$\begin{aligned} \varphi &\rightarrow \varphi' = e^{i\alpha(x)} \varphi, \\ A_\mu &\rightarrow A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \alpha(x). \end{aligned} \quad (3.21)$$

The scalar potential is degenerate with the minima given by

$$|\varphi_0|^2 = \frac{v^2}{2} = \frac{\mu^2}{2\lambda}.$$

Shifting the field φ we choose the parametrization

$$\varphi(x) = \frac{1}{\sqrt{2}} (\eta(x) + v) e^{i\theta(x)/v}.$$

The gauge transformation (3.21) now is

$$\theta(x) \rightarrow \theta'(x) = \theta(x) + v\alpha(x), \quad (3.22)$$

where $\theta(x)$ is the Goldstone boson field. according to eq.(3.32) one can choose the gauge (it is called the unitary gauge), where $\theta'(x) = 0$, i.e. $\alpha(x) = -\theta(x)/v$. Thus, the Goldstone boson disappears, it is not a physical particle. What happens to the degree of freedom then? To answer this question, we define a new set of fields

$$\begin{aligned} \varphi'(x) &= e^{-i\theta(x)/v}\varphi(x) = \frac{1}{\sqrt{2}}(\eta(x) + v), \\ B_\mu(x) &= A_\mu - \frac{1}{ev}\partial_\mu\theta(x). \end{aligned}$$

The derivatives are

$$\begin{aligned} (D_\mu\varphi)' &= UD_\mu\varphi = e^{-i\theta/v}\frac{1}{\sqrt{2}}[\partial_\mu\eta(x) - ieB_\mu(\eta + v)]e^{i\theta/v}, \\ F_{\mu\nu}(A) &= \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu = F_{\mu\nu}(B), \end{aligned}$$

and the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}|\partial_\mu\eta(x) - ieB_\mu(\eta + v)|^2 + \frac{\mu^2}{2}(\eta + v)^2 - \frac{\lambda}{4}(\eta + v)^4 \\ &- \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) \\ &= -\frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) + \frac{1}{2}(\partial_\mu\eta)^2 - \mu^2\eta^2 + \frac{1}{2}(ev)^2B_\mu B^\mu \\ &+ \frac{1}{2}e^2B_\mu B^\mu\eta(2v + \eta) - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4. \end{aligned} \quad (3.23)$$

This Lagrangian describes a **massive** vector boson with the mass $M_B = ev$ and a **massive** scalar η with the mass $m_\eta = \sqrt{2}\mu$. Thus, instead of a massless scalar (the Goldstone boson) we have a third (the longitudinal) degree of freedom of a vector field. Note the conservation of degrees of freedom: 2 real scalar fields (φ_1, φ_2) or (η, θ) plus 2 polarization states of a massless photon versus 1 real scalar Higgs field η plus 3 polarization states of a massive vector field, i.e. $2 + 2 = 3 + 1$. The Goldstone boson is "eaten" by the vector field thus acquiring a mass.

Let us see now how the same phenomenon happens in non-Abelian case which is pretty close to the Standard Model. Consider the SU(2) invariance. Choosing the scalar field in the doublet representation we get the familiar Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + D_\mu\Phi^+ D^\mu\Phi - V(\Phi^+\Phi), \quad (3.24)$$

where

$$V(\Phi^+\Phi) = -\mu^2\Phi^+\Phi + \lambda(\Phi^+\Phi)^2, \quad \mu^2, \lambda > 0.$$

In the unitary gauge one can define

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2}}e^{i\frac{\tau^a\epsilon^a}{v}} \begin{pmatrix} 0 \\ H(x) + v \end{pmatrix}, \\ \Phi'(x) &= U(x)\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ H(x) + v \end{pmatrix}, \\ B_\mu &= U^{-1}A_\mu U + U^{-1}\partial_\mu U. \end{aligned}$$

The Lagrangian is then

$$\mathcal{L} = [D_\mu \Phi']^+ D^\mu \Phi' + \mu^2 \Phi'^+ \Phi' - \lambda (\Phi'^+ \Phi')^2 - \frac{1}{4} F_{\mu\nu}^a(B) F^{\mu\nu a}(B) \quad (3.25)$$

Again the three Goldstone bosons ξ^a have disappeared. The final form of the Lagrangian is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu}^a(B) F^{\mu\nu a}(B) + \frac{1}{2} (\partial_\mu H)^2 - \mu^2 H^2 + \frac{1}{8} (gv)^2 B_\mu^a B^{\mu a} \\ & + \frac{1}{2} g^2 B_\mu^a B^{\mu a} H(2v + H) - \lambda v H^3 - \frac{1}{4} \lambda H^4. \end{aligned} \quad (3.26)$$

It describes a triplet of **massive** vector fields B_μ^a ($a = 1, 2, 3$) with the mass $M_B = \frac{1}{2}gv$ and a **massive** scalar Higgs field H with a mass $m_H = \sqrt{2}\mu$. Note again the conservation of degrees of freedom.

Therefore, as a result we have no massless states in the physical spectrum: the Goldstone bosons are "eaten" by the vector fields and become their longitudinal components. The remaining scalar particle is **massive** and is usually called the **Higgs boson**.

Exactly this effect takes place in the Standard Model, where the W and Z bosons acquire their masses due to the spontaneous symmetry breaking mechanism, the only known mechanism so far which gives the masses to the gauge bosons preserving the gauge invariance at the same time. The conservation of the gauge invariance plays the crucial role in renormalizability of a theory (see below).

4 Lecture IV

Interaction, S-Matrix, Perturbation Theory, Feynman Rules

Up to now we considered the theory of free quantum fields. The equations of motion were linear and their solutions obeying the principle of superposition were the plain waves. The coefficients of the Fourier transform were quantized.

4.1 Interaction

How to introduce an interaction of quantum fields? Interaction means a nonlinear equation of motion which in its turn implies the presence of higher-order polynomials in the Lagrangian. What happens then to the free states which are the eigenvalues of the free Hamiltonian? There are two possibilities: i) strong deformation of the spectrum: new eigenstates, new solutions of equation of motion, etc.; ii) weak deformation of the free theory, i.e. a perturbation of the free motion.

The latter situation is symbolically shown in Fig.7. It is based on the following **hypothesis**: There exist a weak coupling limit and a perturbation theory expansion. If so, then the perturbation theory gives us, like in Quantum mechanics, a regular way to construct scattering amplitudes.

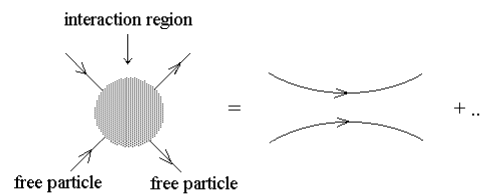


Figure 8:

4.1.1 Interaction Representation

Consider the Schrödinger equation for the amplitude

$$i\frac{\partial\psi(t)}{\partial t} = H_0\psi(t)$$

with a solution

$$\psi(t) = e^{iH_0t}\Phi, \quad (4.1)$$

where Φ is a constant (independent of time). For an interacting system the Schrödinger equation is

$$i\frac{\partial\psi(t)}{\partial t} = (H_0 + H_I)\psi(t).$$

The solution now is given by eq.(4.1), however Φ is no longer a constant but obeys the equation

$$i\frac{\partial\Phi(t)}{\partial t} = e^{iH_0t}H_Ie^{-iH_0t}\Phi(t),$$

or

$$i\frac{\partial\Phi(t)}{\partial t} = H_I(t)\Phi(t), \quad (4.2)$$

where $H_I(t) = \int d\vec{x}\mathcal{H}(x)$ is the Hamiltonian in the so-called **interaction representation**. Eq.(4.2) is called the Schrödinger equation in the interaction representation.

To find the mean value of any operator B one can now use the Schrödinger as well as the interaction representation

$$\overline{B}_t = \psi^*(t)B\psi(t) = \Phi^*(t)e^{iH_0t}Be^{-iH_0t}\Phi(t) = \Phi^*(t)B(t)\Phi(t).$$

4.2 Scattering Matrix (S-matrix)

Suppose that the interaction adiabatically vanishes at $t \rightarrow \pm\infty$. Then the S-matrix is defined by

$$\Phi(\infty) = S\Phi(-\infty). \quad (4.3)$$

The S-matrix as an operator is characterized by matrix elements

$$S_{\alpha\beta} = \langle \beta | S | \alpha \rangle,$$

where $|\alpha\rangle$ and $\langle\beta|$ are in- and out-states, respectively. The transition probability P is given by the modulus squared of the S-matrix elements: $P \sim |S_{\alpha\beta}|^2$.

To find an explicit expression for the S-matrix, consider the Schrödinger equation (4.2) perturbatively. For this purpose define first a finite-time operator $S(t, t_0)$ such that

$$\Phi(t) = S(t, t_0)\Phi(t_0). \quad (4.4)$$

Substituting it into eq.(4.2) we have

$$S(t, t_0) = 1 - i \int_{t_0}^t H(t')dt' + (-i)^2 \int_{t_0}^t H(t')dt' \int_{t_0}^{t'} H(t'')dt'' + \dots, \quad (4.5)$$

where $H(t) \equiv H_I(t)$. Consider the interaction range in the last term of eq.(4.5). It is shown in Fig.8a (a dashed triangle). Changing the order of integration we can rewrite the last term as follows

$$\int_{t_0}^t dt'' \int_{t''}^t dt' H(t')H(t'')$$

or redefining the variables

$$\int_{t_0}^t dt' \int_{t'}^t dt'' H(t'')H(t'),$$

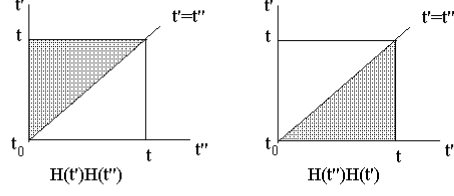


Figure 9:

which corresponds to the integration range shown in Fig.8b. Summing up the two expressions we find

$$S_2 = \frac{(-i)^2}{2} \int_{t_0}^t dt' \left\{ \int_{t_0}^{t'} dt'' H(t')H(t'') + \int_{t'}^t dt'' H(t'')H(t') \right\} \quad (4.6)$$

What is written in the braces is the so-called **chronological** or **T-product** defined by

$$T\{H(t')H(t'')\} = \begin{cases} H(t')H(t''), & t' > t'' \\ H(t'')H(t'), & t' < t'' \end{cases} \quad (4.7)$$

Hence, the second order contribution to the S-matrix can be written as

$$S_2 = \frac{(-i)^2}{2} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 T\{H(t_1)H(t_2)\}. \quad (4.8)$$

The generalization to higher orders is straightforward

$$S = \sum_n S_n, \quad (4.9)$$

$$S_n = \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T\{H(t_1)\dots H(t_n)\}.$$

Formally summing the series (4.9) we get

$$S(t, t_0) = T \left\{ \exp \left[-i \int_{t_0}^t H(t) dt \right] \right\} = T \left\{ \exp \left[-i \int_{t_0}^t dt \int d\vec{x} \mathcal{H}(x) \right] \right\}.$$

In case when an interaction does not contain derivatives there is a simple relation between the Hamiltonian and the Lagrangian of the corresponding field theory

$$H_I(t) = -L_I(t) = - \int d\vec{x} \mathcal{L}_{\mathcal{I}}(x).$$

With the help of this relation and taking the limit $t \rightarrow \infty, t_0 \rightarrow -\infty$, we get the final expression for the S-matrix

$$S = S(\infty, -\infty) = T \exp \left[i \int d^4x \mathcal{L}_{\mathcal{I}}(x) \right], \quad (4.10)$$

that is generally valid. Eq.(4.10) underlines the **Lagrangian perturbation theory**.

Recall some properties of the S-matrix:

1. **Relativistic invariance.** Due to eq.(4.10) this is achieved by the invariance of the Lagrangian discussed above.
2. **Causality.** This property comes from the causal properties of the Green functions or the propagators of free particles.
3. **Unitarity** (Conservation of the norm of the wave function)

$$\Phi^*(\infty)\Phi(\infty) = \Phi^*(-\infty)\Phi(-\infty).$$

For the S-matrix this leads to

$$S^+S = SS^+ = 1,$$

which means that the Lagrangian must be a hermitean function of fields,i.e. $\mathcal{L}^+ = \mathcal{L}$.

4.3 Perturbation Theory

The perturbation theory is constructed on the basis of eq.(4.10). The total Lagrangian is divided into two parts

$$\mathcal{L}_{total} = \mathcal{L}_0 + g\mathcal{L}_I, \quad (4.11)$$

where \mathcal{L}_0 is a free field Lagrangian and \mathcal{L}_I is an interaction. The coupling constant g determines the strength of interaction and serves as an expansion parameter. The solution of eq.(4.2) can be represented as a power series

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n g^n S_n(x_1, \dots, x_n), \quad (4.12)$$

where

$$S_n(x_1, \dots, x_n) = i^n T(\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)).$$

To calculate S_n , consider the T-product more thoroughly. Recall the normal ordering of two operators

$$A(x)B(y) =: A(x)B(y) : + \underline{A(x)B(y)}.$$

The last term is usually called the **normal pairing** and is a c-number (not an operator). The T-product can be expressed in terms of N-products

$$T(A(x)B(y)) = \begin{cases} A \cdot B, & x^0 > y^0 \\ B \cdot A, & x^0 < y^0 \end{cases} = \begin{cases} : AB : + \underline{AB} \\ : BA : + \underline{BA} \end{cases}$$

Note that $: AB := BA :$, hence

$$T(A(x)B(y)) =: A(x)B(y) : + \overline{A(x)B(y)},$$

where

$$\overline{A(x)B(y)} = \begin{cases} A(x)B(y), & x^0 > y^0 \\ \overline{B(y)A(x)}, & x^0 < y^0 \end{cases}$$

is called the **chronological pairing** (a c-number).

Let us find the chronological pairing for the scalar field. According to eq.(1.51) we have

$$\overline{\varphi(x)\varphi(y)} = -iD^-(x-y)\Theta(x^0 - y^0) + iD^+(x-y)\Theta(y^0 - x^0) \equiv -iD^c(x-y),$$

where $D^c(x)$ is a causal Green function.

To reduce the T-product of a number of field operators to the N-product, the following theorem is to be applied:

Wick Theorem:

1. The product of n linear operators is equal to a sum of normal orderings with all possible pairings including the term without pairing.

2. The T-product of n linear operators is equal to a sum of normal orderings with all possible chronological pairings including the term without pairing.

To illustrate this theorem, consider the product of four operators:

$$\begin{aligned} A_1 A_2 A_3 A_4 = & :A_1 A_2 A_3 A_4: + \underbrace{:A_1 A_2 A_3 A_4:}_{12} + \underbrace{:A_1 A_2 A_3 A_4:}_{13} \\ & + \underbrace{:A_1 A_2 A_3 A_4:}_{23} + \underbrace{:A_1 A_2 A_3 A_4:}_{14} + \underbrace{:A_1 A_2 A_3 A_4:}_{24} \\ & + \underbrace{:A_1 A_2 A_3 A_4:}_{34} + \underbrace{:A_1 A_2 A_3 A_4:}_{123} + \underbrace{:A_1 A_2 A_3 A_4:}_{134} \\ & + \underbrace{:A_1 A_2 A_3 A_4:}_{124} \end{aligned}$$

In the case of T-product all pairings should be substituted by chronological pairings.

4.3.1 Causal Green Functions of Free Fields

Consider the free equation of motion in the presence of a source

$$(\square - m^2)\varphi(x) = -J(x), \quad (4.13)$$

where $J(x)$ is an external source of a scalar field. The solution can be found with the help of a Green function

$$\varphi(x) = \int dy G(x-y)J(y),$$

where the Green function $G(x)$ obeys the equation

$$(\square - m^2)G(x) = -\delta(x). \quad (4.14)$$

The solution of eq.(4.14) can be found in momentum representation

$$G(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2} d^4k. \quad (4.15)$$

Eq.(4.15) contains poles when $k_0 = \pm\sqrt{m^2 + \vec{k}^2}$. This means that we have to define the contour of integration so as to avoid the poles. However, different contours

of integration lead to different properties of the Green functions. The causal Green function

$$D^c(x) = \Theta(x^0)D^-(x) - \Theta(-x^0)D^+(x)$$

corresponds to the following choice

$$D^c(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx}}{m^2 - k^2 - i\epsilon} d^4k, \quad \epsilon \rightarrow +0. \quad (4.16)$$

The contour of integration is shown in Fig.10. The causal Green function for spinor

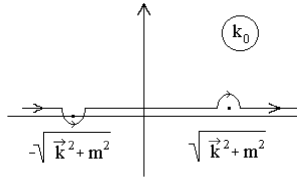


Figure 10:

and vector fields are constructed in full analogy with eq.(4.16)

$$S_{\alpha\beta}^c(x) = (i\hat{\partial} + m)_{\alpha\beta} D^c(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx} (m + \hat{k})_{\alpha\beta}}{m^2 - k^2 - i\epsilon} d^4k, \quad (4.17)$$

$$D_{\mu\nu}^c(x) = (g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}) D^c(x), \quad m \neq 0, \quad (4.18)$$

$$D_{\mu\nu}^c(x) = (g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\partial^2} d_l) D^c(x), \quad m = 0. \quad (4.19)$$

4.4 Feynman Rules

4.4.1 Feynman Rules in x-space

S-matrix elements in perturbation theory admit a very clear graphical interpretation, known as the Feynman rules:

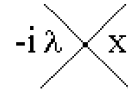
$\overline{u_i(x)u_j(y)}$	\Leftrightarrow	$\frac{i}{x} \text{-----} \frac{j}{y}$	<i>propagator,</i>
$u(x)$	\Leftrightarrow	----- x	<i>External line</i>
$: A(x)B(x)\dots :$	\Leftrightarrow		<i>Vertex.</i>

Example 1 $\mathcal{L}_I = -\lambda : \varphi^4(x) :$

S – matrix element

Graphical Interpretation

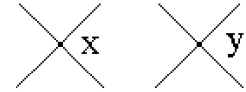
$$S_1(x) = -i\lambda : \varphi^4(x) :$$



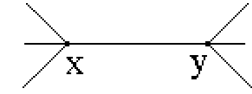
$$S_2(x, y) = i^2 \lambda^2 T \{ : \varphi^4(x) :: \varphi^4(y) : \}$$

(use the Wick theorem)

$$= i^2 \lambda^2 : \varphi^4(x) \varphi^4(y) :$$



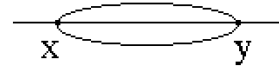
$$+ 16i^2 \lambda^2 \overline{\varphi(x)\varphi(y)} : \varphi^3(x)\varphi^3(y) :$$



$$+ 72i^2 \lambda^2 (\overline{\varphi(x)\varphi(y)})^2 : \varphi^2(x)\varphi^2(y) :$$



$$+ 96i^2 \lambda^2 (\overline{\varphi(x)\varphi(y)})^3 : \varphi(x)\varphi(y) :$$



$$+ 24i^2 \lambda^2 (\overline{\varphi(x)\varphi(y)})^4$$



We take into account the combinatorial factors. The graphical rules can be used to construct the S-matrix elements without explicit use of the Wick theorem. It is usually more useful to draw the graphs by taking into account the symmetry factors.

Example 2 QED $\mathcal{L}_I = e : \bar{\psi} \gamma^\mu A_\mu \psi :$

$$\begin{array}{lcl}
\overline{\psi(x)\bar{\psi}(y)} & \Leftrightarrow & \begin{array}{c} \text{x} \quad \quad \quad \text{y} \\ \longleftarrow \\ \text{x} \quad \quad \quad \text{y} \end{array} & \text{propagator,} \\
\overline{A_\mu(x)A_\nu(y)} & \Leftrightarrow & \begin{array}{c} \text{x} \quad \quad \quad \text{y} \\ \text{~~~~~} \\ \text{x} \quad \quad \quad \text{y} \end{array} & \text{propagator,} \\
e\bar{\psi}\gamma^\mu A_\mu\psi & \Leftrightarrow & \begin{array}{c} \text{x} \\ \swarrow \quad \searrow \\ \text{ie} \end{array} & \text{Vertex}
\end{array}$$

Different kinds of fields are usually denoted by different types of lines. The arrow on a spinor line distinguishes the ψ and $\bar{\psi}$ operators.

Example 3 Yang-Mills Theory

$$\mathcal{L} = -\frac{1}{4} [\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c]^2$$

$$\begin{array}{lcl}
\overline{A_\mu^a(x)A_\nu^b(y)} & \Leftrightarrow & \begin{array}{c} \text{a,}\mu \quad \quad \quad \text{b,}\nu \\ \text{x} \quad \quad \quad \text{y} \\ \text{~~~~~} \\ \text{x} \quad \quad \quad \text{y} \end{array} \\
gf^{abc}A_\mu^b A_\nu^c \partial_\mu A_\nu^a & \Leftrightarrow & \begin{array}{c} \text{a,}\nu \\ \swarrow \\ \mu \quad \text{ie} \text{gf}^{abc} \\ \searrow \\ \text{b,}\mu \quad \quad \quad \text{c,}\nu \end{array} \\
g^2 f^{abc} A_\mu^b A_\nu^c f^{ade} A_\mu^d A_\nu^e & \Leftrightarrow & \begin{array}{c} \text{b,}\mu \quad \quad \quad \text{c,}\nu \\ \swarrow \quad \searrow \\ \text{ie} \text{g}^2 \text{f}^{abc} \text{f}^{ade} \\ \swarrow \quad \searrow \\ \text{d,}\mu \quad \quad \quad \text{e,}\nu \end{array}
\end{array}$$

A crossed line means the derivative in the corresponding Lagrangian.

4.4.2 Feynman Rules in p-space

It is sometimes more useful to perform calculations directly in momentum space. To calculate the matrix element, one has to sandwich the S-operator between the state vectors

$$\Phi_{\dots k \dots} = a_1^+(\vec{k}_1) \dots a_s^+(\vec{k}_s) \Phi_0. \tag{4.20}$$

In n-th order of perturbation theory

$$S_n = \sum \int dx_1 \dots dx_n K(x_1, \dots, x_n) : u_i(x_i) \dots u_j(x_j) \dots :$$

Thus the amplitude is proportional to

$$M \sim \Phi_{\dots k \dots}^* : u_i(x_i) \dots u_j(x_j) \dots : \Phi_{\dots k \dots}$$

Dividing $u_i(x_i)$ into two parts $u_i = u_i^+ + u_i^-$ and taking into account the properties of the vacuum

$$u^- \Phi_0 = 0 = \Phi_0^* u^+,$$

we conclude that $M \neq 0$ only if all u^\pm cancel with the corresponding a^\mp from eq.(4.20) producing the commutators. Otherwise, they will act on the vacuum giving a vanishing result.

Using the general expression for the Fourier transform

$$u^\pm(x_j) = \frac{1}{(2\pi)^{3/2}} \int e^{\pm i p x_j} \sum_{\sigma} v_{\sigma}^{\pm}(\vec{p}) a_{\sigma}^{\pm}(\vec{p}) d\vec{p}, \quad (4.21)$$

where the coefficients $v_{\sigma}^{\pm}(\vec{p})$ depend on the field under consideration (cf. eqs. (1.13), (1.22) and (1.31)) and taking into account the commutation relations, eqs (1.39), (1.52) and (1.54), we finally get

$$M \sim \int \prod_{in-particles} (2\pi)^{-3/2} e^{-ik_i x_i} v_{\sigma}^{-}(\vec{k}_i) \prod_{out-particles} (2\pi)^{-3/2} e^{-ik_f x_f} v_{\sigma}^{+}(\vec{k}_f) \cdot K(x_1, \dots, x_n) dx_1 \dots dx_n, \quad (4.22)$$

where the coefficient function $K(x_1, \dots, x_n)$ is a product of the corresponding D - functions. Evaluating the integral over $\{dx\}$ we get the contraction of the Fourier transforms of the D^c -functions.

The above procedure leads to the following rules to evaluate the matrix elements in the momentum space (**Feynman Rules**):

1. Draw the corresponding Feynman diagrams by taking into account all topological possibilities (the Wick theorem);
2. Introduce the momentum variables for every line;
3. Any external line corresponding to an initial (final) state is associated with a factor

$$(2\pi)^{-3/2} v_{\sigma}^{-}(\vec{p}_{in}) \quad ((2\pi)^{-3/2} v_{\sigma}^{+}(\vec{p}_{out})).$$

4. Any internal line with momentum q is associated with

$$(2\pi)^{-4} D_{\alpha\beta}^c(q).$$

5. Any vertex gives

$$igQ_{\alpha}(2\pi)^4 \delta(\sum p_i),$$

where g is the coupling constant, Q is the matrix (operator) of the corresponding Lagrangian, all p_i are the ingoing momenta.

6. Integrate over all the momenta;
7. In the case of Fermi-fields take into account an additional multiplier $(-)^c$, where c is the number of closed fermion loops;
8. Take into account a symmetry factor.

4.5 Illustration

Example 1 $\mathcal{L}_I = -\lambda : \varphi^4(x) :$

In this case

$$v^-(\vec{p}) = v^+(\vec{p}) = (2p_0)^{-1/2},$$

$$D^c(q) = \frac{1}{i} \frac{1}{m^2 - q^2 - i\epsilon}, \quad Q = 1.$$

Consider the amplitude for 2-particle elastic scattering in the second order of perturbation theory. The Feynman graphs are shown in Fig.11. We consider more thoroughly

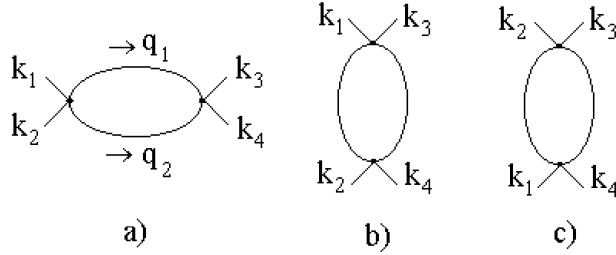


Figure 11:

the contribution of the graph shown in Fig.11a. According to the rules written above we have

$$M_a = [(2\pi)^{12} 2k_1^0 2k_2^0 2k_3^0 2k_4^0]^{-1/2} \frac{1}{i} \frac{1}{(2\pi)^4} \int \frac{d^4 q_1}{m^2 - q_1^2 - i\epsilon} \frac{1}{i} \frac{1}{(2\pi)^4} \int \frac{d^4 q_2}{m^2 - q_2^2 - i\epsilon} \cdot i^2 \lambda^2 (2\pi)^8 \delta(k_1 + k_2 - q_1 - q_2) \delta(q_1 + q_2 - k_3 - k_4) \frac{4!}{2(4!)^2} \quad (4.23)$$

Evaluating the integral with the help of the δ -function and collecting the factors we get from eq.(4.23)

$$M_a = \frac{\lambda^2}{2} \frac{\delta(k_1 + k_2 - k_3 - k_4)}{\sqrt{16k_1^0 k_2^0 k_3^0 k_4^0}} \frac{1}{(2\pi)^6} \int \frac{d^4 q}{(m^2 - q^2)(m^2 - (q - k_1 - k_2)^2)}.$$

We shall discuss the evaluation of the remaining integral later. The total second order contribution is

$$M^{(2)} = M_a + M_b + M_c.$$

Example 2 QED

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\hat{\partial} - m)\psi + e\bar{\psi}\gamma^\mu A_\mu\psi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \hat{\partial} \equiv \gamma^\mu \partial_\mu. \end{aligned} \quad (4.24)$$

The corresponding Feynman rules are:

Electron in an initial state

Electron in a final state

Positron in an initial state

Positron in a final state

Photon in an initial or final state with polarization e_μ^ν

Propagation of electron $1 \rightarrow 2$ (or positron $2 \rightarrow 1$)

Propagation of a photon

from μ to ν (Feynman gauge)

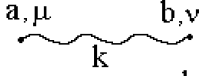
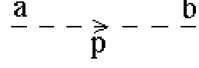
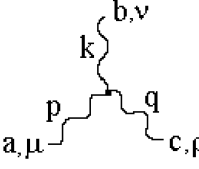
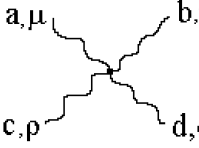
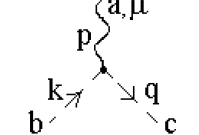
Vertex μ



Example 3 Yang-Mills Theory

$$\begin{aligned}
 \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 - D_\mu \bar{\xi}^a \partial_\mu \xi^a, \\
 F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \\
 D_\mu \bar{\xi}^a &= \partial_\mu \bar{\xi}^a + gf^{abc}\bar{\xi}^b A_\mu^c.
 \end{aligned}
 \tag{4.25}$$

The second term in eq.(4.25) is the gauge-fixing one while ξ are the corresponding ghost fields. The Feynman rules corresponding to eq.(4.25) are:

Vector propagator		$\frac{1}{i} \frac{1}{(2\pi)^4} \delta^{ab} g_{\mu\nu} \frac{-(1-\alpha)k_\mu k_\nu / k^2}{k^2 + i\epsilon}$
Ghost propagator field		$i \frac{1}{(2\pi)^4} \frac{\delta^{ab}}{k^2 + i\epsilon}$
3 - Vertex		$ig f^{abc} (2\pi)^4 \delta(p + k + q) \cdot$ $\cdot [(p - k)_\rho g_{\mu\nu} + (k - q)_\mu g_{\nu\rho} +$ $(q - p)_\nu g_{\mu\rho}]$
4 - Vertex		$-ig^2 \{ f^{abe} f^{cde} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$ $+ \text{permutations} \} \cdot (2\pi)^4 \delta(\sum p_i)$
ghost - vector vertex		$\frac{i}{2} g f^{abc} q_\mu (2\pi)^4 \delta(k + q + p)$

4.6 Probabilities of Scattering Processes

The described technique allows us to calculate the amplitude of a process

$$\Phi_{p'}^* S \Phi_p = \delta(\sum p - \sum p') F(p, p').$$

To find out the probability, we have to take the square of the modulus

$$W = \frac{|\Phi_{p'}^* S \Phi_p|^2}{\Phi_{p'}^* \Phi_p} \sim \underbrace{(2\pi)^{-4} VT}_{\delta(0)} \delta(\sum p - \sum p') |F(p, p')|^2. \quad (4.26)$$

It is proportional to $\delta(0) = (2\pi)^{-4} VT$ and goes to infinity. However, we are interested in the probability of the process when we have in an initial state s particles with momenta $\vec{k}_1, \dots, \vec{k}_s$ and quantum numbers $\sigma_1, \dots, \sigma_s$ scattered into an interval of momenta $\vec{p}_1, \dots, \vec{p}_r$ per unit time per unit volume:

$$w = (2\pi)^{3s-4} n_1 \dots n_s |F(p, p')|^2 \delta(\sum p - \sum p') dp_1 \dots dp_r, \quad (4.27)$$

where $n_1 \dots n_s$ is the average number of particles in the initial state per unit volume. The amplitude $F(p, p')$ is calculated by the Feynman rules.

4.6.1 Two-particle scattering

Consider the two-particle scattering ($s = r = 2$). Note that $w \neq 0$ even if $S = 1$, i.e. in the absence of interaction (see Fig.8). To select the effect of interaction, we consider the matrix element

$$\Phi_{\dots p \dots}^* (S - 1) \Phi_{\dots k \dots}.$$

Define the scattering amplitude (we limit ourselves to spinless particles, for simplicity)

$$\begin{aligned} \Phi_0^* a_1^-(\vec{p}_1) a_2^-(\vec{p}_2) (S-1) a_1^+(\vec{k}_1) a_2^+(\vec{k}_2) \Phi_0 &= \\ &= i \frac{\delta^{(4)}(p_1 + p_2 - k_1 - k_2)}{2\pi \sqrt{p_1^0 p_2^0 k_1^0 k_2^0}} f(p_1, p_2, k_1, k_2). \end{aligned}$$

The effective differential cross-section according to eq.(4.27) is

$$d\sigma = (2\pi)^2 n_1 n_2 |f|^2 \frac{d\vec{p}_1 d\vec{p}_2}{p_1^0 p_2^0 k_1^0 k_2^0} \delta^{(4)}(p_1 + p_2 - k_1 - k_2).$$

Integrating over \vec{p}_1, \vec{p}_2 we get the scattering probability into a solid angle $d\Omega$

$$\frac{d\sigma}{d\Omega_{p'}} = \frac{p_0'(p')^3}{p[(p')^2 Q_0 - p_0'(\vec{p}' \cdot \vec{Q})]} |f|^2, \quad (4.28)$$

where we use the notation

$$\vec{p}_1 = \vec{p}, \quad \vec{p}_2 = \vec{p}', \quad p \equiv |\vec{p}|, \quad p' \equiv |\vec{p}'|, \quad \vec{Q} = \vec{k}_1 + \vec{k}_2.$$

The total cross-section is

$$\sigma = \int_{4\pi} d\Omega \frac{d\sigma}{d\Omega}.$$

4.6.2 Two-particle decay

Another example is a two-particle decay ($s = 1, r = 2$). Symbolically it is shown in Fig.12. Kinemati

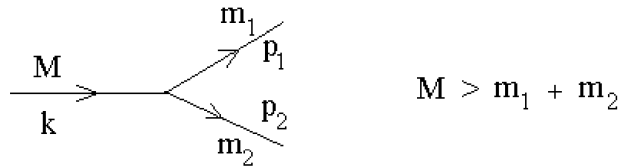


Figure 12:

to

$$w = 2\pi n \frac{|F|^2}{2p_0} \frac{d\vec{p}_1 d\vec{p}_2}{2p_1^0 2p_2^0} \delta^{(4)}(k - p_1 - p_2). \quad (4.29)$$

To find the total probability of a two-particle decay, one has to integrate over \vec{p}_1 and \vec{p}_2 . The result is

$$w_{tot} = \frac{k}{32\pi^2 M^2} \int d\Omega |F(k, \Omega)|^2, \quad (4.30)$$

where $4k^2 = M^2 - 2(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{M^2}$. This probability defines the total width of a particle decay as a sum of partial widths

$$w_{tot} = \Gamma_{tot} = \sum_i \Gamma_i.$$

In its turn, the life-time is given by $\tau = 1/\Gamma_{tot}$ or \hbar/Γ_{tot} if we restore the Planck constant.

5 Lecture V

Radiative Corrections, Renormalization, Model Building

The Feynman rules provide us a straightforward algorithm to calculate the S-matrix elements for different processes. Higher order terms of perturbation theory expansion are usually called the **radiative corrections** (after QED). It happens that the radiative corrections can be calculated not for every Lagrangian, which introduces some severe limitations on the form of interactions.

5.1 Radiative Corrections

Consider the Compton scattering amplitude in QED. Graphically it is shown in Fig.13. So, in perturbation theory the amplitude is given by a power series in $\alpha = e^2/4\pi$, the

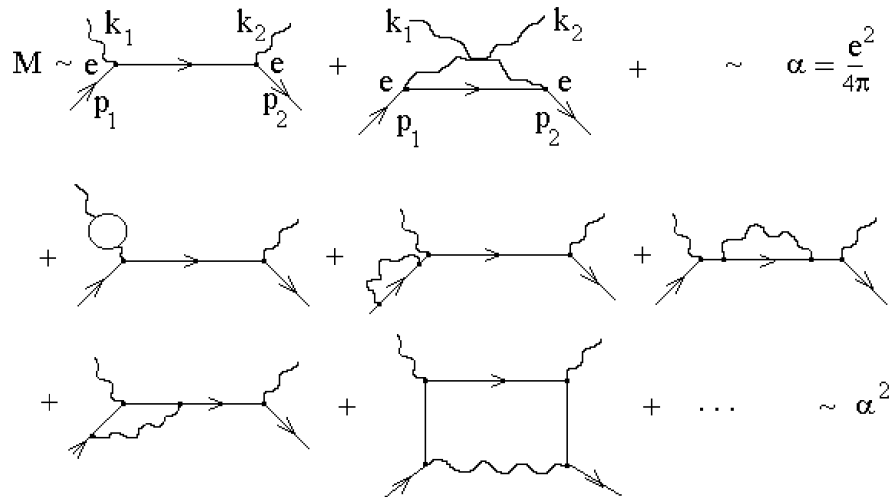


Figure 13:

fine structure constant

$$M = \alpha M_1 + \alpha^2 M_2 + \dots \quad (5.1)$$

Consider the contribution to M_2 from the first diagram with a loop $\sim \alpha^2$ (Fig.14). According to the Feynman rules we have an integral

$$I_{\mu\nu}(p) = \int \frac{d^4q Sp[\gamma_\mu(m + \hat{q})\gamma_\nu(m + \hat{q} - \hat{p})]}{[m^2 - q^2][m^2 - (q - p)^2]}.$$

Consider the ultraviolet behaviour of the integrand ($q \rightarrow \infty$). For $q \gg m, p$ we have the rough estimate

$$\int^\Lambda \frac{d^4q q^2}{(q^2)^2} = \int^\Lambda \frac{d^4q}{q^2} \sim \int^\Lambda dq^2 \sim \Lambda^2 \rightarrow \infty.$$

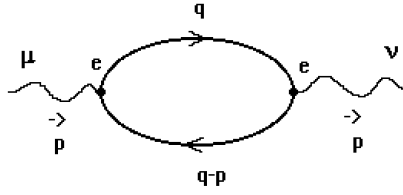


Figure 14:

Here Λ is an ultraviolet cut-off parameter. In the limit $\Lambda \rightarrow \infty$ we have a divergence called the ultraviolet divergence. A more correct calculation (gauge invariant) gives

$$I_{\mu\nu}(p) = (g_{\mu\nu}p^2 - p_\mu p_\nu) \left[\ln \frac{\Lambda^2}{p^2} + \text{finite part} \right]. \quad (5.2)$$

So, the amplitude of Compton scattering is

$$M(\dots) = \alpha R_1 + \alpha^2 \left[a R_1 \ln \frac{\Lambda^2}{p^2} + R_2 \right] + \dots, \quad (5.3)$$

where R_1 and R_2 are some regular (as $\Lambda \rightarrow \infty$) functions of moments, α is the coupling constant and a is a number.

5.1.1 Vacuum Polarization

The appearance of a divergent contribution is the general feature of quantum field theory. Its origin is connected with the properties of the vacuum. The vacuum in QFT is not an empty space. It is a fluctuating medium where the creation and annihilation of virtual particles occur causing the polarization of the medium. The effect of vacuum polarization by the virtual pairs is shown in Fig.15. It makes the "bare" charge unob-

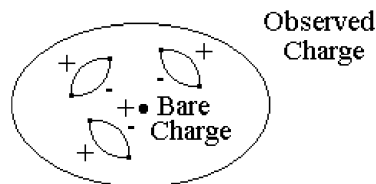


Figure 15:

servable. One cannot avoid interaction with the polarized medium. What we have is an effective charge. The difference is proportional to the polarization effect

$$\sim \alpha \ln \frac{\Lambda^2}{p^2} \rightarrow \infty$$

We see that the divergence is the same as in eq.(5.2), so probably it can be absorbed into the redefining of the coupling.

5.2 Renormalization

5.2.1 The Idea of Renormalization

This is just the idea of renormalization (redefining). Let α_{bare} be singular (infinite) so that $\alpha_{observable}$ be finite and try to expand all the quantities in terms of $\alpha_{observable}$.

Taking the bare coupling to be

$$\alpha_{bare} = \alpha + x\alpha^2 \ln \frac{\Lambda^2}{\mu^2} + \dots, \quad (5.4)$$

where α is the renormalized coupling, we get the amplitude, eq.(5.3), in the form

$$\begin{aligned} M(\dots\alpha_{bare}\dots) &= \widetilde{M}(\dots\alpha\dots) = \\ &= \alpha R_1 + \alpha^2 [aR_1 \ln \frac{\Lambda^2}{p^2} + R_2 + x \ln \frac{\Lambda^2}{\mu^2}] + \dots \end{aligned}$$

Choosing x so that \widetilde{M} to be finite when $\Lambda \rightarrow \infty$ we get $x = -a$, so

$$\widetilde{M}(\dots\alpha\dots) = \alpha R_1 + \alpha^2 [aR_1 \ln \frac{\mu^2}{p^2} + R_2] + \dots \quad (5.5)$$

Now the expression for the amplitude is finite. At first sight it contains a new parameter, μ , that plays the role of some normalization point. However, according to eq.(5.4), the coupling is also μ -dependent: $\alpha = \alpha(\mu)$. So, this dependence is fictitious. The physical result is μ -independent. One can choose any convenient definition. This is the statement of **renormalization group invariance**, which forms the basis of the renormalization group method, a powerful technique used to improve the perturbation theory expansions.

Natural questions arise: Is it always possible to absorb the infinities into the redefining of some couplings? Do all the amplitudes become finite simultaneously? To answer these questions, we need more deep understanding of the renormalization procedure.

5.2.2 Renormalizability

Consider some arbitrary Feynman diagram G , Fig.16 and try to find out whether it is ultraviolet divergent or not. For this purpose we have to count the number of powers of momenta in the integrand: any internal loop gives d^4p , i.e. 4; any derivative in the vertex gives the momentum in p-space, i.e. 1; any internal line gives the propagator that behaves like p^{r_l}/p^2 , i.e. $r_l - 2$, where $r_l = 0, 1, 2$ for various fields. Collecting these numbers together we obtain the quantity called the **index of a diagram** (UV)

$$\omega(G) = 4L + \sum_{vertex} \delta_v + \sum_{internal\ lines} (r_l - 2), \quad (5.6)$$

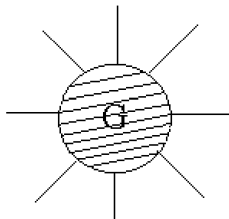


Figure 16:

where L is the number of loops and δ_v is the number of derivatives in a given vertex v .

Now the absence of ultraviolet divergences means $\omega(G) < 0$. However, one should be careful. There may be subdivergences like the one shown in Fig.17. So the necessary

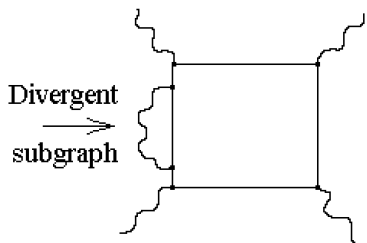


Figure 17:

condition for the finiteness is

$$\omega(\gamma_i) < 0, \quad \forall \gamma_i \subset G,$$

where γ_i are all possible subgraphs of G including G itself.

There exists a more simple way to answer the same question without examining all the diagrams. It can be derived directly from the Lagrangian.

Let us introduce the quantity called the **index of a vertex** (UV)

$$\omega_v = \delta_v + b_v + \frac{3}{2}f_v - 4, \tag{5.7}$$

where δ_v, b_v and f_v are the numbers of derivatives, internal boson and fermion lines, respectively. Then the index of a diagram, eq.(5.6), is equal to

$$\omega(G) = \sum_{vertex} \omega_v + 4, \tag{5.8}$$

where we have used that usually $r_l(boson) = 0$ and $r_l(fermion) = 1$.

Eq.(5.8) tells us that the finiteness ($\omega(G) < 0$) can occur if $\omega_v < 0$. Before formulating any general statements when it really happens, let us introduce some examples.

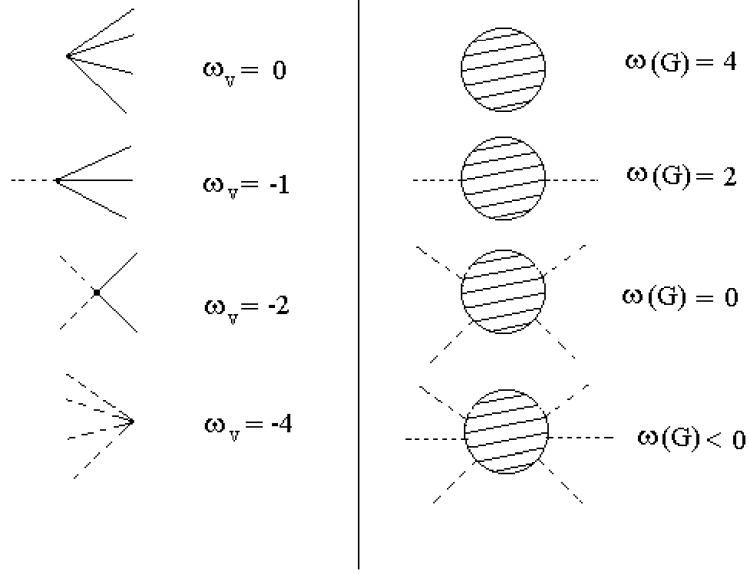


Figure 18:

5.2.3 Example: $\mathcal{L}_i = -\lambda\varphi^4$.

In this case $\delta_v = 0, f_v = 0, \omega_v = b_v - 4$. The situation is shown in Fig.18. Here the solid (dashed) lines denote the internal (external) particles, respectively. We see that there exists a limited number of divergent structures in the φ^4 theory. They are vacuum graphs, two- and four-point functions. All the other diagrams having more than four legs are convergent.

It is useful to introduce the notion of the maximum index of a vertex when all the lines are internal. All the theories can be classified according to the value of ω_v^{\max} :

$$\omega_v^{\max} = \begin{cases} < 0 & \text{Superrenormalizable} & \text{No divergent structures} \\ 0 & \text{Renormalizable} & \text{Finite } \# \text{ of divergent structures} \\ > 0 & \text{Nonrenormalizable} & \text{Infinite } \# \text{ of divergent structures} \end{cases}$$

Only for the first two types of theories we can handle the ultraviolet divergences in perturbation theory. The following theorem is valid:

Theorem: *In any renormalizable theory all the UV divergences can be absorbed into the redefining (renormalization) of a finite number of parameters: the couplings g_i , the fields φ_j and the masses m_j .*

Ignoring the possible mixing these renormalizations are described by the following formulae:

$$\begin{aligned} g_i^{\text{Bare}} &= Z_{g_i}(g, m, \dots)g_i, \\ \varphi_j^{\text{Bare}} &= Z_{\varphi_j}(g, m, \dots)\varphi_j, \\ m_j^{\text{Bare}} &= Z_{m_j}(g, m, \dots)m_j. \end{aligned} \tag{5.9}$$

Eqs.(5.9) are called the **multiplicative renormalization**.

Thus, the theory to be renormalizable should have $\omega_v^{\max} \leq 0$. This requirement can be reformulated in terms of dimensions.

Consider an arbitrary Lagrangian which is a product of the field operators and their derivatives

$$\mathcal{L}_I(x) = g \prod_{i,j} \varphi_i(x) \partial \varphi_j(x). \quad (5.10)$$

The action is the four-dimensional integral

$$A = \int d^4x \mathcal{L}(x). \quad (5.11)$$

Let us calculate the dimensions of the quantities in eqs. (5.10), (5.11). As far as the action is dimensionless (we use the natural units $\hbar = c = 1$)

$$[A] = 0,$$

the dimension of the coupling constant is $[g] = -\omega_v^{\max}$ (in mass units). Thus the renormalizability requirement is:

$$\text{Renormalizability :} \quad [g] \geq 0. \quad (5.12)$$

Let us see when this requirement is satisfied.

5.2.4 Illustration

$$\begin{aligned} \mathcal{L}_{\varphi^4} &= -\lambda \overline{\varphi}^4, \quad [\varphi] = 1 \Rightarrow [\lambda] = 0, & R \\ \mathcal{L}_{QED} &= e \overline{\psi} \gamma^\mu A_\mu \psi, \quad [A_\mu] = 1, \quad [\psi] = 3/2 \Rightarrow [e] = 0, & R \\ \mathcal{L}_{gauge} &= -\frac{1}{4} F_{\mu\nu}^2 = \frac{1}{4} [\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c]^2, \quad [A_\mu] = 1 \Rightarrow [g] = 0, & R \\ \mathcal{L}_{Yukawa} &= y \overline{\psi} \varphi \psi, \quad [\varphi] = 1, \quad [\psi] = 3/2 \Rightarrow [y] = 0. & R \end{aligned}$$

So, all these models are renormalizable.

$$\begin{aligned} \mathcal{L} &= -h \overline{\varphi}^6, \quad [\varphi] = 1 \Rightarrow [h] = -2, & NR \\ \mathcal{L} &= G \overline{\psi} \psi \overline{\psi} \psi, \quad [\psi] = 3/2 \Rightarrow [G] = -2, & NR \\ \mathcal{L} &= \kappa \overline{\psi} \partial_\mu V_\mu \psi, \quad [V_\mu] = 1 \Rightarrow [\kappa] = -1, & NR \\ \mathcal{L} &= \gamma \overline{\psi} \partial_\mu \varphi \gamma^\mu \psi, \quad [\varphi] = 1, \quad [\psi] = 3/2 \Rightarrow [\gamma] = -1. & NR \end{aligned}$$

All these models on the contrary are nonrenormalizable. Note that they include the four-fermion or current-current interaction.

The resume is the following: The only renormalizable interactions are:

- i) φ^4 interaction;
- ii) Yukawa-type interaction;
- iii) Gauge-type interaction.

The φ^3 interaction is superrenormalizable.

If one considers the spins of particles involved, they are severely limited. The renormalizable interactions contain only fields with spin 0, 1/2 and 1. All the models with spin 3/2, 2, etc. are nonrenormalizable. The latter contain also gravity. So, quantum gravity is nonrenormalizable.

Force	Model	Symmetry	Spectrum	Renormalizability
Strong	Quantum Chromodynamics (QCD)	SU(3) gauge unbroken	massive spin 1/2 quarks massless spin 1 gluons	Yes
Weak and Electro-Magnetic	Glashow-Weinberg-Salam model (GWS)	U(1) gauge unbroken SU(2) gauge spontaneously broken	massive spin 1/2 quarks and leptons massive spin 0 Higgses massless spin 1 photon massive spin 1 intermediate bosons	Yes
Gravity	Einstein Gravity	SL(2,C) gauge	massless spin 2 graviton all particles	No

Table 1:

As far as we don't know how to handle the nonrenormalizable interactions because the ultraviolet divergences are out of control, we are left only with three pieces out of which we can construct the Standard Model, namely, the φ^4 , the Yukawa and the gauge interactions with scalar, spinor and vector particles.

Note should be made concerning the vector fields with $M \neq 0$. Recall the propagator for the massive vector field

$$\overline{V_\mu V_\nu} = i \frac{g_{\mu\nu} - k_\mu k_\nu / M^2}{M^2 - k^2 - i\epsilon}.$$

This gives $r_l = 2$, leading to the nonrenormalizability. The only known way out of this puzzle is spontaneous symmetry breaking. Due to the presence of a gauge invariance the propagator is

$$\overline{V_\mu V_\nu} = i \frac{g_{\mu\nu} - (1 - d_l) k_\mu k_\nu / k^2}{M^2 - k^2 - i\epsilon}.$$

So, $r_l = 0$ and the theory remains renormalizable. This mechanism is used in the Standard Model to acquire masses for intermediate vector bosons without destroying the renormalizability.

5.3 Model Building

We now have at our disposal all the necessary ingredients to construct the Standard Model of particle interactions. Four fundamental forces of Nature are described in terms of the Lagrangian quantum field theory (see Table 2.)

Quantum gravity is not constructed up to now. The known versions of the theory are nonrenormalizable. However, there are some new ideas connected with supersym-

metry, the supergravity models, which have better UV properties. This subject is under intensive theoretical investigation.

5.3.1 Finite Theories

One may wonder, is it possible to construct quantum field theory without ultraviolet divergences, i.e. a Finite Field Theory without infinities, without renormalizations, etc.?

Surprisingly the answer is positive. The guiding idea is related with the gauge fields. Recall the vacuum polarization effect. It happens that while the matter fields cause the screening of the bare charge, the gauge fields cause an antiscreening, i.e. produce an opposite effect. Consider the radiative corrections to the propagator of a spinor field. The diagrams are shown in Fig.19. The first diagram contains virtual



Figure 19:

gauge field, while the second contains the virtual scalar one. Both diagrams diverge, but the sum is convergent. The cancellation of divergences occurs if the couplings are equal.

Another example of cancellation of divergences is given by the gauge propagator. The diagrams are shown in Fig.20. One-loop radiative corrections are divergent and

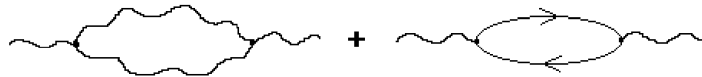


Figure 20:

proportional to

$$\ln \Lambda^2 \left(\frac{11}{3}N - \frac{2}{3}n_f \right),$$

where N corresponds to the $SU(N)$ gauge group and n_f is the number of fermions in the fundamental representation. Cancellation takes place if

$$n_f = \frac{11}{2}N, \quad \left(\frac{33}{2} \text{ for } SU(3) \right).$$

Noninteger number of fermions looks unphysical. Note, however, that 1 Majorana fermion ($C\psi = \psi$) counts as 1/2 of the Dirac fermion field, so

$$\frac{11}{2}N \text{ Dirac Fermions} = 11N \text{ Majorana Fermions.}$$

The Majorana fermions are necessary ingredients of supersymmetry, the symmetry which transforms fermions into bosons and vice versa. This symmetry is now used to construct generalizations of the Standard Model.

The first finite four-dimensional QFT has been constructed within supersymmetric models. This is the so-called N=4 extended SUSY Yang-Mills theory. Unfortunately, it cannot be used to construct realistic models due to unsatisfactory particle content. Later on, the N=2 and N=1 supersymmetric models were elaborated, also free from ultraviolet divergences. The latter models provide us with the possibility to construct finite realistic Grand Unified Theories.

These finite models need no renormalization procedure at all. They are safe at small distances (large momenta). Extremist's point of view is that nature is described by finite theories. There is a very restricted number of models of that sort. They have some remarkable properties: a single coupling constant, fixed number of generations, fixed set of the Higgs fields.

5.4 Concluding Remarks

We presented here a brief discussion of the basic concepts of Quantum Field Theory. This is a formalism needed to construct the Standard Model of particle interactions. Modern attempts to go beyond the Standard Model are also made in the context of Quantum Field Theory. These are the Grand Unified models which unify strong interactions together with electroweak ones, the supersymmetric generalization of the Standard Model and GUTs, and at last the supergravity theories attempted to include the gravitational force into this universal scheme. Quantum field theory promises to be the most natural framework for further development of particle theory.

Still new ideas appeared in recent years. Higher space-time dimensions and the Kaluza-Klein ideology got the new life within supergravity. Even more revolutionary is the development of the string theory. In this approach the local QFT appears to be the low-energy (large distance) limit of some fundamental nonlocal theory, the superstring theory, which pretends to be the ultimate theory of everything. However, even in this case the QFT remains the main tool of the description of physics below the Planck scale.

Suggested Reading

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3) T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford, 1984 (in English);
Mir, Moscow, 1987 (in Russian).