Effective weak Lagrangians in the Standard Model and B decays

Andrey Grozin
A.G.Grozin@inp.nsk.su

Budker Institute of Nuclear Physics

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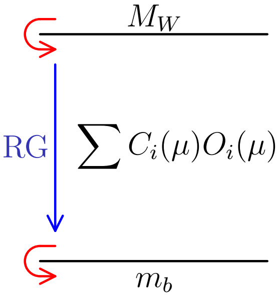
The Lagrangian contains all possible operators (allowed by symmetries). Coefficients of operators of dimension $n+4$ contain $1/M^n$. If M is much larger than energies we are interested in, we can retain only renormalizable terms (dimension 4), and, maybe, a power correction or two.

Introduction

SM



HQET, SCET

$$b \to c l^- \bar{\nu}_l$$

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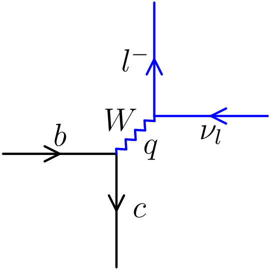
Full theory



$$M = \frac{g_2^2}{2} V_{cb} \frac{1}{M_W^2 - q^2} \times$$
$$\left(\bar{u}_{cL} \gamma^\alpha u_{bL}\right) \left(\bar{u}_{lL} \gamma_\alpha v_{\nu L}\right)$$

$$g_2 = \frac{e}{\sin\theta_W}$$

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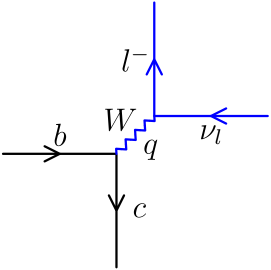

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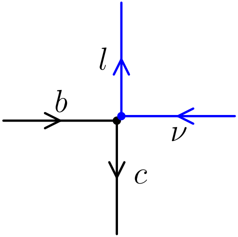
$$\left(\bar{u}_{cL}\gamma^{\alpha} u_{bL}\right)\left(\bar{u}_{lL}\gamma_{\alpha} v_{\nu L}\right)$$

$$g_2 = \frac{e}{\sin\theta_W}$$

Effective theory



$$L = \frac{g_2^2}{2M_W^2} V_{cb} (\bar{c}_L \gamma^{\alpha} b_L)\,(\bar{l}_L \gamma_{\alpha} \nu_L)$$

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Renormalization

$$O_0 = Z(\alpha_s(\mu))O(\mu) \qquad O(\mu) = Z^{-1}(\alpha_s(\mu))O_0$$

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Matrix element

$$<O_0> = Z<O>$$

α_s/ε only from Z

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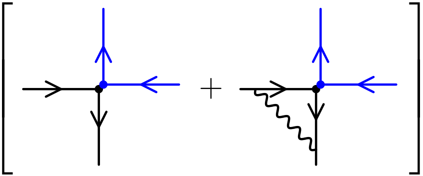
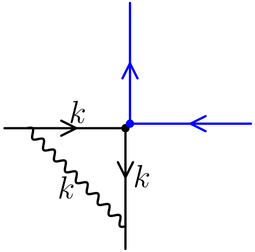
α_s/ε only from Z

$$<O_0> = Z_q$$



in α_s terms retain only $1/\varepsilon$



$$
\begin{aligned}
& -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^3} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right) \gamma^\mu \not{k} \gamma^\alpha \not{k} \gamma^\nu \otimes \gamma_\alpha \\
&= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \left[\frac{1}{d} \gamma^\mu \gamma^\lambda \gamma^\alpha \gamma_\lambda \gamma_\mu - \xi \gamma^\alpha \right] \otimes \gamma_\alpha \\
&= -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \left[\frac{(d-2)^2}{d} - \xi \right] \gamma^\alpha \otimes \gamma_\alpha
\end{aligned}
$$

IR regularization

$$\int \frac{d^d k}{(k^2)^2} \Rightarrow \int \frac{d^d k}{(k^2 - m^2)^2} \Rightarrow \frac{i}{(4\pi)^2} \frac{1}{\varepsilon}$$

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$$<O_0> = \left[1 - C_F \frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\right] \left[1 + C_F \frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\right] = 1$$

The vector current (and the axial current with anticommuting γ_5) does not renormalize
Ward identity — to all orders in α_s

$b \to c l^- \bar{\nu}_l$ to $1/M_W^4$

$$L = \frac{g_2^2}{2M_W^2} V_{cb} (\bar{c}_L \gamma^\alpha b_L) \left(1 - \frac{\partial^2}{M_W^2}\right) (\bar{l}_L \gamma_\alpha \nu_L)$$

$$M = \frac{g_2^2}{2M_W^2} V_{cb} \left(1 + \frac{q^2}{M_W^2}\right) (\bar{u}_{cL} \gamma^\alpha u_{bL}) (\bar{u}_{lL} \gamma_\alpha v_{\nu L})$$

$$b \to c d \bar{u}$$

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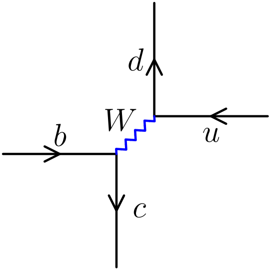

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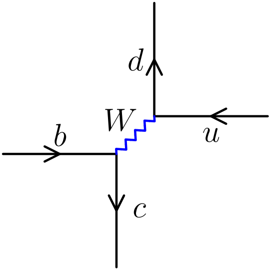

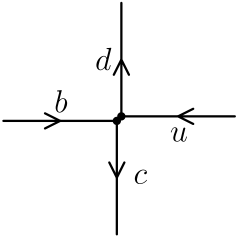

$$L = \frac{g_2^2}{2M_W^2} V_{cb} V_{ud}^* (\bar{c}_L \gamma^\alpha b_L)\,(\bar{d}_L \gamma_\alpha u_L)$$

Operators

$$O_1 = \left(\bar{c}_{Li}\gamma^{\alpha}b^{i}_{L}\right)\left(\bar{d}_{Lj}\gamma_{\alpha}u^{j}_{L}\right)$$
$$O_2 = \left(\bar{c}_{Li}\gamma^{\alpha}b^{j}_{L}\right)\left(\bar{d}_{Lj}\gamma_{\alpha}u^{i}_{L}\right)$$

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Fierz rearrangement $(d = 4)$

$(\bar{\psi}_{1L}\gamma^\alpha\psi_{2L})\,(\bar{\psi}_{3L}\gamma_\alpha\psi_{4L}) = (\bar{\psi}_{3L}\gamma^\alpha\psi_{2L})\,(\bar{\psi}_{1L}\gamma_\alpha\psi_{4L})$

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$$O_2 = (\bar{d}_{Lj}\gamma^\alpha b_L^j)\,(\bar{c}_{Li}\gamma_\alpha u_L^i)$$

$$O_2' = (\bar{c}_L t^a\gamma^\alpha b_L)\,(\bar{d}_L t^a\gamma_\alpha u_L) = T_F\left(O_2 - \frac{O_1}{N_c}\right)$$

Cvitanović algorithm

$$(t^a)^i{}_j (t^a)^k{}_l = a \left[\delta^i_l \delta^k_j - b \delta^i_j \delta^k_l \right]$$

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Multiply by δ^j_i

$$(t^a)^i{}_i (t^a)^k{}_l = 0 = a\left[\delta^k_l - bN_c \delta^k_l\right]$$

$$b = \frac{1}{N_c}$$

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Multiply by $(t^b)^j{}_i$

$$(t^b)^j{}_i (t^a)^i{}_j (t^a)^k{}_l = T_F (t^b)^k{}_l = a\left[(t^b)^k{}_l - \frac{1}{N_c}(t^b)^i{}_i \delta^k_l\right]$$

$$a = T_F$$

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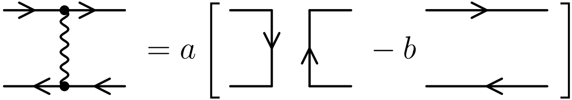
Cvitanović algorithm



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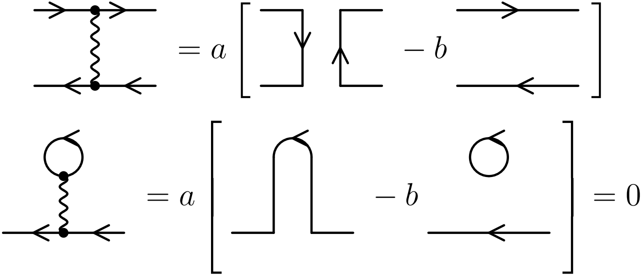

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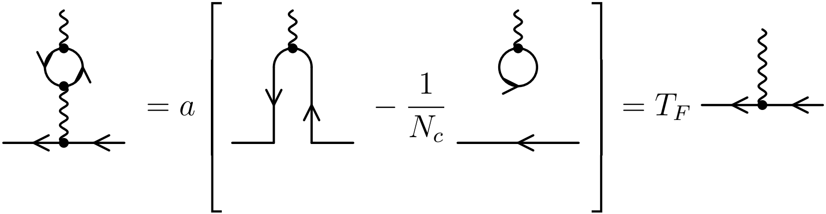
Cvitanović algorithm



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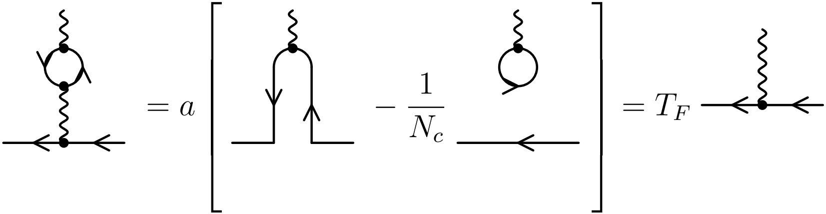

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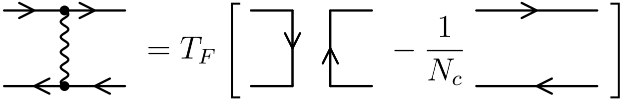

Renormalization

$$O_0 = Z(\alpha_s(\mu))O(\mu) \qquad O(\mu) = Z^{-1}(\alpha_s(\mu))O_0$$

RG equations

$$\frac{dO(\mu)}{d\log\mu} + \gamma(\alpha_s(\mu))O(\mu) = 0$$

Anomalous dimension matrix

$$\gamma = Z^{-1}\frac{dZ}{d\log\mu} = -\frac{dZ^{-1}}{d\log\mu}Z$$

Effective Lagrangian

$$L = \frac{g_2^2}{2M_W^2} V_{cb} V_{ud}^* c_0^T O_0 = \frac{g_2^2}{2M_W^2} V_{cb} V_{ud}^* c^T(\mu) O(\mu)$$

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Matrix form RG equations

$$\frac{dc(\mu)}{d \log \mu} = \gamma^T(\alpha_s(\mu)) c(\mu)$$

Solving RG equations

$$\frac{dc}{d\log\alpha_s} = -\frac{\gamma^T(\alpha_s)}{2\beta(\alpha_s)}c$$

$$\beta(\alpha_s) = \beta_0\frac{\alpha_s}{4\pi} + \cdots \qquad \gamma^T(\alpha_s) = \gamma_0^T\frac{\alpha_s}{4\pi} + \cdots$$

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1 loop — solution with a matrix exponent

$$c(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\gamma_0^T}{2\beta_0}} c(M_W)$$

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1 loop — solution with a matrix exponent

$$c(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\gamma_0^T}{2\beta_0}} c(M_W)$$

If eigenvectors v_i of γ_0^T ($\gamma_0^T v_i = \lambda_i v_i$) form a full basis, then

$$c(\mu) = \sum A_i \left(\frac{\alpha_s(\mu)}{\alpha_s(M)}\right)^{-\frac{\lambda_i}{2\beta_0}} v_i$$

where $c(M_W) = \sum A_i v_i$.

Initial conditions

$c_i(\mu_0)$ are determined by matching — equating some S-matrix elements in the full theory (expanded in p_i/M_W) and in the effective theory. It is most convenient to use $\mu_0 \sim M_W$; then $c_i(\mu_0)$ are given by perturbative series in $\alpha_s(\mu_0)$ containing no large logarithms. They contain all the information about physics at the scale M_W which is important for low-energy processes.

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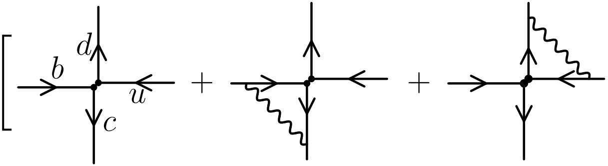
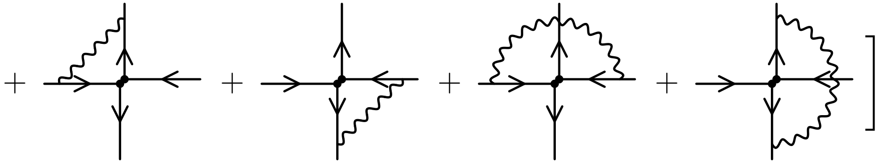
The Wilson coefficients $c_i(\mu)$ at low normalization scales μ are obtained by solving the RG equations. The effective theory knows nothing about M_W; the only information about it is contained in $c_i(\mu)$. When the effective Lagrangian is applied to some physical process with small momenta $p_i \ll M$, it is most convenient to use μ of the order of the characteristic momenta: then the results will contain no large logarithms. This solution of the RG equation sums large logarithmic terms in perturbation series.

1-loop anomalous dimensions

Matrix element

$$\langle O_1^0 \rangle = Z_q^2$$





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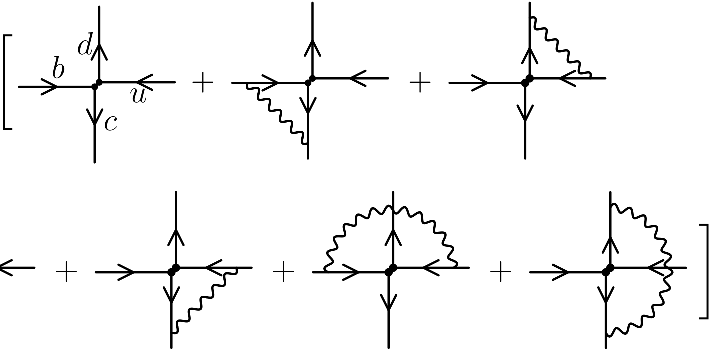
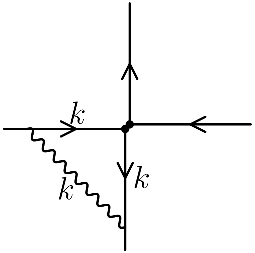
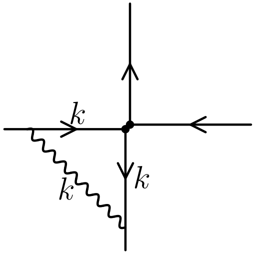


Color structures

$$T_1 = \delta_b^c \delta_u^d \qquad T_2 = \delta_b^d \delta_u^c$$



$$C_F T_1 \frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\gamma^\alpha \otimes \gamma_\alpha$$



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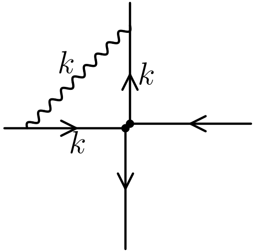
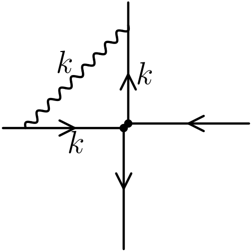


Fierz rearrangement $(d = 4)$

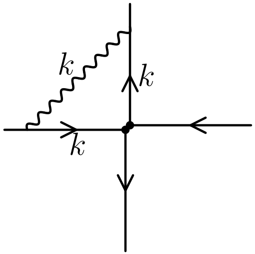
$$T_F \left(T_2 - \frac{T_1}{N_c} \right) \frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\gamma^\alpha \otimes \gamma_\alpha$$



$$T_1 = \delta_b^c \delta_u^d$$

$$T_2 = \delta_b^d \delta_u^c$$

$$T_F \left(T_2 - \frac{T_1}{N_c} \right) \frac{\alpha_s}{4\pi\varepsilon} \left[\frac{1}{d} \gamma^\alpha \gamma^\lambda \gamma^\mu \otimes \gamma_\mu \gamma_\lambda \gamma_\alpha - \xi \gamma^\alpha \otimes \gamma_\alpha \right]$$



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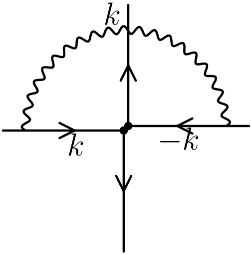
$$T_F \left(T_2 - \frac{T_1}{N_c} \right) \frac{\alpha_s}{4\pi\varepsilon} \left[\frac{1}{d} \gamma^\alpha \gamma^\lambda \gamma^\mu \otimes \gamma_\mu \gamma_\lambda \gamma_\alpha - \xi \gamma^\alpha \otimes \gamma_\alpha \right]$$

Fierz $d = 4$

$$\gamma^\alpha \gamma^\lambda \gamma^\mu \otimes \gamma_\mu \gamma_\lambda \gamma_\alpha = \gamma_\mu \gamma_\lambda \gamma^\alpha \gamma^\lambda \gamma^\mu \otimes \gamma_\alpha = 4 \gamma^\alpha \otimes \gamma_\alpha$$

$$T_F \left(T_2 - \frac{T_1}{N_c} \right) \frac{\alpha_s}{4\pi\varepsilon} (1 - \xi) \gamma^\alpha \otimes \gamma_\alpha$$



$$-T_F\left(T_2-\frac{T_1}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}\left[\frac{1}{d}\gamma^\alpha\gamma^\lambda\gamma^\mu\otimes\gamma_\alpha\gamma_\lambda\gamma_\mu-\xi\gamma^\alpha\otimes\gamma_\alpha\right]$$

$$\gamma_\alpha\gamma_\lambda\gamma_\mu=-\gamma_\mu\gamma_\lambda\gamma_\alpha+2\left(g_{\alpha\lambda}\gamma_\mu-g_{\alpha\mu}\gamma_\lambda+g_{\lambda\mu}\gamma_\alpha\right)$$

$$\gamma^\alpha\gamma^\lambda\gamma^\mu\otimes\gamma_\alpha\gamma_\lambda\gamma_\mu=-\gamma^\alpha\gamma^\lambda\gamma^\mu\otimes\gamma_\mu\gamma_\lambda\gamma_\alpha+2(3d-2)\gamma^\alpha\otimes\gamma_\alpha$$

$$-T_F\left(T_2-\frac{T_1}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}(4-\xi)\gamma^\alpha\otimes\gamma_\alpha$$

Result 1

$$
\begin{aligned}
<O_1^0> = &\left[1 - 2C_F \frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\right] \left[T_1\right. \\
&+ 2C_F T_1 \frac{\alpha_s}{4\pi\varepsilon}(1-\xi) \\
&+ 2T_F \left(T_2 - \frac{T_1}{N_c}\right) \frac{\alpha_s}{4\pi\varepsilon}(1-\xi) \\
&\left. - 2T_F \left(T_2 - \frac{T_1}{N_c}\right) \frac{\alpha_s}{4\pi\varepsilon}(4-\xi)\right] \gamma^\alpha \otimes \gamma_\alpha \\
= &<O_1> - 6T_F \frac{\alpha_s}{4\pi\varepsilon}\left(<O_2> - \frac{1}{N_c}<O_1>\right)
\end{aligned}
$$

Result 2

$$
\begin{aligned}
<O_2^0> = &\left[1 - 2C_F \frac{\alpha_s}{4\pi\varepsilon}(1-\xi)\right] \Bigg[T_2 \\
&+ 2T_F\left(T_1 - \frac{T_2}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}(1-\xi) \\
&+ 2C_F T_2 \frac{\alpha_s}{4\pi\varepsilon}(1-\xi) \\
&- 2T_F\left(T_1 - \frac{T_2}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}(4-\xi)\Bigg]\gamma^\alpha \otimes \gamma_\alpha \\
= &<O_2> - 6T_F\frac{\alpha_s}{4\pi\varepsilon}\left(<O_1> - \frac{1}{N_c}<O_2>\right)
\end{aligned}
$$

Anomalous dimension

$$Z = 1 + 6T_F \frac{\alpha_s}{4\pi\varepsilon} \begin{pmatrix} \frac{1}{N_c} & -1 \\ -1 & \frac{1}{N_c} \end{pmatrix}$$

Anomalous dimension

$$Z = 1 + 6T_F \frac{\alpha_s}{4\pi\varepsilon} \begin{pmatrix} \frac{1}{N_c} & -1 \\ -1 & \frac{1}{N_c} \end{pmatrix}$$

$$Z = 1 + \frac{\alpha_s}{4\pi\varepsilon} z_1$$

$$\frac{d\,Z}{d\,\log\mu} = -2\varepsilon \frac{\alpha_s}{4\pi\varepsilon} z_1 = \gamma_0 \frac{\alpha_s}{4\pi} \qquad \gamma_0 = -2z_1$$

Anomalous dimension

$$Z = 1 + 6T_F \frac{\alpha_s}{4\pi\varepsilon} \begin{pmatrix} \frac{1}{N_c} & -1 \\ -1 & \frac{1}{N_c} \end{pmatrix}$$

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$$\frac{d\,Z}{d\,\log\mu} = -2\varepsilon \frac{\alpha_s}{4\pi\varepsilon} z_1 = \gamma_0 \frac{\alpha_s}{4\pi} \qquad \gamma_0 = -2z_1$$

$$\gamma_0 = -12T_F \begin{pmatrix} \frac{1}{N_c} & -1 \\ -1 & \frac{1}{N_c} \end{pmatrix}$$

Anomalous dimension

$$Z = 1 + 6T_F \frac{\alpha_s}{4\pi\varepsilon} \begin{pmatrix} \frac{1}{N_c} & -1 \\ -1 & \frac{1}{N_c} \end{pmatrix}$$

$$Z = 1 + \frac{\alpha_s}{4\pi\varepsilon} z_1$$

$$\frac{d\,Z}{d\,\log\mu} = -2\varepsilon \frac{\alpha_s}{4\pi\varepsilon} z_1 = \gamma_0 \frac{\alpha_s}{4\pi} \qquad \gamma_0 = -2z_1$$

$$\gamma_0 = -12T_F \begin{pmatrix} \frac{1}{N_c} & -1 \\ -1 & \frac{1}{N_c} \end{pmatrix}$$

$$\gamma_0^T v_\pm = \lambda_\pm v_\pm \qquad v_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \qquad \lambda_\pm = -12T_F \left(\frac{1}{N_c} \mp 1 \right)$$

Running at 1 loop

$$c(M_W) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$c(\mu) = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\lambda_+}{2\beta_0}} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\lambda_-}{2\beta_0}}\right]$$

Running at 1 loop

$$c(M_W) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$c(\mu) = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\lambda_+}{2\beta_0}} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\lambda_-}{2\beta_0}}\right]$$

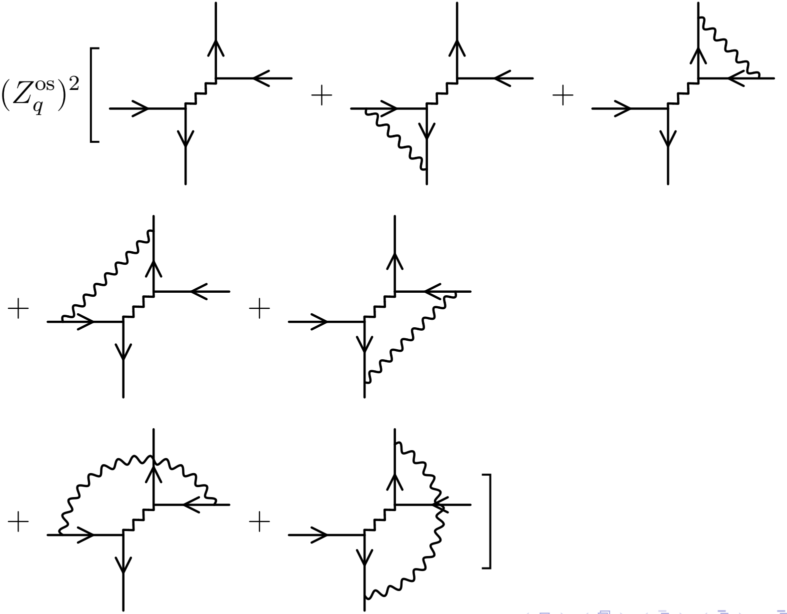
$$O_\pm = O_1 \pm O_2 \qquad L = c_+ O_+ + c_- O_- \qquad c_\pm = \frac{c_1 \pm c_2}{2}$$

$$c_+(M_W) = c_-(M_W) = \frac{1}{2}$$

$$c_+(\mu) = \frac{1}{2}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\lambda_+}{2\beta_0}} \qquad c_-(\mu) = \frac{1}{2}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\lambda_-}{2\beta_0}}$$

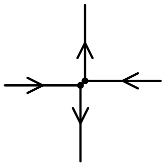
Matching

$(Z_q^{\rm os})^2$



Effective theory

All quark masses and external momenta = 0: $Z_q^{\mathrm{os}} = 1$



Effective theory

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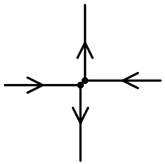
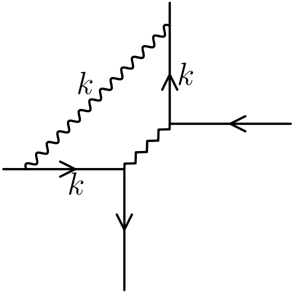


- Full theory UV finite, IR divergence
- Effective theory UV and IR divergences cancel
  IR divergence — as in the full theory

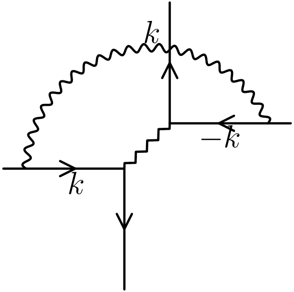
c_i^0 contain UV $1/\varepsilon$



$$- iT_F \left(T_2 - \frac{T_1}{N_c} \right) g_0^2 M_W^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\alpha \not{k} \gamma^\mu \otimes \gamma^\nu \not{k} \gamma_\alpha}{(M_W^2 - k^2)(k^2)^3} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$$= T_F \left(T_2 - \frac{T_1}{N_c} \right) \frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}} I \left[\frac{1}{d} \gamma^\alpha \gamma^\lambda \gamma^\mu \otimes \gamma_\mu \gamma_\lambda \gamma_\alpha - \xi \gamma^\alpha \otimes \gamma_\alpha \right]$$

$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{(M_W^2 - k^2)(-k^2)^2} = I M_W^{d-6}$$



$$-T_F\left(T_2-\frac{T_1}{N_c}\right)\frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}} I\left[\frac{1}{d}\gamma^\alpha\gamma^\lambda\gamma^\mu\otimes\gamma_\alpha\gamma_\lambda\gamma_\mu-\xi\gamma^\alpha\otimes\gamma_\alpha\right]$$

Integtral

$$
\begin{aligned}
I &= \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{(1-k^2)(-k^2)^2} \\
&= \frac{1}{i\pi^{d/2}} \int d^d k \left[\frac{1}{1-k^2} + \frac{1}{(-k^2)^2} - \frac{1}{-k^2} \right] \\
&= \Gamma\left(1 - \frac{d}{2}\right)
\end{aligned}
$$

Full theory result

$$T_1 \gamma^\alpha \otimes \gamma_\alpha + T_F \left(T_2 - \frac{T_1}{N_c} \right) \frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{2}{d} \Gamma\left(1 - \frac{d}{2}\right)$$
$$\times \gamma^\alpha \gamma^\lambda \gamma^\mu \otimes (\gamma_\mu \gamma_\lambda \gamma_\alpha - \gamma_\alpha \gamma_\lambda \gamma_\mu)$$

$$\gamma^\alpha \gamma^\lambda \gamma^\mu \otimes (\gamma_\mu \gamma_\lambda \gamma_\alpha - \gamma_\alpha \gamma_\lambda \gamma_\mu)$$
$$= -6(d-2)\gamma^\alpha \otimes \gamma_\alpha + 2\left(\gamma^\alpha \gamma^\lambda \gamma^\mu \otimes \gamma_\mu \gamma_\lambda \gamma_\alpha - 4\gamma^\alpha \otimes \gamma_\alpha\right)$$

Matching coefficients

$$c_1^0 = 1 - 6\frac{T_F}{N_c}\frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1 + \frac{\varepsilon}{2}\right)$$

$$c_2^0 = 6T_F\frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1 + \frac{\varepsilon}{2}\right)$$

Matching coefficients

$$c_1^0 = 1 - 6\frac{T_F}{N_c}\frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1+\frac{\varepsilon}{2}\right)$$

$$c_2^0 = 6T_F\frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1+\frac{\varepsilon}{2}\right)$$

$$c_1(\mu) = 1 - 12\frac{T_F}{N_c}\frac{\alpha_s(\mu)}{4\pi}\left(\log\frac{\mu}{M_W}+\frac{1}{4}\right)$$

$$c_2(\mu) = 12T_F\frac{\alpha_s(\mu)}{4\pi}\left(\log\frac{\mu}{M_W}+\frac{1}{4}\right)$$

Matching coefficients

$$c_1^0 = 1 - 6\frac{T_F}{N_c}\frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1+\frac{\varepsilon}{2}\right)$$

$$c_2^0 = 6T_F\frac{g_0^2 M_W^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(1+\frac{\varepsilon}{2}\right)$$

$$c_1(\mu) = 1 - 12\frac{T_F}{N_c}\frac{\alpha_s(\mu)}{4\pi}\left(\log\frac{\mu}{M_W}+\frac{1}{4}\right)$$

$$c_2(\mu) = 12T_F\frac{\alpha_s(\mu)}{4\pi}\left(\log\frac{\mu}{M_W}+\frac{1}{4}\right)$$

$$c_1(M_W) = 1 - 3\frac{T_F}{N_c}\frac{\alpha_s(M_W)}{4\pi}$$

$$c_2(M_W) = 3T_F\frac{\alpha_s(M_W)}{4\pi}$$

Physical operators

$$O_1^0 = \left(\bar{c}_{L0i}\gamma^\alpha b_{L0}^i\right)\left(\bar{d}_{L0j}\gamma_\alpha u_{L0}^j\right)$$
$$O_2^0 = \left(\bar{c}_{L0i}\gamma^\alpha b_{L0}^j\right)\left(\bar{d}_{L0j}\gamma_\alpha u_{L0}^i\right)$$

Physical operators

$$O_1^0 = (\bar{c}_{L0i}\gamma^\alpha b^i_{L0})\,(\bar{d}_{L0j}\gamma_\alpha u^j_{L0})$$
$$O_2^0 = (\bar{c}_{L0i}\gamma^\alpha b^j_{L0})\,(\bar{d}_{L0j}\gamma_\alpha u^i_{L0})$$

Evanescent operators

$$E_1^0 = (\bar{c}_{L0i}\gamma^\alpha\gamma^\beta\gamma^\gamma b^i_{L0})\,(\bar{d}_{L0j}\gamma_\gamma\gamma_\beta\gamma_\alpha u^j_{L0}) - 4O_1^0$$
$$E_2^0 = (\bar{c}_{L0i}\gamma^\alpha\gamma^\beta\gamma^\gamma b^j_{L0})\,(\bar{d}_{L0j}\gamma_\gamma\gamma_\beta\gamma_\alpha u^i_{L0}) - 4O_2^0$$

Physical operators

$$O_1^0 = (\bar{c}_{L0i}\gamma^\alpha b^i_{L0})\,(\bar{d}_{L0j}\gamma_\alpha u^j_{L0})$$
$$O_2^0 = (\bar{c}_{L0i}\gamma^\alpha b^j_{L0})\,(\bar{d}_{L0j}\gamma_\alpha u^i_{L0})$$

Evanescent operators

$$E_1^0 = (\bar{c}_{L0i}\gamma^\alpha\gamma^\beta\gamma^\gamma b^i_{L0})\,(\bar{d}_{L0j}\gamma_\gamma\gamma_\beta\gamma_\alpha u^j_{L0}) - 4O_1^0$$
$$E_2^0 = (\bar{c}_{L0i}\gamma^\alpha\gamma^\beta\gamma^\gamma b^j_{L0})\,(\bar{d}_{L0j}\gamma_\gamma\gamma_\beta\gamma_\alpha u^i_{L0}) - 4O_2^0$$

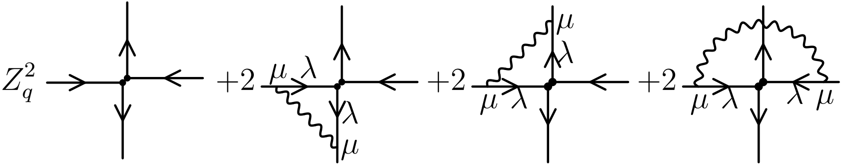
$d = 4$ Fierz

$$\begin{aligned}
&(\bar{c}_{L0i}\gamma^\alpha\gamma^\beta\gamma^\gamma b^i_{L0})\,(\bar{d}_{L0j}\gamma_\gamma\gamma_\beta\gamma_\alpha u^j_{L0})\\
&= (\bar{d}_{L0j}\gamma_\gamma\gamma_\beta\gamma^\alpha\gamma^\beta\gamma^\gamma b^i_{L0})\,(\bar{c}_{L0i}\gamma_\alpha u^j_{L0})\\
&= 4(\bar{d}_{L0j}\gamma^\alpha b^i_{L0})\,(\bar{c}_{L0i}\gamma_\alpha u^j_{L0}) = 4(\bar{c}_{L0i}\gamma^\alpha b^i_{L0})\,(\bar{d}_{L0j}\gamma_\alpha u^j_{L0})
\end{aligned}$$

More evanescent operators

$$F_1^0 = (\bar{c}_{L0i}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\varepsilon b^i_{L0})\,(\bar{d}_{L0j}\gamma_\varepsilon\gamma_\delta\gamma_\gamma\gamma_\beta\gamma_\alpha u^j_{L0}) - 16O_1^0$$
$$F_2^0 = (\bar{c}_{L0i}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\varepsilon b^j_{L0})\,(\bar{d}_{L0j}\gamma_\varepsilon\gamma_\delta\gamma_\gamma\gamma_\beta\gamma_\alpha u^i_{L0}) - 16O_2^0$$

1-loop matrix elements



Color structure T_1

$$\left(1 - 2C_F \frac{\alpha_s}{4\pi\varepsilon}\right) T_1 \Gamma \otimes \bar{\Gamma} + 2C_F T_1 \frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\gamma^\mu\gamma^\lambda\Gamma\gamma_\lambda\gamma_\mu \otimes \bar{\Gamma}$$
$$+ 2T_F \left(T_2 - \frac{T_1}{N_c}\right) \frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\Gamma\gamma^\lambda\gamma^\mu \otimes \gamma_\mu\gamma_\lambda\bar{\Gamma}$$
$$- 2T_F \left(T_2 - \frac{T_1}{N_c}\right) \frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\Gamma\gamma^\lambda\gamma^\mu \otimes \bar{\Gamma}\gamma_\lambda\gamma_\mu$$

Color structure T_2

$$
\left(1 - 2C_F\frac{\alpha_s}{4\pi\varepsilon}\right) T_2\Gamma \otimes \bar{\Gamma} + 2C_F T_2 \frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\Gamma\gamma^\lambda\gamma^\mu \otimes \gamma_\mu\gamma_\lambda\bar{\Gamma}
+ 2T_F\left(T_1 - \frac{T_2}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\gamma^\mu\gamma^\lambda\Gamma\gamma_\lambda\gamma_\mu \otimes \bar{\Gamma}
- 2T_F\left(T_1 - \frac{T_2}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}\frac{1}{d}\Gamma\gamma^\lambda\gamma^\mu \otimes \bar{\Gamma}\gamma_\lambda\gamma_\mu
$$

Color structure T_2

$$\left(1 - 2C_F \frac{\alpha_s}{4\pi\varepsilon}\right) T_2 \Gamma \otimes \bar{\Gamma} + 2C_F T_2 \frac{\alpha_s}{4\pi\varepsilon} \frac{1}{d} \Gamma \gamma^\lambda \gamma^\mu \otimes \gamma_\mu \gamma_\lambda \bar{\Gamma}$$
$$+ 2T_F \left(T_1 - \frac{T_2}{N_c}\right) \frac{\alpha_s}{4\pi\varepsilon} \frac{1}{d} \gamma^\mu \gamma^\lambda \Gamma \gamma_\lambda \gamma_\mu \otimes \bar{\Gamma}$$
$$- 2T_F \left(T_1 - \frac{T_2}{N_c}\right) \frac{\alpha_s}{4\pi\varepsilon} \frac{1}{d} \Gamma \gamma^\lambda \gamma^\mu \otimes \bar{\Gamma} \gamma_\lambda \gamma_\mu$$

$$\hat{O} = \gamma^\alpha \otimes \gamma_\alpha$$
$$\hat{E} = \gamma^\alpha \gamma^\beta \gamma^\gamma \otimes \gamma_\gamma \gamma_\beta \gamma_\alpha - 4\hat{O}$$
$$\hat{F} = \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^\varepsilon \otimes \gamma_\varepsilon \gamma_\delta \gamma_\gamma \gamma_\beta \gamma_\alpha - 16\hat{O}$$

Matrix elements

$$<O_1^0> = T_1\hat{O} + T_F\left(T_2 - \frac{T_1}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}\left(-6\hat{O} + \hat{E}\right)$$

$$<O_2^0> = T_2\hat{O} + C_F T_2 \frac{\alpha_s}{4\pi\varepsilon}\frac{1}{2}\hat{E} + T_F\left(T_1 - \frac{T_2}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}\left(-6\hat{O} + \frac{1}{2}\hat{E}\right)$$

$$<E_1^0> = T_1\hat{E} - C_F T_1 \frac{\alpha_s}{4\pi\varepsilon} 48\varepsilon\hat{O} + T_F\left(T_2 - \frac{T_1}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}\left(-48\varepsilon\hat{O} - 14\hat{E} + \hat{F}\right)$$

$$<E_2^0> = T_2\hat{E} + T_F\left(T_1 - \frac{T_2}{N_c}\right)\frac{\alpha_s}{4\pi\varepsilon}\left(96\varepsilon\hat{O} - 10\hat{E} + \frac{1}{2}\hat{F}\right)$$

Renormalization

$$\begin{pmatrix} O_0 \\ E_0 \end{pmatrix} = Z(\alpha_s(\mu)) \begin{pmatrix} O(\mu) \\ 0 \end{pmatrix} \qquad E(\mu) = 0$$

Renormalization

$$\begin{pmatrix} O_0 \\ E_0 \end{pmatrix} = Z(\alpha_s(\mu)) \begin{pmatrix} O(\mu) \\ 0 \end{pmatrix} \qquad E(\mu) = 0$$

$$\gamma(\alpha_s) = \begin{pmatrix} \gamma_{OO} & \gamma_{OE} \\ 0 & \gamma_{EE} \end{pmatrix}$$

$$\frac{d}{d\,\log\mu} \begin{pmatrix} O(\mu) \\ 0 \end{pmatrix} + \begin{pmatrix} \gamma_{OO} & \gamma_{OE} \\ 0 & \gamma_{EE} \end{pmatrix} \begin{pmatrix} O(\mu) \\ 0 \end{pmatrix}$$

$$\frac{d\,O(\mu)}{d\,\log\mu} + \gamma_{OO}(\alpha_s(\mu))O(\mu) = 0$$

Renormalization

$$\frac{d}{d\,\log\mu}\left(\begin{array}{c} c_O(\mu) \\ c_E(\mu) \end{array}\right) = \left(\begin{array}{cc} \gamma^T_{OO} & 0 \\ \gamma^T_{OE} & \gamma^T_{EE} \end{array}\right)\left(\begin{array}{c} c_O(\mu) \\ c_E(\mu) \end{array}\right)$$

$$\frac{d\,c_O(\mu)}{d\,\log\mu} = \gamma^T_{OO} c_O(\mu)$$

$$\frac{d\,c_E(\mu)}{d\,\log\mu} = \gamma^T_{OE} c_O(\mu) + \gamma^T_{EE} c_E(\mu)$$

Evolution of $c_O(\mu)$ does not involve $c_E(\mu)$
$c_E(\mu) \neq 0$ but they multiply $E(\mu) = 0$

1 loop

$$\begin{pmatrix} \langle O_0 \rangle \\ \langle E_0 \rangle \end{pmatrix} = Z \begin{pmatrix} \langle O_0 \rangle_{\text{tree}} \\ \langle E_0 \rangle_{\text{tree}} \end{pmatrix}$$

$$Z = 1 + \begin{pmatrix} b & c \\ a\varepsilon & d \end{pmatrix} \frac{\alpha_s}{4\pi\varepsilon}$$

UV $1/\varepsilon$ divergence of the loop integrals
does not depend on external momenta, masses
Evanescent Dirac structure $\Rightarrow$ normal Dirac structure $\times\varepsilon$

$$\langle E(\mu) \rangle = \langle E_0 \rangle - a \langle O_0 \rangle \frac{\alpha_s}{4\pi} = 0$$

1 loop

$$Z = 1 + \left(Z_{10} + \frac{Z_{11}}{\varepsilon} \right) \frac{\alpha_s}{4\pi}$$

$$Z_{10} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \qquad Z_{11} = \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}$$

$$\gamma = \gamma_0 \frac{\alpha_s}{4\pi} \qquad \gamma_0 = -2 Z_{11}$$

2 loops

The non-minimal renormalization matrix

$$Z(\alpha_s) = 1 + \left(Z_{10} + \frac{Z_{11}}{\varepsilon} \right) \frac{\alpha_s}{4\pi} + \left(Z_{20} + \frac{Z_{21}}{\varepsilon} + \frac{Z_{22}}{\varepsilon^2} \right) \left(\frac{\alpha_s}{4\pi} \right)^2$$

2 loops

The non-minimal renormalization matrix

$$Z(\alpha_s) = 1 + \left(Z_{10} + \frac{Z_{11}}{\varepsilon} \right) \frac{\alpha_s}{4\pi} + \left(Z_{20} + \frac{Z_{21}}{\varepsilon} + \frac{Z_{22}}{\varepsilon^2} \right) \left(\frac{\alpha_s}{4\pi} \right)^2$$

The anomalous dimension matrix must be finite at $\varepsilon \to 0$

$$Z_{22} = \frac{1}{2} Z_{11}(Z_{11} - \beta_0) = \frac{1}{2} \begin{pmatrix} b(b-\beta_0) & bc + cd - \beta_0 c \\ 0 & d(d-\beta_0) \end{pmatrix}$$

$$\gamma(\alpha) = -2Z_{11}\frac{\alpha}{4\pi} - 2(2Z_{21} - Z_{10}Z_{11} - Z_{11}Z_{10} + \beta_0 Z_{10}) \left(\frac{\alpha}{4\pi} \right)^2$$

The 1-loop $\mathcal{O}(\varepsilon^0)$ term

$$Z_{10} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

$1/\varepsilon$ divergences of 1-loop integrals
(momentum-independent) times ε from γ-matrix algebra

The 1-loop $\mathcal{O}(\varepsilon^0)$ term

$$Z_{10} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

$1/\varepsilon$ divergences of 1-loop integrals
(momentum-independent) times ε from γ-matrix algebra

$$Z_{11} = \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}$$

The 1-loop $\mathcal{O}(\varepsilon^0)$ term

$$Z_{10} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

$1/\varepsilon$ divergences of 1-loop integrals
(momentum-independent) times ε from γ-matrix algebra

$$Z_{11} = \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}$$

$$Z_{21} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

g — $1/\varepsilon^2$ divergences of 2-loop integrals
(momentum-independent) times ε from γ-matrix algebra
The lower left corner of γ must vanish

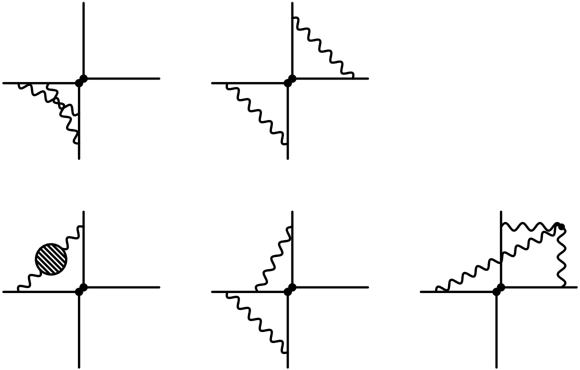
$$g = \frac{1}{2}(ab + da - \beta_0 a)$$

Evolution of the physical operators — the upper left corner of γ. At 2 loops

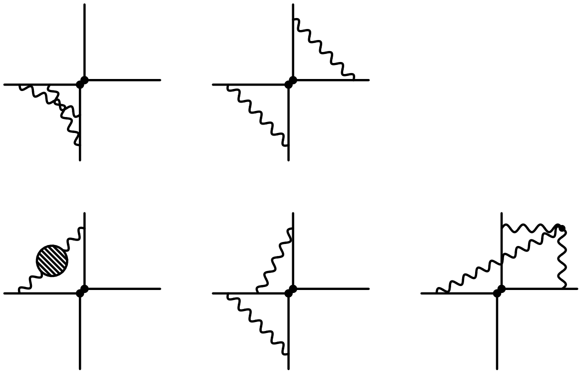
$$-2(2e + ca)\left(\frac{\alpha}{4\pi}\right)^2$$

e — the $1/\varepsilon$ part of 2-loop diagrams with the insertion of a physical operator

2-loop anomalous dimensions

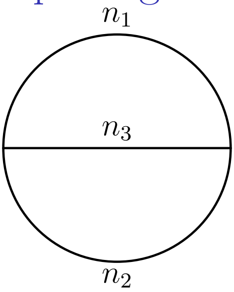


2-loop anomalous dimensions



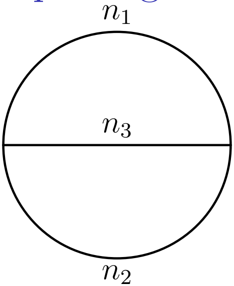
External momenta = 0
All denominators with a small mass m
Infrared rearrangement

2-loop integrals



$$- \frac{1}{\pi^d} \int \frac{d^d k_1 \, d^d k_2}{(k_1^2 + m^2)^{n_1} (k_2^2 + m^2)^{n_2} ((k_1 - k_2)^2 + m^2)^{n_3}}$$
$$= I_{n_1 n_2 n_3} m^{2(d - n_1 - n_2 - n_3)}$$

2-loop integrals



$$- \frac{1}{\pi^d} \int \frac{d^d k_1 \, d^d k_2}{(k_1^2 + m^2)^{n_1} (k_2^2 + m^2)^{n_2} ((k_1 - k_2)^2 + m^2)^{n_3}}$$
$$= I_{n_1 n_2 n_3} m^{2(d - n_1 - n_2 - n_3)}$$

Integration by parts

$$\left[d - 3n_1 + 3n_1 \mathbf{1}^+ + n_2 \mathbf{2}^+ (\mathbf{3}^- - \mathbf{1}^-) + n_3 \mathbf{3}^+ (\mathbf{2}^- - \mathbf{1}^-) \right] I = 0$$

I_{111} known

Running of c at NLO

$$c(\mu) = U(\mu, \mu_0) c(\mu_0)$$

$$\frac{d\,U(\mu, \mu_0)}{d \log \mu} = \gamma^T(\alpha_s(\mu)) U(\mu, \mu_0)$$

Running of c at NLO

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$$\frac{d\,U(\mu, \mu_0)}{d\log\mu} = \gamma^T(\alpha_s(\mu))U(\mu, \mu_0)$$

Eigenbasis of γ_0^T

$$\tilde{\gamma}_0 = V^{-1}\gamma_0^T V = \text{diagonal}$$
$$\tilde{\gamma}_1 = V^{-1}\gamma_1^T V$$
$$\tilde{U}(\mu, \mu_0) = V^{-1}U(\mu, \mu_0)V$$

Transformation matrix $\gamma_0^T V = V\tilde{\gamma}_0$
i-th column $\gamma_0^T v_i = \tilde{\gamma}_{0i} v_i$

Running of c at NLO

$$\frac{d\,\tilde{U}}{d\log\alpha_s} = -\frac{1}{2}\frac{\tilde{\gamma}_0 + \tilde{\gamma}_1\frac{\alpha_s}{4\pi}}{\beta_0 + \beta_1\frac{\alpha_s}{4\pi}}\tilde{U}$$

Running of c at NLO

$$\frac{d\,\tilde{U}}{d\log\alpha_s} = -\frac{1}{2}\frac{\tilde{\gamma}_0 + \tilde{\gamma}_1\frac{\alpha_s}{4\pi}}{\beta_0 + \beta_1\frac{\alpha_s}{4\pi}}\tilde{U}$$

LO

$$\tilde{U}_0(\mu, \mu_0) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{\tilde{\gamma}_0}{2\beta_0}} = \text{diagonal}$$

Running of c at NLO

$$\frac{d\,\tilde{U}}{d\log\alpha_s} = -\frac{1}{2}\frac{\tilde{\gamma}_0 + \tilde{\gamma}_1\frac{\alpha_s}{4\pi}}{\beta_0 + \beta_1\frac{\alpha_s}{4\pi}}\tilde{U}$$

LO

$$\tilde{U}_0(\mu,\mu_0) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{\tilde{\gamma}_0}{2\beta_0}} = \text{diagonal}$$

NLO

$$\tilde{U}(\mu,\mu_0) = \left[1 + \tilde{J}\frac{\alpha_s(\mu)}{4\pi}\right]\tilde{U}_0(\mu,\mu_0)\left[1 - \tilde{J}\frac{\alpha_s(\mu_0)}{4\pi}\right]$$

Running of c at NLO

Equation

$$2\beta_0 \tilde{J} + \tilde{\gamma}_0 \tilde{J} - \tilde{J} \tilde{\gamma}_0 = \frac{\beta_1}{\beta_0} \tilde{\gamma}_0 - \tilde{\gamma}_1$$

Running of c at NLO

Equation

$$2\beta_0 \tilde{J} + \tilde{\gamma}_0 \tilde{J} - \tilde{J}\tilde{\gamma}_0 = \frac{\beta_1}{\beta_0}\tilde{\gamma}_0 - \tilde{\gamma}_1$$

Solution

$$\tilde{J}_{ij} = \frac{\beta_1}{2\beta_0^2}\tilde{\gamma}_{0i}\delta_{ij} - \frac{\tilde{\gamma}_{1ij}}{2\beta_0 + \tilde{\gamma}_{0i} - \tilde{\gamma}_{0j}}$$

Running of c at NLO

Equation

$$2\beta_0 \tilde{J} + \tilde{\gamma}_0 \tilde{J} - \tilde{J}\tilde{\gamma}_0 = \frac{\beta_1}{\beta_0}\tilde{\gamma}_0 - \tilde{\gamma}_1$$

Solution

$$\tilde{J}_{ij} = \frac{\beta_1}{2\beta_0^2}\tilde{\gamma}_{0i}\delta_{ij} - \frac{\tilde{\gamma}_{1ij}}{2\beta_0 + \tilde{\gamma}_{0i} - \tilde{\gamma}_{0j}}$$

In the original basis

$$\begin{aligned} U(\mu,\mu_0) &= V\tilde{U}(\mu,\mu_0)V^{-1} \\ &= \left[1 + J\frac{\alpha_s(\mu)}{4\pi}\right] U_0(\mu,\mu_0) \left[1 - J\frac{\alpha_s(\mu_0)}{4\pi}\right] \\ U_0(\mu,\mu_0) &= V\tilde{U}_0(\mu,\mu_0)V^{-1} \\ J &= V\tilde{J}V^{-1} \end{aligned}$$

$$b \to s$$

Dipole operator

$$B^0 \leftrightarrow \bar{B}^0$$