Z_n graded extensions of KdV equations and fractional supersymmetry

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Abstract

We construct Z_n graded extentions of the KdV equations which are invariant under the fractional supersymmetry transformations.

1 Introduction

Generalized Grassmann algebras and fractional supersymmetry (as well as some other related problems) have been considered in [5] – [13]. Note that generalized grassmann algebras can be cosidered as an example of nonsimple Volichenko algebras [14].

2 Generalized grassmann algebras and fractional superfields

The generalized grassmann algebra with two generators θ and ∂ is defined by relations

$$\partial\,\theta - q\,\theta\,\partial = 1\ , \quad \theta^{p+1} = 0 = \partial^{p+1}\ , \tag{2.1}$$ td1

where q is a primitive root of unity: $q^{p+1} = 1$ and we chose for convenience $q = exp(i\,2\pi/(p+1))$. For $p = 1$ and $p = \infty$ we obtain from (2.1) the fermionic and bosonic Heisenberg algebras, respectively. One can deduce from (2.1) the identity

$$\partial_\theta\,\theta^n = q^n\,\theta^n\,\partial_\theta + (n)_q\,\theta^{n-1}\ , \tag{2.2}$$ td1a

where $(n)_q = (q^n - 1)/(q-1)$. It is convenient also to introduce the operator

$$g = \partial_\theta \, \theta - \theta \, \partial_\theta \qquad (2.3) \quad \texttt{td02}$$

which satisfy the relations

$$g\,\theta = q\,\theta\,g \;, \quad g\,\partial_\theta = q^{-1}\,\partial_\theta\,g \;, \quad g^{p+1} = 1 \;, \qquad (2.4) \quad \texttt{td2}$$

$$\partial_\theta\,\theta = \frac{1-q\,g}{1-q} \;, \quad \theta\,\partial_\theta = \frac{1-g}{1-q} \;.$$

Then we consider a complex fractional superplane with two coordinates $\{x, \theta\}$ ($x \in \mathbf{C}$) and introduce the functions on it (Z_{p+1}- graded superfields)

$$u(x,\theta) = u_0(x) + \theta\,u_1(x) + \theta^2\,u_2(x) + \ldots + \theta^p\,u_p(x) \;, \qquad (2.5) \quad \texttt{td3}$$

where in general the coefficient functions $u_k(x)$ are noncommutative objects (e.g. they possess nontrivial commutation relations with θ and ∂_θ). We specify the algebra of coefficients u_k latter.

The action of the differential operators ∂_θ and ∂_x on fields (2.5) is

$$(\partial_x\,u(x,\theta)) = \sum_{i=0}^{p} \theta^i\,u_i'(x) \;, \quad (\partial_\theta\,u(x,\theta)) = \sum_{i=1}^{p} (i)_q\,\theta^{i-1}\,u_{i-1}(x) \;, \qquad (2.6) \quad \texttt{td33}$$

(here we have used the identity (2.2)). Introduce a set of important differential operators D_n ($n = 0, 1, \ldots, p$)

$$D_0 = \partial_\theta + \frac{1}{(p)_q!}\,\theta^p\,\partial_x \;, \quad D_n = g^n\,D_0 \;\; (n = 1, \ldots, p) \;, \qquad (2.7) \quad \texttt{new1}$$

where $(n)_q! = (1)_q (2)_q \cdots (n)_q$. Operators D_n satisfy relations

$$(1-q)\theta(\sum_{k=0}^{p} q^k D_k) = (1-q)\theta\partial_\theta(\sum_{k=0}^{p} g^k) = (1-g)(\sum_{k=0}^{p} g^k) = 1 - g^{p+1} = 0 \;, \qquad (2.8) \quad \texttt{td5b}$$

$$D_n^{p+1} = (-1)^{pn}\,\partial_x \;, \quad D_k\,D_n = q^{n-k}\,D_n\,D_k \;, \qquad (2.9) \quad \texttt{td5}$$

where we have used the agreement $q^{\frac{p+1}{2}} = -1$. One can interprete D_n as a Z_{p+1}-graded generalizations of covariant derivatives and supercharges. We note that the operator $Q := D_p$ can be written in the form:

$$Q = \overline{\partial}_\theta + \frac{q}{(p)_q!}\,\theta^p\,\partial_x \qquad (Q\,D_n = q^{n+1}\,D_n\,Q) \;, \qquad (2.10) \quad \texttt{td4}$$

where derivative $\overline{\partial}_\theta = g^{-1}\,\partial_\theta$ obeys $\overline{\partial}_\theta\,\theta - q^{-1}\theta\,\overline{\partial}_\theta = 1$, $\overline{\partial}_\theta^{p+1} = 0$; i.e. we have

$$Q\,\theta - q^{-1}\theta\,Q = 1 \;. \qquad (2.11) \quad \texttt{td4b}$$

In general we have the following formula (Leibniz rule for derivatives D_n)

$$D_n\theta = 1 + \theta \left((q-1) \sum_{k=0}^{n} q^k D_k + q^n\, D_n \right) \ , \tag{2.12}$$ new2

which can be deduced from the identity

$$(D - D_n)\theta = \theta \left(D + (1-q) \sum_{k=1}^{n} q^k D_k - q^n D_n \right) \ . \tag{2.13}$$ new23

This identity can be proved as following. For the right hand side we have

$$\theta \left(1 + (1-q) \sum_{k=1}^{n} q^k g^k - q^n g^n \right) \partial_\theta = \left(1 + (1-q) \sum_{k=1}^{n} g^k - g^n \right) \theta\partial_\theta =$$

$$= \frac{(1-g^n)(1-qg)}{(1-g)} \theta\partial_\theta = \frac{(1-g^n)(1-qg)}{(1-q)} \ ,$$

and the last expression coincides with the left hand side of (2.13). Note that eq. (2.12) for $n = p$ reproduces (2.11) in view of eq. (2.8).

The matrix representation of the algebra (2.1) is constructed in the standard way [1]. Indeed, arbitrary function $u(\theta)$ (2.5) can be realized as $(p+1)$ dimensional vector

$$u = \begin{pmatrix} u_0 \\ \vdots \\ u_p \end{pmatrix} \ , \quad u_k = \langle k|u\rangle \ . \tag{2.14}$$ mrepr

To reproduce the actions of θ and ∂ to the function $u(\theta)$ one can use the following matrix representation of θ and ∂_θ

$$\theta_{km} = \langle k|\theta|m\rangle = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} , \ (\partial_\theta)_{km} = \langle k|\partial_\theta|m\rangle = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & (2)_q & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & (p)_q \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} . \tag{2.15}$$ threpr

The corresponding matrix representation of g (2.4) is

$$g_{km} = \langle k|g|m\rangle = \begin{pmatrix} 1 & 0 & . & 0 \\ 0 & q & . & . \\ . & . & . & . \\ 0 & . & 0 & q^p \end{pmatrix} \ . \tag{2.16}$$ ggm

In eqs. (2.14) – (2.16) we have introduced the ladder of $p+1$ states $|m\rangle$ ($m = 0, \dots, p$) defined by: $\partial|0\rangle = 0$, $\theta|m\rangle = |m+1\rangle$, $\partial|m\rangle = (m)_q\,|m-1\rangle$. The dual states $\langle k|$ satisfy the orthogonality condition $\langle k|m\rangle = \delta_{km}$. For the entries of the matrix θ we have $\theta_{km} = \delta_{k,m+1}$ and in general $(\theta^n)_{km} = \delta_{k,m+n}$, while for the matrix ∂ we have $(\partial^n)_{km} = (k)_q \cdots (k+n-1)_q \, \delta_{k+n,m}$.

Example. Consider the fermionic case $p = 1$ ($q = -1$) of the algebra (2.1)

$$\begin{gathered} \partial\,\theta + \theta\,\partial = 1 \ , \quad \theta^2 = 0 = \partial^2 \ , \\ g\partial_\theta = \partial_\theta \ , \quad g\,\theta = -\theta \ . \end{gathered} \tag{2.17}$$ td1aa

In this case the superfield (2.5) is $u(x,\theta) = u_0(x) + \theta u_1(x)$. Two differential operators (2.7) have the form

$$D_0 = \partial_\theta + \theta\partial_x = D \ , \quad D_1 = g(\partial_\theta + \theta\partial_x) = \partial_\theta - \theta\partial_x = Q \ ,$$

and D is a covariant derivative while Q is a supercharge. The matrix representation of this algebra is

$$\theta_{km} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad (\partial_\theta)_{km} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad g_{km} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

3 Theory of Z_{p+1}-graded pseudodifferential operators

Now we would like to define the algebra of Z_{p+1}-graded pseudodifferential operators

$$L = \sum_{k \le m} u^{(k)}(x,\,\theta)\, D^k \ , \tag{3.1}$$ td6

where the summation may include the negative integers k.

First of all we need to know how D acts on the products of superfields $u^{(k)}(x,\,\theta)$ of arbitrary grading. To find this action we postulate the homogeneous formula

$$D\, u^{(k)} = q^{\alpha_k}\, u^{(k)}\, D + (D\, u^{(k)}) \ , \tag{3.2}$$ td7

where $(D\,u^{(k)})$ denotes the left action of D on $u^{(k)}$ and α_k are some constants which have to be specified.

Considering the action of the second part of D (which is $(\theta^p/(p)_q!)\partial_x$) on the superfield $u^{(k)}(x,\theta)$ we deduce from (3.2) that the first component $u_0^{(k)}$ (2.5) of $u^{(k)}$ possess the following commutation relation with θ

$$u_0^{(k)}\, \theta = q^{\alpha_k}\, \theta\, u_0^{(k)} \ . \tag{3.3}$$ td8

For the first part of D we have from (3.2) the equation

$$\partial_\theta\, u^{(k)} = q^{\alpha_k}\, u^{(k)}\, \partial_\theta + (\partial_\theta\, u^{(k)}) \ ,$$

which leads to the relations

$$\partial_\theta\, u_n^{(k)} = q^{\alpha_k - n}\, u_n^{(k)}\, \partial_\theta \ . \tag{3.4}$$ td9

Now if we postulate the q-commutation relations for θ

$$\theta\, u_n^{(k)} = q^{\beta}\, u_n^{(k)}\, \theta\ , \tag{3.5}$$ td10

then we obtain from (2.1) and (3.4) that $\beta = n - \alpha_k$ which corresponds to (3.3) for $n = 0$.

It is natural to put $\alpha_0 = 0$ for bosonic superfield $u^{(0)}$. Then taking into account the relation $u^{(-k)} \sim (D^k\, u^{(0)}) = (k)_q!\, u_k^{(0)} + \ldots$ (here $\sim$ denotes that the superfields $u^{(-k)}$ and $(D^k\, u^{(0)})$ have the same grading), we obtain that $\alpha_k = k$. Thus, we have for (3.2) and (3.5)

$$D\, u^{(k)} = q^k\, u^{(k)}\, D + (D\, u^{(k)})\ , \quad \theta\, u_n^{(k)} = q^{n-k}\, u_n^{(k)}\, \theta\ , \tag{3.6}$$ td11

(one can consider this identity as a definition of $(Du^{(k)})$). It is natural to say that the grading of the fractional superfield $u^{(k)}$ is equal to k.

The ring $\mathcal{D}$ of formal pseudodifferential operators consist of the formal series $L = \sum_{k\le m} u^{(k)}\, D^k$. To describe the multiplication in $\mathcal{D}$ we find from (3.6) (for $n \geq 0$)

$$D^n\, u^{(k)} = \sum_{i=0}^{n} q^{i\,k}\, \frac{(n)_{q^{-1}}!}{(n-i)_{q^{-1}}!\,(i)_{q^{-1}}!}\, D^{n-i}(u^{(k)})\, D^i\ . \tag{3.7}$$ td17

Consider the expansion

$$\begin{aligned} D^{-n}\, u^{(k)} = W_1^{(k)}(n)\, D^{-n} + W_2^{(k-1)}(n)\, D^{-n-1} + W_3^{(k-2)}(n)\, D^{-n-2} + \ldots = \\ = \textstyle\sum_{i=0}^{\infty}\, W_{1+i}^{(k-i)}(n)\, D^{-n-i}\ . \end{aligned} \tag{3.8}$$ td18

First, we take $n = 1$ in (3.8), act on it by D from the left and use (3.6). As a result we get the recursion relations

$$q^k\, W_1^{(k)}(1) = u^{(k)}\ , \quad (D\, W_1^{(k)}(1)) + q^{k-1}\, W_2^{(k-1)}(1) = 0\ , \ldots,$$
$$(D\, W_{i+1}^{(k-i)}(1)) + q^{k-i-1}\, W_{i+2}^{(k-i-1)} = 0\ , \ldots$$

which can be solved as

$$W_{i+1}^{(k-i)}(1) = (-1)^i\, q^{-k(i+1)+(i+1)i/2}\, D^i(u^{(k)})\ ,$$

and thus

$$D^{-1}(u^{(k)}) = \sum_{i=0}^{\infty} (-1)^i\, q^{-k(i+1)+(i+1)i/2}\, D^i(u^{(k)})\, D^{-i-1}\ . \tag{3.9}$$ td19

Now using this equation and equation

$$\begin{aligned} D^{-1}\, D^{-n}\, u^{(k)} = D^{-1}\, \Big(W_1^{(k)}(n)\, D^{-n} + W_2^{(k-1)}(n)\, D^{-n-1} + W_3^{(k-2)}(n)\, D^{-n-2} + \ldots\Big) = \\ = W_1^{(k)}(n+1)\, D^{-n-1} + W_2^{(k-1)}(n+1)\, D^{-n-2} + W_3^{(k-2)}(n+1)\, D^{-n-3} + \ldots\ , \end{aligned}$$

we obtain general recurent relations for the coeficients $W_{i+1}^{(k-i)}(n)$:

$$q^{-k}\, W_1^{(k)}(n) = W_1^{(k)}(n+1)\ ,$$

$$-q^{-2k+1}\,(D\,W_1^{(k)}(n)) + q^{-k+1}\,W_2^{(k-1)}(n) = W_2^{(k-1)}(n+1)\ ,\ldots,$$

$$\textstyle\sum_{l=0}^{i}\,(-1)^{i-l}\,q^{-(k-l)(i+1-l)+(i+1-l)(i-l)/2}(D^{i-l}\,W_{1+l}^{(k-l)}(n)) = W_{i+1}^{(k-i)}(n+1)\ .$$

The solution of these relations is

$$W_{i+1}^{(k-i)}(n) = (-1)^i\,q^{-(i+n)k+i(i+1)/2}\,\frac{(i+n-1)_q!}{(n-1)_q!\,(i)_q!}\,D^i(u^{(k)})\ . \qquad (3.10)\quad \texttt{td20}$$

and its substitution in (3.8) gives

$$D^{-n}\,u^{(k)} = \sum_{i=0}^{\infty}(-1)^i\,q^{-(i+n)k+i(i+1)/2}\,\frac{(i+n-1)_q!}{(n-1)_q!\,(i)_q!}\,D^i(u^{(k)})\,D^{-n-i} =$$

$$= \sum_{m\le -n}(-1)^{m+n}\,q^{m\,k-m-n}\,\frac{(-m-1)_q!}{(n-1)_q!\,(-n-m)_{q^{-1}}!}\,D^{-n-m}(u^{(k)})\,D^m\ . \qquad (3.11)\quad \texttt{td21}$$

One can write down formulas (3.7) and (3.11) as unique relation

$$D^n\,u^{(k)} = \sum_{m\le n} q^{m\,k}\left(\begin{array}{c} n \\ m \end{array}\right)_{q^{-1}} D^{n-m}(u^{(k)})\,D^m\ , \qquad (3.12)\quad \texttt{td22}$$

where

$$\left(\begin{array}{c} n \\ m \end{array}\right)_{q^{-1}} = \left(\begin{array}{ll} \frac{(n)_{q^{-1}}!}{(m)_{q^{-1}}!\,(n-m)_{q^{-1}}!} & \text{for } 0\le m\le n,\ \ m\le n<0, \\ \\ \quad 0 & \qquad\text{for } m<0\le n, \end{array}\right.$$

and we use the relation

$$\frac{(n)_{q^{-1}}!}{(m)_{q^{-1}}!} \equiv (-q)^{n-m}\frac{(-m-1)_{q^{-1}}!}{(-n-1)_{q^{-1}}!}\ , \qquad (3.13)\quad \texttt{td23}$$

for $n\le -1$, $m\le -1$. Relation (3.13) is in an agreement with the relation for q-gamma functions

$$\frac{\Gamma_q(-n)}{\Gamma_q(-m)} = (-q)^{n-m}\,\frac{\Gamma_{q^{-1}}(m+1)}{\Gamma_{q^{-1}}(n+1)}\ ,\quad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\,(1-q)^{1-x}\ ,$$

where $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$. We note that if we put in (3.12) $q=-1$ we obtain the formulas from [4].

Let the operator L (3.1) be of the form

$$L = D^m + \sum_{k<m} u^{(k-m)}(x,\,\theta)\,D^k\ .$$

The grading of this operator is $(-m)$. Then, from (3.12) we conclude that there is a unique operator $\Lambda = D + \sum_{k\leq 0} v^{(k-1)}\, D^k$ such that $\Lambda^m = L$. The operator Λ should play the essential role in the defining of Z_{p+1} graded KP hierarchy

$$D_{(j)}\Lambda = [\Lambda^j_-,\, \Lambda]\;,$$

where as usual we have introduced the notaions:

$$L_+ = \sum_{0\leq k\leq m} u^{(k-m)}\, D^k\;,\quad L_- = \sum_{k<0} u^{(k-m)}\, D^k\;.$$

The algebra of derivatives $D_{(j)}$ over a set of graded times t_j (and their explicit forms) should be specified.

4 Fractional supersymmetry

To demonstrate how the fractional supersymmetry acts we consider fractional superfield $u^{(k)}$ of grading k. The variation of $u^{(k)}$ is

$$\delta\, u^{(k)} = (\epsilon\, Q\, u^{(k)}) \tag{4.14}$$ td12

where ϵ is a parameter of fractional supersymmetry. To find the commutation relations of ϵ with coefficients of $u^{(k)}$ we require identity

$$\delta(u^{(k)}\, D^k) = (\epsilon\, Q)(u^{(k)}\, D^k) - (u^{(k)}\, D^k)\,(\epsilon\, Q) = (\epsilon\, Q\, u^{(k)})\, D^k \tag{4.15}$$ td13

which guarantee the fractional supersymmetry in the theory of pseudo-differential operators. To prove this identity we consider the chain of equalities

$$\begin{aligned} &\epsilon\, Q\, u^{(k)}\, D = \epsilon\, Q\,(u^{(k)}_0 + \theta\, u^{(k)}_1 + \ldots + \theta^p\, u^{(k)}_p)\, D = \\ &= \epsilon\,(Q\, u^{(k)})\, D + \epsilon\,(q^k\, u^{(k)}_0 + q^{k-2}\,\theta\, u^{(k)}_1 + \ldots + q^{k-2p}\,\theta^p\, u^{(k)}_p)\, q\, D\, Q\;, \end{aligned} \tag{4.16}$$ td14

If we postulate here only quadratic commutation relations ϵ with θ and $u^{(k)}_n$

$$\epsilon\,\theta^n\, u^{(k)}_n = q^{2n+\alpha}\,\theta^n\, u^{(k)}_n\,\epsilon\;,\quad \epsilon\,\theta = q^{-\beta}\,\theta\,\epsilon\;,\quad \epsilon\, D = q^{\beta}\, D\,\epsilon\;,$$

and take into account that $u^{(k)}_k$ is a bosonic coefficient $[\epsilon,\, u^{(k)}_k] = 0$, then we deduce

$$\epsilon\,\theta^k\, u^{(k)}_k = q^{2k+\alpha}\,\theta^k\, u^{(k)}_k\,\epsilon = q^{-k\beta}\,\theta^k\, u^{(k)}_k\,\epsilon\;\Rightarrow\;\alpha + k(\beta+2) = 0 \tag{4.17}$$ td15

and from (4.16) we have

$$(\epsilon\, Q)\,(u^{(k)}\, D^k) = \delta(u^{(k)}\, D^k) + q^{(1+\beta)(1-k)}\,(u^{(k)}\, D^k)\,(\epsilon\, Q)$$

which proves (4.15) for $\beta = -1$ and $\forall k$.

Summarizing, we have

$$\epsilon\, u^{(k)}_n = q^{n-k}\, u^{(k)}_n\,\epsilon\;,\quad \theta\, u^{(k)}_n = q^{n-k}\, u^{(k)}_n\,\theta\;,\quad \epsilon\,\theta = q\,\theta\,\epsilon\;.$$

In addition we obtain the condition of the covariance of the derivative D

$$\epsilon\, D = q^{-1}\, D\,\epsilon \Longrightarrow [\epsilon Q,\, D] = 0\;.$$

5 Z_{p+1} - graded extension of KdV equation

Let us cosider the differential operator

$$
\begin{aligned}
L &= D^{2p+2} + v^{(p)}(x,\,\theta)\,D^{p} + \ldots + v^{(1)}(x,\,\theta)\,D + v^{(0)}(x,\,\theta) = \\
&= D^{2p+2} + \textstyle\sum_{k=0}^{p}\, v^{(k)}\,D^{k} \equiv \partial_x^2 + V(x,\theta,D)\;,
\end{aligned}
\qquad (5.18) \quad \texttt{td31}
$$

(if we put the dimension of operator ∂ as $(\partial)=1$, then we obtain $(v_n^{(k)}) = 2 + (n-k)/3$) and find the pseudodifferential operator

$$
L^{1/2} = D^{p+1} + \sum_{k\le p} w^{(k)}\,D^{k}\;, \qquad (5.19) \quad \texttt{td31a}
$$

such that $(L^{1/2})^2 = L$. First of all we note that in view of (5.18) we have to put $w^{(k)}=0$ for $k=0,1,\ldots,p$. Then we have

$$
\begin{aligned}
(L^{1/2})^2 &= D^{2p+2} + D^{p+1}\sum_{k<0} w^{(k)}\,D^{k} + \sum_{k<0} w^{(k)}\,D^{k+p+1} + (\sum_{k<0} w^{(k)}\,D^{k})^2 = \\
&= D^{2p+2} + \sum_{k<0}(\partial_x\,w^{(k)})\,D^{k} + 2\sum_{k<0} w^{(k)}\,D^{k+p+1} + \\
&+ \sum_{n<0,l<0} w^{(n)} \sum_{k\le n+l} q^{(k-l)\,l} \left(\begin{array}{c} n \\ k-l \end{array}\right)_{q^{-1}} (D^{n-k+l}\,w^{(l)})\,D^{k}\;,
\end{aligned}
$$

and comparing with (5.18) we deduce ($u' \equiv \partial_x(u)$)

$$
\begin{aligned}
w^{(k-p-1)} &= \frac{1}{2}\,v^{(k)} \quad (0\le k\le p), \\
-\,2\,w^{(k-p-1)} &= (w^{(k)})' + \sum_{\substack{n,l<0 \\ k\le n+l}} w^{(n)} q^{(k-l)\,l} \left(\begin{array}{c} n \\ k-l \end{array}\right)_{q^{-1}} (D^{n-k+l}\,w^{(l)}) \quad (k\le -1)\;,
\end{aligned}
\qquad (5.20) \quad \texttt{td32}
$$

In particular (for $k\le -1$) we have

$$
\begin{aligned}
& w^{(-p-2)} = -\frac{1}{4}\,(v^{(p)})'\;, \quad w^{(-p-3)} = -\frac{1}{4}(v^{(p-1)})' - \frac{1}{8}\,q\,(v^{(p)})^2\;, \\
& w^{(-p-4)} = -\frac{1}{4}\,(v^{(p-2)})' + \frac{1}{8}\,q^3\,v^{(p)}\,(D\,v^{(p)}) - \frac{1}{8}\,q^2\,(v^{(p)}\,v^{(p-1)} + v^{(p-1)}\,v^{(p)})\;.
\end{aligned}
\qquad (5.21) \quad \texttt{td33}
$$

Taking into account (5.19) - (5.21) one can write (5.19) in the form

$$
L^{1/2} = \partial_x + \frac{1}{2}\,V\,\partial_x^{-1} - \frac{1}{4}\,V'\,\partial_x^{-2} + \sum_{k\le -p-2} \widetilde{w}^{(k)}\,D^{k}\;, \qquad (5.22) \quad \texttt{td34}
$$

where $\widetilde{w}^{(k)} = w^{(k)} + (1/4)\,(v^{(k+2p+2)})'$ for $(-2p-2 \le n \le -p-2)$ and $\widetilde{w}^{(k)} = w^{(k)}$ for $n < -2p-2$. Using (5.22) we find

$$
\begin{aligned}
(L^{3/2})_+ = (L\,L^{1/2})_+ &= \left((\partial_x^2 + V)\,(\partial_x + \frac{1}{2}\,V\,\partial_x^{-1} - \frac{1}{4}\,V'\,\partial_x^{-2} + \sum_{k\le -p-2} \widetilde{w}^{(k)}\,D^k) \right)_+ = \\
&= \partial_x^3 + \frac{3}{4}\,V' + \frac{3}{2}\,V\,\partial_x + \sum_{0\le k\le p} \widetilde{w}^{(k-2p-2)}\,D^k + \\
&+ \sum_{n,k=0}^{p} v^{(n)} \sum_{p+1-k\le m\le n} q^{m\,k} \left(\begin{array}{c} n \\ m \end{array} \right)_{q^{-1}} (D^{n-m} v^{(k)})\,D^{m+k-p-1}\,.
\end{aligned}
\qquad (5.23)\ \ \texttt{td35}
$$

Now one can obtain Z_{p+1} - graded KdV equation as the Lax equation $L_t = [(L^{3/2})_+, L]$. Here we present the explicite form for Z_{p+1}- graded KdV equations for two cases:
1.) Z_3 - case, $p=2$ and $q^3 = 1$,
2.) $v^{(k)} = 0$ $(\forall k \neq 1)$, $v^{(1)}(x,\theta) \neq 0$..

1. Z_3 case.
In this case we find coeficients $\widetilde{w}^{(n)}$ for the formula (5.22) (see (5.21))

$$
\begin{aligned}
&\widetilde{w}^{(-4)} = 0\,, \quad \widetilde{w}^{(-5)} = -\frac{1}{8}\,q\,(v^{(2)})^2\,, \quad \widetilde{w}^{(-6)} = \frac{1}{8}\,v^{(2)}\,(D\,v^{(2)}) - \frac{1}{8}\,q^2\,(v^{(2)}\,v^{(1)} + v^{(1)}\,v^{(2)})\,, \\
&w^{(-7)} = \frac{1}{8}\,\Big((v^{(2)})'' - (v^{(2)}\,v^{(0)} + v^{(0)}\,v^{(2)}) + \\
&+ v^{(2)}\,\Big(q^2\,(D\,v^{(1)}) - (D^2\,v^{(2)}) \Big) - v^{(1)}\,\Big((D\,v^{(2)}) + (v^{(1)}) \Big) \Big)\,, \\
&w^{(-8)} = \frac{1}{8}\,\Big((v^{(1)})'' - (v^{(1)}\,v^{(0)} + v^{(0)}\,v^{(1)}) + \\
&+ v^{(2)}\,\Big(q\,(v^{(2)})' + q\,(D\,v^{(0)}) - (D^2\,v^{(1)}) \Big) - v^{(1)}\,(D\,v^{(1)}) + q\,(v^{(2)} v^{(2)})' \Big)\,,
\end{aligned}
\qquad (5.24)\ \ \texttt{td36}
$$

Substitution of these expresions into (5.23) gives

$$
\begin{aligned}
(L^{3/2})_+ &= \partial_x^3 + \frac{3}{4}\,V' + \frac{3}{2}\,V\,\partial_x + \frac{3}{8}\,\Big(q\,(v^{(2)})^2\,D + q^2\,(v^{(2)}\,v^{(1)} + v^{(1)}\,v^{(2)}) - v^{(2)}\,(Dv^{(2)}) \Big) = \\
&= \partial_x^3 + \frac{3}{4}\,V' + \frac{3}{2}\,V\,\partial_x + \frac{3}{8}\,q\,(v^{(2)})^2\,D + \frac{3}{8}\,V_0\,,
\end{aligned}
\qquad (5.25)\ \ \texttt{td37}
$$

where

$$
V_0 = q^2\,(v^{(2)}\,v^{(1)} + v^{(1)}\,v^{(2)}) - v^{(2)}\,(Dv^{(2)})\,, \qquad (5.26)\ \ \texttt{v0}
$$

and residue of the operator $L^{3/2}$ (which is proportional to D^{-1}) is

$$
\begin{aligned}
res_D(L^{3/2}) &= \frac{1}{8}\,(v^{(2)})'' + \frac{3}{8}\,\Big(v^{(2)}\,(v^{(0)} - q^2\,(D\,v^{(1)})) + q\,(v^{(1)})^2 + \\
&+ v^{(2)}\,(D^2 v^{(2)})) + v^{(1)}\,(D\,v^{(2)}) + v^{(0)}\,v^{(2)} \Big) =
\end{aligned}
\qquad (5.27)\ \ \texttt{td38}
$$

$$= \frac{1}{8}\,(v^{(2)})'' + \frac{3}{8}\,\left(v^{(2)}\,v^{(0)} - q^2\,v^{(2)}\,(D\,v^{(1)}) + q\,(v^{(1)})^2 + V(D)\,(v^{(2)})\right)\ .$$

Now the Lax pair equation $L_t = [(L^{3/2})_+, L]$ for Z_3 case is equivalent to

$$\partial_t v^{(2)} = 2\,\partial_x\left(res_D(L^{3/2})\right)\ , \qquad (5.28) \quad \texttt{td39}$$

$$\partial_t v^{(1)} = \frac{1}{4}\partial_x^3(v^{(1)}) - \frac{3q}{8}\partial_x^2((v^{(2)})^2) + \frac{3}{4}\partial_x\left((v^{(1)}\,v^{(0)} - q\,v^{(2)}\,(D\,v^{(0)}) + V(D)\,(v^{(1)})\right) +$$
$$+\frac{3}{8}\left(q^2\,[v^{(2)},\,(v^{(1)})^2] + q\,[(v^{(2)})^2,\,v^{(0)}] +\right.$$
$$\left. +v^{(2)}\left([(Dv^{(1)},\,v^{(2)}] - q[(D^2v^{(2)}),\,v^{(2)}]\right) - q[v^{(1)}(Dv^{(2)}),\,v^{(2)}]\right)\ , \qquad (5.29) \quad \texttt{td40}$$

$$\partial_t v^{(0)} = \frac{1}{4}\partial_x^3(v^{(0)}) + \frac{3}{4}\partial_x\left(V(D)\,(v^{(0)})\right) - \frac{3}{8}\,\left(\partial_x^2 + V(D)\right)(V_0) + \frac{3}{8}\left(q\,(v^{(2)})^2\,D + V_0\right)\,v^{(0)} = \qquad (5.30) \quad \texttt{v00}$$
$$= \frac{1}{4}\partial_x^3(v^{(0)}) + \frac{3}{8}\partial_x^2\left(v^{(2)}(Dv^{(2)}) - q^2\,[v^{(2)},\,v^{(1)}]_+\right) + \frac{3}{4}\partial_x\left(V(D)\,(v^{(0)})\right) +$$
$$+\frac{3q^2}{8}\left([v^{(1)},\,v^{(2)}]_+\,v^{(0)} - V(D)\,([v^{(1)},\,v^{(2)}]_+)\right)$$
$$+\frac{3}{8}\left(V(D)\,\left(v^{(2)}(Dv^{(2)})\right) - v^{(2)}(Dv^{(2)})\,v^{(0)} + q\,(v^{(2)})^2(Dv^{(0)})\right) \qquad (5.31) \quad \texttt{td41}$$

We have obtained these equations with the help of REDUCE program of analytical calculations.

The function V_0 in (5.30) is defined in (5.26). It is clear from (5.30) that reduction $v^{(0)} = 0$ leads to the constraint

$$\left(\partial_x^2 + V(D)\right)(V_0) = \left(\partial_x^2 + V(D)\right)\left(q^2\,(v^{(2)}\,v^{(1)} + v^{(1)}\,v^{(2)}) - v^{(2)}\,(Dv^{(2)})\right) = 0\ .$$

Now we note that the system of equations (5.28) - (5.31) can be reduced to the unique Z_3-graded KdV equation

$$\partial_t v^{(1)} = \frac{1}{4}\partial_x^3(v^{(1)}) + \frac{3}{4}\,\partial_x\left(v^{(1)}\,(D\,v^{(1)})\right)\ , \qquad (5.32) \quad \texttt{td42}$$

if we fix $v^{(0)} = 0 = v^{(2)}$ and $\partial_x\,(v^{(1)}\,v^{(1)}) = 0$. From the dimension consideration it is natural to put

$$v^{(1)}\,v^{(1)} = 0\ \Rightarrow$$
$$(v_0^{(1)})^2 = 0\ ,\quad v_1^{(1)}\,v_0^{(1)} + q\,v_0^{(1)}\,v_1^{(1)} = 0\ ,\quad v_2^{(1)}\,v_0^{(1)} + q^2\,v_0^{(1)}\,v_2^{(1)} = -(v_1^{(1)})^2\ . \qquad (5.33) \quad \texttt{td43}$$
$$v_2^{(1)}\,v_1^{(1)} + q\,v_1^{(1)}\,v_2^{(1)} = 0$$

The last relation is a consistency condition of the relations (5.33). We note that dimmensions of the fields $v_\alpha^{(1)}$ are (see (5.18))

$$\left(v^{(1)}(x,\theta)\right) = (v_0^{(1)}) = \frac{5}{3}\ ,\quad (v_1^{(1)}) = 2\ ,\quad (v_2^{(1)}) = \frac{7}{3}\ ,$$

where $(\partial_x) = 1$, $(D) = 1/3$ and therefore $v_1^{(1)}$ can be related with Virassoro field. Note that in view (5.32) and (5.33) we obtain new relations on the superfield $v^{(1)}$

$$\partial_t(v^{(1)}\,v^{(1)}) = \frac{1}{4}\,\left(v^{(1)}\,(v^{(1)})''' + (v^{(1)})'''\,v^{(1)}\right) +$$

$$+\frac{3}{4}\,\left(v^{(1)}\,\left(v^{(1)}\,(D\,v^{(1)})\right)' + \left(v^{(1)}\,(D\,v^{(1)})\right)'\,v^{(1)}\right) = 0\;, \qquad (5.34)\quad \texttt{td44}$$

and so on. We do not know how it is possible to write the whole relations (5.33), (5.34) as unique operator product expansion relation of the type

$$v^{(1)}(x_1,\theta_1)\,v^{(1)}(x_2,\theta_2) = ... + \left((x_1-x_2) + q\theta_1^2\theta_2 - q^2\theta_1\theta_2^2\right)^3\;v^{(1)}\,(v^{(1)})''' + ...$$

For writing such a product expansion we have to use the generalized Taylor expansion [11]

$$F(\mathbf{z}_1) = \sum_n\;I_n(\mathbf{z}_1,\mathbf{z}_2)\,D^n\,F(\mathbf{z}_2)\;,$$

where $\mathbf{z} = (x,\theta)$ and intervals I_n are defined by [11]

$$D_{\mathbf{z}_1}\,I_{n+1}(\mathbf{z}_1,\mathbf{z}_2) = I_n(\mathbf{z}_1,\mathbf{z}_2)\;,\quad I_0 = 1\;.$$

Let us note that the reduction $v^{(2)} = 0 = v^{(0)}$ (in the system (5.28) - (5.31)) is not quite interesting since one can not obtain the usual KdV equation from (5.32) in view of (5.33) which requires $v_1^{(1)} = 0$ for $v_0^{(1)} = 0 = v_2^{(1)}$. Another reduction is $v^{(1)} = 0 = v^{(0)}$ which leads to the Z_3 - KdV equation

$$\partial_t v^{(2)} = \frac{1}{4}\partial_x^3(v^{(2)}) + \frac{3}{4}\,\partial_x\left(v^{(2)}\,(D^2\,v^{(2)})\right)\;, \qquad (5.35)\quad \texttt{td45}$$

with additional relations

$$\left((v^{(2)})^2\right)''+v^{(2)}\,[(D^2v^{(2)}),\,v^{(2)}]_- = 0 \Rightarrow \left(\partial_x^2 + v^{(2)}D^2\right)(v^{(2)})^2 = -v^{(2)}D\left(v^{(2)}(D\,v^{(2)})\right)\;,$$

$$(\partial_x^2 + v^{(2)}\,D^2)\,\left(v^{(2)}\,(D\,v^{(2)})\right) = 0\;. \qquad (5.36)\quad \texttt{v22}$$

We also don't know how to write these relations in compact form, but we strees that component $v_0^{(2)}$ of the superfield $v^{(2)}$ has the dimension $\frac{4}{3}$ which coinsides with the dimension of paraconformal currents which have been introduced in [2].

One of the resolution of (5.36) is

$$D\,\left(v^{(2)}\,(D\,v^{(2)})\right) = 0\;,\quad \left(v^{(2)}\,v^{(2)}\right) = 0\;.$$

These relations are equivalent to

$$\partial_x(v_0^{(2)}\,v_1^{(2)}) = 0\;,\quad (v_1^{(2)})^2 = v_0^{(2)}\,v_2^{(2)}\;,\quad v_2^{(2)}\,v_1^{(2)} - q\,v_1^{(2)}\,v_2^{(2)} = v_0^{(2)}\,\partial_x(v_0^{(2)})\;,$$

$$(v_0^{(2)})^2 = 0\;,\quad v_1^{(2)}\,v_0^{(2)} + q\,v_0^{(2)}\,v_1^{(2)} = 0\;,\quad v_0^{(2)}\,v_2^{(2)} + q\,v_2^{(2)}\,v_0^{(2)} + (v_1^{(2)})^2 = 0\;. \qquad (5.37)\quad \texttt{cons}$$

The equations (5.35) in components are

$$(\partial_t - \frac{1}{4}\partial_x^3)(v_0^{(2)}) = \frac{3}{4}\,\partial_x\left((1+q)\,v_0^{(2)}\,v_2^{(2)}\right)\ ,$$

$$(\partial_t - \frac{1}{4}\partial_x^3)(v_1^{(2)}) = \frac{3}{4}\,\partial_x\left(q^{-2}\,v_0^{(2)}\,\partial_x(v_0^{(2)}) + (1+q)\,v_1^{(2)}\,v_2^{(2)}\right)\ ,$$

$$(\partial_t - \frac{1}{4}\partial_x^3)(v_2^{(2)}) = \frac{3}{4}\,\partial_x\left(-v_0^{(2)}\,\partial_x(v_1^{(2)}) + q^{-1}\,v_1^{(2)}\,\partial_x(v_0^{(2)}) + v_2^{(2)}\,v_2^{(2)}\right)\ .$$

Let us note that if we put $v_0^{(2)} = 0$ then constraints (5.37) give conditions

$$(v_1^{(2)})^2 = 0\ ,\quad v_2^{(2)}\,v_1^{(2)} = q\,v_1^{(2)}\,v_2^{(2)}$$

which should be consistent with the reduced system of equations

$$(\partial_t - \frac{1}{4}\partial_x^3)(v_1^{(2)}) = \frac{3(1+q)}{4}\,\partial_x\left(v_1^{(2)}\,v_2^{(2)}\right)\ , \tag{5.38}$$ 1z

$$(\partial_t - \frac{1}{4}\partial_x^3)(v_2^{(2)}) = \frac{3}{4}\,\partial_x\left(v_2^{(2)}\,v_2^{(2)}\right)\ . \tag{5.39}$$ 2z

Note that one can combine these equation in such a way that $u = v_2^{(2)} + \alpha\,v_1^{(2)}$ satisfy KdV equation for arbitrary parameter α. Thus, the system of eqs. (5.38), (5.39) can be splitted into two identical KdV equations for the fields $v_2^{(2)}$ and $u = v_2^{(2)} + \alpha\,v_1^{(2)}$.

2. The Z_{p+1}- graded case $v^{(k)} = 0\ \forall k \neq 1$.
Let us consider

$$L = \partial_x^2 + v^{(1)}(x,\,\theta)\,D\ , \tag{5.40}$$ td46

and find the pseudodifferential operator

$$L^{1/2} = \partial_x + \tfrac{1}{2}\,v^{(1)}\,\partial_x^{-1}\,D - \tfrac{1}{4}\,(v^{(1)})'\,\partial_x^{-2}\,D + \\ + \tfrac{1}{8}\left((v^{(1)})'' - v^{(1)}\,(D\,v^{(1)})\right)\,\partial_x^{-3}\,D + (\ldots)d_x^{-4}\,D^2 + \ldots\ , \tag{5.41}$$ td47

which satisfy to $(L^{1/2})^2 = L$ for $(v^{(1)})^2 = 0$. Then we have

$$(L^{3/2})_+ = \partial_x^3 + \frac{3}{4}\,(v^{(1)})'\,D + \frac{3}{2}\,v^{(1)}\,\partial_x\,D\ ,$$

and Z_{p+1}-graded KdV equation $\partial_t\,L = [(L^{3/2})_+,\,L]_-$ is (for $(v^{(1)})^2 = 0$)

$$\partial_t v^{(1)} = \frac{1}{4}\,(v^{(1)})''' + \frac{3}{4}\,\left(v^{(1)}\,(D\,v^{(1)})\right)'\ , \tag{5.42}$$ td48

which coinsides with (5.32). In components this equation is represented as

$$\partial_t v_k^{(1)} = \tfrac{1}{4}\,(v_k^{(1)})''' + \tfrac{3}{4}\,\textstyle\sum_{n=0}^{k}(k+1-n)_q q^{(n-1)(n-k)}\,v_n^{(1)}\,v_{k+1-n}^{(1)}\ ,\ (0 \leq k < p)$$

$$\partial_t v_p^{(1)} = \tfrac{1}{4}\,(v_p^{(1)})''' + \tfrac{3}{4}\,\textstyle\sum_{n=1}^{p}(-n)_q q^{(n-1)(n+1)}\,v_n^{(1)}\,v_{p+1-n}^{(1)} + \tfrac{3q^{-1}}{4(p)_q!}\,v_0^{(1)}\,(v_0^{(1)})'\ . \tag{5.43}$$ td50

6 Hamiltonian Structure for Z_{p+1}- graded KdV Equation

7 Z_3 Superconformal Non-algebra

Let us consider bosonic operator

$$T = \mathcal{F}\partial + \alpha\, D(\mathcal{F})\, D^2 + \overline{\mathcal{F}}\, D\,, \quad D\overline{\mathcal{F}} = D(\overline{\mathcal{F}}) + q^{-2}\, \overline{\mathcal{F}}\, D$$

where α is a constant, $\partial = D^3$ and

$$\overline{\mathcal{F}}(\theta, x) \sim D^2(\mathcal{F}) + (D(\mathcal{F}))^2\,,$$

is a superfield of the order 2. We calculate the commutator

$$[T,\ \mathcal{H}\, D] = [\mathcal{F}\partial + \alpha\, D(\mathcal{F})\, D^2 + \overline{\mathcal{F}}\, D\,,\ \mathcal{H}\, D] =$$
$$= (\mathcal{F}\mathcal{H} - \mathcal{H}\mathcal{F})\partial\, D + \Big(-(1+\alpha\, q^{-1})\, \mathcal{H}\, D(\mathcal{F}) + \alpha\, D(\mathcal{F})\, \mathcal{H}\Big)\ \partial +$$
$$+ \Big(\alpha\, (1+q^{-1})\, D(\mathcal{F})\, D(\mathcal{H}) - \alpha\, \mathcal{H}\, D^2(\mathcal{F}) + \overline{\mathcal{F}}\, \mathcal{H} - q^{-2}\, \mathcal{H}\, \overline{\mathcal{F}}\Big)\ D^2 +$$
$$+ \Big(\mathcal{F}\, \partial(\mathcal{H}) + \alpha\, D(\mathcal{F})\, D^2(\mathcal{H}) + \overline{\mathcal{F}}\, D(\mathcal{H}) - \mathcal{H}\, D(\overline{\mathcal{F}})\Big)\ D\,.$$

The algebra will be correct if we put the first three terms are equal to zero. It meens that we have to fix

$$\alpha = \frac{1}{1 - q^{-1}}\,, \quad [\mathcal{F},\ \mathcal{H}] = 0 = [\mathcal{H},\ D(\mathcal{F})]\,, \qquad (7.44) \quad \texttt{sc1}$$

$$(1 + q^{-1})\, D(\mathcal{F})\, D(\mathcal{H}) = \mathcal{H}\, D^2(\mathcal{F}) - (1 - q^{-1})\, \overline{\mathcal{F}}\, \mathcal{H} + (q-1)\, \mathcal{H}\, \overline{\mathcal{F}} = \qquad (7.45) \quad \texttt{sc2}$$
$$= \mathcal{H}\, D^2(\mathcal{F}) + (q-1)\ \Big(\mathcal{H}\, \overline{\mathcal{F}} - q^{-1}\, \overline{\mathcal{F}}\, \mathcal{H}\Big)\,.$$

We consider

$$\mathcal{F} = f_0 + \theta\, f_1 + \theta^2\, f_2\,, \quad \mathcal{H} = h_0 + \theta\, h_1 + \theta^2\, h_2\ \Rightarrow$$
$$\Rightarrow\ D(\mathcal{F}) = f_1 + 2_q\, \theta\, f_2 + \frac{1}{2_q}\, \theta^2\, f_0\,'\,, \quad 2_q = 1 + q\,,$$

as a superfields of degree zero and from (7.44) we obtain

$$[f_0,\ h_0] = 0\,, \quad [f_1,\ h_0] = [h_1,\ f_0] = 0\,,$$
$$h_1\, f_1 - q^{-1}\, f_1\, h_1 = (1+q)\, [f_2,\ h_0]\,, \quad f_1\, h_1 - q^{-1}\, h_1\, f_1 = (1+q)\, [h_2,\ f_0]\,,$$
$$[h_0,\ f_0\,'] = 0\,, \quad h_2\, f_1 - q\, f_1\, h_2 + f_2\, h_1 - q\, h_1\, f_2 = 0$$

We substitute in (7.45) the expression

$$\overline{\mathcal{F}} = \beta\, D^2(\mathcal{F}) + \gamma\, \mathcal{F}^{-1}\, (D(\mathcal{F}))^2 =$$
$$= \beta\, \left((1+q)\, f_2 + \theta\, f_0\,' + \frac{1}{1+q}\, \theta^2\, f_1\,'\right) + \gamma\, \mathcal{F}^{-1}\, (D(\mathcal{F}))^2\,,$$

as result we deduce

8 Lax operators for $N2$ super KdV

These Lax operators are (see Popowicz, Krivonos, Sorin)

$$L_1 = D(\partial_x + J)\overline{D}\ , \quad L_2 = \partial_x + [D, \overline{D}]\, \partial_x^{-1} J\ ,$$

$$L_3 = [D, \frac{1}{(\partial_x + J)}\, \overline{D}(\partial_x + J)]_+\ ,$$

where $J = v + \theta\bar{\psi} + \bar{\theta}\psi + \theta\bar{\theta}u$ (u is Virasorro field) and

$$D = \partial_\theta + \frac{1}{2}\bar{\theta}\partial_x\ , \quad \overline{D} = \partial_{\bar{\theta}} + \frac{1}{2}\theta\partial_x\ ,$$

The integral over a generic algebra (in particular over generalized grassmann algebras) has been considered in [16]

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