Soft-Collinear Effective Theory: $B \to \gamma$ Form Factors

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Part I: a SCET primer

- Basic idea
- label vs coordinate space formalisms
- getting in touch with the objects that we will need to manipulate in the factorization proof

1. Motivations

Processes involving heavy mesons and energetic particles with $E \gg \Lambda_{QCD}$:

$B \to \gamma e \nu$, $B \to \gamma\gamma$, $B \to \gamma ee$

$B \to (\rho, K^*)\gamma$, $B \to K^* ee$

$B \to D\pi$, $B \to \pi\pi$, $B \to K\pi$, ...

$B \to X_s \gamma$, $B \to X_u e\nu$, ...

DIS , Jet production, ...

2. Kinematics

- We have collinear particles in the reference frame of the heavy quark (π in $B \to D\pi$, ρ in $B \to \rho\gamma$, X_s in $B \to X_s\gamma$, the spectator quark struck by the photon in $B \to \gamma e \nu$)

  introduce light cone vectors: $n = n_+ = (1001)$, $\bar{n} = n_- = (100-1)$

  $n^2 = 0 = \bar{n}^2 \quad n\bar{n} = 2$

  $$\begin{cases} p^\mu = \bar{n}p \, \dfrac{n^\mu}{2} + np \, \dfrac{\bar{n}^\mu}{2} + p_\perp^\mu = (p_-, p_+, p_\perp) \\ p^2 = \bar{n}p \, np - p_\perp^2 \end{cases}$$

  n-collinear $\Rightarrow$ $p_c \sim Q(1, \lambda^2, \lambda)$ with $\lambda \ll 1$

  There are 2 types of collinear fields:

  hard-collinear : $p^2 = \Lambda m_b$ $\Rightarrow$ $\lambda = \sqrt{\frac{\Lambda}{m_b}}$ (spectator hit by γ in $b \to \gamma e \nu$)

  collinear : $p^2 = \Lambda^2$ $\Rightarrow$ $\lambda = \frac{\Lambda}{m_b}$ (quarks inside fast light mesons)

- We have soft particles : $p_s = (\Lambda, \Lambda, \Lambda)$

- Heavy quark : $p_h = m_b v + (\Lambda, \Lambda, \Lambda)$

  $v = (1,0,0,0)$

3. Basic idea

The relevant scales are: $\underbrace{m_W^2,\ m_b^2,\ \Lambda m_b}_{\text{perturbative} \Rightarrow \text{int. out}},\ \underbrace{\Lambda^2}_{\text{non perturbative}}$

Q: How to integrate out the perturbative modes?

- m_W^2: standard
- $m_b^2, \Lambda m_b$: more problematic because the scale m_b appears in the initial and final states.

$\boxed{\text{I}}$ diagrammatic approach: order by order in perturbation theory one identifies the momentum regions that contribute to the various Feynman diagrams and uses the threshold expansion to isolate their contributions

* expansion in $\lambda = \frac{\Lambda}{m}$ (limits the total # of regions)

* dimensional regularization (allows to separate the regions)

equivalent

$\boxed{\text{II}}$ SCET: Effective field theory in which different regions become independent degrees of freedom

- $m_W^2 \gg m_b^2$: $A(B \to X) = \langle X | H_{eff} | \bar{B} \rangle = \sum_i \underbrace{C_i(\mu_b)}_{\text{SD}} \underbrace{\langle X | O_i(\mu_b) | \bar{B} \rangle}_{\text{LD}}$

- $m_b^2 \gg \Lambda m_b \gg \Lambda_{QCD}^2$:

$$\langle X | O(\mu_b) | B \rangle = \underbrace{C(\sqrt{\Lambda m_b})}_{\text{WC's}} \cdot \underbrace{J(\Lambda)}_{\text{Jet functions}} * \underbrace{\langle X | \bar{O}(\Lambda) | B \rangle}_{\substack{\text{Form Factors} \\ \text{LCDA's}}} + \underbrace{O\left(\frac{\Lambda}{m_b}\right)}_{\substack{\text{power} \\ \text{corrections}}}$$

- the approach is rigorous (SCET is the effective theory of QCD)

- the $\frac{\Lambda}{m_b}$ expansion is necessary to separate the regions and to write the effective theory at all.

- the above formula can <u>always</u> be written ($\bar{O} \ni s, c$)

- sometimes $\langle X | \bar{O} | B \rangle$ factorizes in simple objects:

  LCDA's of the B or of the light mesons

  Form Factors that obey symmetry relations

★ Advantages of SCET over QCD factorization:

- symmetries are more explicit (Gauge, Reparametrization, ...)
- $\frac{\Lambda}{m_b}$ power counting is simplified
- one can use RGE methods to resum Sudakov-type logs

★ The main problem is to identify the regions at all order in α_s and at a given order in $\frac{\Lambda}{m_b}$.

They might depend on the IR regulator chosen.

4. Fields and Scaling

We choose $\lambda = \sqrt{\frac{\Lambda}{m_b}}$ and force the kinetic terms to be $O(1)$ by assigning λ-scaling to the fields

hard-collinear: $\left\{ \begin{array}{l} \xi_{hc} \\ A^\mu_{hc} \end{array} \right. \quad p \sim m_b (1 \;\; \lambda^2 \;\; \lambda) \quad \begin{array}{c} \lambda \\ (1 \;\; \lambda^2 \;\; \lambda) \end{array}$

collinear: $\left\{ \begin{array}{l} \xi_{c} \\ A^\mu_{c} \end{array} \right. \quad p \sim m_b (1 \;\; \lambda^4 \;\; \lambda^2) \quad \begin{array}{c} \lambda^2 \\ (1 \;\; \lambda^4 \;\; \lambda^2) \end{array}$

soft: $\left\{ \begin{array}{l} h_v, \; q_s \\ A^\mu_{s} \end{array} \right. \quad p \sim m_b (\lambda^2 \lambda^2 \lambda^2) \quad \begin{array}{c} \lambda^3 \\ \lambda^2 \end{array}$

5. General procedure to obtain $\mathcal{L}_{SCET}$

- $q_{QCD} = q_{hc} + q_s \; ; \; A^\mu_{QCD} = A^\mu_{hc} + A^\mu_s$

- $\mathcal{L}_{QCD} \longrightarrow \mathcal{L}_{QCD}\Big|_{\substack{q = q_{hc} + q_s \\ A^\mu = A^\mu_{hc} + A^\mu_s}}$

- Expand in λ

  q_{hc} has two small and two large components:

  $$\begin{cases} \xi_{hc} = \dfrac{\not{n}\not{\bar{n}}}{4} q_{hc} \longrightarrow O(\lambda) \\ \eta_{hc} = \dfrac{\not{\bar{n}}\not{n}}{4} q_{hc} \longrightarrow O(\lambda^2) \end{cases}$$

  η_{hc} is eliminated at tree level using the equations of motion.

- The resulting Lagrangian is <u>**NOT renormalized**</u> by hard modes ($\sim m_b^2$) at all orders in α_s !!
  (without the heavy quark Lagrangian, $\mathcal{L}_{SCET}$ is equivalent to QCD).

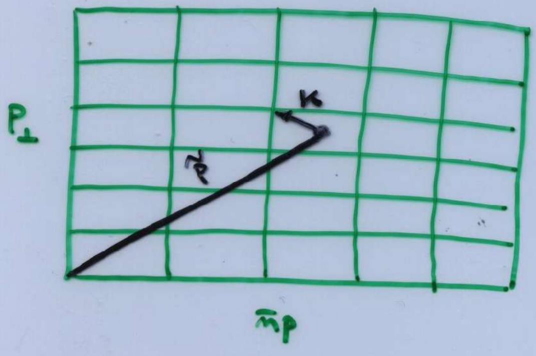
6. Formalisms : coordinate space vs label

$$\xi_{hc}(x) = \sum_{\tilde{p}} e^{-i\tilde{p}x}\,\xi_{\tilde{p}}(x)$$

$$A^{\mu}_{hc}(x) = \sum_{\tilde{p}} e^{-i\tilde{p}x}\,A^{\mu}_{\tilde{p}}(x)$$

the typical momentum of a collinear particle is written as

$p = \tilde{p} + k$ where $\underbrace{\tilde{p}^{\mu} = n^{\mu}\frac{\bar{n}\tilde{p}}{2} + \tilde{p}_{\perp}}_{\text{large fluctuations}}$ and k is soft



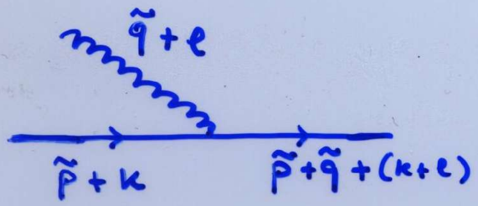
introduce a distinct collinear field for each bin in the $(\bar{n}p, p_\perp)$ plane.

$$\Longrightarrow \quad \partial\xi_p \sim \partial A^{\mu}_p \sim \partial q_s \sim \partial A^{\mu}_s \sim \partial h_v \sim \lambda^2$$

All the remaining fields are homogeneous.

- In the label formalism, product of soft and hard-collinear fields are homogeneous
- In coordinate space one has to multiple expand. Failing to do so would invalidate the separation of the regions.

■ problem: labels change in $O(\lambda^0)$ collinear interactions



$$\Rightarrow \quad \bar{\xi}_{\tilde{p}+\tilde{q}} \; A^{\mu}_{\tilde{q}} \; \xi_{\tilde{p}}$$

The SCET lagrangian is non-local and labels appear explicitely (from derivatives).

[Q: How to write this kind of interactions in closed form?
A: **label operators** [Bauer, Stewart] : $\mathcal{P}^\mu$, $\mathcal{P}^{\mu\dagger}$

$$\mathcal{P}^\mu \phi_{p_1} \cdots \phi_{p_m} \equiv (\tilde{p}_1^\mu + \cdots + \tilde{p}_m^\mu)\, \phi_{p_1} \cdots \phi_{p_m}$$

$$\phi^\dagger_{p_1} \cdots \phi^\dagger_{p_m} \mathcal{P}^{\mu\dagger} \equiv (\tilde{p}_1^\mu + \cdots + \tilde{p}_m^\mu)\, \phi^\dagger_{p_1} \cdots \phi^\dagger_{p_m}$$

* $f(\mathcal{P}^\mu)\, \phi_{p_1} \cdots \phi_{p_m} = f(\tilde{p}_1^\mu + \cdots + \tilde{p}_m^\mu)\, \phi_{p_1} \cdots \phi_{p_m}$

* $i\partial_\mu \sum_p e^{-ipx} \xi_p(x) = \sum_p e^{-ipx} (\mathcal{P}^\mu + i\partial^\mu)\, \xi_p(x) \equiv (\mathcal{P}^\mu + i\partial^\mu)\, \xi_p$

  i.e.: we don't need to keep all the e^{-ipx} that appear in the expansion of the QCD Lagrangian because all these factors can be pulled to the left of any operator and become trivial

* Wilson coefficients of SCET operators are functions of the labels.
  By gauge invariance, only combination of labels picked up by the label operators can appear.
  $\Rightarrow$ WC's are functions of the label operators

Final notes on the 2 formalisms:

- After the introduction of the label operators, lagrangian terms and effective currents look very similar
- Treatment of Gauge invariance is more elegant and simple in coordinate space
- We will see that the LCDA of the B meson appears in a somewhat different way
- At sub-leading order in λ there is a reshuffling of terms between the SCET lagrangian and the effective currents.

  e.g.: currents involving soft-collinear fields at subleading order

  in coordinate space $\longrightarrow$ extra terms due to the multipole expansion

  label $\longrightarrow$ extra $O(\lambda)$ corrections to the collinear propagator.

7. Wilson lines and LO Lagrangian

Non localities lead to the appearance of collinear and soft Wilson lines: W and S

$$[W]_{\text{coord}} = \underset{\text{path ordered!!}}{P} \exp\left[-ig\int_{-\infty}^{0} ds \; \bar{n} A_{hc}(x^\mu + s\,\bar{n}^\mu)\right]$$

path ordered !!

$$[W]_{\text{label}} = \left[\exp\left[-g \frac{1}{\bar{n}\underset{\text{label operator!!}}{P}} \bar{n} A_{hc}\right]\right]$$

label operator !!

label operators that appear in the expansion of the exponential do not act outside the [......]

$$S = P \exp\left[ig\int_{-\infty}^{0} dt \; n\cdot A_s(x^\mu + t\,n^\mu)\right]$$

The LO hard-collinear lagrangian is:

$$\mathcal{L}^{(0)}_{\xi\xi} = \bar{\xi}_{p'} \frac{\not{\bar{n}}}{2} \Big\{ \boxed{i\, n\cdot\partial} + g\, n\cdot A_s + g\, n\cdot A_q + \quad \text{the only derivative term}$$

$$+ \left(\not{P}_\perp + g \not{A}_{\perp q}\right) \frac{1}{\bar{n}P + g\bar{n}A_{q'}} \left(\not{P}_\perp + g\not{A}_{\perp q''}\right) \Big\} \xi_p$$

$$\equiv \bar{\xi}_{p'} \frac{\not{\bar{n}}}{2} \Big\{ i n\cdot D + i \not{D}_\perp \frac{1}{i\bar{n}D} \not{D}_\perp \Big\} \xi_p$$ ← same as in coord. space. No $i\epsilon$ in the denominator

$$= \bar{\xi}_{p'} \frac{\not{\bar{n}}}{2} \Big\{ i n\cdot D + i \not{D}_\perp W \frac{1}{\bar{n}P} W^+ \not{D}_\perp \Big\} \xi_p$$

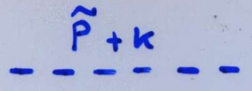
The NLO soft-hard-collinear lagrangian is:

$$\mathcal{L}^{(1)}_{\xi q} = \bar{q}\, W^+ i \not{D}_\perp \xi_p$$ → starts with 1 ~~[illegible]~~ gluon

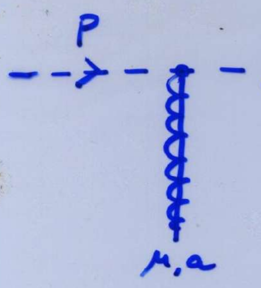
The transformation $\xi_p \to S\,\xi_p^{(0)}$, $A^\mu_p \to S A^{\mu(0)}_p S^+$ decouples soft gluons from $\mathcal{L}^{(0)}_{\xi\xi}$.

$$\mathcal{L}^{(0)}_{\xi\xi} \longrightarrow \bar{\xi}^{(0)}_{p'} \frac{\not{\bar{n}}}{2} \Big\{ i n\cdot D^{(0)}_c + \left(\not{P}_\perp + g\not{A}^{(0)}_{\perp q}\right) W^{(0)} \frac{1}{\bar{n}P} W^{+(0)} \left(\not{P}_\perp + g \not{A}^{(0)}_{\perp q'}\right)\Big\} \xi^{(0)}_p$$

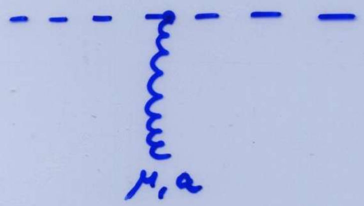
8. Some Feynman rules



$$= i\,\frac{\not{n}}{2}\,\frac{\bar{n}\cdot\tilde{p}}{\bar{n}\cdot\tilde{p}\;\underline{n\cdot k}-\tilde{p}_\perp^2}$$

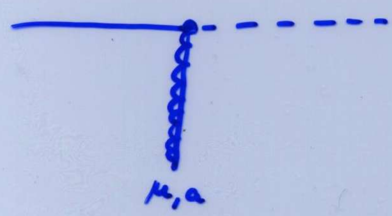


$$= i g T^a \frac{\not{n}}{2}\left[n^\mu+\frac{\gamma_\perp^\mu \not{p}_\perp}{\bar{n}p}+\frac{\not{p}'_\perp\gamma_\perp^\mu}{\bar{n}p'}-\frac{\not{p}'_\perp\not{p}_\perp}{\bar{n}p\,\bar{n}p'}\,\bar{n}^\mu\right]$$



$$= -i g T^a \frac{\not{n}}{2}\, n^\mu$$

From the Wilson line



$$= i g T^a\left(\gamma_\perp^\mu-\bar{n}^\mu\,\frac{\not{k}_\perp}{\bar{n}k}\right)$$

Part II : $B \to \gamma$ form factors in the SM

- Introduction to $B \to \gamma e \nu$, $B \to \gamma\gamma$, $B \to \gamma ee$

- Parametrization of the form factors

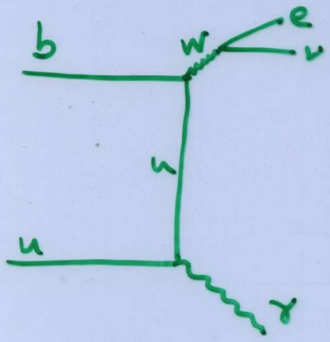
1. $B \to \gamma e \nu$

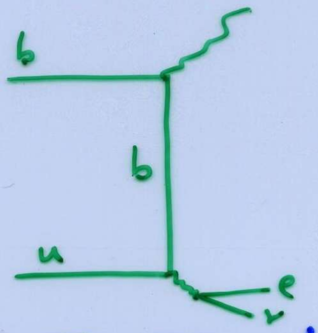
- The effective hamiltonian is:

$$H_{eff} = 4 \frac{G_F}{\sqrt{2}} \underline{V_{ub}} \left(\bar{u} \gamma^\mu P_L b\right)\left(\bar{e} \gamma_\mu P_L \nu\right)$$

and receives tree level SM contributions (W exchange).

- There are 2 tree level SM diagrams:





power suppressed

- The amplitude for this decay is:

$$A(B \to \gamma e \nu) = \underbrace{\frac{4 G_F V_{ub}}{\sqrt{2}}}_{m_W^2} \underbrace{\langle \gamma | \bar{u} \gamma^\mu P_L b | \bar{B} \rangle}_{m_b^2,\ \Lambda m_b,\ \Lambda^2} \bar{u}_e \gamma^\mu P_L u_\nu$$

- If we refuse to power expand and use SCET, we stop here and parametrize the form factors:

$$\frac{1}{e} \langle \gamma(q, \epsilon) | \bar{u} \gamma_\mu b | \bar{B}(v) \rangle = i\, \varepsilon_{\mu\alpha\beta\delta}\, \epsilon^\alpha v^\beta q^\delta\, f_V(E_\gamma)$$

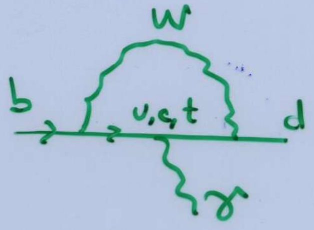
$$\frac{1}{e} \langle \gamma | \bar{u} \gamma_\mu \gamma_5 b | \bar{B} \rangle = \left(q_\mu\, v\cdot\epsilon - \epsilon_\mu\, v\cdot q\right) f_A(E_\gamma) + \underbrace{v^\mu \frac{v\cdot\epsilon}{v\cdot q} f_B m_B}_{\text{cancels emission from the electron}}$$

2. $B \to \gamma\gamma$

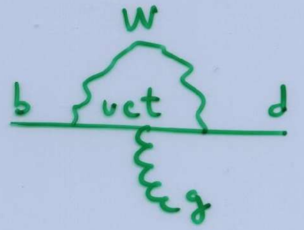
The effective hamiltonian contains many more operaters:

$$H_{eff} = \frac{4 G_F}{\sqrt{2}} V_{tb} V_{td}^{*} \sum_{i=1}^{8} C_i(\mu)\, O_i(\mu)$$

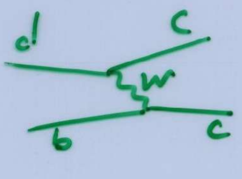
$$O_7 = \frac{e}{16\pi^2} m_b\, \bar{d}_L \sigma^{\mu\nu} b_R\, F_{\mu\nu}$$



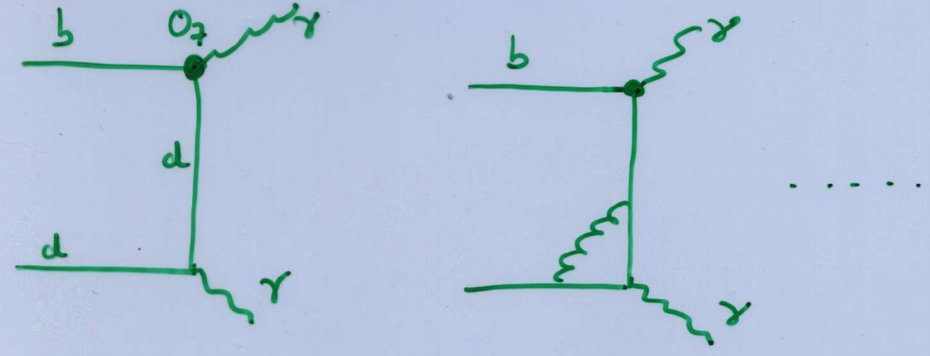
$$O_8 = \frac{g_s}{16\pi^2} m_b\, \bar{d}_L \sigma^{\mu\nu} T^a b_R\, G^a_{\mu\nu}$$



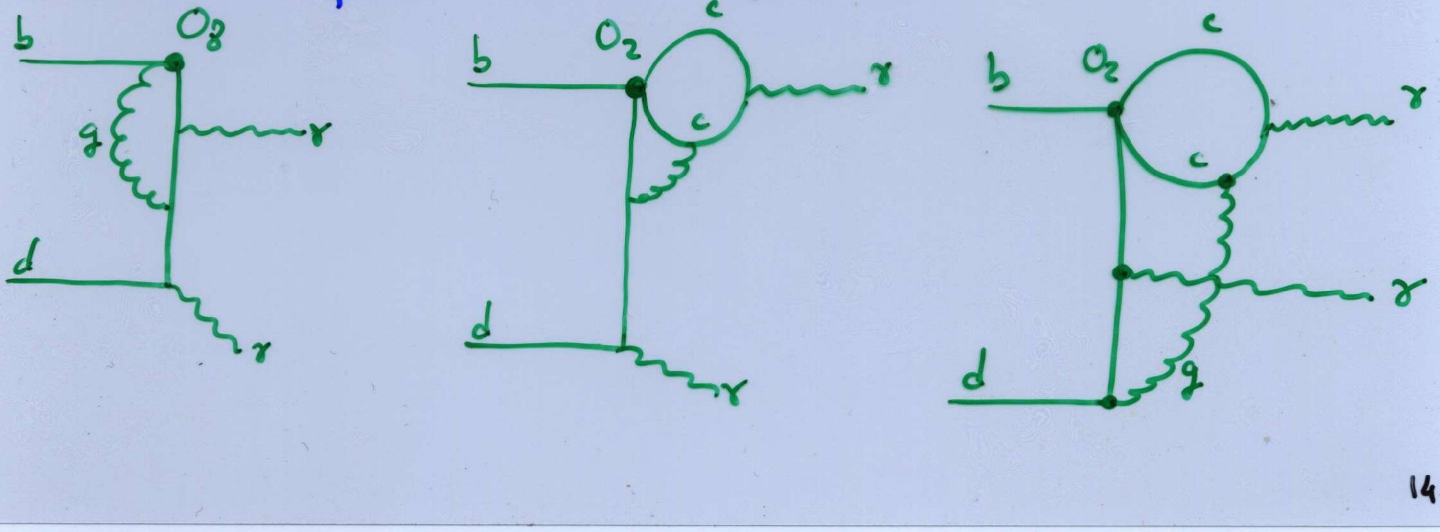
$$O_2 = (\bar{d}\, \gamma^\mu P_L\, \bar{c})(\bar{c}\, \gamma^\mu P_L\, b)$$



- contributions from O_7



- contributions from the other operators:



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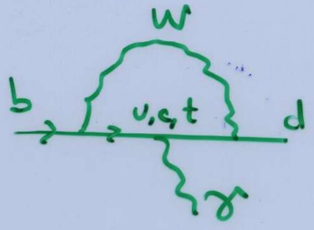
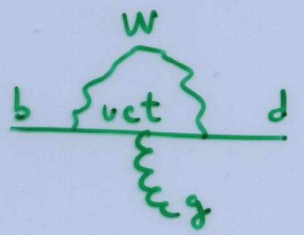
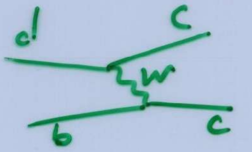


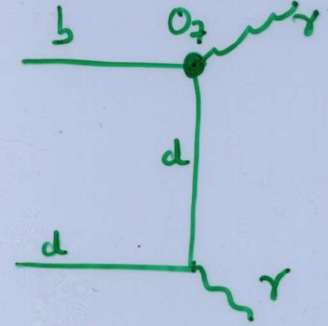
$$O_8 = \frac{g_s}{16\pi^2} m_b\, \bar{d}_L \sigma^{\mu\nu} T^a b_R\, G^a_{\mu\nu}$$

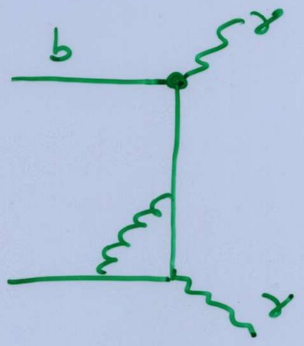


$$O_2 = (\bar{d}\, \gamma^\mu P_L\, \bar{c})(\bar{c}\, \gamma^\mu P_L\, b)$$



- contributions from O_7





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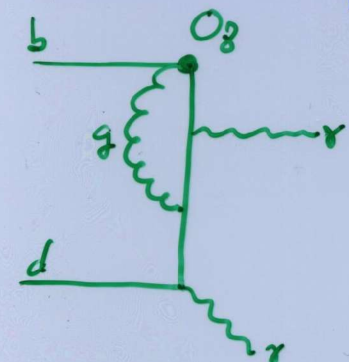
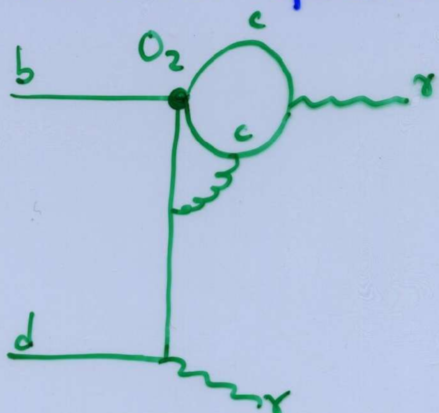
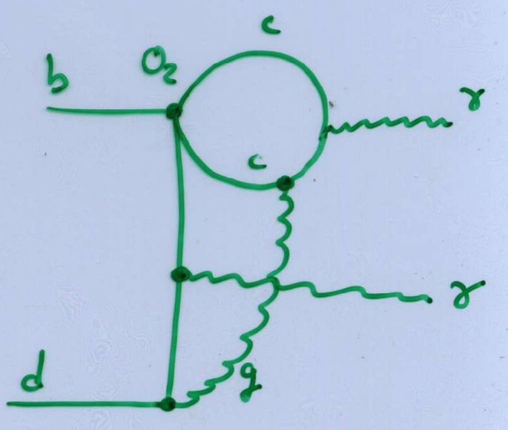
- contributions from the other operators:







- our approach applies trivially to O_7:

$$\langle \gamma\gamma | O_7 | B \rangle \propto \langle \gamma | \bar{d}\, \sigma^{\mu\nu} b | B \rangle$$

$$\frac{1}{e} \langle \gamma | \bar{d}\, \sigma^{\mu\nu} b | B \rangle = i\, \varepsilon_{\mu\nu\alpha\beta}\, \epsilon^{*\alpha} (p+q)^\beta\, g_+(E_\gamma) + i\, \varepsilon_{\mu\nu\alpha\beta}\, \epsilon^{*\alpha} (p-q)^\beta\, g_-(E_\gamma) - 2i\, \varepsilon_{\mu\nu\alpha\beta}\, \epsilon^* \cdot p\; p^\alpha q^\beta\, h(E_\gamma)$$

- Descotes-Genon - Sachrajda showed that, at 1-loop, contributions of O_2 and O_8 are reduced to the O_7 one up to power corrections:

$$\langle \gamma\gamma | O_{2,8} | B \rangle = \underbrace{(**)}_{m_b^2} \langle \gamma\gamma | O_7 | B \rangle + O\left(\frac{\Lambda}{m_b}\right)$$

- From our analysis we find:
  - the 3 form factors factorize and the Jet function is universal:

$$g_+, g_-, h = C_{g_+, g_-, h} \cdot \underbrace{\int d\xi\; J(E_\gamma, \xi)\, \phi_B(\xi)}_{\text{same as in } f_V \text{ and } f_A}$$

  - $$g_-(E_\gamma) = -\, g_+(E_\gamma) + O(\alpha_s^2)$$

    $$h(E_\gamma) = O(\alpha_s^2)$$

    $$\frac{g_+(E_\gamma)}{f_V(E_\gamma)} = \frac{1}{2} \frac{Q_d}{Q_u} \left(1 + O(\alpha_s)\right)$$

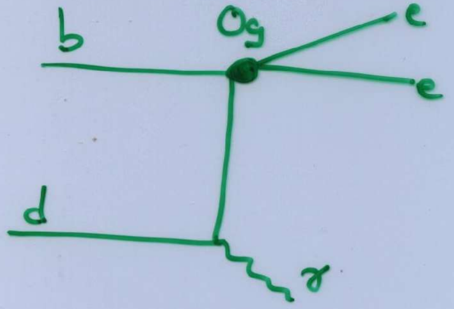
3. $B \to \gamma e e$

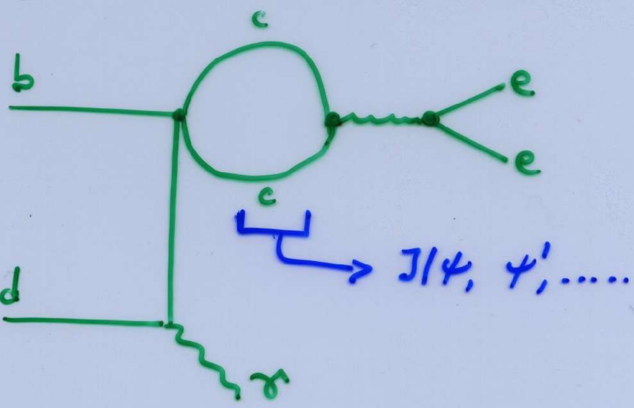
We have 2 more operators:

$$O_9 = \frac{\alpha_e}{4\pi} \, \bar{d} \gamma^\mu P_L b \;\; \bar{e} \gamma^\mu e$$

$$O_{10} = \frac{\alpha_e}{4\pi} \, \bar{d} \gamma^\mu P_L b \;\; \bar{e} \gamma^\mu \gamma^5 e$$

* presence of $c\bar{c}$ resonances in the e-e invariant mass spectrum





One has to put a cut on $s = (P_{e_1}^2 + P_{e_2}^2)$:

$s < m_{J/\psi}^2$ (low-s region) $\Longrightarrow$ the photon is energetic and SCET is applicable

$s > m_{\psi'}^2$ (high-s region) $\Longrightarrow$ the photon is soft

- The amplitude is:

$$A(B \to \gamma e e) = \frac{4 G_F}{\sqrt{2}} \frac{\alpha_e}{4\pi} V_{tb} V_{td}^* \Big[\big(C_9 \, \bar{e} \gamma^\mu e + C_{10} \, \bar{e} \gamma^\mu \gamma^5 e \big) \cdot$$

$$\cdot \overbrace{\langle \gamma | \, \bar{d} \gamma^\mu P_L b \, | B \rangle}^{f_V,\, f_A}$$

$$- 2 C_7 \frac{m_b}{q^2} q^\nu \underbrace{\langle \gamma | \bar{d}_L \sigma_{\mu\nu} b_R | B \rangle}_{g_+,\, g_-,\, h} \bar{e} \gamma^\mu e \Big]$$

Part III : SCET analysis

- Integration of hard modes (matching QCD $\longrightarrow$ $SCET_I$)
- Resummation of Sudakov logs ($\ln \frac{E_\gamma}{\mu}$)
- Integration of hard-collinear modes (matching $SCET_I \longrightarrow SCET_{II}$)
- Proof of the factorization theorem
- Resummation of the residual Sudakov logs
- Failure of factorization at subleading order

Our task is the calculation of the $B \to \gamma$ matrix elements of the currents $\bar{q}\,\Gamma\, b$ with $\Gamma: \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}$

Electromagnetic interactions are perturbative, hence:

$$\langle \gamma(q,\epsilon) |\ \bar{q}\,\Gamma b(0)\ | \bar{B}(v) \rangle =$$

$$i e Q_q \int d^4x\, e^{iqx} \langle 0 |\, T\, (\bar{q}\,\Gamma b)(0)\ (\bar{q}\, \not{\epsilon}^*_\perp q)(x) | \bar{B} \rangle$$

Steps:

1. Matching onto $SCET_I$

$$\bar{q}\,\Gamma\, b \longrightarrow \sum_i C_i(\mu)\ \bar{\xi}_p W \Gamma'_i h_v$$

$$\bar{q}\, \not{\epsilon}^*_\perp q \longrightarrow \epsilon_\perp^{*\mu} J_\mu$$

2. Resum large logs $[\ln E_\gamma/\mu]$

$$C_i(\mu) \longrightarrow C_i(\mu_{\Lambda m_b})$$

the e.m. current is not renormalized

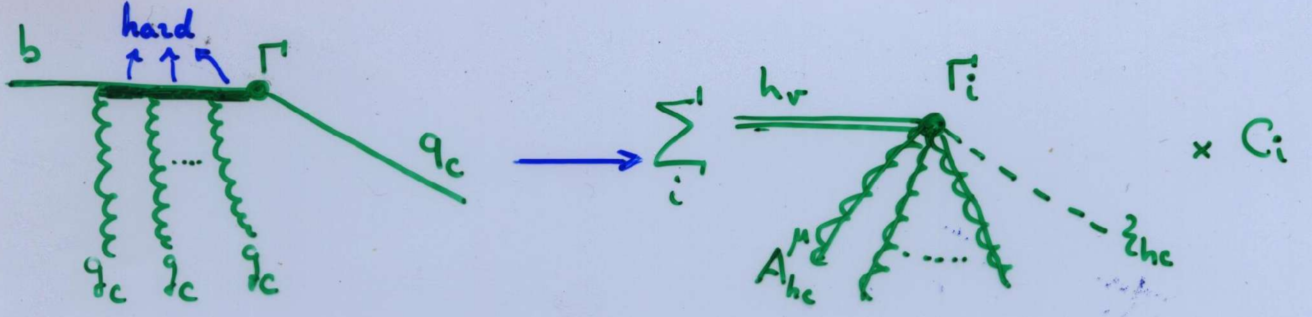
3. decoupling of soft gluons

$$\langle 0 | T \underbrace{O_c^{(0)}}_{\xi_p^{(0)},\, A_q^{(0)\mu}}\ \underbrace{O_s}_{q_s h_v A_s^\mu} | \bar{B} \rangle$$

4. factorization and matching onto HQET

$$\underbrace{\langle 0 | T\, O_c^{(0)} | 0 \rangle}_{\text{Jet function}} \underbrace{\langle 0 | T\, O_s | \bar{B} \rangle}_{\text{LCDA of B meson}}$$

1. Matching of the heavy-to-light current



QCD

SCET$_{\mathrm{I}}$

$$\bar{q}\,\Gamma\, b = \sum C_i(\bar{n}p)\,[\bar{\xi}\,W]_p\,\Gamma_i\, h_v + O(\lambda)$$

we use the projection properties of ξ and h to reduce the number of Γ_i.

$$\bar{q}\gamma^\mu b = \underline{C_1^{(V)}}\,\bar{\xi}W\gamma_\mu h + \underline{C_2^{(V)}}\,\bar{\xi}W v_\mu h + \underline{C_3^{(V)}}\,\bar{\xi}W n_\mu h$$

$$\bar{q}\gamma^\mu\gamma^5 b = \underline{C_1^{(A)}}\,\bar{\xi}W\gamma_\mu\gamma_5 h + \underline{C_2^{(A)}}\,\bar{\xi}W v_\mu\gamma_5 h + \underline{C_3^{(V)}}\,\bar{\xi}W n_\mu\gamma_5 h$$

$$\bar{q}\, i\sigma^{\mu\nu} b = \underline{C_1^{(T)}}\,\bar{\xi}W i\sigma^{\mu\nu} h + \underline{C_2^{(T)}}\,\bar{\xi}W(n_\mu\gamma_\nu - n_\nu\gamma_\mu)h$$
$$+ \underline{C_3^{(T)}}\,\bar{\xi}W(v_\mu\gamma_\nu - v_\nu\gamma_\mu)h + \underline{C_4^{(T)}}\,\bar{\xi}W(n_\mu v_\nu - n_\nu v_\mu)h$$

at tree level: $C_1^{(V)} = C_1^{(A)} = C_1^{(T)} = 1$

$$C_{2,3}^{(V,A)} = C_{2,3,4}^{(T)} = 0$$

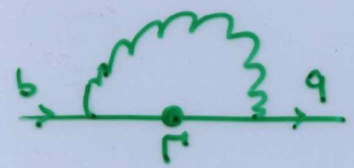
matching at 1-loop:

- calculate perturbative renormalized matrix elements in the full and effective theories
- WC's don't depend on external states $\Rightarrow$ free quarks
- all long distance physics is reproduced in the effective theory
- the difference determines the WC's.

trick: we can replace the current $[\bar{\xi} W]_p \Gamma_i h_v$ with the simpler $\bar{\xi}_p \Gamma_i h_v$ because the WC's are universal (they are functions of the label operators)

- we regularize everything in dim-reg

- full theory side:



$+$ UV counterterms

example: $\Gamma = \gamma^\mu$

$$\langle q | \bar{q} \gamma^\mu b | b \rangle_R = \bar{u} \gamma^\mu u \left\{ 1 + \frac{\alpha_s C_F}{4\pi} \Bigg[\overbrace{-\frac{1}{\epsilon^2} - \frac{5}{2\epsilon} - \frac{2}{\epsilon} \ln \frac{\mu}{\bar{n} p}}^{\text{IR divergences}} + \dots \Bigg] \right\}$$

$$+ \bar{u} u \frac{\alpha_s C_F}{4\pi} \frac{p^\mu}{m_b} [\dots] + O(\lambda^2)$$

spinors in the full theory to be expanded in λ

- effective theory side:

We take on shell external quarks in dim-reg $\Rightarrow$ all diagrams vanish! In fact there is a cancellation between <u>IR</u> poles and <u>UV</u> poles.

* After adding the effective theory counterterms we can extract the WC's

* IR divergences in the full theory $= -$ UV divergences in E.T.

- Extraction of the WC's:

$$\underbrace{C_1^{(V)} = C_1^{(A)}}_{\text{valid at all orders}} = 1 - \frac{\alpha_s C_F}{4\pi}\left\{ \frac{1}{2}\ln^2\frac{\mu^2}{m_b^2} + \frac{5}{2}\ln\frac{\mu^2}{m_b^2} - 2\ln\frac{\bar{n}p}{m_b}\cdot\ln\frac{\mu^2}{m_b^2} \right.$$

$$+ 2\log^2\frac{\bar{n}p}{m_b} + 2\,\mathrm{Li}_2\left(1-\frac{\bar{n}p}{m_b}\right) + \log\frac{\bar{n}p}{m_b}\,\frac{3\bar{n}p - m_b}{m_b - \bar{n}p}$$

$$\left. + \frac{\pi^2}{12} + 6 \right\}$$

$$C_1^{(T)} = 1 - \frac{\alpha_s C_F}{4\pi}\left\{\ldots\right\}$$

$$C_2^{(T)} = \frac{\alpha_s C_F}{4\pi}\left\{\ldots\right\}$$

$$C_{3,4}^{(T)} = 0$$

2. Renormalization-Group improvement

The Wilson coefficients contain logs of $\frac{\mu^2}{m_b^2}$ that become large for $\mu^2 \sim \Lambda m_b$.

Bosch, Hill, Lange and Neubert have shown that these WC's obey a renormalization group equation of the form:

$$\frac{dC(\mu)}{d\ln\mu} = \left[-\Gamma_{cusp}(\alpha_s)\,\ln\frac{\mu}{\bar{n}p} + \gamma(\alpha_s)\right]C(\mu)$$

where:

- Γ_{cusp} is the universal cusp anomalous dimension from the theory of renormalization of Wilson loops

$$\Gamma_{cusp} = \frac{\alpha_s C_F}{\pi} + \frac{\alpha_s^2 C_F}{4\pi}\left[\left(\frac{67}{9}-\frac{\pi^2}{3}\right)C_A - \frac{10}{9}n_f\right] + \dots$$

- the RGE contains a single $\log\frac{\mu}{\bar{n}p}$ at all orders
- $\gamma(\alpha_s)$ has to be found by direct calculation. From the explicit expression for the WC's we obtain

$$\gamma = -\frac{5\alpha_s C_F}{4\pi} + \dots$$

- The Wilson coefficients contain 2 types of logs:

$$\alpha_s \ln^2\frac{\mu^2}{m_b^2} \longrightarrow \alpha_s^n \ln^{n+1}\frac{\mu^2}{m_b^2} \longrightarrow \gamma^{LO} = -\Gamma^{(1)}_{cusp}\ln\frac{\mu}{\bar{n}p}$$

$$\alpha_s \ln\frac{\mu^2}{m_b^2} \longrightarrow \alpha_s^n \ln^n\frac{\mu^2}{m_b^2} \longrightarrow \gamma^{NLO} = -\Gamma^{(2)}_{cusp}\ln\frac{\mu}{\bar{n}p} + \gamma^{(1)}$$

★ The solution of the RGE is:

$$C_i(\mu) = C_i(m_b)\exp\left[\frac{f_0(\eta)}{\alpha_s(m_b)} + f_1(\eta)\right] \qquad \eta = \frac{\alpha_s(\mu)}{\alpha_s(m_b)}$$

3. Matching of the electromagnetic current

In order to derive e.m. interactions of a collinear $\perp$ photon with soft and collinear fields, we consider invariance under the combined $SU(3) \times U(1)_{e.m.}$ collinear gauge transformations

- $A^{\perp}_{em}\, q_s\, \xi_{hc}$: $\quad e \sum_p e^{-ipx} \left[\bar{\xi} W\right]_p \not{\epsilon}^*_\perp q \qquad$ [from $\mathcal{L}^{(1)}_{\xi q}$]

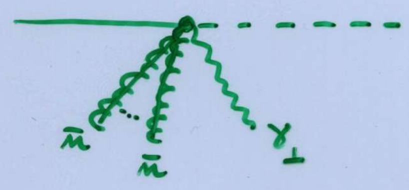
- $A^{\perp}_{em}\, \xi_{hc}\, \xi_{hc}$:

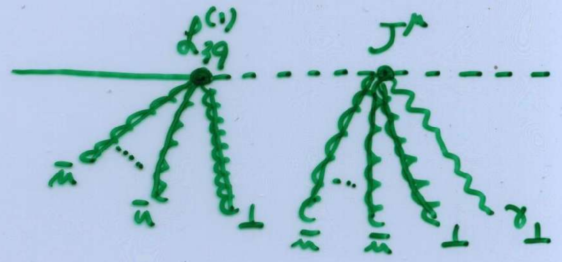
$$e \sum_{p_1 p_2} e^{i(p_2 - p_1)x} \Big\{ \left[\bar{\xi} W\right]_{p_2} \not{\epsilon}^*_\perp \frac{1}{\bar{n}\cdot p} \left[W^\dagger (\not{P}_\perp + g \not{A}_\perp) \frac{\not{\bar{n}}}{2} \xi\right]_{p_1} + \left[\bar{\xi} \frac{\not{\bar{n}}}{2} W (\not{P}^\dagger_\perp + g \not{A}^\dagger_\perp)\right]_{p_2} \frac{1}{\bar{n} p - \bar{n} q} \not{\epsilon}^*_\perp \left[W^\dagger \xi\right]_{p_1} \Big\} \equiv e\, \epsilon^{*\mu}_\perp J_\mu$$

[from $\mathcal{L}^{(0)}_{\xi\xi}$]

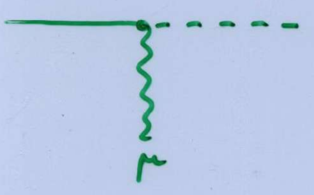
$$\bar{q}_s \not{\epsilon}^*_\perp q_c = \underbrace{\sum_p e^{-ipx}\, \bar{q}\, \not{\epsilon}^*_\perp \left[W^\dagger \xi\right]_p}_{\text{local term}} + \underbrace{i \int d^4y\; T\left(\epsilon^{*\mu}_\perp J_\mu(x)\, \mathcal{L}^{(1)}_{\xi q}(y)\right)}_{\text{non local term}}$$

$[\lambda^3 \quad \lambda] = \lambda^4 \qquad [\lambda^{-4} \quad \lambda^3 \quad \lambda^5] = \lambda^4$

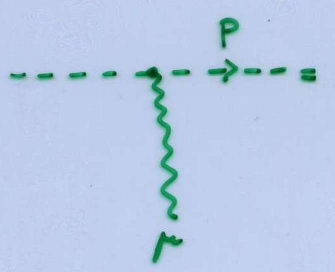




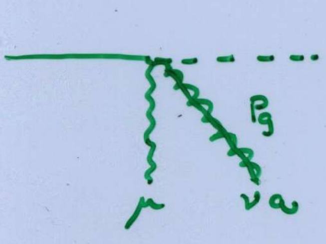
New Feynman rules:



$= -ie\, \gamma^\perp_\mu$



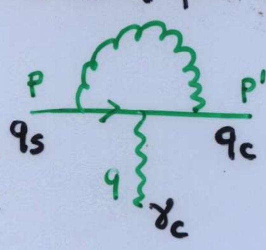
$= e \frac{\not{\bar{n}}}{2}\left[\gamma^\perp_\mu \frac{\not{p}_\perp}{\bar{n}p} + \frac{\not{p}_\perp \gamma^\perp_\mu}{\bar{n}(p-q)}\right]$



$= e\, \gamma^\perp_\mu\, g_s T^a \frac{\bar{n}^\nu}{\bar{n}\cdot p_g}$

1- loop check ✓

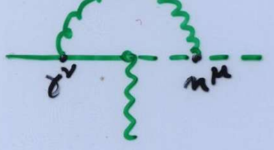
QCD ($p^2 = m^2$; $p'^2 \neq 0$)



$$= -i g_s C_F \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu (\not{p}' + \not{k} + m)\, \not{\epsilon}^*_\perp\, (\not{p} + \not{k} + m)\, \gamma_\mu}{[(p+k)^2 - m^2]\; k^2\; [(p'+k)^2 - m^2]}$$

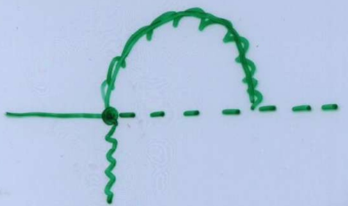
$$= \frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon} + \log\frac{-p'^2}{\mu^2} - 2\log\frac{m^2}{\mu^2}\right]\not{\epsilon}^*_\perp + O\left(\frac{m^2}{(\bar{n}q)^2}, \frac{p'^2}{(\bar{n}q)^2}\right)$$

Soft modes



$$= \frac{\alpha_s C_F}{4\pi}\not{\epsilon}^*_\perp\left[\frac{2}{\epsilon} - \underline{2\log\frac{m^2}{\mu^2}} + 4\right]$$

Collinear modes (local)



$$= -2 i g_s^2 C_F \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{\bar{n}(k-q)\;\not{\epsilon}^*_\perp}{(p-q+k)^2\;\bar{n}k\;k^2}$$

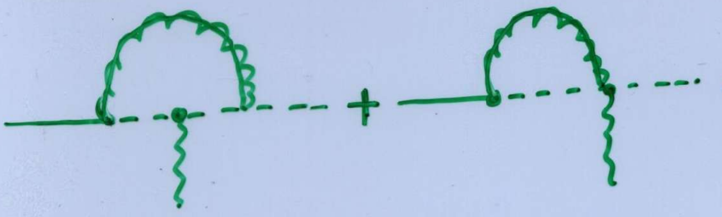
technique: $d^dk = \frac{d\bar{n}k\; dnk}{2}\, d^{d-2}k_\perp$

1. nk integration (nk appears <u>only</u> in denominators)
2. $k_\perp$ integration is standard
3. $\bar{n}k$ integration usually leads to Γ functions

$$= \frac{\alpha_s C_F}{4\pi}\epsilon^*_\perp\left(\frac{2}{\epsilon^2} - \frac{2L-2}{\epsilon} + L^2 - 2L - \frac{\pi^2}{6} + 4\right)$$

with $L = \ln -p'^2/\mu^2$

Collinear modes (non local):



$$= \frac{\alpha_s C_F}{4\pi}\not{\epsilon}^*_\perp\left(-\frac{2}{\epsilon^2} - \frac{3-2L}{\epsilon} - L^2 + 3L + \pi^2/6 - 8\right)$$

$$\Gamma_c = \Gamma_c^{\text{local}} + \Gamma_c^{\text{non-local}} = \frac{\alpha_s C_F}{4\pi}\not{\epsilon}^*_\perp\left(-\frac{1}{\epsilon} + \underline{\log\frac{-p'^2}{\mu^2}} - 4\right)$$

4. Proof of the factorization theorem

$$\mathcal{A} \cong C_i \int d^4x\, e^{i(q-p_2)x} \langle 0| T\left([\bar\xi W]_{p_1} \Gamma_i h_v\right)(0)\, \left(\bar q\, \not\epsilon_\perp^{*} [W^\dagger \xi]_{p_2}\right)(x)\, |\bar B\rangle$$

$$+\, C_i \int d^4x\, d^4y\, e^{iqx} \langle 0| T\left([\bar\xi W]_{p_1} \Gamma_i h_v\right)(0)\, \epsilon_\perp^{*\mu} J_\mu(x)\, \mathcal{L}^{(1)}_{\xi q}(y)\, |\bar B\rangle$$

$\Rightarrow$ decoupling transformation:

$$\xi_p \longrightarrow S\, \xi_p^{(0)}, \quad A^\mu \longrightarrow S A^{(0)\mu} S^\dagger, \quad W \longrightarrow S W^{(0)} S^\dagger$$

- $$\mathcal{A}_{local} \cong \sum C_i \int d^4x\, e^{i(q-p_2)x} \langle 0| T \left\{ [W^{(0)\dagger} \xi^{(0)}]_{p_2}(x)\, [\bar\xi^{(0)} W^{(0)}]_{p_1}(0) \right\}^{\alpha\beta}$$

$$\cdot\, \Gamma_i^{\beta\alpha} \cdot \left\{ S^\dagger h_v(0)\, \bar q S(x) \right\}^{\alpha\gamma} \cdot \not\epsilon_\perp^{*\,\gamma\alpha}\, |\bar B\rangle$$

$$= \sum C_i \int d^4x\, e^{i(q-p_2)x}\, \mathrm{tr}\Big[\langle 0| T\, [W^{(0)\dagger} \xi^{(0)}]_{p_2}(x)\, [\bar\xi^{(0)} W^{(0)}]_{p_1}(0) \underline{|0\rangle}$$

$$\cdot\, \Gamma_i \cdot \underline{\langle 0|} T\, S^\dagger h(0)\, \bar q S(x) |\bar B\rangle \cdot \not\epsilon_\perp^{*} \Big]$$

$$\langle 0| T [W^{(0)\dagger}\xi^{(0)}]_{p_2}(x)\, [\bar\xi^{(0)} W^{(0)}]_{p_1}(0) |0\rangle \equiv i\, \delta_{\bar n p_1, \bar n p_2} \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \frac{\not n}{2}\, J_e(\bar n p_1, k)$$

$$= i \sum C_i \int d^4x\, \underbrace{e^{i(q-p_2)x}}_{\Rightarrow \text{ label conserv.}}\, e^{-ikx} \frac{d^4k}{(2\pi)^4}\, J_e(\bar n p_1, k)\, \mathrm{tr}\Big[\frac{\not n}{2} \Gamma_i \langle 0| T S^\dagger h(0)\, \bar q S(x) |\bar B\rangle \cdot \not\epsilon_\perp^{*} \Big]$$

- $$\mathcal{A}_{non\ local} = \mathcal{A}_{local}\Big|_{J_e \to J_{ne}}$$

$$J_{ne}(\bar n p, k) \longrightarrow \langle 0| T [W^{(0)\dagger} (\not P_\perp + g \not A_\perp^{(0)}) \frac{\not n}{2} \xi^{(0)}]\underline{(y)} \cdot [\bar\xi^{(0)} W]\, \underline{(0)}$$

$$\cdot \Big[[\bar\xi^{(0)} W^{(0)}] \not\epsilon_\perp^{*} \frac{1}{\bar n P} [W^{(0)\dagger} (\not P_\perp + g \not A_\perp^{(0)}) \frac{\not n}{2} \xi^{(0)}]$$

$$+ [\bar\xi^{(0)} \frac{\not n}{2} (\not P_\perp^{\dagger} + g \not A_\perp^{(0)\dagger}) W^{(0)}] \frac{1}{\bar n P - n q} \not\epsilon_\perp^{*} [W^{(0)\dagger} \xi^{(0)}] \Big]\underline{(x)}$$

$$|0\rangle$$

the contractions start at the 1. loop order

What forces us on the lightcone?

The collinear Lagrangian at LO contains only $n\cdot\partial$ derivatives.

$\Longrightarrow$ any $\langle 0|\ldots|0\rangle$ of collinear fields can only depend on $n\cdot k$ at all orders in perturbation theory and at LO in λ.

$$\boxed{J_e(\bar{n}p, k) + J_{ne}(\bar{n}p, k) = J(\bar{n}p, k) = J(\bar{n}p, nk)}$$

$$A = \sum C_i \int \underbrace{d^4x \, \frac{d^4k}{(2\pi)^4} \, e^{-ikx}}_{\delta^2(x_\perp)\,\delta(nx)\,\frac{d\,nk}{2\pi}\,e^{-i\,nk\,\bar{n}x}} \, J(\bar{n}p, \underline{nk}) \, \mathrm{tr}\left[\frac{\not{n}}{2}\Gamma_i \langle 0|T S^\dagger h(0)\, \bar{q}S(x)|\bar{B}\rangle \not{\epsilon}_\perp^*\right]$$

$$= \sum C_i \int d\,nk \; J(\bar{n}p, nk) \int \frac{d\bar{n}x}{2\pi} \, e^{-i\,nk\,\bar{n}x} \, \mathrm{tr}\left[\ldots \; \bar{q}S\left(n^\mu\frac{\bar{n}x}{2}\right)\ldots\right]$$

$$\psi_B(\bar{n}x) \equiv \langle 0|T\, \bar{q}\left(n^\mu\frac{\bar{n}x}{2}\right) S\left(n^\mu\frac{\bar{n}x}{2}, 0\right) h_v(0)|\bar{B}\rangle$$

$$= \int \frac{d\,nk'}{4\pi} \, e^{-i\frac{nk'\,\bar{n}x}{2}} \, \psi_B(nk')$$

$$= \int \frac{d\,nk'}{4\pi} \, e^{-i\frac{nk'\,\bar{n}x}{2}} \, \frac{1+\not{v}}{2}\left[\frac{\not{n}\bar{\not{n}}}{4}\psi_+(nk') + \cancel{\frac{\bar{\not{n}}\not{n}}{4}\psi_-(nk')}\right]\gamma_5$$

$$= \sum \underbrace{C_i}_{m_b^2} \int \frac{d\,nk}{4\pi} \, \underbrace{J(2E_\gamma; nk)}_{\Lambda m_b} \, \underbrace{\psi_+(nk)}_{\Lambda^2} \, \mathrm{tr}\left[\frac{\not{n}}{2}\Gamma_i \frac{1+\not{v}}{2}\gamma_5 \not{\epsilon}_\perp^*\right]$$

Which WC's do contribute ?

$$\Gamma_i = \begin{array}{llll} \gamma^\mu , & v^\mu , & n^\mu & \\ \gamma^\mu\gamma^5 , & v^\mu\gamma^5 , & n^\mu\gamma^5 & \\ \sigma^{\mu\nu} , & n^\mu\gamma^\nu - n^\nu\gamma^\mu , & v_\mu\gamma_\nu - v_\nu\gamma_\mu , & n_\mu v_\nu - n_\nu v_\mu \\ \hline 1 & 2 & 3 & 4 \end{array} \quad \begin{array}{l} V \\ A \\ T \end{array}$$

Only $\Gamma_1^V , \Gamma_1^A , \Gamma_{123}^T$ survive in the trace $\mathrm{tr}\left[\frac{\not n}{2} \Gamma^i \frac{1+\not v}{2} \gamma_5 \not\epsilon_\perp \right]$

Let's define :

$$I(E_\gamma, \mu) = \int \frac{dnk}{4\pi} \, J(2E_\gamma, nk) \, \psi_+(nk)$$

The form factors are :

$$f_V(E_\gamma) = \frac{Q_q}{E_\gamma} \, C_1^{(V)} \, I(E_\gamma, \mu)$$

$$f_A(E_\gamma) = \frac{Q_q}{E_\gamma} \, C_1^{(A)} \, I(E_\gamma, \mu)$$

$$g_+(E_\gamma) = \frac{1}{2} \left\{ C_1^{(T)} + C_2^{(T)} + \left(1 - \frac{E_\gamma}{m_b}\right) C_3^{(T)} \right\} \frac{Q_q}{E_\gamma} \, I(E_\gamma, \mu)$$

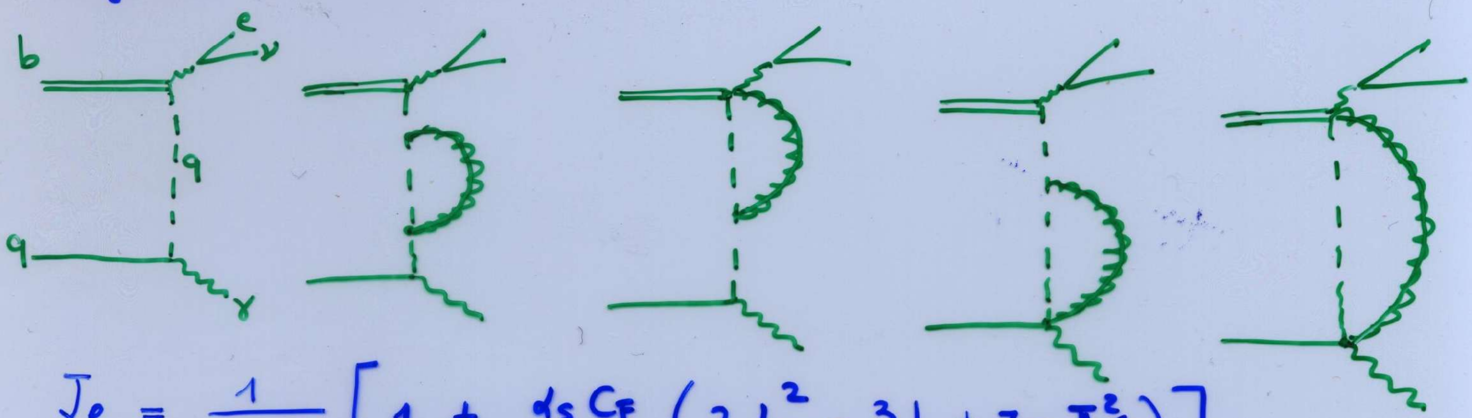
$$g_-(E_\gamma) = -\frac{1}{2} \left\{ C_1^{(T)} + C_2^{(T)} + \left(1 + \frac{E_\gamma}{m_b}\right) C_3^{(T)} \right\} \frac{Q_q}{E_\gamma} \, I(E_\gamma, \mu)$$

$$h(E_\gamma) = \frac{1}{2 m_B^2} \, C_3^{(T)} \, \frac{Q_q}{E_\gamma} \, I(E_\gamma, \mu)$$

moreove $C_1^{(V)} = C_1^{(A)}$ at all orders in perturbation theory

Calculation of the Jet functions

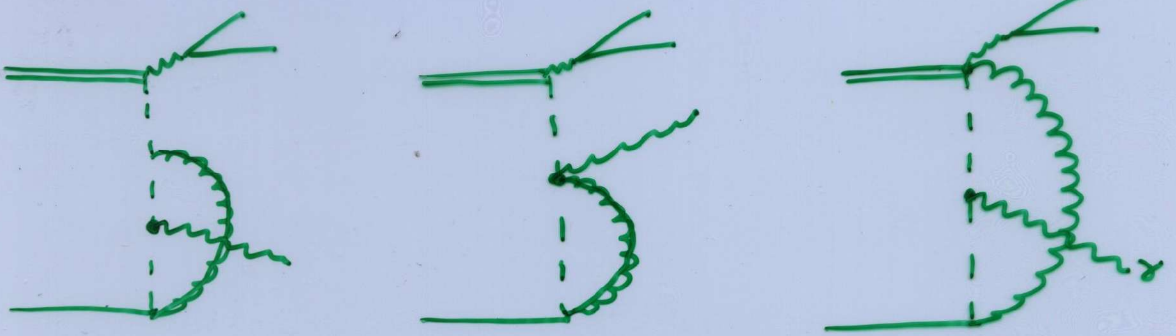
- diagrams that contribute to J_e :



$$J_e = \frac{1}{nk}\left[1 + \frac{\alpha_s C_F}{4\pi}\left(2L^2 - 3L + 7 - \frac{\pi^2}{3}\right)\right]$$

$$L = \log\left(\frac{2E_\gamma\, nk}{\mu^2}\right)$$

- diagrams that contribute to J_{ne} :



$$J_{ne} = \frac{1}{nk}\,\frac{\alpha_s C_F}{4\pi}\left(-L^2 + 3L - 8 + \frac{\pi^2}{6}\right)$$

- the residual logs are small for $\mu \sim O(\Lambda m_b)$ as expected.

Completing the factorization proof

- It is essential to show that the convolution $I(2E_\gamma, \mu)$ is <u>free of endpoint singularities.</u>

  Their appearance signals an incorrect identification of the IR degrees of freedom.

  Their absence can be taken as a proof that everything has been accounted for.

- If $I(E_\gamma, \mu)$ were log-divergent at $nk \to 0$ we would have leading power corrections from regions where intermediate propagators are collinear ($p^2 \sim \lambda^2$).

  We should then include collinear fields in the analysis and match $SCET_I \longrightarrow SCET_{II}$ (instead of HQET); the form factors would become sensitive to the hadronic structure of the photon and factorization would break down.

- Let's consider $\int dnk\, J(nk)\, \underbrace{\phi_+(nk)}_{\phi_+ \sim nk \text{ for } nk \to 0}$

  - at $nk \to \infty$ we cannot have problems because of confinement
  - at $nk \to 0$ we must study the function $J(\bar{n}p, nk)$.

    $\langle 0 | \xi\, \bar{\xi} | 0 \rangle$ has dim $= 3$ and λ-scaling $= \lambda^2$

    $$\cong \int \underbrace{d^4k}_{\substack{\text{dim}=4 \\ \lambda\text{-scal}=\lambda^4}}\, J\left(\frac{p^2}{\mu^2}, nk\right) e^{ikx}$$

    $$\Longrightarrow \begin{cases} \text{dim}(J) = -1 \\ \lambda\text{-scal}(J) = \lambda^{-2} \end{cases}$$

    using nk and $\frac{2E_\gamma nk}{\mu^2} \sim O(1)$, the only possibility is

    $$J(2E_\gamma, nk) \sim \frac{\log^n \frac{2E_\gamma nk}{\mu^2}}{nk} \Longrightarrow \text{no endpoint singularities}$$

5. Resummation of the remaining logs

If we are interested in the evaluation of the form factors using a model for the LCDA of the B meson it might be useful to lower the scale μ from $\Lambda m_b \longrightarrow$ few $\times \Lambda_{QCD}$ and make it independent of the b-quark mass and the photon energy.

- **basic idea:** [Bosch-Hill-Lange-Neubert]

  The decay amplitude $C_i \; J * \phi_{B+}$ is scale independent. This links the scale dependence of $T = C_i \cdot J$ to the one of the LCDA ϕ_+ (Grozin-Neubert):

$$\frac{dT(nk,\mu)}{d\ln\mu} = \left[\Gamma_{cusp}(\alpha_s) \; \ln\frac{\mu}{nk} + \underbrace{\gamma(\alpha_s)}\right] T + \int_0^{\infty} d\omega \; nk \; \underbrace{\Gamma(\omega, nk)} \; T(\omega)$$

direct calculation

One has to solve the RGE for the WC's C_i, run them down to $\mu \simeq \Lambda m_b$, multiply by J and use this hard scattering as initial condition for the running $\mu \simeq \Lambda m_b \longrightarrow \mu \simeq$ few $\times \Lambda$