For the following functions:

$$
\begin{aligned}
\mathrm{Ls}_n(\theta) &= -\int_0^\theta \ln^{n-1}\left|2\sin\frac{\phi}{2}\right| d\phi, \\
\mathrm{Ls}_n^{(m)}(\theta) &= -\int_0^\theta \phi^m \ln^{n-m-1}\left|2\sin\frac{\phi}{2}\right| d\phi.
\end{aligned}
$$

the following usefull relations are given:

$$
\begin{aligned}
\mathrm{Ls}_3^{(1)}(\theta) &= \theta \mathrm{Cl}_2(\theta) + \mathrm{Cl}_3(\theta) - \zeta_3, \\
\mathrm{Ls}_4^{(2)}(\theta) &= \theta^2 \mathrm{Cl}_2(\theta) + 2\theta \mathrm{Cl}_3(\theta) - 2\mathrm{Cl}_4(\theta), \\
\mathrm{Ls}_5^{(3)}(\theta) &= \theta^3 \mathrm{Cl}_2(\theta) + 3\theta^2 \mathrm{Cl}_3(\theta) - 6\theta \mathrm{Cl}_4(\theta) - 6\mathrm{Cl}_5(\theta) + 6\zeta_5, \\
\mathrm{Ls}_6^{(4)}(\theta) &= 24\mathrm{Cl}_6(\theta) - 24\theta \mathrm{Cl}_5(\theta) - 12\theta^2 \mathrm{Cl}_4(\theta) + 4\theta^3 \mathrm{Cl}_3(\theta) + \theta^4 \mathrm{Cl}_2(\theta),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cl}_{2l+1}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{2}\left(1-2^{-2l}\right)\left(1-3^{-2l}\right)\ \zeta_{2l+1}, \\
\mathrm{Cl}_{2l+1}\left(\tfrac{2\pi}{3}\right) &= -\tfrac{1}{2}\left(1-3^{-2l}\right)\ \zeta_{2l+1}, \\
\mathrm{Cl}_{2l}\left(\tfrac{\pi}{3}\right) &= \left(1+2^{1-2l}\right)\ \mathrm{Cl}_{2l}\left(\tfrac{2\pi}{3}\right)\ . \\
\mathrm{Ls}_3\left(\tfrac{\pi}{3}\right) &= -\tfrac{7}{18}\pi\zeta_2, \\
\mathrm{Ls}_4^{(1)}\left(\tfrac{\pi}{3}\right) &= -\tfrac{17}{72}\zeta_4\ . \\
\mathrm{Cl}_4\left(\tfrac{\pi}{3}\right) &= \tfrac{2}{9}\mathrm{Ls}_4\left(\tfrac{\pi}{3}\right) - \tfrac{1}{9}\pi\zeta_3, \\
\mathrm{Cl}_6\left(\tfrac{\pi}{3}\right) &= \tfrac{2}{135}\mathrm{Ls}_6\left(\tfrac{\pi}{3}\right) - \tfrac{1}{9}\pi\zeta_5 - \tfrac{7}{81}\pi\zeta_2\zeta_3\ . \\
\mathrm{Ls}_5^{(2)}\left(\tfrac{\pi}{3}\right) &= \tfrac{2}{3}\mathrm{Ls}_5\left(\tfrac{\pi}{3}\right) + \tfrac{253}{54}\pi\zeta_4\ , \\
\mathrm{Ls}_6^{(2)}\left(\tfrac{\pi}{3}\right) &= -\tfrac{4}{15}\mathrm{Ls}_6\left(\tfrac{\pi}{3}\right) + \tfrac{2}{3}\zeta_2\mathrm{Ls}_4\left(\tfrac{\pi}{3}\right) + \tfrac{5}{6}\pi\zeta_5 + \tfrac{11}{3}\pi\zeta_2\zeta_3, \\
\mathrm{Ls}_6^{(3)}\left(\tfrac{\pi}{3}\right) &= \tfrac{2}{3}\pi\mathrm{Ls}_5\left(\tfrac{\pi}{3}\right) + \tfrac{18887}{432}\zeta_6 + 4\zeta_3^2\ .
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ls}_5^{(1)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{3}\pi\mathrm{Ls}_4\left(\tfrac{\pi}{3}\right) - \tfrac{19}{4}\zeta_5 - \tfrac{1}{2}\zeta_2\zeta_3, \\
\mathrm{Ls}_6^{(1)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{3}\pi\mathrm{Ls}_5\left(\tfrac{\pi}{3}\right) + \tfrac{2029}{96}\zeta_6 + 2\zeta_3^2, \\
\mathrm{Ls}_7^{(1)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{3}\pi\mathrm{Ls}_6\left(\tfrac{\pi}{3}\right) - \tfrac{2465}{32}\zeta_7 - \tfrac{15}{2}\zeta_5\zeta_2 - \tfrac{205}{8}\zeta_4\zeta_3 \\
\mathrm{Ls}_8^{(1)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{3}\pi\mathrm{Ls}_7\left(\tfrac{\pi}{3}\right) + \tfrac{140879}{256}\zeta_8 + \tfrac{285}{2}\zeta_5\zeta_3 + \tfrac{15}{2}\zeta_3^2\zeta_2 + \tfrac{21}{2}\zeta_{5,3}, \\
\mathrm{Ls}_9^{(1)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{3}\pi\mathrm{Ls}_8\left(\tfrac{\pi}{3}\right) - \tfrac{487235}{192}\zeta_9 - \tfrac{945}{4}\zeta_7\zeta_2 - \tfrac{71015}{64}\zeta_6\zeta_3 - \tfrac{12915}{16}\zeta_5\zeta_4 - 35\zeta_3^3, \\
\mathrm{Ls}_{10}^{(1)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{3}\pi\mathrm{Ls}_9\left(\tfrac{\pi}{3}\right) + \tfrac{14059623}{512}\zeta_{10} + \tfrac{51765}{8}\zeta_7\zeta_3 + \tfrac{57105}{16}\zeta_5^2 + \tfrac{4305}{4}\zeta_3^2\zeta_4 + 630\zeta_2\zeta_3\zeta_5 + \tfrac{3075}{16}\zeta_{7,3}, \\
\mathrm{Ls}_{11}^{(1)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{3}\pi\mathrm{Ls}_{10}\left(\tfrac{\pi}{3}\right) - \tfrac{76115403}{512}\zeta_{11} - \tfrac{26775}{2}\zeta_9\zeta_2 - \tfrac{8875377}{128}\zeta_8\zeta_3 - \tfrac{348705}{8}\zeta_7\zeta_4 - \tfrac{1917405}{32}\zeta_6\zeta_5 \\
&\quad - \tfrac{17955}{2}\zeta_5\zeta_3^2 - 315\zeta_3^3\zeta_2 - \tfrac{1323}{2}\zeta_{5,3}\zeta_3 - \tfrac{1323}{2}\zeta_{3,5,3},
\end{aligned}
$$

As an example of application of the *even* basis, we consider

$$U_{a,b} \equiv \zeta(a,b;-1,-1)\ .$$

The PSLQ-analysis, yields the following relation:

$$\begin{aligned}
U_{5,1} &= \tfrac{8}{39}\mathrm{Ls}_6^{(1)}\left(\tfrac{\pi}{2}\right) - \tfrac{4}{39}\pi\mathrm{Ls}_5\left(\tfrac{\pi}{2}\right) + \tfrac{10}{13}\mathrm{Li}_6\left(\tfrac{1}{2}\right) + \tfrac{10}{13}\mathrm{Li}_5\left(\tfrac{1}{2}\right)\ln 2 + \tfrac{5}{13}\mathrm{Li}_4\left(\tfrac{1}{2}\right)\ln^2 2 \\
&\quad + \tfrac{5}{468}\ln^6 2 - \tfrac{1913}{208}\zeta_6 - \tfrac{31}{104}\zeta_3^2 + \tfrac{31}{16}\zeta_5\ln 2 + \tfrac{35}{312}\zeta_3\ln^3 2 - \tfrac{5}{104}\zeta_2\ln^4 2\ .
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ls}_4^{(1)}\left(\tfrac{\pi}{2}\right) &= -\tfrac{5}{96}\ln^4 2 + \tfrac{5}{16}\zeta_2\ln^2 2 - \tfrac{35}{32}\zeta_3\ln 2 + \tfrac{125}{32}\zeta_4 + \tfrac{1}{2}\pi\mathrm{Ls}_3\left(\tfrac{\pi}{2}\right) - \tfrac{5}{4}\mathrm{Li}_4\left(\tfrac{1}{2}\right), \\
\mathrm{Ls}_5^{(1)}\left(\tfrac{\pi}{2}\right) &= -\tfrac{1}{16}\ln^5 2 + \tfrac{5}{16}\zeta_2\ln^3 2 - \tfrac{105}{128}\zeta_3\ln^2 2 - \tfrac{15}{8}\mathrm{Li}_4\left(\tfrac{1}{2}\right)\ln 2 - \tfrac{9}{8}\zeta_2\zeta_3 \\
&\quad + \tfrac{1}{2}\pi\mathrm{Ls}_4\left(\tfrac{\pi}{2}\right) - \tfrac{1209}{256}\zeta_5 - \tfrac{15}{8}\mathrm{Li}_5\left(\tfrac{1}{2}\right), \\
\mathrm{Ls}_4^{(1)}(\pi) &= -\tfrac{1}{6}\ln^4 2 + \zeta_2\ln^2 2 - \tfrac{7}{2}\zeta_3\ln 2 + \tfrac{19}{8}\zeta_4 - 4\mathrm{Li}_4\left(\tfrac{1}{2}\right), \\
\mathrm{Ls}_5^{(1)}(\pi) &= -\tfrac{2}{5}\ln^5 2 + 2\zeta_2\ln^3 2 - \tfrac{21}{4}\zeta_3\ln^2 2 - 12\mathrm{Li}_4\left(\tfrac{1}{2}\right)\ln 2 + \tfrac{9}{2}\zeta_2\zeta_3 + \tfrac{93}{32}\zeta_5 - 12\mathrm{Li}_5\left(\tfrac{1}{2}\right), \\
\mathrm{Ls}_5^{(2)}(\pi) &= -\tfrac{1}{3}\pi\ln^4 2 + 2\pi\zeta_2\ln^2 2 - 7\pi\zeta_3\ln 2 + \tfrac{27}{4}\pi\zeta_4 - 8\pi\mathrm{Li}_4\left(\tfrac{1}{2}\right).
\end{aligned}$$

It is useful to define another generalization of the log-sine integrals as

$$\mathrm{Lsc}_{i,j}(\theta) = -\int_0^\theta \mathrm{d}\phi \ln^{i-1}\left|2\sin\frac{\phi}{2}\right| \ln^{j-1}\left|2\cos\frac{\phi}{2}\right|\ .$$

When $i=1$ or $j=1$, it reduces to ordinary Ls functions,

$$\mathrm{Lsc}_{i,1}(\theta) = \mathrm{Ls}_i(\theta)\ ,\quad \mathrm{Lsc}_{1,j}(\theta) = -\mathrm{Ls}_j(\pi-\theta) + \mathrm{Ls}_j(\pi)\ .$$

There exists an obvious symmetry property,

$$\mathrm{Lsc}_{i,j}(\theta) = -\mathrm{Lsc}_{j,i}(\pi-\theta) + \mathrm{Lsc}_{j,i}(\pi)\ ,$$

which implies that the functions $\mathrm{Lsc}_{i,j}(\theta)$ with $i>j$ are not independent. The values of $\mathrm{Lsc}_{i,j}(\pi)$ can be extracted from Lewin's book:

$$\mathrm{Lsc}_{2,2}(\pi) = \tfrac{1}{4}\pi\zeta_2,\quad \mathrm{Lsc}_{2,3}(\pi) = -\tfrac{1}{4}\pi\zeta_3,\quad \mathrm{Lsc}_{3,3}(\pi) = -\tfrac{9}{16}\pi\zeta_4,\quad \mathrm{Lsc}_{2,4}(\pi) = \tfrac{21}{16}\pi\zeta_4\ .$$

For particular values of θ we have:

$$\begin{aligned}
\mathrm{Lsc}_{2,3}\left(\tfrac{\pi}{3}\right) &= -\tfrac{59}{108}\pi\zeta_3 - \tfrac{2}{27}\mathrm{Ls}_4\left(\tfrac{\pi}{3}\right) + \tfrac{1}{2}\mathrm{Ls}_4\left(\tfrac{2\pi}{3}\right), \\
\mathrm{Lsc}_{2,3}\left(\tfrac{2\pi}{3}\right) &= -\tfrac{8}{27}\pi\zeta_3 + \tfrac{7}{27}\mathrm{Ls}_4\left(\tfrac{\pi}{3}\right), \\
\mathrm{Lsc}_{2,4}\left(\tfrac{\pi}{3}\right) &= \tfrac{595}{192}\pi\zeta_4 + \tfrac{3}{4}\zeta_2\mathrm{Ls}_3\left(\tfrac{2\pi}{3}\right) - \tfrac{3}{8}\pi\mathrm{Ls}_4^{(1)}\left(\tfrac{2\pi}{3}\right) - \tfrac{1}{24}\mathrm{Ls}_5\left(\tfrac{\pi}{3}\right) + \tfrac{3}{8}\mathrm{Ls}_5\left(\tfrac{2\pi}{3}\right) + \tfrac{9}{32}\mathrm{Ls}_5^{(2)}\left(\tfrac{2\pi}{3}\right), \\
\mathrm{Lsc}_{2,4}\left(\tfrac{2\pi}{3}\right) &= \tfrac{421}{384}\pi\zeta_4 + \tfrac{27}{8}\zeta_2\mathrm{Ls}_3\left(\tfrac{2\pi}{3}\right) - \tfrac{27}{16}\pi\mathrm{Ls}_4^{(1)}\left(\tfrac{2\pi}{3}\right) - \tfrac{13}{48}\mathrm{Ls}_5\left(\tfrac{\pi}{3}\right) + \tfrac{81}{64}\mathrm{Ls}_5^{(2)}\left(\tfrac{2\pi}{3}\right), \\
\mathrm{Lsc}_{3,3}\left(\tfrac{\pi}{3}\right) &= -\tfrac{589}{576}\pi\zeta_4 + \tfrac{7}{4}\zeta_2\mathrm{Ls}_3\left(\tfrac{2\pi}{3}\right) - \tfrac{7}{8}\pi\mathrm{Ls}_4^{(1)}\left(\tfrac{2\pi}{3}\right) - \tfrac{23}{72}\mathrm{Ls}_5\left(\tfrac{\pi}{3}\right) + \tfrac{21}{32}\mathrm{Ls}_5^{(2)}\left(\tfrac{2\pi}{3}\right), \\
\mathrm{Lsc}_{2,3}\left(\tfrac{\pi}{2}\right) &= \tfrac{1}{8}\pi\zeta_3 - \tfrac{1}{3}\mathrm{Ls}_4\left(\tfrac{\pi}{2}\right) + 2\mathrm{Cl}_4\left(\tfrac{\pi}{2}\right).
\end{aligned} \tag{1}$$