Bloch–Nordsieck effective theory and HQET

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Velocity

$$\vec{v} = \frac{\partial \varepsilon(\vec{p})}{\partial \vec{p}} = \frac{\vec{p}}{M} \to 0$$

Lagrangian

$$L = h^{+} i\partial_0 h$$

equation of motion

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$$\varepsilon = -eA_0$$

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Not Lorentz-invariant; gauge invariant

Lagrangian

+ Lagrangian of the photon field

$$\partial_\mu F^{\mu\nu} = j^\nu$$
$$j^0 = -eh^+h$$

The electron produces the Coulomb field

Spin symmetry

At the leading order in $1/M$, the electron spin does not interact with electromagnetic field
We can rotate it without affecting physics
In addition to the $U(1)$ symmetry $h \to e^{i\alpha} h$,
also the $SU(2)$ spin symmetry

$$h \to U h$$

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The electron magnetic moment $\vec{\mu} = \mu\vec{\sigma}$
interacts with magnetic field: $-\vec{\mu} \cdot \vec{B}$
By dimensionality $\mu \sim e/M$
(Bohr magneton $e/(2M)$ up to radiative corrections)

$$L_m = -\frac{e}{2M} h^+ \vec{B} \cdot \vec{\sigma} h$$

Violates the $SU(2)$ spin symmetry at the $1/M$ level

Spin-flavour symmetry

n_f flavours of heavy fermions

$$L = \sum_{i=1}^{n_f} h_i^+ i D_0 h_i$$

$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction

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$U(1) \times SU(2n_f)$ symmetry
Broken at $1/M_i$ by kinetic energy and magnetic interaction
At the leading order in $1/M$, not only the spin direction but also its magnitude is irrelevant
We can, for example, switch the electron spin off:

$$L = \varphi^* iD_0 \varphi$$

Superflavour symmetry

The scalar and the spinor fields together

$$L = \varphi^* i D_0 \varphi + h^+ i D_0 h$$

$U(1) \times SU(3)$ symmetry

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The superflavour $SU(3)$ symmetry:

- $\varphi \to e^{2i\alpha}\varphi$, $h \to e^{-i\alpha} h$
- $SU(2)$ spin rorations
- $$\delta \begin{pmatrix} \varphi \\ h \end{pmatrix} = i \begin{pmatrix} 0 & \varepsilon^+ \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ h \end{pmatrix}$$

  ε — an infinitesimal spinor

Broken at $1/M$

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Broken at $1/M$
We can consider, e. g., spins $\frac{1}{2}$ and 1
$SU(5)$ superflavour symmetry

Feynman rules

Leading order in $1/M$

$$L = \varphi_0^* i D_0 \varphi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2$$

The usual photon propagator

Feynman rules

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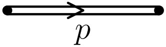
The momentum-space free electron propagator



$$= i S_0(p) \qquad S_0(p) = \frac{1}{p_0 + i0}$$

depends only on p_0, not on $\vec{p}$

(spin-$\frac{1}{2}$ field h_0 — the unit 2×2 spin matrix)

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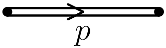

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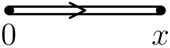
The coordinate-space propagator



$$= iS_0(x) \qquad S_0(x) = S_0(x_0)\delta(\vec{x}\,) \qquad S_0(t) = -i\theta(t)$$

Static electron does not move

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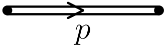

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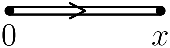

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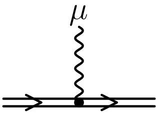
Solving the equation

$$i\partial_0 S_0(x) = \delta(x)$$

Feynman rules

Vertex



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$$v^\mu = (1, \vec{0}\,)$$

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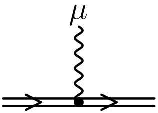

$= ie_0 v^\mu$

$v^\mu = (1, \vec{0}\,)$

The static field φ_0 (or h_0) describes only particles, there are no antiparticles.
No loops formed by static-electron propagators.
The electron propagates only forward in time;
the product of θ functions for a loop vanishes.
In the momentum space: all poles of the propagators are in the lower p_0 half-plane;
closing the integration contour upwards, we get 0.

Wilson line

In an external field $A^\mu(x)$

$$iD_0 S(x,x') = (i\partial_0 + e_0 A^0(x))S(x,x') = \delta(x-x')$$

Solution

$$S(x,x') = S(x_0,x'_0)\delta(\vec{x}-\vec{x}\,') \qquad S(x_0,x'_0) = S_0(x_0-x'_0)W(x_0,x'_0)$$

Wilson line from x' to x (along v)

$$W(x_0,x'_0) = \exp ie_0 \int_{x'_0}^{x_0} A^\mu(t,\vec{x}\,)v_\mu dt$$

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Quantum field (operator $A_0^\mu(x)$): $P\exp$

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The HQET Lagrangian has been introduced as a device to investigate of Wilson lines

Gauge $A^0 = 0$

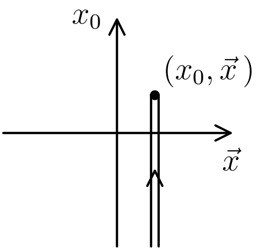
The field $\varphi_0(x)$ does not interact with the electromagnetic field (and thus becomes free).
However, this gauge is rather pathological.
The static electron creates the Coulomb electric field $\vec{E}$.
In the $A^0 = 0$ gauge, $\vec{A}$ has to depend on t linearly.

Gauge $A^0 = 0$

We can formally express the field $\varphi_0(x)$ in any gauge via a free field $\varphi^{(0)}(x)$:

$$\varphi_0(x) = W(x)\varphi^{(0)}(x)$$

$$W(x_0, \vec{x}) = P \exp i \int_{-\infty}^{x_0} A_0^{\mu}(t, \vec{x})v_\mu dt$$



Then $W^{-1}(x)D_0W(x) = \partial_0$, and

$$L = \varphi^{(0)+} i\partial_0 \varphi^{(0)}$$

Residual momentum

The full-theory energy M is the HEET zero level

$$E = M + \varepsilon$$

ε — the residual energy

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$$P^\mu = M v^\mu + p^\mu$$

- P^μ — 4-momentum of some state (containing a single electron) in the full theory
- p^μ — its momentum in HEET (the residual momentum)

v^μ — 4-velocity of a reference frame in which the electron always stays approximately at rest

Reparametrization invariance

HEET is applicable if there exists such v that

$$p^\mu \ll M \qquad p^\mu_{\gamma i} \ll M$$

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This condition does not fix v uniquely: $v \to v + \delta v$, $\delta v \sim p/M$.
Effective theories corresponding to different choices of v must produce identical physical predictions: reparametrization invariance.
Relations between quantities at different orders in $1/M$.

Relativistic notation

Lagrangian

$$L = \varphi_0^* i v \cdot D \varphi_0 + (\text{light fields})$$

Free propagator

$$S_0(p) = \frac{1}{p \cdot v + i0}$$

Mass shell

$$p \cdot v = 0$$

Spin $\frac{1}{2}$

4-component spinor field

$$\not{v} h_v = h_v$$

Lagrangian

$$L = \bar{h}_{v0} i v \cdot D h_{v0} + (\text{light fields})$$

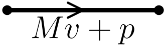
Propagator

$$S_0(p) = \frac{1 + \not{v}}{2} \frac{1}{p \cdot v + i0}$$

Vertex $i e_0 v^\mu$

QED

$$S_0(Mv+p) = \frac{M + M\not{v} + \not{p}}{(Mv+p)^2 - M^2 + i0} = \frac{1+\not{v}}{2}\,\frac{1}{p\cdot v + i0} + \mathcal{O}\left(\frac{p}{M}\right)$$



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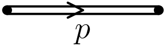

$+\,\mathcal{O}\left(\frac{p}{M}\right)$

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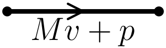

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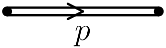

$+\,\mathcal{O}\left(\frac{p}{M}\right)$

$$\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}$$

We may insert the projectors $(1+\not{v})/2$ before $u(P_i)$ and after $\bar{u}(P_i)$, too, because

$$\not{v}u(Mv+p) = u(Mv+p) + \mathcal{O}\left(\frac{p}{M}\right)$$

QED

We have derived the HEET Feynman rules from the QED ones at $M \to \infty$. Therefore, we again arrive at the HEET Lagrangian which corresponds to these Feynman rules.

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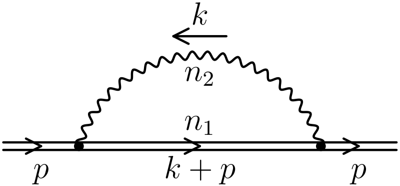
We have derived the HEET Feynman rules from the QED ones at $M \to \infty$. Therefore, we again arrive at the HEET Lagrangian which corresponds to these Feynman rules. We have thus proved that at the tree level any QED diagram is equal to the corresponding HEET diagram up to $\mathcal{O}(p/M)$ corrections. This is not true at loops, because loop momenta can be arbitrarily large. Renormalization properties of HEET (anomalous dimensions, etc.) differ from those in QED.

One-loop diagrams



$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{\left[-2(k+p)_0 - i0\right]^{n_1} \left[-k^2 - i0\right]^{n_2}}$$
$$= I(n_1, n_2)(-2\omega)^{d-n_1-2n_2}$$

Depends only on $\omega = p_0$, not $\vec{p}$

$\omega > 0$ — real pair production, cut

- integer $n_1 \leq 0$ — massless vacuum diagram
  $I(n_1, n_2) = 0$
- integer $n_2 \leq 0$ — HEET loop $I(n_1, n_2) = 0$

Coordinate space

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(-2\omega - i0)^n} \frac{d\omega}{2\pi} = \frac{i}{2\Gamma(n)} \left(\frac{it}{2}\right)^{n-1} e^{-0t}\theta(t)$$

$$\int_{0}^{\infty} e^{(i\omega-0)t} \left(\frac{it}{2}\right)^{n-1} dt = -\frac{2i\Gamma(n)}{(-2\omega - i0)^n}$$

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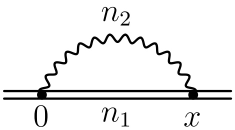
$$\int \frac{e^{-ip\cdot x}}{(-p^2 - i0)^n} \frac{d^d p}{(2\pi)^d} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - n)}{\Gamma(n)} \left(\frac{4}{-x^2 + i0} \right)^{d/2-n}$$

$$\int \left(\frac{4}{-x^2 + i0} \right)^n e^{ip\cdot x} d^d x = -i(4\pi)^{d/2} \frac{\Gamma(d/2 - n)}{\Gamma(n)} \frac{1}{(-p^2 - i0)^{d/2-n}}$$

Coordinate space



Product of propagators ($x = vt$, $-x^2/4 = -t^2/4 = (it/2)^2$)

$$-\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}\left(\frac{it}{2}\right)^{n_1+2n_2-d-1}\theta(t)$$

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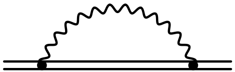

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Inverse Fourier transform

$$\frac{i}{(4\pi)^{d/2}}I(n_1,n_2)(-2\omega)^{d-n_1-2n_2}$$

$$I(n_1,n_2)=\frac{\Gamma(n_1+2n_2-d)\Gamma\left(\frac{d}{2}-n_2\right)}{\Gamma(n_1)\Gamma(n_2)}$$

α parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha \, \alpha^{n-1} \, e^{-a\alpha}$$

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$$\frac{1}{\Gamma(n_1)\Gamma(n_2)} \int d\alpha\, \alpha^{n_2-1}\, d\beta\, \beta^{n_1-1}\, d^d k\, e^X$$

$$X = \alpha k^2 + 2\beta(k+p)\cdot v$$

α has dimensionality $1/m^2$, β — $1/m$

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Wick rotation $k_0 = i k_{E0}$
Euclidean momentum space ($k^2 = -k_E^2$)

$$\int d^d k\, e^{\alpha k^2} = i \int d^d k_E\, e^{-\alpha k_E^2} = i \left(\frac{\pi}{\alpha}\right)^{d/2}$$

α parametrization

$$(-2\omega)^{d-n_1-2n_2} I(n_1,n_2) = \frac{1}{\Gamma(n_1)\Gamma(n_2)}$$

$$\times \int d\alpha\, \alpha^{n_2-1}\, d\beta\, \beta^{n_1-1}\, \alpha^{-d/2} \exp\left(-\frac{\beta^2}{\alpha} + 2\beta\omega\right)$$

α parametrization

$$(-2\omega)^{d-n_1-2n_2} I(n_1,n_2) = \frac{1}{\Gamma(n_1)\Gamma(n_2)}$$
$$\times \int d\alpha\,\alpha^{n_2-1}\,d\beta\,\beta^{n_1-1}\,\alpha^{-d/2}\exp\left(-\frac{\beta^2}{\alpha}+2\beta\omega\right)$$

$\beta = \alpha y$, integrate in α

$$\frac{\Gamma\left(n_1+n_2-\frac{d}{2}\right)}{\Gamma(n_1)\Gamma(n_2)} \int_0^\infty dy\, y^{n_1-1} \left[y(y-2\omega)\right]^{d/2-n_1-n_2}$$

HQET Feynman parameter y has the dimensionality of energy and varies from 0 to ∞

HQET Feynman parametrization

$$\frac{1}{a^n b^m} = \frac{1}{\Gamma(n)\Gamma(m)} \int d\alpha\, \alpha^{n-1}\, d\beta\, \beta^{m-1}\, e^{-a\alpha - b\beta}$$

$\beta = \alpha y$, integrate in α

$$\frac{1}{a^n b^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^\infty \frac{y^{m-1} dy}{(a+by)^{n+m}}$$

HQET Feynman parametrization

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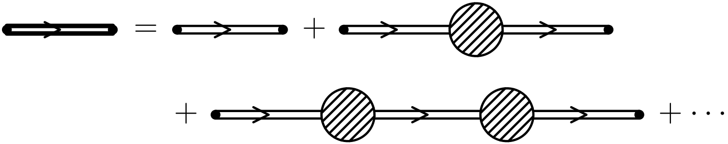
$$\frac{1}{a^n b^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^\infty \frac{y^{m-1} dy}{(a+by)^{n+m}}$$

$$\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int \frac{y^{n_1-1}\, dy\, d^d k}{(-k^2 - 2y(k+p)\cdot v)^{n_1+n_2}}$$

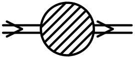
$k = k' - yv$

Static-electron propagator

The full propagator $S(p)$ depends only $\omega = p_0$, not $\vec{p}$



Static-electron self-energy



$= -i\Sigma(\omega)$

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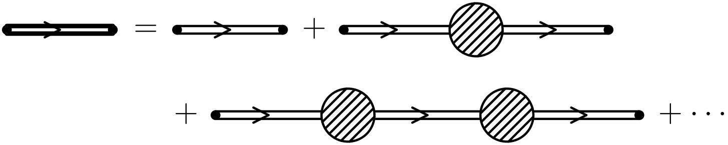

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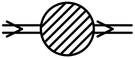


$= -i\Sigma(\omega)$

$$iS(\omega) = iS_0(\omega) + iS_0(\omega)(-i)\Sigma(\omega)iS_0(\omega) + iS_0(\omega)(-i)\Sigma(\omega)iS_0(\omega)(-i)\Sigma(\omega)iS_0(\omega) + \cdots$$

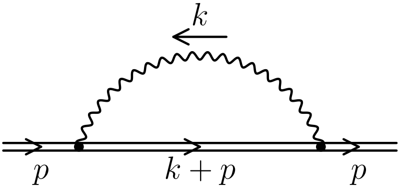
$S_0(\omega) = 1/\omega$ — free propagator

Static-electron propagator

$$S(\omega) = S_0(\omega) + S_0(\omega)\Sigma(\omega)S(\omega)$$
$$S^{-1}(\omega) = S_0^{-1}(\omega) - \Sigma(\omega)$$

$$S(\omega) = \frac{1}{\omega - \Sigma(\omega)}$$

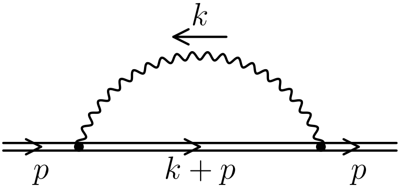
1 loop



$$\Sigma(\omega) = i \int \frac{d^d k}{(2\pi)^d} i e_0 v^\mu \frac{1}{k_0 + \omega} i e_0 v^\nu \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$$\xi = 1 - a_0$$

1 loop



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$\xi = 1 - a_0$
Numerator $(k \cdot v)^2 = (k_0 + \omega - \omega)^2 \to \omega^2$

$$\Sigma(\omega) = \frac{e_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \left[2I(1,1) + \frac{\xi}{2} I(1,2) \right]$$
$$= \frac{e_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(\xi + \frac{2}{d-3} \right)$$

Vanishes in the d-dimensional Yennie gauge

$$a_0 = \frac{2}{d-3} + 1$$

Vanishes in the d-dimensional Yennie gauge

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x space

Free photon propagator

$$D^0_{\mu\nu}(x) = \frac{i\Gamma(d/2-1)}{8\pi^{d/2}} \frac{(1+a_0)x^2 g_{\mu\nu} + (d-2)(1-a_0)x_\mu x_\nu}{(-x^2+i0)^{d/2}}$$

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$$\begin{aligned}
\Sigma(x) &= -e_0^2 D^0_{\mu\nu}(vt) v^\mu v^\nu \theta(t) \\
&= ie_0^2 \frac{\Gamma(d/2-1)}{8\pi^{d/2}} (d-3) \left(\xi + \frac{2}{d-3}\right) (it)^{2-d} \theta(t)
\end{aligned}$$

Transform to p space

Propagator to 1 loop

$$S(\omega) = S_0(\omega)\left[1 - \frac{e_0^2(-2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{2\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(\xi + \frac{2}{d-3}\right) + \mathcal{O}(e_0^4)\right]$$

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x space

$$S(t) = S_0(t)\left[1 - \frac{e_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon}\Gamma(-\varepsilon)\left(\xi + \frac{2}{d-3}\right) + \mathcal{O}(e_0^4)\right]$$

$(S_0(t) = -i\theta(t))$
Real in the Euclidean space $t = -i\tau$

Renormalization

$$S(\omega) = S_0(\omega)\left[1 + \frac{\alpha}{4\pi\varepsilon} e^{-2L\varepsilon}\left(3 - a + 4\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \mathcal{O}(\alpha^2)\right]$$

$$L = \log\frac{-2\omega}{\mu}$$

Renormalization

$$S(\omega) = S_0(\omega)\left[1 + \frac{\alpha}{4\pi\varepsilon}e^{-2L\varepsilon}\left(3 - a + 4\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \mathcal{O}(\alpha^2)\right]$$

$$L = \log\frac{-2\omega}{\mu}$$

Should be $Z_h(\alpha(\mu), a(\mu))S_r(\omega;\mu)$

$$Z_h(\alpha, a) = 1 - (a-3)\frac{\alpha}{4\pi\varepsilon} + \mathcal{O}(\alpha^2)$$

Renormalization

$$S(\omega) = S_0(\omega)\left[1 + \frac{\alpha}{4\pi\varepsilon} e^{-2L\varepsilon}\left(3 - a + 4\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \mathcal{O}(\alpha^2)\right]$$

$$L = \log\frac{-2\omega}{\mu}$$

Should be $Z_h(\alpha(\mu), a(\mu)) S_r(\omega; \mu)$

$$Z_h(\alpha, a) = 1 - (a-3)\frac{\alpha}{4\pi\varepsilon} + \mathcal{O}(\alpha^2)$$

Anomalous dimension of the static electron field

$$\gamma_h(\alpha, a) = 2(a-3)\frac{\alpha}{4\pi} + \mathcal{O}(\alpha^2)$$

Vanishes in the Yennie gauge $a = 3$

x space

$$S(t) = S_0(t)\left[1 + \frac{\alpha}{4\pi\varepsilon} e^{2L_t\varepsilon}\left(3 - a + 4\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \mathcal{O}(\alpha^2)\right]$$

$$L_t = \log\frac{i\mu t}{2} + \gamma_E$$

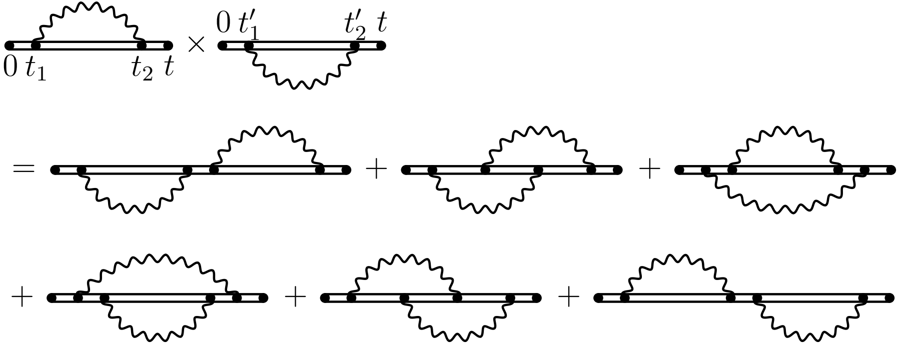
Exponentiation

1-loop correction to x-space propagator, multiply by itself

Integral in t_1, t_2, t'_1, t'_2 with $0 < t_1 < t_2 < t$, $0 < t'_1 < t'_2 < t$

Ordering of primed and non-primed t's can be arbitrary

6 regions corresponding to 6 diagrams



Exponentiation

This is 2× the 2-loop correction
1-loop correction cubed is 3!× the 3-loop correction, . . .

$$S(t) = S_0(t)\exp\left[-\frac{e_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon}\Gamma(-\varepsilon)\left(\xi+\frac{2}{d-3}\right)\right]$$

In the d-dimensional Yennie gauge the exact propagator is free

Exponentiation

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$$S(t) = S_0(t) \exp\left[-\frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{it}{2}\right)^{2\varepsilon} \Gamma(-\varepsilon) \left(\xi + \frac{2}{d-3}\right)\right]$$

In the d-dimensional Yennie gauge the exact propagator is free
No corrections to the photon propagator: $Z_A = 1$,
Therefore, the photon field is not renormalized: $Z_A = 1$, $a = a_0$

Electron density

$$J_0 = \varphi^* \varphi$$

$$Q_0 = \int J_0(x_0, \vec{x}) d^3\vec{x} = Z_J(\alpha(\mu))Q(\mu) = 1$$

$$Z_j = 1$$

Electron density

$$J_0 = \varphi^* \varphi$$

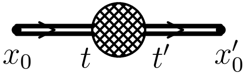
$$Q_0 = \int J_0(x_0, \vec{x}) d^3\vec{x} = Z_J(\alpha(\mu)) Q(\mu) = 1$$

$$Z_j = 1$$

Ward identity: coordinate space

Green function

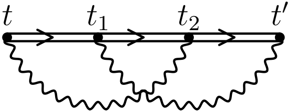
$$<0|\varphi_0^*(x) J_0(0) \varphi(x')|0> = \delta(\vec{x})\delta(\vec{x}\,')G(x_0, x_0') =$$



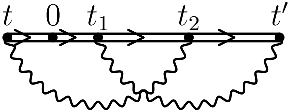
Vertex $\Gamma(t, t') = \delta(t' - t) + \Lambda(t, t')$ and 2 propagators

Ward identity: coordinate space

Starting from each diagram for Σ, we can obtain a set of diagrams for Λ



$\Rightarrow$



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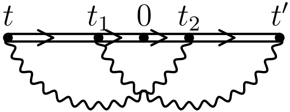

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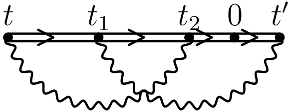
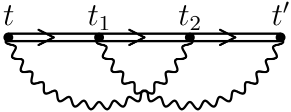


$= \theta(-t)\theta(t')$



Integration regions $t \leq 0 \leq t_1 \leq t_2 \leq t'$, $t \leq t_1 \leq 0 \leq t_2 \leq t'$, $t \leq t_1 \leq t_2 \leq 0 \leq t' \Rightarrow t \leq t_1 \leq t_2 \leq t'$

Ward identity: coordinate space

$$\Lambda(t,t') = -i\theta(-t)\theta(t')\Sigma(t'-t)$$

Similarly

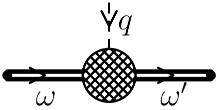
$$G(t,t') = i\theta(-t)\theta(t')S(t'-t)$$

Renormalization

$$Z_h Z_J G_r = Z_h S_r$$

$Z_J = \text{finite} = 1$

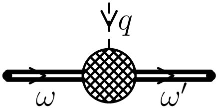
Momentum space

$G(\omega, \omega') =$  $= iS(\omega)\,\Gamma(\omega, \omega')\,iS(\omega')$

$q_0 = \omega' - \omega,\ \Gamma(\omega, \omega') = 1 + \Lambda(\omega, \omega')$

Momentum space

$G(\omega,\omega') =$

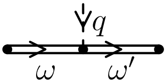


$= iS(\omega)\,\Gamma(\omega,\omega')\,iS(\omega')$

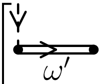
$q_0 = \omega' - \omega$, $\Gamma(\omega,\omega') = 1 + \Lambda(\omega,\omega')$

Starting from each diagram for $\Sigma(\omega)$, we can obtain a set of diagrams for $\Lambda(\omega,\omega')$

Elementary identity



$= -\dfrac{i}{\omega' - \omega}\Bigg[$



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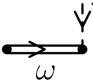
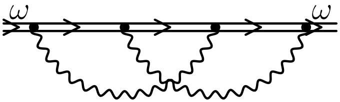
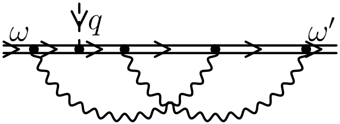


$\Bigg]$

each diagram is a difference



$\Rightarrow$



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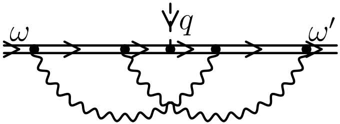

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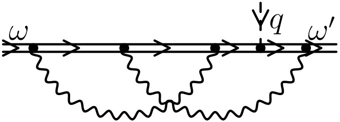
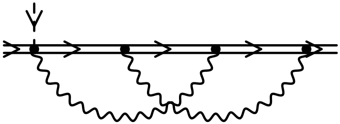


$=$

$$-\frac{i}{\omega' - \omega}\Bigg[$$



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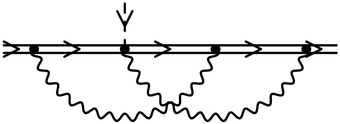

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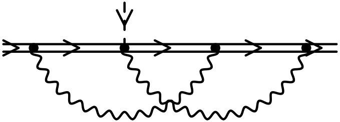

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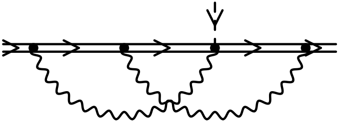

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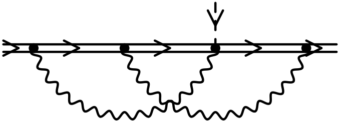

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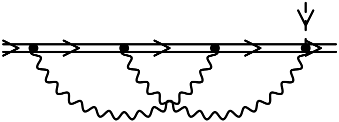
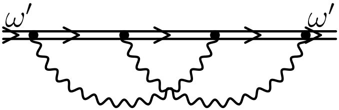


$$\Bigg] =$$

$$-\frac{i}{\omega' - \omega}\Bigg[$$



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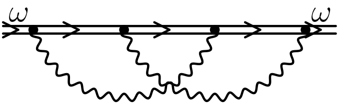


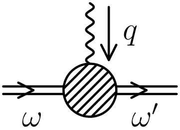
$$\Bigg]$$

Momentum space

$$\Lambda(\omega,\omega') = -\frac{\Sigma(\omega') - \Sigma(\omega)}{\omega' - \omega} \quad \text{or} \quad \Gamma(\omega,\omega') = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega}$$

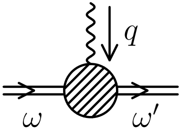
$$G(\omega,\omega') = \frac{S(\omega') - S(\omega)}{\omega' - \omega}$$

Vertex



$$= ie_0 v^\mu \Gamma(\omega, \omega') \qquad \Gamma(\omega, \omega') = 1 + \Lambda(\omega, \omega')$$

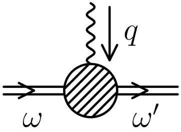
Vertex



$$= ie_0 v^\mu \Gamma(\omega, \omega') \qquad \Gamma(\omega, \omega') = 1 + \Lambda(\omega, \omega')$$

Ward identity $Z_\Gamma Z_h = 1$
$Z_\alpha = (Z_\Gamma Z_h)^{-2} Z_A^{-1} = Z_A^{-1} = 1$ — the electron charge is not renormalized in HEET

Vertex



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Ward identity $Z_\Gamma Z_h = 1$
$Z_\alpha = (Z_\Gamma Z_h)^{-2} Z_A^{-1} = Z_A^{-1} = 1$ — the electron charge is not renormalized in HEET
$e_0 \to e$, $a_0 \to a$ in the bare propagator

$$Z_h = \exp\left[-(a-3)\frac{\alpha}{4\pi\varepsilon}\right]$$

$$\gamma_h = 2(a-3)\frac{\alpha}{4\pi}$$

Vanishes in the Yennie gauge

Operators

Full QED operators — series in $1/M$
via HEET operators

$$O(\mu) = C(\mu)\tilde{O}(\mu) + \frac{1}{2M}\sum_i B_i(\mu)\tilde{O}_i(\mu) + \cdots$$

Matching on-shell matrix elements

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2} h_{v0}(x) + \cdots\right]$$

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2} h_{v0}(x) + \cdots\right]$$

On-shell matrix elements

$$<0|\psi_0|e(p)> = \left(Z_\psi^{\text{os}}\right)^{1/2} u(p)$$
$$<0|h_{v0}|e(p)> = (Z_h^{\text{os}})^{1/2} u_v(k)$$

Bare matching coefficient $(Z_h^{\text{os}} = 1)$

$$z_0 = \frac{Z_\psi^{\text{os}}(e_0^{(1)})}{Z_h^{\text{os}}(e_0^{(0)})}$$

Electron field

$$\psi_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2} h_{v0}(x) + \cdots\right]$$

On-shell matrix elements

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$$<0|h_{v0}|e(p)> = (Z_h^{\text{os}})^{1/2} u_v(k)$$

Bare matching coefficient $(Z_h^{\text{os}} = 1)$

$$z_0 = \frac{Z_\psi^{\text{os}}(e_0^{(1)})}{Z_h^{\text{os}}(e_0^{(0)})}$$

Renormalized matching coefficient

$$z(\mu) = \frac{Z_h(\alpha^{(0)}(\mu), a^{(0)}(\mu))}{Z_\psi(\alpha_s^{(1)}(\mu), a^{(1)}(\mu))} z_0$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$S(x) = S_L(x)$$

Gauge dependence of QED propagators

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x)\, e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}$$

$$\tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

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Landau, Khalatnikov (1955)
Fradkin (1955)
Zumino (1960)
Fukuda, Kubo, Yokoyama (1980)
Bogoliubov, Shirkov (1980)

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x)e^{\sigma(x)}$$

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$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^\varepsilon \Gamma(-\varepsilon)$$

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$
\begin{aligned}
S(x) &= S_0(x) e^{\sigma(x)} \\
\sigma(x) &= \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4} \right)^\varepsilon \Gamma(-\varepsilon) \\
&= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4} \right)^\varepsilon e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)
\end{aligned}
$$

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&= \log Z_\psi + \sigma_r
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\log Z_\psi(\alpha, a) &= \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}
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\log Z_\psi(\alpha, a) &= \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon} \\
\gamma_\psi(\alpha, a) &= 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)
\end{aligned}
$$

$d \log(a(\mu)\alpha(\mu))/d \log \mu = -2\varepsilon$ exactly

$\gamma_L(\alpha)$ starts from α^2

Gauge dependence of Z_ψ, γ_ψ

Massless electron

$$S(x) = S_0(x) e^{\sigma(x)}$$

$$\sigma(x) = \sigma_L(x) + a_0 \frac{e_0^2}{(4\pi)^{d/2}} \left(\frac{-x^2}{4}\right)^\varepsilon \Gamma(-\varepsilon)$$

$$= \sigma_L(x) + a(\mu) \frac{\alpha(\mu)}{4\pi} \left(\frac{-\mu^2 x^2}{4}\right)^\varepsilon e^{\gamma_E \varepsilon} \Gamma(-\varepsilon)$$

$$= \log Z_\psi + \sigma_r$$

$$\log Z_\psi(\alpha, a) = \log Z_L(\alpha) - a \frac{\alpha}{4\pi\varepsilon}$$

$$\gamma_\psi(\alpha, a) = 2a \frac{\alpha}{4\pi} + \gamma_L(\alpha)$$

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$\gamma_L(\alpha)$ starts from α^2

4 loops: Chetyrkin, Rétey (2000)

Gauge independence of $z(\mu)$ in QED

- $z_0 = Z_\psi^{\text{os}}$ gauge invariant
- $\log Z_h = (3 - a^{(0)}) \dfrac{\alpha^{(0)}}{4\pi\varepsilon}$
  $\alpha^{(0)} = \alpha_{\text{os}} \approx 1/137$
- $\log Z_\psi = -a^{(1)}(\mu) \dfrac{\alpha^{(1)}(\mu)}{4\pi\varepsilon} + (\text{gauge invariant})$
- Decoupling $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$
  Gauge dependence cancels in $\log(\tilde{Z}_\psi / Z_\psi)$

Result

$$z(M) = 1 - \frac{\alpha}{\pi} + \left(\pi^2 \log 2 - \frac{3}{2}\zeta_3 - \frac{55}{48}\pi^2 + \frac{5957}{1152}\right)\left(\frac{\alpha}{\pi}\right)^2 + \cdots$$

Power counting

Small parameter (p — residual momentum)

$$\lambda \sim \frac{p}{M}$$

Soft fields: $\partial \sim \lambda$, $A \sim \lambda$, $D \sim \lambda$

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$$<T\{\varphi(x)\varphi^+(0)\}> \sim \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{1}{k\cdot v + i0}$$

$$\varphi \sim \lambda^{3/2}$$

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$$\varphi \sim \lambda^{3/2}$$

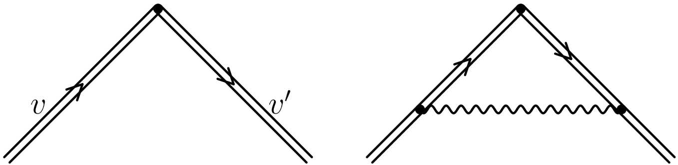
$$\varphi^+ iD_0 \varphi \sim \lambda^4$$

$$\varphi^+ \vec{D}^{\,2} \varphi \sim \lambda^5 \qquad \varphi^+ \vec{B}\cdot\vec{\sigma}\varphi \sim \lambda^5$$

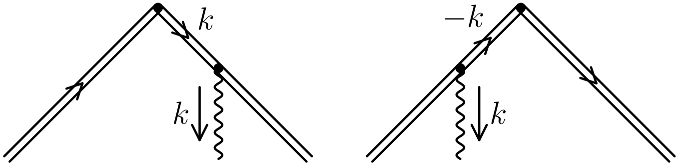
Action: main ~ 1, corrections $\sim \lambda$

Heavy–heavy current

$J_0 = \varphi^*_{v'0}\varphi_{v0} \qquad J(\mu) = Z_J^{-1}(\vartheta) J_0 \qquad \cosh\vartheta = v \cdot v'$

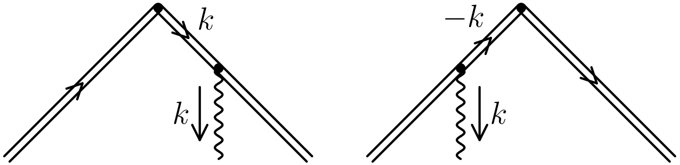


Real radiation



$$M^\mu = e\left(\frac{v^\mu}{k\cdot v} - \frac{v'^\mu}{k\cdot v'}\right)$$

Real radiation



$$M^\mu = e\left(\frac{v^\mu}{k\cdot v} - \frac{v'^\mu}{k\cdot v'}\right)$$

$$F(\omega) = -e^2 \int \frac{d^dk}{(2\pi)^d}\, 2\pi\delta(k^2)\,\delta(k\cdot v - \omega)\left(\frac{v}{k\cdot v} - \frac{v'}{k\cdot v'}\right)^2$$

Real radiation

$$F(\omega) = -\frac{2}{\Gamma(1-\varepsilon)}\frac{e^2}{(4\pi)^{d/2}}\frac{1}{\omega^{1+2\varepsilon}} \times \int_{-1}^{+1} dc \left[1 + \frac{2\coth\vartheta}{c-\coth\vartheta} + \frac{1}{\sinh^2\vartheta}\frac{1}{(c-\coth\vartheta)^2}\right]$$

Real radiation

$$
\begin{aligned}
F(\omega) = & -\frac{2}{\Gamma(1-\varepsilon)} \frac{e^2}{(4\pi)^{d/2}} \frac{1}{\omega^{1+2\varepsilon}} \\
& \times \int_{-1}^{+1} dc \left[1 + \frac{2\coth\vartheta}{c-\coth\vartheta} + \frac{1}{\sinh^2\vartheta} \frac{1}{(c-\coth\vartheta)^2} \right] \\
= & \frac{8}{\Gamma(1-\varepsilon)} \frac{e^2}{(4\pi)^{d/2}} \frac{\vartheta\coth\vartheta - 1}{\omega^{1+2\varepsilon}}
\end{aligned}
$$

Soft radiation function in classical electrodynamics

Bjorken sum rule

ξ — amplitude *not* to emit a photon

$$\xi^2 + \int_0^\infty F(\omega)\, d\omega = 1$$

Bjorken sum rule

ξ — amplitude *not* to emit a photon

$$\xi^2 + \int_0^\infty F(\omega)\, d\omega = 1$$

IR regularization, UV $1/\varepsilon$ only

$$\xi = 1 - \frac{1}{2} \int_\lambda^\infty F(\omega)\, d\omega = 1 - 2 \frac{\alpha}{4\pi\varepsilon} (\vartheta \coth \vartheta - 1)$$

Cusp anomalous dimension

$$Z_J = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta\coth\vartheta - 1)$$

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$$\Gamma(\vartheta) = (\vartheta \coth \vartheta - 1)\frac{\alpha}{\pi}$$

given by the classical soft radiation function

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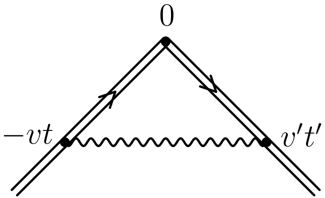
given by the classical soft radiation function

The Guinness book of records

The anomalous dimension known for the longest time (definitely $>$ 100 years)

Coordinate space



$$\Lambda(t,t';\vartheta) = ie^2 D^0_{\mu\nu}(x) v^\mu v'^\nu \theta(t)\theta(t')$$

$$= -\frac{e^2}{8\pi^{d/2}}\Gamma(1-\varepsilon)\theta(t)\theta(t')$$

$$\times \frac{(1+a)x^2\cosh\vartheta + (d-2)(1-a)(t+t'\cosh\vartheta)(t'+t\cosh\vartheta)}{(-x^2)^{d/2}}$$

$$x = vt + v't'$$

$$\Lambda(\omega,\omega';\vartheta) = \int dt\,dt'\,e^{i\omega t + i\omega' t'}\Lambda(t,t';\vartheta)$$

$$t = \tau \frac{1+\xi}{2} \qquad t' = \tau \frac{1-\xi}{2}$$

$$\Lambda(0,0;\vartheta) = -\frac{e^2}{16\pi^{d/2}}\Gamma(1-\varepsilon) \int_0^T \frac{d\tau}{\tau^{1-2\varepsilon}} \int_{-1}^{+1} d\xi$$

$$\times \frac{(1+a)\cosh\vartheta(c^2 - s^2\xi^2) + (d-2)(1-a)(c^4 - s^4\xi^2)}{(-c^2 + s^2\xi^2)^{d/2}}$$

$$c = \cosh(\vartheta/2) \qquad s = \sinh(\vartheta/2)$$

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$$c = \cosh(\vartheta/2) \qquad s = \sinh(\vartheta/2)$$

$$Z_\Gamma(\vartheta) = 1 - \frac{\alpha}{4\pi\varepsilon}\int_{-1}^{+1} d\xi \left[(1+a)\frac{\cosh\vartheta}{2\cosh^2(\vartheta/2)}\frac{1}{1-\xi^2\tanh^2(\vartheta/2)} \right.$$

$$\left. + (1-a)\frac{1-\xi^2\tanh^4(\vartheta/2)}{[1-\xi^2\tanh^2(\vartheta/2)]^2}\right]$$

$$\xi = \frac{\tanh\psi}{\tanh(\vartheta/2)}$$

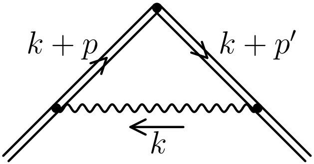
$$Z_\Gamma(\vartheta) = 1 - \frac{\alpha}{4\pi\varepsilon}\int_{-\vartheta/2}^{+\vartheta/2} d\psi\ \left[2\coth\vartheta + \frac{1-a}{\sinh\vartheta}\cosh 2\psi\right]$$
$$= 1 - \frac{\alpha}{4\pi\varepsilon}(2\vartheta\coth\vartheta + 1 - a)$$

$$\xi = \frac{\tanh\psi}{\tanh(\vartheta/2)}$$

$$\begin{aligned} Z_\Gamma(\vartheta) &= 1 - \frac{\alpha}{4\pi\varepsilon}\int_{-\vartheta/2}^{+\vartheta/2} d\psi\ \left[2\coth\vartheta + \frac{1-a}{\sinh\vartheta}\cosh 2\psi\right] \\ &= 1 - \frac{\alpha}{4\pi\varepsilon}(2\vartheta\coth\vartheta + 1 - a) \end{aligned}$$

$$Z_J(\vartheta) = Z_\Gamma(\vartheta) Z_h = 1 - 2\frac{\alpha}{4\pi\varepsilon}(\vartheta\coth\vartheta - 1)$$

Momentum space



$\omega = p \cdot v$, $\omega' = p' \cdot v'$ IR reg. $\omega = \omega'$

$$\Lambda(\omega,\omega;\vartheta) = -ie_0^2 v \cdot v' \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k\cdot v + \omega)(k \cdot v' + \omega) k^2}$$

$$= 4I(1,1,1)\cosh\vartheta \frac{e_0^2(-2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}}$$

$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = (-2\omega)^{d-n_1-n_2-2n_3} I(n_1,n_2,n_3)$$

$$D_1 = -2(k\cdot v + \omega) \quad D_2 = -2(k \cdot v' + \omega) \quad D_3 = -k^2$$

HQET Feynman parametrization

$$
\begin{aligned}
&(-2\omega)^{-2\varepsilon} I(1,1,1) \\
&= 2 \int \frac{d^d k}{i\pi^{d/2}} \frac{dy\, dy'}{\left[-k^2 - 2y(k \cdot v + \omega) - 2(k \cdot v' + \omega)\right]^3} \\
&= \Gamma(1+\varepsilon) \int \frac{dy\, dy'}{\left[(yv + y'v')^2 - 2\omega(y + y')\right]^{1+\varepsilon}}
\end{aligned}
$$

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&= \Gamma(1+\varepsilon) \int \frac{dy\,dy'}{\left[(yv + y'v')^2 - 2\omega(y+y')\right]^{1+\varepsilon}}
\end{aligned}
$$

$y = zx$, $y' = z(1-x)$

$$
I(1,1,1) = \Gamma(2\varepsilon)\Gamma(1-\varepsilon) \int_0^1 \frac{dx}{A^{1-\varepsilon}}
$$

$$
\begin{aligned}
A &= x^2 + (1-x)^2 + 2x(1-x)\cosh\vartheta\\
&= \left[1 - (1-e^{\vartheta})x\right]\left[1 - (1-e^{-\vartheta})x\right]
\end{aligned}
$$

$$\frac{I(1,1,1)}{\Gamma(2\varepsilon)\Gamma(1-\varepsilon)} = {}_2F_1\left(\begin{array}{c} 1, 1-\varepsilon \\ 3/2 \end{array} \middle| \frac{1-\cosh\vartheta}{2} \right) = \frac{\vartheta}{\sinh\vartheta} + \mathcal{O}(\varepsilon)$$

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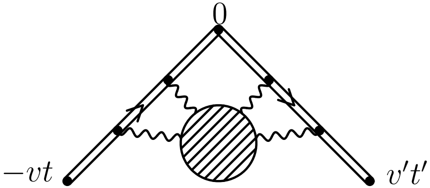
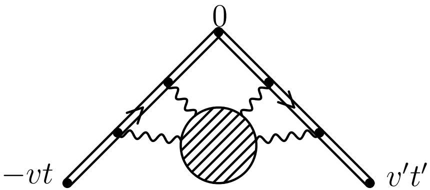
$$\Lambda(\omega,\omega;\vartheta) = 2\frac{\alpha}{4\pi\varepsilon}\vartheta\coth\vartheta + \mathcal{O}(1)$$

Exponentiation



$$G(t,t';\vartheta) = \theta(t)\theta(t')\exp\left[\frac{e^2}{(4\pi)^{d/2}}F(t,t';\vartheta)\right]$$

Exponentiation



$$G(t,t';\vartheta) = \theta(t)\theta(t')\exp\left[\frac{e^2}{(4\pi)^{d/2}}F(t,t';\vartheta)\right]$$

Divide by $G(t,t';\vartheta=0) = iS(t+t')\theta(t)\theta(t')$

$$\begin{aligned}\mathcal{G}(t,t';\vartheta) &= \frac{G(t,t';\vartheta)}{iS(t+t')} \\ &= \exp\left[\frac{e^2}{(4\pi)^{d/2}}\left(\mathcal{F}(t,t';\vartheta) - \mathcal{F}(t,t';\vartheta=0)\right)\right]\end{aligned}$$

Exponentiation

Should be $Z_J(\vartheta)\mathcal{G}_r(t,t';\vartheta)$:

$$Z_J(\vartheta) = \exp\left[\frac{\alpha}{4\pi\varepsilon}\left(f(\vartheta) - f(\vartheta = 0)\right)\right]$$
$$\varepsilon e^{\gamma\varepsilon}\mathcal{F}(t,t';\vartheta) = f(\vartheta) + \mathcal{O}(\varepsilon)$$

Exponentiation

Should be $Z_J(\vartheta)\mathcal{G}_r(t,t';\vartheta)$:

$$Z_J(\vartheta) = \exp\left[\frac{\alpha}{4\pi\varepsilon}\left(f(\vartheta) - f(\vartheta = 0)\right)\right]$$
$$\varepsilon e^{\gamma\varepsilon}\mathcal{F}(t,t';\vartheta) = f(\vartheta) + \mathcal{O}(\varepsilon)$$

$$\Gamma(\vartheta) = (\vartheta\coth\vartheta - 1)\frac{\alpha}{\pi}$$

exactly

Limiting cases

$\Gamma(-\vartheta) = \Gamma(\vartheta)$
$\vartheta \to 0$

$$\Gamma(\vartheta) = \Gamma_0(\alpha)\vartheta^2 + \mathcal{O}(\vartheta^4)$$
$$\Gamma_0(\alpha) = \frac{\alpha}{3\pi}$$

Limiting cases

$\Gamma(-\vartheta) = \Gamma(\vartheta)$

$\vartheta \to 0$

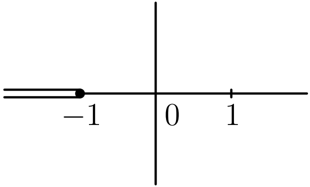
$$\Gamma(\vartheta) = \Gamma_0(\alpha)\vartheta^2 + \mathcal{O}(\vartheta^4)$$
$$\Gamma_0(\alpha) = \frac{\alpha}{3\pi}$$

$\vartheta \to \infty$

$$\Gamma(\vartheta) = \Gamma_\infty(\alpha)\vartheta + \mathcal{O}(\vartheta^0)$$
$$\Gamma_\infty(\alpha) = \frac{\alpha}{\pi}$$

Analytical structure

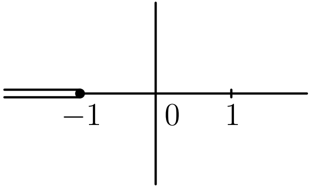
Euclidean ϑ_E: $\cos\vartheta_E \in [-1,1]$



$$\Gamma_E(\vartheta_E) = (\vartheta_E \cot\vartheta_E - 1)\frac{\alpha}{\pi}$$

Analytical structure

Euclidean ϑ_E: $\cos\vartheta_E \in [-1,1]$



$$\Gamma_E(\vartheta_E) = (\vartheta_E \cot\vartheta_E - 1)\frac{\alpha}{\pi}$$

Cut $\cos\vartheta_E \le -1$: production (or annihilation) of a heavy particle–antiparticle pair

The physical side $v \cdot v' - i0$, $\vartheta = \vartheta_0 - i\pi$

$$\Gamma(\vartheta_0 - i\pi) = \left[(\vartheta_0 - i\pi)\coth\vartheta_0 - 1\right]\frac{\alpha}{\pi}$$

$$\operatorname{Im}\Gamma(\vartheta_0 - i\pi) = -\alpha\coth\vartheta_0$$

RG equation $\Rightarrow$ Coulomb phase factors

Potential

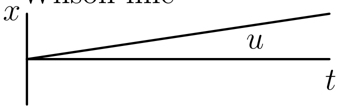
$\vartheta_0 = u \ll 1$ — nonrelativistic relative velocity

$$\operatorname{Im}\Gamma(u - i\pi) = -\frac{\alpha}{u} + \mathcal{O}(u^0)$$

Coulomb gauge: instantaneous Coulomb potential

$$V(r) = -\frac{e^2}{4\pi r^{1-2\varepsilon}}$$

Wilson line



$$W = \exp\left[-i \int_0^T V(ut)\, dt\right]$$

$1/\varepsilon$ divergences — renormalization constant

$$Z = \exp\left[i\frac{\alpha}{2u\varepsilon}\right]$$

No self-energy corrections

$$\Gamma = \frac{d\,\log Z}{d\,\log\mu} = -i\frac{\alpha}{u}$$

The $1/u$ term in $\operatorname{Im}\Gamma(u - i\pi)$ is determined by the particle–antiparticle potential
$u = i\delta \Rightarrow$ the Euclidean result

$1/M$ corrections: spin 0

Kinetic energy

$$L = L_0 + \frac{1}{2M} C_k^0 O_k^0 = L_0 + \frac{1}{2M} C_k(\mu) O_k(\mu)$$

$$O_k^0 = \varphi_0^* \vec{D}^{\,2} \varphi_0 = -\varphi_0^* D_\perp^2 \varphi_0 = Z_k(\mu) O_k(\mu)$$

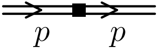
$1/M$ corrections: spin 0

Kinetic energy

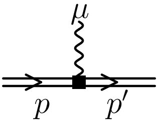
$$L = L_0 + \frac{1}{2M} C_k^0 O_k^0 = L_0 + \frac{1}{2M} C_k(\mu) O_k(\mu)$$

$$O_k^0 = \varphi_0^* \vec{D}^{\,2} \varphi_0 = -\varphi_0^* D_\perp^2 \varphi_0 = Z_k(\mu) O_k(\mu)$$

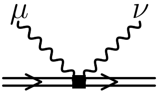
New vertices ($g_\perp^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu$)



$$= i \frac{C_k^0}{2M} p_\perp^2$$



$$= i \frac{C_k^0}{2M} e_0 (p + p')_\perp^\mu$$



$$= i \frac{C_k^0}{M} e_0^2 g_\perp^{\mu\nu}$$

The sum of 1PI self-energy diagrams at $1/M$:
$-i(C_k^0/(2M))\Sigma_k(\omega, p_\perp^2)$

- 0-photon vertex: quadratic in $p_\perp$
- 1-photon vertex: linear in $p_\perp$
- 2-photon vertex: independent of $p_\perp$

Linear terms vanish
The coefficient of $p_\perp^2$ is $iC_k^0/(2M)$ times the sum of the leading-order diagrams with a unit operator insertion:

$$\Sigma_k(\omega, p_\perp^2) = \frac{d\Sigma(\omega)}{d\omega} p_\perp^2 + \Sigma_{k0}(\omega)$$

Another derivation

Variation of Σ for $v \to v + \delta v$

- Propagators $1/(p \cdot v + i0) \Rightarrow$ insertions $ip_i \cdot \delta v$
- Vertices $\Rightarrow ie_0' \delta v^\mu$

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- Vertices $\Rightarrow ie_0' \delta v^\mu$

Variation of Σ_k for $p_\perp \to p_\perp + \delta p_\perp$

- 0-photon vertex: $i(C_k^0/M) p_i \cdot \delta p_\perp$
- 1-photon vertex: $i(C_k^0/M) e_0' \delta p_\perp^\mu$
- 2-photon vertex: 0

Another derivation

Variation of Σ for $v \to v + \delta v$

- Propagators $1/(p \cdot v + i0) \Rightarrow$ insertions $ip_i \cdot \delta v$
- Vertices $\Rightarrow ie'_0 \delta v^\mu$

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- 2-photon vertex: 0

$$\frac{\partial \Sigma_k}{\partial p_\perp^\mu} = 2 \frac{\partial \Sigma}{\partial v^\mu}$$

Ward identity of reparametrization invariance

Another derivation

Variation of Σ for $v \to v + \delta v$

- Propagators $1/(p \cdot v + i0) \Rightarrow$ insertions $ip_i \cdot \delta v$
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Ward identity of reparametrization invariance

$\partial \Sigma_k / \partial p_\perp^\mu = 2(\partial \Sigma_k / \partial p_\perp^2) p_\perp^\mu$, $\partial \Sigma / \partial v^\mu = (d\Sigma / d\omega) p_\perp^\mu$

$$\frac{\partial \Sigma_k}{\partial p_\perp^2} = \frac{d\Sigma}{d\omega}$$

Mass shell

QED

$$P_0 = \sqrt{M^2 + \vec{P}^{\,2}} \qquad p_0 = \frac{\vec{p}^{\,2}}{2M}$$

Mass shell

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$$P_0 = \sqrt{M^2 + \vec{P}^{\,2}} \qquad p_0 = \frac{\vec{p}^{\,2}}{2M}$$

HEET

$$\omega - \Sigma(\omega) - \frac{C_k^0}{2M}\left[\vec{p}^{\,2} - \frac{d\Sigma(\omega)}{d\omega}\vec{p}^{\,2} + \Sigma_{k0}(\omega)\right] = 0$$

Expand in ω up to linear terms: $\Sigma(0) = 0$, $(d\Sigma(\omega)/d\omega)_{\omega=0} = 0$, $\Sigma_{k0}(0) = 0$ (no-scale)

$$\omega = \frac{C_k^0}{2M}\vec{p}^{\,2}$$

Mass shell

QED

$$P_0 = \sqrt{M^2 + \vec{P}^{\,2}} \qquad p_0 = \frac{\vec{p}^{\,2}}{2M}$$

HEET

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Expand in ω up to linear terms: $\Sigma(0) = 0$, $(d\Sigma(\omega)/d\omega)_{\omega=0} = 0$, $\Sigma_{k0}(0) = 0$ (no-scale)

$$\omega = \frac{C_k^0}{2M}\vec{p}^{\,2}$$

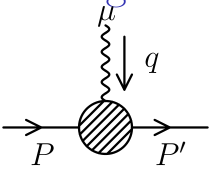
$C_k^0 = Z_k^{-1}(\mu) C_k(\mu) = 1$

$$Z_k(\mu) = 1 \qquad \gamma_k = 0$$

$$C_k(\mu) = C_k^0 = 1$$

Reparametrization invariance

Scattering in external field: QED



$$e_{\rm os}\,\varphi^*(P')F(q^2)(P+P')^\mu\varphi(P)\,,$$

$e_{\rm os} = e'_{\rm os} = e'_0$
Current $J^\mu = 2P^\mu|\varphi(P)|^2$; $J^0 = 1 \Rightarrow \varphi(P) = 1/\sqrt{2E}$

$$F(q^2) = 1 + F'(0)\frac{q^2}{M^2} + \cdots$$

$F(0) = 1$ due to Ward identity, $<r^2> = 6F'(0)/M^2$
$P^{(\prime)} = Mv + p^{(\prime)}$ $(p\cdot v = p'\cdot v = 0)$

$$e_{\rm os}\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M}\right]$$

Scattering in external field: HEET

Loop corrections vanish (no scale)

$$e'_0 \left[v^\mu + \frac{C_k^0}{2M} (p + p')_\perp^\mu \right]$$

$1/M$ corrections: spin 1/2

$$
\begin{aligned}
L &= L_0 + \frac{1}{2M} C_k^0 O_k^0 + \frac{1}{2M} C_m^0 O_m^0 \\
&\qquad = L_0 + \frac{1}{2M} C_k(\mu) O_k(\mu) + \frac{1}{2M} C_m(\mu) O_m(\mu) \\
O_k^0 &= h_0^+ \vec{D}^{\,2} h_0 = -\bar{h}_{v0} D_\perp^2 h_{v0} = Z_k(\mu) O_k(\mu) \\
O_m^0 &= -e_0 h_0^+ \vec{B}_0 \cdot \vec{\sigma} h_0 = \frac{1}{2} e_0 \bar{h}_{v0} F^0_{\mu\nu} \sigma^{\mu\nu} h_{v0} = Z_m(\mu) O_m(\mu)
\end{aligned}
$$

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
$Z_k(\mu) = 1$, $C_k(\mu) = C_k^0 = 1$.

$1/M$ corrections: spin $1/2$

$$
\begin{aligned}
L &= L_0 + \frac{1}{2M} C_k^0 O_k^0 + \frac{1}{2M} C_m^0 O_m^0 \\
&= L_0 + \frac{1}{2M} C_k(\mu) O_k(\mu) + \frac{1}{2M} C_m(\mu) O_m(\mu)
\end{aligned}
$$

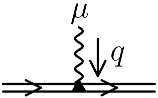
$$O_k^0 = h_0^+ \vec{D}^{\,2} h_0 = -\bar{h}_{v0} D_\perp^2 h_{v0} = Z_k(\mu) O_k(\mu)$$

$$O_m^0 = -e_0 h_0^+ \vec{B}_0 \cdot \vec{\sigma} h_0 = \frac{1}{2} e_0 \bar{h}_{v0} F^0_{\mu\nu} \sigma^{\mu\nu} h_{v0} = Z_m(\mu) O_m(\mu)$$

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

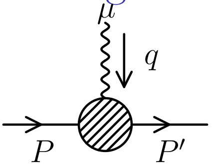
$Z_k(\mu) = 1$, $C_k(\mu) = C_k^0 = 1$.

The magnetic interaction breaks the spin symmetry



$$= \frac{i e_0' C_m^0}{2M} [\not{q}, \gamma^\mu]$$

Scattering in external field: QED

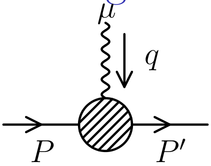


$$e_{\mathrm{os}}\,\bar{u}'(P')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

$$= e_{\mathrm{os}}\,\bar{u}'(P')\left[\left(F_1(q^2)+F_2(q^2)\right)\gamma^\mu - F_2(q^2)\frac{(P+P')^\mu}{2M}\right]u(P)$$

$$= e_{\mathrm{os}}\,\bar{u}'(P')\left[F_1(q^2)\frac{(P+P')^\mu}{2M} + \left(F_1(q^2)+F_2(q^2)\right)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

Scattering in external field: QED



$$e_{\rm os}\,\bar{u}'(P')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

$$= e_{\rm os}\,\bar{u}'(P')\left[\left(F_1(q^2)+F_2(q^2)\right)\gamma^\mu - F_2(q^2)\frac{(P+P')^\mu}{2M}\right]u(P)$$

$$= e_{\rm os}\,\bar{u}'(P')\left[F_1(q^2)\frac{(P+P')^\mu}{2M} + \left(F_1(q^2)+F_2(q^2)\right)\frac{[\not{q},\gamma^\mu]}{4M}\right]u(P)$$

$$F_1(q^2) = 1 + F_1'(0)\frac{q^2}{M^2} + \cdots$$

$$F_m(q^2) = F_1(q^2) + F_2(q^2) = F_m(0) + \cdots$$

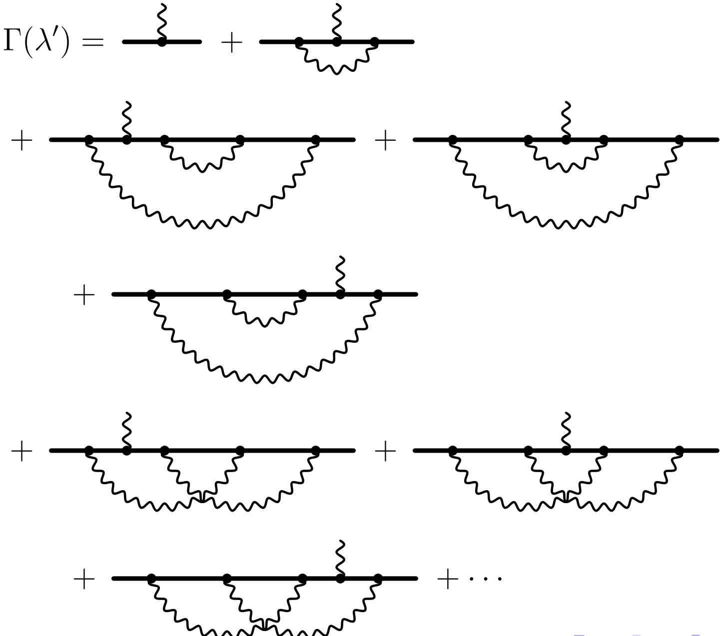
Electron magnetic moment is IR finite

$\Gamma(\lambda)$ — the vertex (expanded in q up to the linear term and projected onto the magnetic-moment structure) with an IR cutoff $\lambda \ll M$

$$\mu(\lambda) = Z_\psi^{\text{os}}(\lambda)\Gamma(\lambda)$$

$\mu(\lambda')$ for $\lambda' \ll \lambda$

Essential contributions to $\Gamma(\lambda')$



When the momenta k_i of all L soft photon lines $\to 0$, the residual momenta p_i of all $2L$ soft electron lines also $\to 0$, thus producing an IR divergence.

- Each photon propagator is $Z_A^{\mathrm{os}}(\lambda) D_0(k_i)$
- Each electron propagator is $Z_\psi^{\mathrm{os}}(\lambda) S_0(Mv + p_i)$
- Each virtual-photon vertex is $Z_\Gamma^{\mathrm{os}}(\lambda) e_0 \gamma^\mu$
- Each external-photon vertex is $\Gamma(\lambda)$

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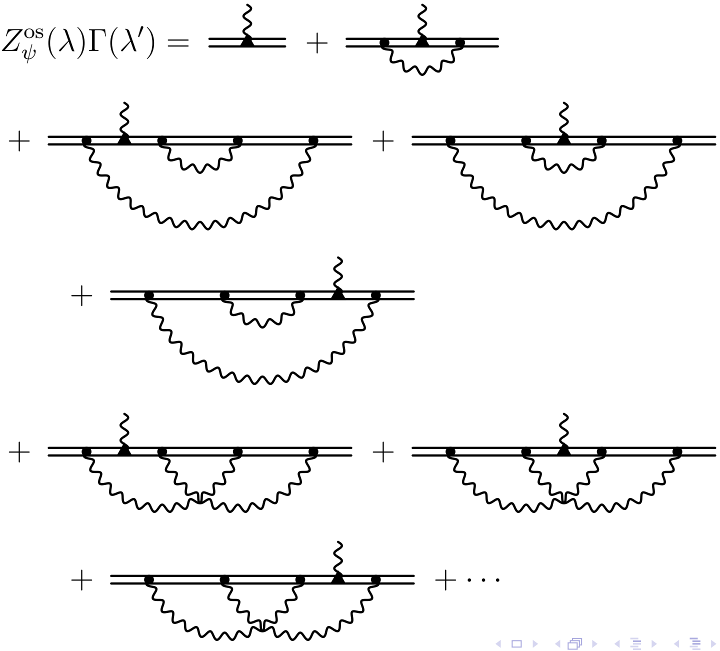
- Each virtual-photon vertex is $Z_\Gamma^{\text{os}}(\lambda) e_0 \gamma^\mu$
- Each external-photon vertex is $\Gamma(\lambda)$

Let's multiply $\Gamma(\lambda')$ by $Z_\psi^{\text{os}}(\lambda)$

- Each virtual-photon vertex
  $Z_\psi^{\text{os}}(\lambda) \left[Z_A^{\text{os}}(\lambda)\right]^{1/2} Z_\Gamma^{\text{os}}(\lambda) = e_{\text{os}}(\lambda)$
- The external-photon vertex $Z_\psi^{\text{os}}(\lambda) \Gamma(\lambda) = \mu(\lambda)$

IR divergences can be reproduced in the effective theory

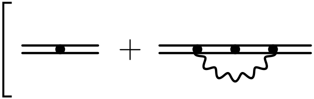
$Z_\psi^{\mathrm{os}}(\lambda)\Gamma(\lambda') =$

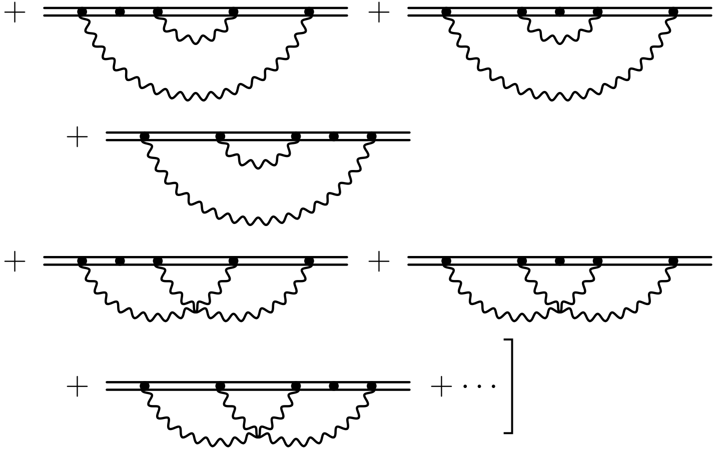


- propagators are free
- virtual-photon vertices contain $e_{\text{os}}(\lambda)$
- external-photon vertices contain $\mu(\lambda)$

The electron lines attached to the external-photon vertex may be considered on-shell and the polarizations physical
This vertex can be written via 2 form factors
We select the magnetic-moment structure $i\sigma^{\mu\nu}q_\nu/(2M)$
It contains q; hence q may be neglected everywhere else

$$Z_\psi^{\mathrm{os}}(\lambda)\Gamma(\lambda') = \mu(\lambda)$$





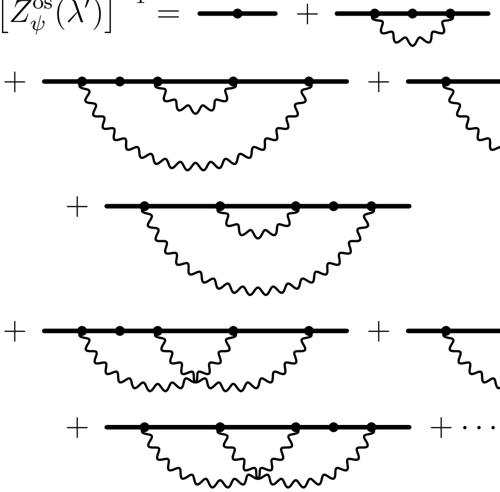
We also need $Z_\psi^{\rm os}(\lambda')$

The on-shell electron wave-function renormalization is

$$Z_\psi^{\rm os} = \frac{1}{1-\Sigma_0'(0)} \qquad \Sigma_0(\omega) = \frac{1}{4}\,{\rm Tr}(1+\not{v})\Sigma((M+\omega)v)$$

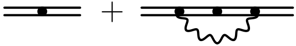
Essential contributions to $\left[Z_\psi^{\rm os}(\lambda')\right]^{-1}$

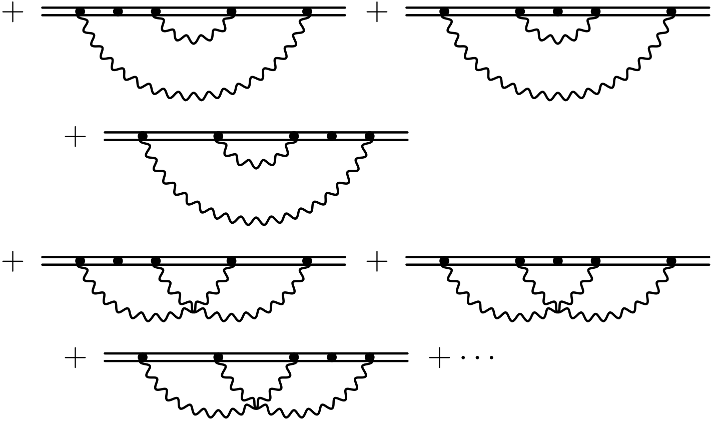
$\left[Z_\psi^{\rm os}(\lambda')\right]^{-1} =$



Let's multiply it by $Z_\psi^{\text{os}}(\lambda)$

$$Z_\psi^{\text{os}}(\lambda)\left[Z_\psi^{\text{os}}(\lambda')\right]^{-1} =$$





Electron magnetic moment is IR finite

(propagators are free, vertices contain $e_{\rm os}(\lambda)$)

$$\mu(\lambda') = \frac{Z_\psi^{\rm os}(\lambda)\Gamma(\lambda')}{Z_\psi^{\rm os}(\lambda)\left[Z_\psi^{\rm os}(\lambda')\right]^{-1}} = \mu(\lambda)$$

Scattering in external field

QED

$$e_{\rm os}\,\bar{u}'_v(p')\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M} + (1+F_2(0))\,\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u_v(p)$$

Scattering in external field

QED

$$e_{\rm os}\,\bar{u}'_v(p')\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M} + (1+F_2(0))\,\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u_v(p)$$

HEET No loop corrections (no scale)

$$e'_0\,\bar{u}'_v(p')\left[v^\mu + \frac{C_k^0}{2M}(p+p')^\mu_\perp + \frac{C_m^0}{2M}i\sigma^{\mu\nu}q_\nu\right]u_v(p)$$

Scattering in external field

QED

$$e_{\mathrm{os}}\,\bar{u}'_v(p')\left[v^\mu + \frac{(p+p')_\perp^\mu}{2M} + (1+F_2(0))\,\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u_v(p)$$

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$$C_k^0 = 1 \qquad C_m^0 = 1 + F_2(0)$$

C_m^0 is finite at $\varepsilon \to 0 \Rightarrow Z_m(\mu) = 1$, $\gamma_m = 0$
$C_m(\mu) = C_m^0 = 1 + F_2(0)$ is the full electron magnetic moment (in Bohr magnetons)
contains all orders in α, not fixed by reparametrization invariance

$1/M^2$ corrections

$$L = L_0 + \frac{1}{2M} C_k^0 O_k^0 + \frac{1}{2M} C_m^0 O_m^0 + \frac{1}{4M^2} C_s^0 O_s^0 + \frac{1}{4M^2} C_d^0 O_d^0$$

$$O_s^0 = -\frac{i}{2} e_0' h_0^+ \left(\vec{D} \times \vec{E}_0 - \vec{E}_0 \times \vec{D} \right) \cdot \vec{\sigma} h_0$$

$$= -\frac{i}{2} e_0' \bar{h}_{v0} \left[D_\perp^\mu, F_0^{\lambda\nu} \right]_+ v_\lambda \sigma_{\mu\nu} h_{v0} \, ,$$

$$O_d^0 = \frac{1}{2} e_0' h_0^+ \left(\vec{D} \cdot \vec{E}_0 - \vec{E}_0 \cdot \vec{D} \right) h_0$$

$$= \frac{1}{2} e_0' \bar{h}_{v0} v^\mu \left[D_\perp^\nu, F_{0\mu\nu} \right] h_{v0}$$

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$$L = L_0 + \frac{1}{2M} C_k^0 O_k^0 + \frac{1}{2M} C_m^0 O_m^0 + \frac{1}{4M^2} C_s^0 O_s^0 + \frac{1}{4M^2} C_d^0 O_d^0$$

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$$O_d^0 = \frac{1}{2} e_0' h_0^+ \left(\vec{D} \cdot \vec{E}_0 - \vec{E}_0 \cdot \vec{D} \right) h_0$$

$$= \frac{1}{2} e_0' \bar{h}_{v0} v^\mu \left[D_\perp^\nu, F_{0\mu\nu} \right] h_{v0}$$

The scattering amplitude (no loop corrections)

$$e_0' \, \bar{u}_v'(p') \Bigg[v^\mu + C_k^0 \frac{(p+p')_\perp^\mu}{2M} + C_m^0 \frac{[\not{q}, \gamma^\mu]}{4M}$$

$$+ C_d^0 \frac{q^2}{8M^2} v^\mu + C_s^0 \frac{[\not{p}, \not{p}']}{8M^2} v^\mu \Bigg] u_v(p)$$

Foldy–Wouthuysen transformation

$P = Mv + p$ on shell $(p \cdot v = -p^2/(2M))$
A QED Dirac spinor $u(P)$ is related to the corresponding HEET spinor $u_v(p)$ by the Foldy–Wouthuysen transformation
The 2-component spinor $u_v(p)$ is just the Dirac spinor in the P rest frame, and hence the Foldy–Wouthuysen transformation is simply the boost to the v rest frame

$$u(P) = c\left(1 + \frac{\not{p}}{2M}\right) u_v(p)$$

c: the particle density in the v rest frame $\bar{u}\not{v}u$ is $\bar{u}_v u_v$

$$c = \left[\left(1 - \frac{p^2}{2M^2}\right)\left(1 - \frac{p^2}{4M^2}\right)\right]^{-1/2} = 1 + \frac{3}{8}\frac{p^2}{M^2} + \cdots$$

QED scattering amplitude

$$e_{\rm os}\,\bar{u}'_v(p')\Big\{F_1(q^2)\left[v^\mu+\frac{(p+p')^\mu_\perp}{2M}-\frac{q^2+[\not p,\not p']}{8M^2}\right]$$
$$+\big(F_1(q^2)+F_2(q^2)\big)\left[\frac{[\not q,\gamma^\mu]}{4M}+\frac{q^2+[\not p,\not p']}{4M^2}\right]\Big\}u_v(p)$$

QED scattering amplitude

$$e_{\rm os}\,\bar{u}'_v(p')\Bigg\{F_1(q^2)\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M} - \frac{q^2+[\not{p},\not{p}']}{8M^2}\right] + \big(F_1(q^2)+F_2(q^2)\big)\left[\frac{[\not{q},\gamma^\mu]}{4M} + \frac{q^2+[\not{p},\not{p}']}{4M^2}\right]\Bigg\}u_v(p)$$

$$C_k^0 = 1 \qquad C_m^0 = 1 + F_2(0)$$

$$C_s^0 = 1 + 2F_2(0) \qquad C_d^0 = 1 + 2F_2(0) + 8F_1'(0)$$

QED scattering amplitude

$$e_{\mathrm{os}}\,\bar{u}'_v(p')\Big\{F_1(q^2)\left[v^\mu + \frac{(p+p')^\mu_\perp}{2M} - \frac{q^2+[\not p,\not p']}{8M^2}\right] + \big(F_1(q^2)+F_2(q^2)\big)\left[\frac{[\not q,\gamma^\mu]}{4M} + \frac{q^2+[\not p,\not p']}{4M^2}\right]\Big\}u_v(p)$$

$$C_k^0 = 1 \qquad C_m^0 = 1 + F_2(0)$$
$$C_s^0 = 1 + 2F_2(0) \qquad C_d^0 = 1 + 2F_2(0) + 8F_1'(0)$$

Reparametrization invariance

$$C_k = 1 \qquad C_s = 2C_m - 1$$

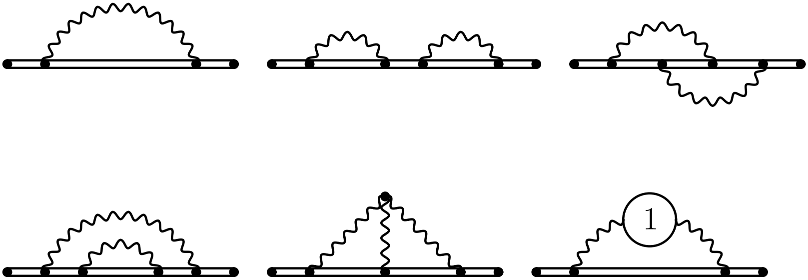
Qualitative explanation

Spin–orbit interaction: a moving electron in an electric field

$$C_s = 2C_m - 1$$

- In the electron rest frame there is magnetic field (Lorentz transformation), and the electron magnetic moment C_m interacts with it.
- Kinematical effect — Thomas precession, no radiative corrections. If we neglect corrections to C_m, it compensates $1/2$ of the first term.

HQET propagator



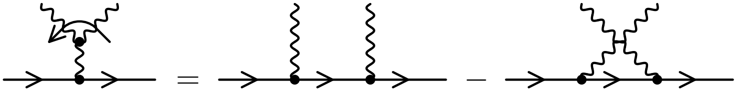
Non-abelian exponentiation

Unlike the abelian case

$$
\begin{aligned}
S(t) = -i\theta(t)\exp\Bigg[& C_F \frac{g_0^2}{(4\pi)^{d/2}} \left(\frac{it}{2}\right)^{2\varepsilon} S_F \\
& + C_F \frac{g_0^4}{(4\pi)^{d}} \left(\frac{it}{2}\right)^{4\varepsilon} (C_A S_{FA} + T_F n_l S_{Fl}) \\
& + C_F \frac{g_0^6}{(4\pi)^{3d/2}} \left(\frac{it}{2}\right)^{6\varepsilon} \Big(C_A^2 S_{FAA} \\
& \qquad + C_F T_F n_l S_{FFl} + C_A T_F n_l S_{FAl} + (T_F n_l)^2 S_{Fll} \Big) \\
& + \cdots \Bigg]
\end{aligned}
$$

Non-abelian exponentiation

If the colour factors of 3 diagrams were the same as that of the one-particle-reducible diagram, i. e. equal to the square of the colour factor C_F of the one-loop diagram (as in the abelian case), then the sum of these diagrams would be equal to $\frac{1}{2}$ of the square of the one-loop correction S_F. However, the colour factor of 1 diagram differs from C_F^2 by $-C_F C_A/2$, which is the colour factor the diagram with a 3-gluon vertex:

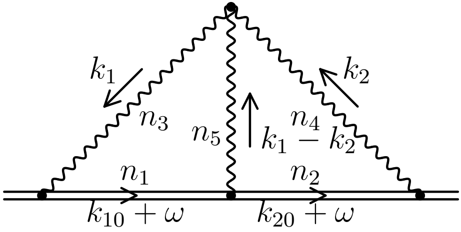


$$[t^a, t^b] = i f^{abc} t^c$$

Non-abelian exponentiation

We should include this contribution with $-C_F C_A/2$ instead of its full colour factor into the term S_{FA} (maximally non-abelian or colour-connected part). Of course, the diagram with 3-gluon vertex also contributes to S_{FA}. The diagrams with the one-loop gluon self-energy contribute to S_{Fl} (quark loop) and S_{FA} (gluon and ghost loops).

Diagram 1



Symmetry $1 \leftrightarrow 2$, $3 \leftrightarrow 4$

0 if two adjacent indices ≤ 0

$$\frac{1}{(i\pi^{d/2})^2} \int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} =$$

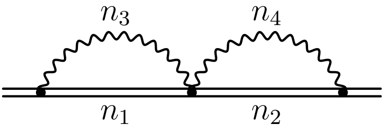
$$I(n_1, n_2, n_3, n_4, n_5)(-2\omega)^{2d - n_1 - n_2 - 2(n_3 + n_4 + n_5)}$$

$$D_1 = -2(k_{10} + \omega) \qquad D_2 = -2(k_{20} + \omega)$$

$$D_3 = -k_1^2 \qquad D_4 = -k_2^2 \qquad D_5 = -(k_1 - k_2)^2$$

Trivial case $n_5 = 0$

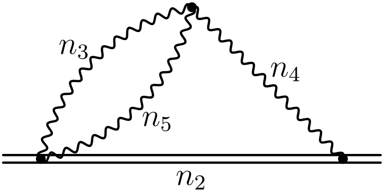
$$I(n_1, n_2, n_3, n_4, 0) =$$

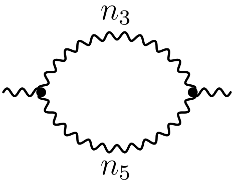


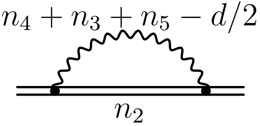
$$= I(n_1, n_3) I(n_2, n_4)$$

Trivial case $n_1 = 0$

Inner loop $G(n_3, n_5)(-p^2)^{d/2-n_3-n_5}$

$I(0, n_2, n_3, n_4, n_5) =$ 

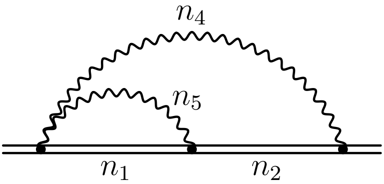
$=$ 

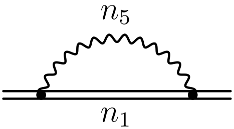
$\times$ 

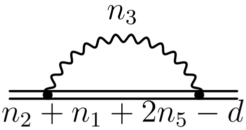
$= G(n_3, n_5) I(n_2, n_4 + n_3 + n_5 - d/2)$

Trivial case $n_3 = 0$

Inner loop $I(n_1, n_5)(-2\omega)^{d-n_1-2n_5}$

$I(n_1, n_2, 0, n_4, n_5) =$ 

$=$ 

$\times$ 

$= I(n_1, n_5) I(n_2 + n_1 + 2n_5 - d, n_4)$

Integration by parts

When applied to the integrand

$$\frac{\partial}{\partial k_2} \to \frac{n_2}{D_2} 2v + \frac{n_4}{D_4} 2k_2 + \frac{n_5}{D_5} 2(k_2 - k_1)$$

Integration by parts

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$$\frac{\partial}{\partial k_2} \to \frac{n_2}{D_2} 2v + \frac{n_4}{D_4} 2k_2 + \frac{n_5}{D_5} 2(k_2 - k_1)$$

Applying $(\partial / \partial k_2) \cdot k_2$, $(\partial / \partial k_2) \cdot (k_2 - k_1)$ and using $2k_2 \cdot v = -D_2 - 2\omega$, $2(k_2 - k_1) \cdot k_2 = D_3 - D_4 - D_5$:

$$d - n_2 - n_5 - 2n_4 - 2\omega \frac{n_2}{D_2} + \frac{n_5}{D_5}(D_3 - D_4)$$

$$d - n_2 - n_4 - 2n_5 + \frac{n_2}{D_2} D_1 + \frac{n_4}{D_4}(D_3 - D_5)$$

Integration by parts

$$\left[d - n_2 - n_5 - 2n_4 + n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{4}^-)\right] I = 0$$

$$\left[d - n_2 - n_4 - 2n_5 + n_2\mathbf{2}^+\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)\right] I = 0$$

Integration by parts

$$\left[d - n_2 - n_5 - 2n_4 + n_2\mathbf{2}^+ + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{4}^-)\right] I = 0$$
$$\left[d - n_2 - n_4 - 2n_5 + n_2\mathbf{2}^+\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)\right] I = 0$$

Applying $(\partial/\partial k_2) \cdot v$

$$\left[-2n_2\mathbf{2}^+ + n_4\mathbf{4}^+(\mathbf{2}^- - 1) + n_5\mathbf{5}^+(\mathbf{2}^- - \mathbf{1}^-)\right] I = 0$$

Homogeneity

Applying $\omega(d/d\omega)$ to the integral

$$\left[2(d-n_3-n_4-n_5)-n_1-n_2+n_1\mathbf{1}^+ + n_2\mathbf{2}^+\right] I = 0$$

This is the sum of the $(\partial/\partial k_2)\cdot k_2$ relation
and the symmetric $(\partial/\partial k_1)\cdot k_1$ one

Homogeneity

Applying $\omega(d/d\omega)$ to the integral

$$\left[2(d-n_3-n_4-n_5)-n_1-n_2+n_1\mathbf{1}^+ + n_2\mathbf{2}^+\right] I = 0$$

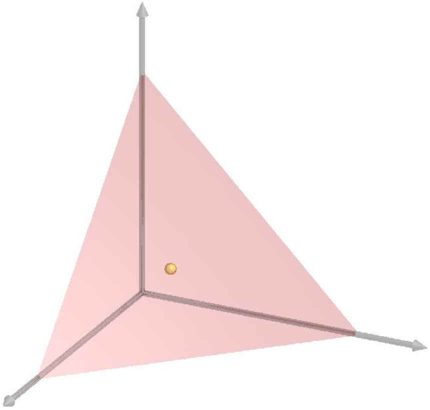
This is the sum of the $(\partial/\partial k_2)\cdot k_2$ relation
and the symmetric $(\partial/\partial k_1)\cdot k_1$ one
The $(\partial/\partial k_2)\cdot(k_2-k_1)$ relation minus $\mathbf{1}^-$ times the homogeneity relation:

$$\Big[d-n_1-n_2-n_4-2n_5+1 \\ -\big(2(d-n_3-n_4-n_5)-n_1-n_2+1\big)\mathbf{1}^- \\ +n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)\Big] I = 0$$

Integration by parts

$$I = \frac{(2(d-n_3-n_4-n_5)-n_1-n_2+1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-)}{d-n_1-n_2-n_4-2n_5+1} I$$

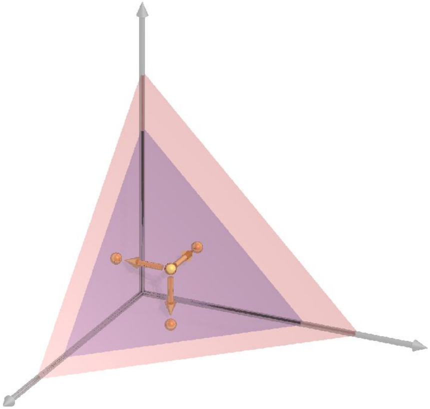
$n_1 + n_3 + n_5$ reduces by 1



Integration by parts

$$I = \frac{(2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-)}{d - n_1 - n_2 - n_4 - 2n_5 + 1} I$$

$n_1 + n_3 + n_5$ reduces by 1



Integration by parts

$$I = \frac{(2(d-n_3-n_4-n_5)-n_1-n_2+1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-)}{d-n_1-n_2-n_4-2n_5+1} I$$

$n_1+n_3+n_5$ reduces by 1

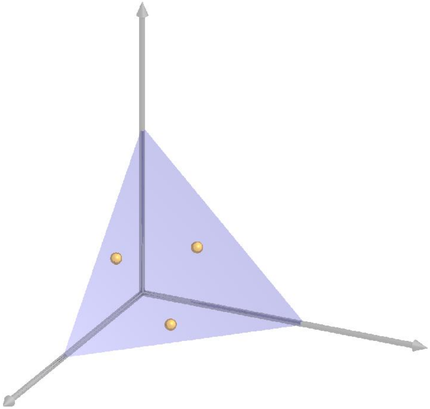
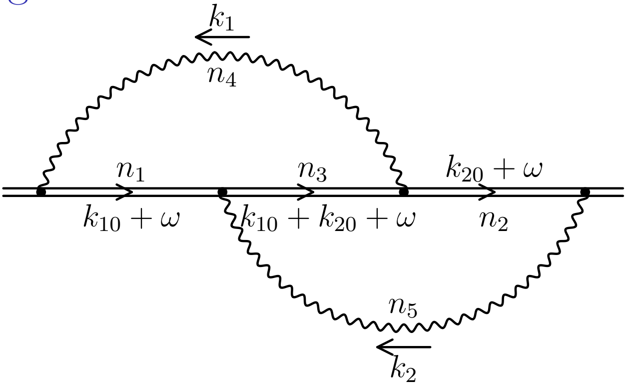


Diagram 2



Symmetry

$1 \leftrightarrow 2$, $4 \leftrightarrow 5$

0 if $n_4 \le 0$

or $n_5 \le 0$

or 2 adjacent

$n_{1\ldots3} \le 0$

$$\frac{1}{(i\pi^{d/2})^2}\int\frac{d^dk_1\,d^dk_2}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}} =$$

$$J(n_1,n_2,n_3,n_4,n_5)(-2\omega)^{2d-n_1-n_2-n_3-2(n_4+n_5)}$$

$$D_1 = -2(k_{10}+\omega) \qquad D_2 = -2(k_{20}+\omega)$$

$$D_3 = -2(k_{10}+k_{20}+\omega) \qquad D_4 = -k_1^2 \qquad D_5 = -k_2^2$$

Partial fractioning

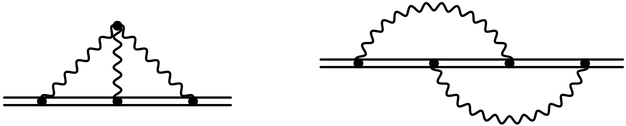
Trivial cases: $n_3 = 0$, $n_{1,2} = 0$
Denominators are linearly dependent:

$$D_1 + D_2 - D_3 = -2\omega$$
$$J = (\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^-)J$$

$n_1 + n_2 + n_3$ reduces by 1
Numerator $(k_1 \cdot k_2)^n$ — not a problem

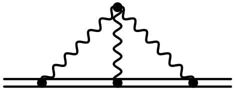
2 loops: summary

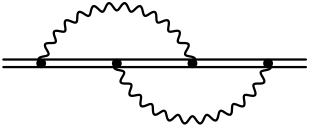
2 generic topologies (for all integer n_i)



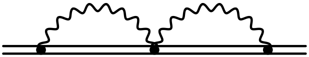
2 loops: summary

2 generic topologies (for all integer n_i)

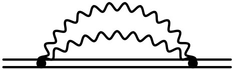




Basis (all $n_i = 1$)



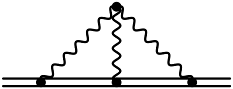
$= I_1^2$

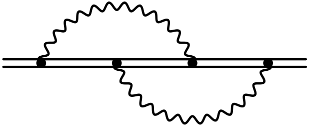


$= I_2$

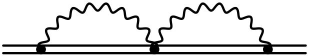
2 loops: summary

2 generic topologies (for all integer n_i)

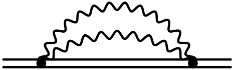




Basis (all $n_i = 1$)

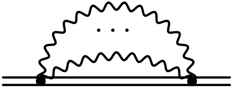


$= I_1^2$



$= I_2$

Sunset



$$= I_n = \frac{\Gamma(1+2n\varepsilon)\Gamma^n(1-\varepsilon)}{(1-n(d-2))_{2n}}$$

HQET self-energy

$$\begin{aligned}
\Sigma(\omega) &= -C_F \frac{g_0^2(-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}}(d-3)I_1 A \\
&+ C_F \frac{g_0^4(-2\omega)^{1-4\varepsilon}}{(4\pi)^d}\left\{-16\frac{(d-2)(2d-5)}{(d-4)(d-6)}I_2 P\right. \\
&\quad - C_F \frac{4(d-3)^2(2d-5)}{d-4} I_2 A^2 \\
&\quad + \left(C_F - \frac{C_A}{2}\right) 2(d-3)\left[(d-3)I_1^2 - 2(2d-5)I_2\right] A^2 \\
&\quad \left. - C_A(d-3)\left[(d-3)I_1^2 + 2\frac{2d-5}{d-4}I_2\right] A(1-a_0)\right\}
\end{aligned}$$

HQET self-energy

$$A = a_0 - 1 - \frac{2}{d-3} \qquad \xi = 1 - a_0$$

$$P = T_F n_l - \frac{3d - 2 + (d-1)(2d-7)\xi - \frac{1}{4}(d-1)(d-4)\xi^2}{4(d-2)} C_A$$

HQET propagator

$$
\begin{aligned}
\omega S(\omega) = 1 &+ C_F \frac{g_0^2(-2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}} 2(d-3) I_1 A \\
&+ C_F \frac{g_0^4(-2\omega)^{-4\varepsilon}}{(4\pi)^d} \Bigg\{ 32 \frac{(d-2)(2d-5)}{(d-4)(d-6)} T_F n_l I_2 \\
&\quad + 8 \frac{(d-3)(2d-5)(2d-7)}{d-4} A^2 C_F I_2 \\
&\quad - 4(d-3) A C_A I_1^2 \\
&\quad + 8 \frac{(2d-5)(2d-7)}{(d-3)(d-4)(d-6)} \Bigg[\frac{(d-2)^2(d-5)}{(d-3)(2d-7)} \\
&\qquad + (d^2-4d+5)A - \frac{1}{4}(d-3)(d^2-9d+16)A^2 \Bigg] C_A I_2 \Bigg\}
\end{aligned}
$$

HQET propagator

$$\begin{aligned}
S(t) = {} & -i\theta(t)\exp\Bigg\{C_F\frac{g_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon}\Gamma(-\varepsilon)A \\
& + C_F\frac{g_0^4}{(4\pi)^{d}}\left(\frac{it}{2}\right)^{4\varepsilon}\Gamma^2(-\varepsilon)\Bigg[2\frac{d-2}{(d-3)(d-6)(2d-7)}T_F n_l \\
& \quad + \frac{1}{2(d-3)^2(d-6)}\Bigg(\frac{(d-2)^2(d-5)}{(d-3)(2d-7)} \\
& \qquad + (d^2-4d+5)A - \frac{1}{4}(d-3)(d^2-9d+16)A^2\Bigg)C_A \\
& \quad - \frac{A}{d-3}\frac{\Gamma^2(1+2\varepsilon)}{\Gamma(1+4\varepsilon)}C_A\Bigg]\Bigg\}
\end{aligned}$$

Anomalous dimension

Re-expressing the propagator $S(t)$ via $\alpha_s(\mu)$, $a(\mu)$ — still has the exponential form with the same colour structures
Same for Z_h

$$\gamma_h = 2C_F(a-3)\frac{\alpha_s}{4\pi} + C_F\left[C_A\left(\frac{a^2}{2} + 4a - \frac{179}{6}\right) + \frac{32}{3}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2$$

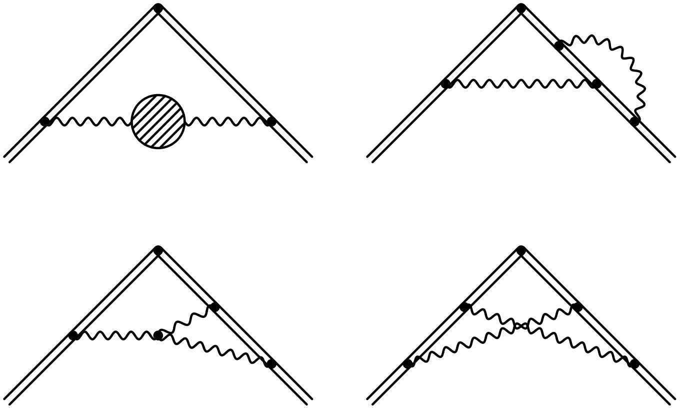
Heavy–heavy currents

The Green function with a heavy–heavy current insertion (Wilson line with an angle)

$$G(t, t'; \vartheta) = \exp\left[C_F \frac{g_0^2}{(4\pi)^{d/2}} F + C_F \frac{g_0^4}{(4\pi)^d} \left(C_A F_A + T_F n_l F_l\right) + \cdots\right]$$

Z_J similar
No C_F^2 in $\Gamma(\vartheta)$ at 2 loops

Heavy–heavy currents



Heavy–heavy currents

$$\Gamma(\vartheta) = (\vartheta\coth\vartheta - 1)C_F\frac{\alpha_s}{\pi} + \Bigg\{ -\frac{5}{9}(\vartheta\coth\vartheta - 1)T_F n_l$$

$$+ \Bigg[-\frac{1}{2}\coth^2\vartheta\Big(\zeta_3 - \zeta_2\vartheta - \vartheta\,\mathrm{Li}_2\left(e^{-2\vartheta}\right) - \mathrm{Li}_3\left(e^{-2\vartheta}\right)\Big)$$

$$- \frac{1}{2}\coth\vartheta\Bigg(\zeta_2 + \left(2\zeta_2 - \frac{67}{18}\right)\vartheta + \vartheta^2 + \frac{\vartheta^3}{3}$$

$$+ 2\vartheta\log\left(1 - e^{-2\vartheta}\right) - \mathrm{Li}_2\left(e^{-2\vartheta}\right)\Bigg)$$

$$+ \zeta_2 - \frac{49}{36} + \frac{\vartheta^2}{2}\Bigg] C_A \Bigg\} C_F\left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$$

Limiting cases

$\vartheta \to 0$

$$\Gamma_0 = \frac{4}{3} C_F \frac{\alpha_s}{4\pi} + C_F \left[C_A \left(\frac{376}{27} - \frac{8}{9}\pi^2 \right) - \frac{80}{27} T_F n_l \right] \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots$$

Limiting cases

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$$\Gamma_0 = \frac{4}{3} C_F \frac{\alpha_s}{4\pi} + C_F \left[C_A \left(\frac{376}{27} - \frac{8}{9}\pi^2 \right) - \frac{80}{27} T_F n_l \right] \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots$$

$\vartheta \to \infty$

$$\Gamma_\infty = 4 C_F \frac{\alpha_s}{4\pi} + C_F \left[C_A \left(\frac{268}{9} - \frac{4}{3}\pi^2 \right) - \frac{80}{9} T_F n_l \right] \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots$$

Related to the asymptotics of the evolution kernel $P_{qq}(z)$ at $z \to 1$:

$$P_{qq}(z) = \Gamma_\infty \left(\frac{1}{1-z} \right)_+ + C\delta(1-z) + \mathcal{O}((1-z)^0)$$

known to 3 loops

The asymptotics of $V_{qq}(x,y)$ at $x - y \to 0$ is also governed by Γ_∞

Limiting cases

$\vartheta \to 0$

$$\Gamma_0 = \frac{4}{3} C_F \frac{\alpha_s}{4\pi} + C_F \left[C_A \left(\frac{376}{27} - \frac{8}{9}\pi^2 \right) - \frac{80}{27} T_F n_l \right] \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots$$

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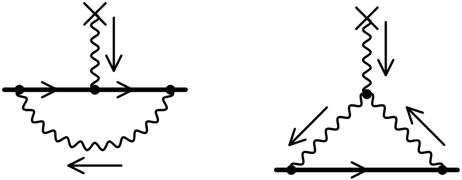
known to 3 loops

The asymptotics of $V_{qq}(x,y)$ at $x - y \to 0$ is also governed by Γ_∞

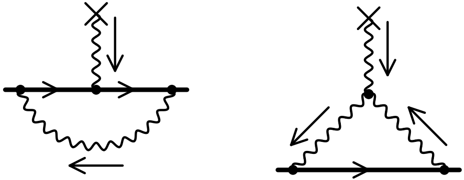
The imaginary part of $\Gamma(\delta - i\pi)$ at $\delta \to 0$ is determined by the quark–antiquark potential (known to 3 loops)

Chromomagnetic interaction



$$F_2(0) = \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(\varepsilon)}{2(d-3)} \left[2(d-4)(d-5)C_F - (d^2 - 8d + 14)C_A\right]$$

Chromomagnetic interaction



$$F_2(0) = \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(\varepsilon)}{2(d-3)} \left[2(d-4)(d-5)C_F - (d^2 - 8d + 14)C_A\right]$$

$$C_m^0 = 1 + F_2(0)$$

$$C_m(\mu) Z_m^{-1}(\alpha_s(\mu)) = 1 + \frac{\alpha_s(\mu)}{4\pi} e^{-2L\varepsilon} \left[2C_F + \left(\frac{1}{\varepsilon} + 2\right) C_A\right]$$

$$L = \log \frac{M}{\mu}$$

Chromomagnetic interaction

$$\gamma_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A \left(17C_A - 13T_F n_l\right) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

Chromomagnetic interaction

$$\gamma_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A \left(17C_A - 13T_F n_l\right) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$M_{B^*} - M_B = \frac{2}{3M_b} C_m(\mu) \mu_G^2(\mu) + \mathcal{O}\left(\frac{1}{M_b^2}\right)$$

$$\mu_G^2(\mu) = \hat{\mu}_G^2 \alpha_s(\mu)^{\gamma_{m0}/(2\beta_0)} \left[1 + \mathcal{O}(\alpha_s)\right]$$

Chromomagnetic interaction

$$\gamma_m = 2C_A \frac{\alpha_s}{4\pi} + \frac{4}{9} C_A \left(17 C_A - 13 T_F n_l\right) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$

$$M_{B^*} - M_B = \frac{2}{3M_b} C_m(\mu) \mu_G^2(\mu) + \mathcal{O}\left(\frac{1}{M_b^2}\right)$$

$$\mu_G^2(\mu) = \hat{\mu}_G^2 \alpha_s(\mu)^{\gamma_{m0}/(2\beta_0)} \left[1 + \mathcal{O}(\alpha_s)\right]$$

$$\frac{M_{B^*}^2 - M_B^2}{M_{D^*}^2 - M_D^2} = \left(\frac{\alpha_s(M_b)}{\alpha_s(M_c)}\right)^{\gamma_{m0}/(2\beta_0)} \left[1 + \mathcal{O}\left(\alpha_s, \frac{\Lambda_{\text{QCD}}}{M_{c,b}}\right)\right]$$

Electron propagator near the mass shell

On-shell mass $M = M_0 + \delta M$, $\omega \ll M$

$$p = (M + \omega)v \qquad \Sigma(p) = \Sigma_0(\omega) + \Sigma_1(\omega)(\not{v} - 1)$$

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$$\begin{aligned} S(p) &= \frac{1}{\not{p} - M_0 - \Sigma(p)} \\ &= \frac{1}{[M + \omega - \Sigma_1(\omega)]\not{v} - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)} \end{aligned}$$

Electron propagator near the mass shell

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$$S(p) = \frac{1}{\rlap{/}p - M_0 - \Sigma(p)}$$
$$= \frac{1}{[M+\omega-\Sigma_1(\omega)]\,\rlap{/}v - M + \delta M - \Sigma_0(\omega) + \Sigma_1(\omega)}$$

The denomunator

$$[M+\omega-\Sigma_1(\omega)]^2 - [M - \delta M + \Sigma_0(\omega) - \Sigma_1(\omega)]^2$$

should vanish at $\omega = 0$:

$$\delta M = \Sigma_0(0)$$

Electron propagator near the mass shell

$$S(p) = \frac{1}{[M + \omega - \Sigma_1(\omega)]\,\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M + \omega - \Sigma_1(\omega)]\,\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M + \omega - \Sigma_1(\omega)]^2 - [M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)]^2}$$

Electron propagator near the mass shell

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$$= \frac{[M+\omega-\Sigma_1(\omega)]\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M+\omega-\Sigma_1(\omega)]^2 - [M+\Sigma_0(\omega)-\Sigma_0(0)-\Sigma_1(\omega)]^2}$$

The denominator at $\omega \to 0$

$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

Electron propagator near the mass shell

$$S(p) = \frac{1}{[M + \omega - \Sigma_1(\omega)]\, \not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M + \omega - \Sigma_1(\omega)]\, \not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M + \omega - \Sigma_1(\omega)]^2 - [M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)]^2}$$

The denominator at $\omega \to 0$

$$[M - \Sigma_1(0) + \omega - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

Electron propagator near the mass shell

$$S(p) = \frac{1}{[M + \omega - \Sigma_1(\omega)]\,\not{v} - M - \Sigma_0(\omega) + \Sigma_0(0) + \Sigma_1(\omega)}$$
$$= \frac{[M + \omega - \Sigma_1(\omega)]\,\not{v} + M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)}{[M + \omega - \Sigma_1(\omega)]^2 - [M + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega)]^2}$$

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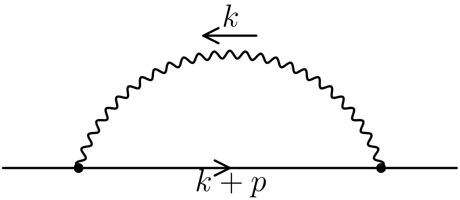
$$- [M - \Sigma_1(0) + \Sigma_0(\omega) - \Sigma_0(0) - \Sigma_1(\omega) + \Sigma_1(0)]^2$$
$$\approx 2\,(M - \Sigma_1(0))\,[\omega - \Sigma_0(\omega) + \Sigma_0(0)]$$

The numerator at $\omega \to 0$

$$(M - \Sigma_1(0))\,(\not{v} + 1)$$

$$S(p) \approx \frac{\not{v} + 1}{2} \frac{1}{\omega - \Sigma_0(\omega) + \Sigma_0(0)}$$

Electron self-energy



$$p = (M+\omega)v$$
$$D_1 = M^2 - (k+p)^2$$
$$D_2 = -k^2$$

$$\Sigma_0(\omega) = \frac{1}{4}\operatorname{Tr}(\not{v}+1)\Sigma(p) = -ie_0^2 \int \frac{d^d k}{(2\pi)^d}\frac{1}{2D_1D_2}$$
$$\left[(d+2)M - (d-2)\omega - (d-2)\frac{D_2+M^2}{M+\omega} + \frac{\xi\omega^2}{D_2}\frac{D_2+4M\omega+\omega^2}{M+\omega}\right]$$

Hard contribution $k \sim M$

$$D_1 = D_h - (D_2 - D_h + 2M^2)\frac{\omega}{M} - \omega^2$$
$$D_h = M^2 - (k + Mv)^2$$

$D_h \sim M^2$, $D_2 \sim M^2$; Taylor series in ω; single scale M

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$D_h \sim M^2$, $D_2 \sim M^2$; Taylor series in ω; single scale M

$$\Sigma_h(\omega) = \frac{e_0^2 M^{1-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} \left(1 - \frac{\omega}{M} + \cdots\right)$$

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$$D_1 = D_h - (D_2 - D_h + 2M^2)\frac{\omega}{M} - \omega^2$$
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On-shell mass renormalization (gauge invariant)

$$\delta M = M\left[\frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\frac{d-1}{d-3} + \cdots\right]$$

Hard contribution $k \sim M$

$$D_1 = D_h - (D_2 - D_h + 2M^2)\frac{\omega}{M} - \omega^2$$
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On-shell mass renormalization (gauge invariant)

$$\delta M = M \left[\frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} + \cdots\right]$$

On-shell wave-function renormalization
(gauge invariant in QED)

$$Z_\psi^{\mathrm{os}} = \frac{1}{1 - \Sigma_0'(0)} = 1 - \frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{d-1}{d-3} + \cdots$$

Soft contribution $k \sim \omega$

$$D_1 = M D_s - (k + \omega v)^2 \qquad D_s = -2(k \cdot v + \omega)$$

$D_s \sim \omega$, $D_2 \sim \omega^2$; Taylor series in $1/M$; single scale ω

Soft contribution $k \sim \omega$

$$D_1 = M D_s - (k + \omega v)^2 \qquad D_s = -2(k \cdot v + \omega)$$

$D_s \sim \omega$, $D_2 \sim \omega^2$; Taylor series in $1/M$; single scale ω

$$\Sigma_s(\omega) = \Sigma(\omega) \left(1 + \mathcal{O}\left(\frac{\omega}{M}\right)\right)$$
$$\Sigma(\omega) = \frac{e_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(\xi + \frac{2}{d-3}\right)$$

$\Sigma(\omega)$ — HEET self-energy

Electron propagator in QED and HEET

$$S(p) = \frac{1+\not{v}}{2} \frac{1}{\omega - \Sigma_h'(0)\omega - \Sigma_s(\omega)} = z_0 S(\omega)$$

$$z_0 = Z_\psi^{\text{os}} = \frac{1}{1 - \Sigma_h'(0)}$$

$$S(\omega) = \frac{1+\not{v}}{2} \frac{1}{\omega - \Sigma(\omega)}$$

$S(\omega)$ — HEET propagator

Electron propagator in QED and HEET

$$S(p) = \frac{1+\rlap/v}{2} \frac{1}{\omega - \Sigma_h'(0)\omega - \Sigma_s(\omega)} = z_0 S(\omega)$$

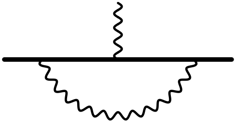
$$z_0 = Z_\psi^{\text{os}} = \frac{1}{1 - \Sigma_h'(0)}$$

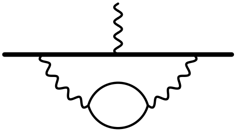
$$S(\omega) = \frac{1+\rlap/v}{2} \frac{1}{\omega - \Sigma(\omega)}$$

$S(\omega)$ — HEET propagator

- Higher terms in $\Sigma_h \Rightarrow$ corrections to ψ_0 via h_{v0}
- Higher terms in $\Sigma_s \Rightarrow$ corrections to $S(\omega)$ due to $1/M$ terms in the HEET Lagrangian

Muon magnetic moment





Form factors

$$F_1(q^2) = \frac{Z_\psi^{\text{os}}}{2(d-2)(1+t)^2} \times \frac{1}{4}\,\text{Tr}\big[(d-1)v_\mu - (1+t)\gamma_\mu\big](\not{v}'+1)\Gamma^\mu(\not{v}+1)$$

$$F_2(q^2) = \frac{Z_\psi^{\text{os}}}{2(d-2)t(1+t)^2} \times \frac{1}{4}\,\text{Tr}\big[\big(1-(d-2)t\big)v_\mu - (1+t)\gamma_\mu\big](\not{v}'+1)\Gamma^\mu(\not{v}+1)$$

$$t = -\frac{q^2}{4M^2}$$

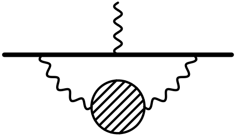
Form factors

$$F_1(0) = 1$$

$$F_2(0) = \frac{Z_\psi^{\text{os}}}{d-2}\left[\frac{1}{4}\,\text{Tr}(\gamma_\mu - d v_\mu)\Gamma_0^\mu(\rlap{/}v + 1) + \frac{2}{d-1}\frac{1}{4}\,\text{Tr}\left(\gamma_\mu\gamma_\nu + \gamma_\mu v_\nu - \gamma_\nu v_\mu - v_\mu v_\nu\right)\Gamma_1^{\mu\nu}(\rlap{/}v + 1)\right]$$

$$\Gamma^\mu(Mv, Mv+q) = \Gamma_0^\mu + \Gamma_1^{\mu\nu}\frac{q_\nu}{M} + \cdots$$

Muon magnetic moment

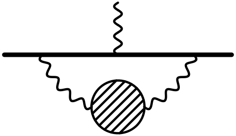


$$D_1 = M^2 - (Mv + k)^2$$

$$D_2 = -k^2$$

$$\mu = -i\frac{e_0^2}{d-1}\int\frac{d^dk}{(2\pi)^d}\Pi(k^2)\Bigg[\frac{16}{d-2}\frac{M^2}{D_1^3} - 4\frac{d-3}{d-2}\frac{D_2}{D_1^3} - \frac{D_2^2}{M^2D_1^3}$$
$$- 2\frac{2d^2-9d+13}{(d-2)D_1^2} - \frac{(d+2)(d-3)D_2}{2M^2D_1^2}$$
$$+ 2\frac{d^2-4d+5}{(d-2)D_1D_2} + \frac{d^2-d-3}{M^2D_1} - \frac{(d+1)(d-2)}{2M^2D_2}\Bigg]$$

Muon magnetic moment



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$$- 2\frac{2d^2-9d+13}{(d-2)D_1^2} - \frac{(d+2)(d-3)D_2}{2M^2D_1^2}$$
$$\left. + 2\frac{d^2-4d+5}{(d-2)D_1D_2} + \frac{d^2-d-3}{M^2D_1} - \frac{(d+1)(d-2)}{2M^2D_2}\right]$$

$\Pi(k^2) = 1 \Rightarrow 1$ loop

$$\mu_0 = -2\frac{d-5}{d-3}\frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(1+\varepsilon)$$

Hard contribution $k \sim M$

$D_1 \sim D_2 \sim M^2$; Taylor series in m^2; single scale M

$$\Pi(k^2) = -2\frac{d-2}{d-1}\frac{e_0^2(-k^2)^{-\varepsilon}}{(4\pi)^{d/2}}G_1\left[1+\mathcal{O}\left(\frac{m^2}{k^2}\right)\right]$$

$$G_1 = -\frac{2}{(d-3)(d-4)}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

Hard contribution $k \sim M$

$D_1 \sim D_2 \sim M^2$; Taylor series in m^2; single scale M

$$\Pi(k^2) = -2\frac{d-2}{d-1}\frac{e_0^2(-k^2)^{-\varepsilon}}{(4\pi)^{d/2}}G_1\left[1+\mathcal{O}\left(\frac{m^2}{k^2}\right)\right]$$

$$G_1 = -\frac{2}{(d-3)(d-4)}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\frac{\mu_h}{\mu_0} = 32\frac{(d-2)(d^2-7d+11)}{(d-1)(d-4)(d-5)(3d-8)(3d-10)}\frac{e_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(1+\varepsilon)R$$

$$R = \frac{\Gamma(1+2\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(1-4\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1-3\varepsilon)} = 1+\mathcal{O}(\varepsilon^2)$$

Hard contribution $k \sim M$

Re-expressing via renormalized $\alpha(\mu)$:

$$\mu_0 + \mu_h = \frac{\alpha(M)}{2\pi}\left[1 - \frac{25}{18}\frac{\alpha}{\pi}\right]$$

$$\alpha(M) = \alpha(m)\left(1 + \frac{2}{3}\frac{\alpha}{\pi}\log\frac{M}{m}\right)$$

$$\mu_0 + \mu_h = \frac{\alpha}{2\pi}\left[1 + \frac{2}{3}\frac{\alpha}{\pi}\left(\log\frac{M}{m} - \frac{25}{12}\right)\right]$$

Soft contribution $k \sim m$

$$D_1 = M D_s \qquad D_s = -2k \cdot v$$

$D_s \sim m$, $D_2 \sim m^2$; Taylor series in $1/M$; single scale m

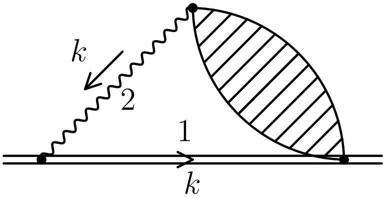
Soft contribution $k \sim m$

$$D_1 = M D_s \qquad D_s = -2k \cdot v$$

$D_s \sim m$, $D_2 \sim m^2$; Taylor series in $1/M$; single scale m

$$\mu_s = \frac{-2ie_0^2}{(d-1)(d-2)M} \int \frac{d^d k}{(2\pi)^d} \Pi(k^2) \left[\frac{8}{D_s^3} + \frac{d^2 - 4d + 5}{D_s D_2} \right]$$

On-shell HQET diagrams with mass

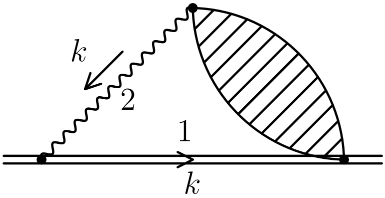


$$F(n_1,n_2) = \int \frac{\Pi(k^2)\, d^d k}{D_1^{n_1} D_2^{n_2}}$$

$$D_1 = -2k \cdot v - i0$$

$$D_2 = -k^2 - i0$$

On-shell HQET diagrams with mass



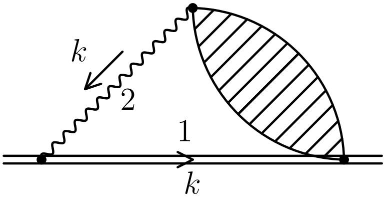
$$F(n_1, n_2) = \int \frac{\Pi(k^2)\, d^d k}{D_1^{n_1} D_2^{n_2}}$$

$$D_1 = -2k \cdot v - i0$$

$$D_2 = -k^2 - i0$$

$$\frac{\partial}{\partial k} \cdot \left(k - 2\frac{D_2}{D_1} v \right) \frac{\Pi(k^2)}{D_1^{n_1} D_2^{n_2}} = \left[d - n_1 - 2 - 4(n_1 + 1)\frac{D_2}{D_1^2} \right] \frac{\Pi(k^2)}{D_1^{n_1} D_2^{n_2}}$$

On-shell HQET diagrams with mass



$$F(n_1,n_2) = \int \frac{\Pi(k^2)\,d^dk}{D_1^{n_1} D_2^{n_2}}$$

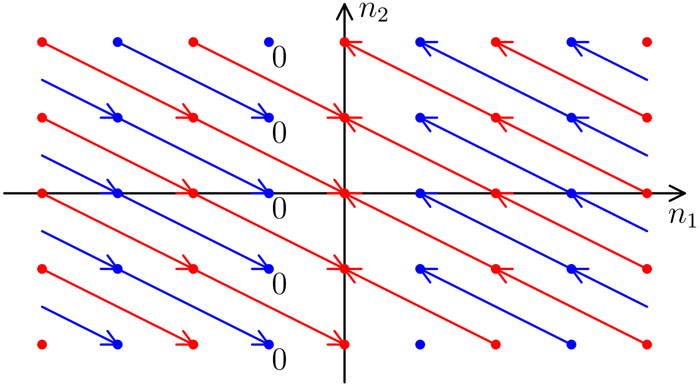
$$D_1 = -2k\cdot v - i0$$

$$D_2 = -k^2 - i0$$

$$\frac{\partial}{\partial k}\cdot\left(k - 2\frac{D_2}{D_1}v\right)\frac{\Pi(k^2)}{D_1^{n_1}D_2^{n_2}} = \left[d - n_1 - 2 - 4(n_1+1)\frac{D_2}{D_1^2}\right]\frac{\Pi(k^2)}{D_1^{n_1}D_2^{n_2}}$$

$$(d - n_1 - 2)F(n_1,n_2) = 4(n_1+1)\mathbf{1}^{++}\mathbf{2}^{-}F(n_1,n_2)$$

On-shell HQET diagrams with mass



On-shell HQET diagrams with mass

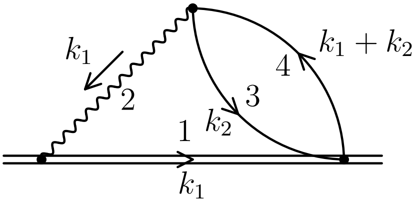
$$F(n_1,n_2)=\begin{cases}(-4)^{-n_1/2}\dfrac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-n_1}{2}\right)}\dfrac{\Gamma\left(\frac{1-n_1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}F\left(0,n_2+\dfrac{n_1}{2}\right) & \text{even } n_1\\ 2^{1-n_1}\dfrac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma\left(\frac{d-n_1}{2}\right)}F\left(1,n_2+\dfrac{n_1-1}{2}\right) & \text{odd } n_1>0\\ 0 & \text{odd } n_1<0\end{cases}$$

On-shell HQET diagrams with mass

$$F(n_1,n_2)=\begin{cases} (-4)^{-n_1/2}\dfrac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-n_1}{2}\right)}\dfrac{\Gamma\left(\frac{1-n_1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}F\left(0,n_2+\dfrac{n_1}{2}\right) & \text{even } n_1\\ 2^{1-n_1}\dfrac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma\left(\frac{d-n_1}{2}\right)}F\left(1,n_2+\dfrac{n_1-1}{2}\right) & \text{odd } n_1>0\\ 0 & \text{odd } n_1<0 \end{cases}$$

$n_1<0$: $i0\Rightarrow 0$ in $D_1^{-n_1}$, averaging over k directions

2 loops



Symmetry $3 \leftrightarrow 4$

0 if $n_3 \leq 0$ or $n_4 \leq 0$

$$F(n_1, n_2, n_3, n_4) = \frac{1}{(i\pi^{d/2})^2} \int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4}}$$

$$D_1 = -2k_1 \cdot v - i0 \qquad D_2 = -k_1^2 - i0$$

$$D_3 = 1 - k_2^2 - i0 \qquad D_4 = 1 - (k_1 + k_2)^2 - i0$$

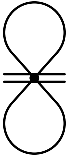
$$F(n_1,n_2,n_3,n_4) = \frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{d-n_1}{2}-n_2\right)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)} \times$$

$$\frac{\Gamma\left(\frac{n_1-d}{2}+n_2+n_3\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_4\right)\Gamma\left(\frac{n_1}{2}+n_2+n_3+n_4-d\right)}{\Gamma\left(\frac{d-n_1}{2}\right)\Gamma(n_1+2n_2+n_3+n_4-d)}$$

$$F(n_1,n_2,n_3,n_4) = \frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{d-n_1}{2}-n_2\right)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)} \times$$

$$\frac{\Gamma\left(\frac{n_1-d}{2}+n_2+n_3\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_4\right)\Gamma\left(\frac{n_1}{2}+n_2+n_3+n_4-d\right)}{\Gamma\left(\frac{d-n_1}{2}\right)\Gamma(n_1+2n_2+n_3+n_4-d)}$$

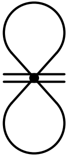
Apparently even $\Rightarrow F(0, n_2+n_1/2, n_3, n_4)$ (vacuum)

$I_0^2 =$ 

$$F(n_1,n_2,n_3,n_4) = \frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{d-n_1}{2}-n_2\right)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)} \times$$

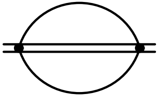
$$\frac{\Gamma\left(\frac{n_1-d}{2}+n_2+n_3\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_4\right)\Gamma\left(\frac{n_1}{2}+n_2+n_3+n_4-d\right)}{\Gamma\left(\frac{d-n_1}{2}\right)\Gamma(n_1+2n_2+n_3+n_4-d)}$$

Apparently even $\Rightarrow F(0, n_2+n_1/2, n_3, n_4)$ (vacuum)

$I_0^2 =$ 

Apparently odd

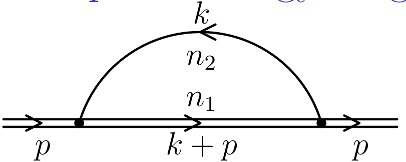
- $n_1 \le 0 \Rightarrow 0$
- $n_1 > 0 \Rightarrow F(1, n_2+(n_1-1)/2, n_3, n_4)$

$J_0 =$  $= 2^{4d-9}\pi^2 \frac{\Gamma(5-2d)}{\Gamma^2(2-d/2)}$

Soft contribution $k \sim m$

$$\mu_s = \frac{-2ie_0^2}{M} \int \frac{d^d k}{(2\pi)^d} \frac{\Pi(k^2)}{D_s D_2} = \frac{\alpha^2}{4} \frac{m}{M}$$

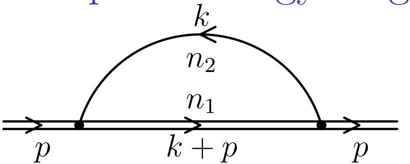
One-loop self-energy diagram with mass



$$I_{n_1 n_2}(m, p_0) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D_1^{n_1} D_2^{n_2}}$$

$$D_1 = -2(k+p)_0 - i0 \qquad D_2 = m^2 - k^2 - i0$$

One-loop self-energy diagram with mass



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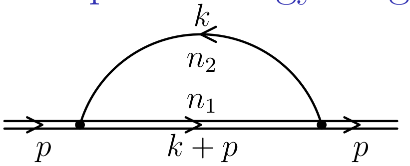
Cut from the threshold $\omega = m$ to $+\infty$
Integer $n_2 \le 0$: vanishes (HEET loop)
Integer $n_1 \le 0$: vacuum diagram, e. g,

$$I_{0n}(m, \omega) = m^{d-2n} V(n)$$

One-loop self-energy diagram with mass



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Integer $n_2 \leq 0$: vanishes (HEET loop)
Integer $n_1 \leq 0$: vacuum diagram, e. g,

$$I_{0n}(m, \omega) = m^{d-2n} V(n)$$

$$\lim_{m \to 0} I_{n_1 n_2}(m, \omega) = I(n_1, n_2)(-2\omega)^{d-2n_1-n_2} \qquad \text{if} \qquad n_2 < \frac{d}{2}$$

HQET Feynman parametrization

$$I_{n_1 n_2}(m,\omega) = \frac{\Gamma\left(n_1 + n_2 - \frac{d}{2}\right)}{\Gamma(n_1)\Gamma(n_2)} \int\limits_0^\infty y^{n_1-1}(y^2-2\omega y+m^2)^{d/2-n_1-n_2}\,dy$$

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$$I_{n_1 n_2}(m,0) = I_0(n_1,n_2) m^{d-n_1-2n_2}$$

$$I_0(n_1,n_2) = \frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_1-d}{2}+n_2\right)}{2\Gamma(n_1)\Gamma(n_2)} = \frac{\pi^{1/2}}{2^{n_1}} \frac{\Gamma\left(\frac{n_1-d}{2}+n_2\right)}{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma(n_2)}$$

vanishes at odd negative integer n_1 — odd in k

Result

$\omega < 0$

$$I_{n_1 n_2}(m,\omega) = m^{d-n_1-2n_2} \frac{\Gamma\left(n_1+n_2-\frac{d}{2}\right)\Gamma(n_1+2n_2-d)}{\Gamma(n_2)\Gamma(2(n_1+n_2)-d)} \times {}_2F_1\left(\begin{array}{c} \frac{n_1}{2}, \frac{n_1-d}{2}+n_2 \\ n_1+n_2-\frac{d-1}{2} \end{array} \middle| 1-\frac{\omega^2}{m^2}\right)$$

The point $\omega = 0$ is regular; when we go from a small $\omega < 0$ to $\omega > 0$ along some path in the complex plane, we make a full cycle around the branch point of the hypergeometric function, and arrive at another Riemann sheet.

$\omega \ll m$

One integration region $k \sim m$
Expand $D_1^{-n_1}$ in ω

$$I_{n_1 n_2}(m,\omega) = m^{d-n_1-2n_2} \sum_{n=0}^{\infty} I_0(n_1+n, n_2) \frac{(n_1)_n}{n!} \left(\frac{2\omega}{m}\right)^n$$

$$= m^{d-n_1-2n_2} I_0(n_1,n_2) \left[{}_2F_1\left(\begin{array}{c} \frac{n_1}{2}, \frac{n_1-d}{2}+n_2 \\ \frac{1}{2} \end{array} \middle| \frac{\omega^2}{m^2} \right) \right.$$

$$\left. + \frac{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma\left(\frac{n_1-d+1}{2}+n_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_1-d}{2}+n_2\right)} \frac{2\omega}{m} \, {}_2F_1\left(\begin{array}{c} \frac{n_1+1}{2}, \frac{n_1-d+1}{2}+n_2 \\ \frac{3}{2} \end{array} \middle| \frac{\omega^2}{m^2} \right) \right]$$

Regular Taylor series in ω,
power of m — dimension counting

$\omega \gg m$

OPE

- hard $k \sim \omega$ — the (1-loop) coefficient function of the unit operator
- soft $k \sim m$ — the series of perturbative (1-loop) vacuum averages of local operators (with $2n$ derivatives) accompanied by their tree-level coefficient functions

$$I_{n_1 n_2}(m, \omega) = I_h + I_s$$

Hard $k \sim \omega$

Expand $D_2^{-n_2}$ in m^2

$$I_h = (-2\omega)^{d-n_1-2n_2} \sum_{n=0}^{\infty} I(n_1, n_2+n) \frac{(n_2)_n}{n!} \left(-\frac{m^2}{4\omega^2} \right)^n$$
$$= (-2\omega)^{d-n_1-2n_2} I(n_1, n_2)\, {}_2F_1 \left(\begin{array}{c} \frac{n_1-d}{2} + n_2, \frac{n_1-d+1}{2} + n_2 \\ n_2 + 1 - \frac{d}{2} \end{array} \middle| \frac{m^2}{\omega^2} \right)$$

Regular Taylor series in m^2,
powers of -2ω — dimension counting

Soft $k \sim m$

Expand $D_1^{-n_1}$ in k (all odd terms vanish)

$$
\begin{aligned}
I_s &= m^{d-2n_2}(-2\omega)^{-n_1} \sum_{n=0}^{\infty} I_0(-2n, n_2) \frac{(n_1)_{2n}}{(2n)!} \left(\frac{m^2}{4\omega^2} \right)^n \\
&= m^{d-2n_2}(-2\omega)^{-n_1} V(n_2)\, {}_2F_1 \left(\left. \begin{array}{c} \frac{n_1}{2}, \frac{n_1+1}{2} \\ \frac{d}{2} - n_2 + 1 \end{array} \right| \frac{m^2}{\omega^2} \right)
\end{aligned}
$$

Regular Taylor series in ω (after extracting $(-2\omega)^{-n_1}$), powers of m — dimension counting

Soft $k \sim m$

Expand $D_1^{-n_1}$ in k (all odd terms vanish)

$$
\begin{aligned}
I_s &= m^{d-2n_2}(-2\omega)^{-n_1} \sum_{n=0}^{\infty} I_0(-2n, n_2) \frac{(n_1)_{2n}}{(2n)!} \left(\frac{m^2}{4\omega^2}\right)^n \\
&= m^{d-2n_2}(-2\omega)^{-n_1} V(n_2)\, {}_2F_1\left(\left. \begin{array}{c} \frac{n_1}{2}, \frac{n_1+1}{2} \\ \frac{d}{2} - n_2 + 1 \end{array} \right| \frac{m^2}{\omega^2} \right)
\end{aligned}
$$

Regular Taylor series in ω (after extracting $(-2\omega)^{-n_1}$), powers of m — dimension counting

The leading term in I_h dominates over the leading term in I_s at $m \to 0$ if $n_2 < d/2$

Mellin–Barnes

$$\frac{1}{(a+b)^n} = \frac{a^{-n}}{\Gamma(n)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \, \Gamma(-z)\Gamma(n+z) \left(\frac{b}{a}\right)^z$$

- all poles of $\Gamma(\cdots + z)$ are to the left of the contour
- all poles of $\Gamma(\cdots - z)$ are to the right of it

Mellin–Barnes

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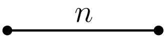
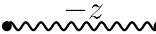
- all poles of $\Gamma(\cdots + z)$ are to the left of the contour
- all poles of $\Gamma(\cdots - z)$ are to the right of it
- closing the contour to the right — the expansion in b/a
- closing it to the left — the expansion in a/b

Massive propagator via massless one

$$\frac{1}{(m^2-p^2)^n} = \frac{m^{-2n}}{\Gamma(n)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(-z)\Gamma(n+z)\left(\frac{-p^2}{m^2}\right)^z$$



$$= \frac{m^{-2n}}{\Gamma(n)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(-z)\Gamma(n+z)m^{-2z}$$



Massive diagram via massless one

$$I_{n_1 n_2}(m,\omega) = \frac{m^{-2n_2}(-2\omega)^{d-n_1}}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz$$

$$\Gamma(n_1 - d - 2z)\Gamma\left(\tfrac{d}{2} + z\right)\Gamma(n_2 + z)\left(\frac{-2\omega}{m}\right)^{2z}$$

Massive diagram via massless one

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$$\Gamma(n_1 - d - 2z)\Gamma\left(\tfrac{d}{2} + z\right)\Gamma(n_2 + z)\left(\frac{-2\omega}{m}\right)^{2z}$$

- Close the contour to the right: the sum over residues of the right poles — the expansion in ω/m.
  1 series of right poles $z_n = (n + n_1 - d)/2$ ($n = 0, 1, 2\ldots$) — the small ω result
- Close the contour to the left: the sum over residues of the left poles — the expansion in m/ω (analytical continuation to large ω)
  2 series of left poles: $z_n^h = -n - n_2$ and $z_n^s = -n - \frac{d}{2}$ — I_h and I_s

Another derivation at tree level

$$\psi(x) = e^{-iMv\cdot x}\left(h_v(x) + H_v(x)\right)$$

$$h_v(x) = e^{iMv\cdot x}\frac{1+\not{v}}{2}\psi(x) \qquad H_v(x) = e^{iMv\cdot x}\frac{1-\not{v}}{2}\psi(x)$$

$$\not{v}h_v(x) = h_v(x) \qquad \not{v}H_v(x) = -H_v(x)$$

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$$\begin{aligned} L &= \bar{\psi}\left(i\not{D} - M\right)\psi \\ &= \bar{h}_v iv\cdot D h_v + \bar{H}_v\left(-iv\cdot D - 2M\right)H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v \end{aligned}$$

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Solution of the equation of motion

$$H_v = \frac{1}{2M + iv\cdot D}\not{D}_\perp h_v = \frac{1}{2M}i\not{D}_\perp h_v - \frac{iv\cdot D}{(2M)^2}i\not{D}_\perp h_v + \cdots$$

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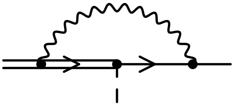
$$L = \bar{h}_v\left[iv\cdot D - \frac{D_\perp^2}{2M} + \frac{eF_{\mu\nu}\sigma^{\mu\nu}}{4M} + \cdots\right]h_v$$

HQET heavy–light current



$$\tilde{\jmath}_0 = \bar{q}_0 \varphi_0 = \tilde{Z}_j \tilde{\jmath}(\mu)$$

$$\tilde{\Lambda} = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu \not{k} v^\nu \left[g_{\mu\nu} - (1-a_0) k_\mu k_\nu / k^2\right]}{(k^2)^2 k_0}$$

$$= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_0 \not{k} - (1-a_0) k_0}{(k^2)^2 k_0}$$

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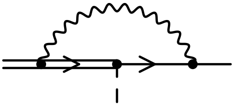

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$$= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_0 \not{k} - (1-a_0) k_0}{(k^2)^2 k_0}$$

$$\not{k} = k_0 \gamma_0 - \vec{k} \cdot \vec{\gamma}$$

$$\tilde{\Lambda} = -iC_F g_0^2 a_0 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2}$$

$$\tilde{Z}_\Gamma = 1 + C_F a \frac{\alpha_s}{4\pi\varepsilon} \qquad \tilde{Z}_j = Z_q^{1/2} Z_\varphi^{1/2} \tilde{Z}_\Gamma = 1 + \frac{3}{2} C_F \frac{\alpha_s}{4\pi\varepsilon}$$

HQET heavy–light current

$$\tilde{\gamma}_j = -3C_F\frac{\alpha_s}{4\pi}$$
$$+ C_F\left[\left(-\frac{8}{3}\pi^2 + \frac{5}{2}\right)C_F + \left(\frac{2}{3}\pi^2 - \frac{49}{6}\right)C_A + \frac{10}{3}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2$$
$$+ \cdots$$

QCD/HQET matching

$$j(\mu) = C_\Gamma(\mu, \mu')\tilde{\jmath}(\mu') + \frac{1}{2M}\sum_i B_i^\Gamma(\mu, \mu')\tilde{O}_i(\mu') + \mathcal{O}\left(\frac{1}{M^2}\right)$$

Matrix elements of $j(\mu)$, in situations amenable to HQET treatment, after expansion to a given order in $1/M$, coincide with the corresponding matrix elements of the right-hand side of this equation.

QCD/HQET matching

Let's consider the decay of a heavy quark into a light quark with energy $\omega \ll M$ via a heavy–light weak current. The matrix element in QCD depends on two widely separated large scales $M \gg \omega$ and the renormalization scale μ (if the current has a non-zero anomalous dimension). For no choice of μ can we get rid of large logarithmic corrections. When we go to HQET, all M-dependence is isolated in the matching coefficient of the heavy–light current C_Γ. The HQET matrix element knows nothing about M, and depends only on ω and μ', where the μ'-dependence is determined by the anomalous dimension of the HQET heavy–light current. If μ' is chosen to be of the order of ω, then there are no large logarithmic corrections.

QCD/HQET matching

On-shell matrix element of $j(\mu)$

$$M(P,p',\mu) = (Z_q^{\mathrm{os}})^{1/2} (Z_Q^{\mathrm{os}})^{1/2} Z_j^{-1}(\mu) \Gamma(P,p')$$

should be equal to

$$C_\Gamma(\mu,\mu') \tilde{M}(p,p',\mu') + \mathcal{O}((p,p')/M)$$

where

$$\tilde{M}(p,p',\mu') = (Z_q'^{\mathrm{os}})^{1/2} (\tilde{Z}_Q^{\mathrm{os}})^{1/2} \tilde{Z}_j^{-1}(\mu') \tilde{\Gamma}(p,p')$$

Both matrix elements are UV-finite; their IR divergences coincide, because HQET coincides with QCD in the IR region.

QCD/HQET matching

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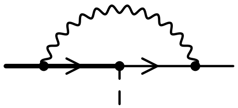
Both matrix elements are UV-finite; their IR divergences coincide, because HQET coincides with QCD in the IR region.

$$C_\Gamma(\mu,\mu') = \left(\frac{Z_q^{\text{os}}}{Z_q'^{\text{os}}}\right)^{1/2} \left(\frac{Z_Q^{\text{os}}}{\tilde{Z}_Q^{\text{os}}}\right)^{1/2} \frac{\tilde{Z}_j(\mu')}{Z_j(\mu)} \frac{\Gamma(P,p')}{\tilde{\Gamma}(p,p')} + \mathcal{O}\left(\frac{p,p'}{m}\right)$$

$p = p' = 0$

QCD/HQET matching



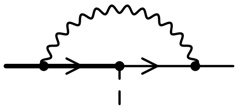
$$\Lambda = -C_F \frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \frac{(1-h)(d-2+2h)}{(d-2)(d-3)}$$

$$\not{v}\Gamma = \sigma\Gamma\not{v} \qquad \sigma = \pm 1$$

$$\gamma^\mu \Gamma \gamma_\mu = 2\sigma h(d)\Gamma$$

$$h(d) = \eta\left(n - \frac{d}{2}\right) \qquad \eta = (-1)^{n+1}\sigma$$

QCD/HQET matching



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$$\gamma^\mu \Gamma \gamma_\mu = 2\sigma h(d) \Gamma$$

$$h(d) = \eta \left(n - \frac{d}{2} \right) \qquad \eta = (-1)^{n+1} \sigma$$

$$C_\Gamma(M, M) = 1 + C_F \frac{\alpha_s(M)}{4\pi} \left[3(n-2)^2 + (2-\eta)(n-2) - 4 \right]$$

f_B

$$<0|\bar{b}\gamma^\mu\gamma_5^{\mathrm{AC}} q|B(P)> = i f_B P^\mu$$

$$<B(P')|B(P)> = 2P^0(2\pi)^3\delta\left(\vec{P}' - \vec{P}\right)$$

$${}_{\mathrm{nr}}<B(P')|B(P)>_{\mathrm{nr}} = \delta\left(\vec{P}' - \vec{P}\right)$$

$$<0|\bar{b}\gamma^0\gamma_5^{\mathrm{AC}} q|B(Mv)>_{\mathrm{nr}} = iF = i\frac{M_B}{\sqrt{2M_B}} f_B$$

$$f_B = \sqrt{\frac{2}{M_B}} F(M_b)$$

$$\frac{f_B}{f_D} = \sqrt{\frac{M_c}{M_b}} \left(\frac{\alpha_s(M_b)}{\alpha_s(M_c)}\right)^{\tilde{\gamma}_{j0}/(2\beta_0)} \left[1 + \mathcal{O}\left(\alpha_s, \frac{\Lambda_{\mathrm{QCD}}}{M_{c,b}}\right)\right]$$

f_B

$$
\begin{aligned}
\frac{f_{B^*}}{f_B} &= \frac{C_{\gamma^1}(M,M)}{C_{\gamma^0}(M,M)} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{M_b}\right) \\
&= 1 - 2C_F\frac{\alpha_s(M)}{4\pi} + \mathcal{O}\left(\alpha_s^2, \frac{\Lambda_{\text{QCD}}}{M_b}\right)
\end{aligned}
$$

Larin factors

$$Z_P(\alpha_s(\mu)) = \frac{C_{\gamma_5^{\mathrm{AC}}}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}}(\mu,\mu')} = \frac{C_1(\mu,\mu')}{C_{\gamma^0\gamma^1\gamma^2\gamma^3}(\mu,\mu')}$$

$$Z_A(\alpha_s(\mu)) = \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^0}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^0}(\mu,\mu')} = \frac{C_{\gamma^0}(\mu,\mu')}{C_{\gamma^1\gamma^2\gamma^3}(\mu,\mu')}$$

$$= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^3}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^3}(\mu,\mu')} = \frac{C_{\gamma^3}(\mu,\mu')}{C_{\gamma^0\gamma^1\gamma^2}(\mu,\mu')}$$

$$Z_T(\alpha_s(\mu)) = \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^0\gamma^1}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^0\gamma^1}(\mu,\mu')} = \frac{C_{\gamma^0\gamma^1}(\mu,\mu')}{C_{\gamma^1\gamma^2}(\mu,\mu')}$$

$$= \frac{C_{\gamma_5^{\mathrm{AC}}\gamma^2\gamma^3}(\mu,\mu')}{C_{\gamma_5^{\mathrm{HV}}\gamma^2\gamma^3}(\mu,\mu')} = \frac{C_{\gamma^2\gamma^3}(\mu,\mu')}{C_{\gamma^0\gamma^1}(\mu,\mu')} = 1$$

Equation of motion

$$i\partial_\alpha j_0^\alpha = i\partial_\alpha j^\alpha = M_0 j_0 = M(\mu) j(\mu)$$

Matrix element from $Q(Mv)$ to $q(0)$

$$MC_{\gamma^0}(\mu, \mu') = M(\mu) C_1(\mu, \mu')$$

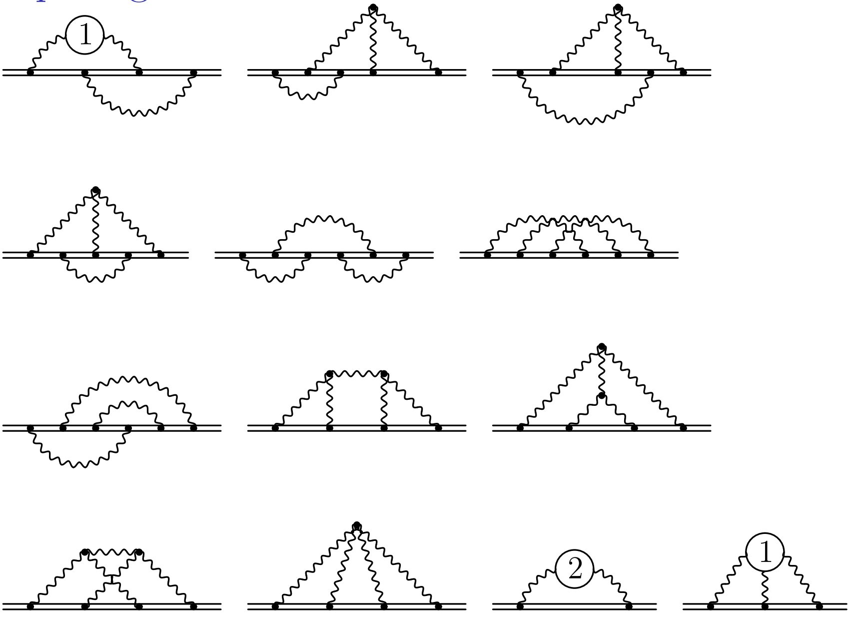
3 loops: 2 c-webs

Let's multiply the 1-loop correction and the last diagram. We can imagine that this set is obtained from the one-particle-reducible diagram by allowing the gluon – heavy-quark vertices to slide along the heavy-quark line, crossing each other. These diagrams are said to contain two connected webs. Everything is already accounted for by the product of the one-loop correction and the part of) two-loop correction in the expansion of the exponent, except the contribution of the first diagram (and its mirror-symmetric), taken with the maximally non-abelian part of its colour factor. It contributes to the three-loop correction in the exponent.

3 loops: 2 c-webs

Similarly, out of all the diagrams with two connected webs, only 3 diagrams contribute to S_{FAA}, with the maximally non-abelian part of their colour factors. This part appears, in the first case, for example, when we commute t^a matrices to obtain the colour structure of the reducible diagram; it is identical to the colour factor of the ladder diagram equal to $C_F C_A^2/4$.

3-loop diagrams



3 loops: 3 c-webs

We move the vertices along the heavy-quark line in such a way as to disentangle those c-webs. While doing so, we get extra terms from the commutators, having colour structures of the corresponding diagrams with the three-gluon vertex. These diagrams have fewer c-webs, which are more complicated. Finally, each colour factor can be expressed as a linear combination of three ones:

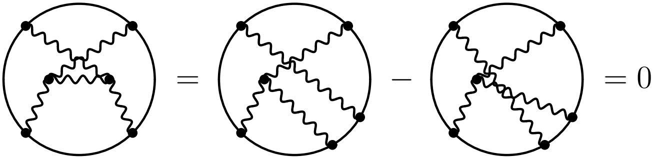
- C_F^3 (3 1-loop c-webs)
- $-C_F^2 C_A/2$ (2 c-webs: 1-loop and with 3-gluon vertex)
- $C_F C_A^2/4$ (1 c-web: the ladder diagram)

3 loops: 3 c-webs

- Occurs with the unit coefficient in all 15 colour factors. The sum of the corresponding contributions is just the term with the cube of the 1-loop correction in the expansion of the exponent.
- Occurs in the diagrams obtained by multiplying the 1-loop correction and the third diagram, the sum of the corresponding contributions is contained in the product of the one-loop term and the two-loop one in the expansion of the exponent.
- We are left with the colour-connected parts of the colour factors (a single c-web contributions). They are present in 3 diagrams, and contribute to S_{FAA}.

3 loops: 1 c-web

- 2 diagrams have equal colour factors (just close the quark line), they contribute to S_{FAA}.
- 1 diagram has 0 colour factor:



- The diagram with the four-gluon vertex can be decomposed into three terms, with colour factors of the previous 3 ones.

3 loops: 1 c-web

- 2-loop gluon self-energy corrections, including one-particle-reducible ones; it contributes to S_{FAA}, S_{FFl}, S_{FAl}, S_{Fll}.
- 1-loop corrections to the three-gluon vertex, including one-particle-reducible ones (i. e., one-loop self-energy corrections to each gluon propagator); it contributes to S_{FAA}, S_{FAl}.

Scattering in an external field

Background field $A_0^\mu \to \bar{A}_0^\mu + A_0^\mu$

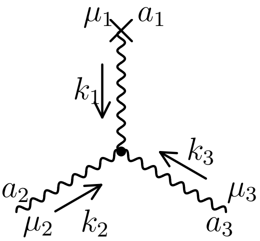
$$L = \sum_i \bar{q}_{i0}(i\not{D}_0 - m_{i0})q_{i0} - \frac{1}{4}G^a_{0\mu\nu}G_0^{a\mu\nu}$$

$$- \frac{1}{2a_0}\left(\bar{D}_\mu A_0^\mu\right)^2 + (\bar{D}_\mu \bar{c}_0^a)(D_0^\mu c_0^a)$$

Scattering in an external field

Background field $A_0^\mu \to \bar{A}_0^\mu + A_0^\mu$

$$L = \sum_i \bar{q}_{i0}(i\not{D}_0 - m_{i0})q_{i0} - \frac{1}{4}G^a_{0\mu\nu}G_0^{a\mu\nu} - \frac{1}{2a_0}\left(\bar{D}_\mu A_0^\mu\right)^2 + (\bar{D}_\mu \bar{c}_0^a)(D_0^\mu c_0^a)$$



$$g_0 f^{a_1 a_2 a_3}\Bigg[(k_2 - k_3)^{\mu_1}\, g^{\mu_2\mu_3} + \left(k_3 - k_1 + \frac{1}{a_0}k_2\right)^{\mu_2} g^{\mu_3\mu_1} + \left(k_1 - k_2 - \frac{1}{a_0}k_3\right)^{\mu_3} g^{\mu_1\mu_2}\Bigg]$$