Introduction to non-perturbative Heavy Quark Effective Theory

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Helmholtz International Summer School
HEAVY QUARK PHYSICS

Bogoliubov Laboratory of Theoretical Physics
JINR, Dubna, Russia

June 6 – 16, 2005

Outline

LECTURE 1: *Non-perturbative formulation of HQET*

- Motivation
- Basics of HQET as an *effective* theory of QCD
- Non-perturbative formulation of HQET
- Matching of HQET and QCD in finite volume

LECTURE 2: *Applications*

- Tests of HQET in finite volume
- Advances in B-physics applications: $M_{\rm b}$ and $F_{\rm B_s}$
- Status of (quenched) physics results
- Perspectives

LECTURE 1

Non-perturbative formulation of HQET

B-physics from the lattice ...
... and the need for recoursing to an *effective* theory

Lattice QCD calculations with b-quarks

- valuably contribute to precision CKM-physics (unitarity triangle)
- provide an ‘ab initio’ approach to determine experimentally inaccessible key parameters such as
  - the b-quark mass, M_{b}
  - B-meson decay constants, e.g.

$$\langle \mathrm{B_s}(p) \,|\, [\,\overline{\psi}_{\mathrm{s}}\gamma_{\mu}\gamma_5\psi_{\mathrm{b}}\,](0)\,|\, 0 \,\rangle = ip_{\mu}F_{\mathrm{B_s}}$$

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Challenge of a realistic treatment of lattice B-systems:

- The b-quark is too heavy $\Leftrightarrow$ highly localized
  - Very fine lattice resolutions (*not* $m_{\mathrm{b}}^{-1} \simeq (4\,\mathrm{GeV})^{-1} < a \simeq 0.07\,\mathrm{fm}$)
  - in (at the same time) physically large volumes required

$\Rightarrow$ Direct numerical simulation still beyond today's computing resources

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Viable framework for heavy quarks in the lattice regularization:
Effective theories $\longrightarrow$ NRQCD
HQET (even took its origin for the lattice [Eichten, 1988])

Lattice QCD

'Ab initio' approach to determine phenomenologically relevant key parameters

$$\mathcal{L}_{\mathrm{QCD}}\left[g_0, m_f\right] = -\frac{1}{2g_0^2}\,\mathrm{Tr}\left\{F_{\mu\nu}F_{\mu\nu}\right\} + \sum_{f=\mathrm{u,d,s},\ldots} \overline{\psi}_f\left\{\gamma_\mu\left(\partial_\mu + g_0 A_\mu\right) + m_f\right\}\psi_f$$

$$\underbrace{\begin{bmatrix} F_\pi \\ m_\pi \\ m_{\mathrm{K}} \\ m_{\mathrm{D}} \\ m_{\mathrm{B}} \end{bmatrix}}_{\text{Experiment}} \quad \overset{\mathcal{L}_{\mathrm{QCD}}[g_0,m_f]}{\Longrightarrow} \quad \underbrace{\begin{bmatrix} \Lambda_{\mathrm{QCD}} \\ \frac{1}{2}\left(M_{\mathrm{u}} + M_{\mathrm{d}}\right) \\ M_{\mathrm{s}} \\ M_{\mathrm{c}} \\ M_{\mathrm{b}} \end{bmatrix}}_{\text{QCD parameters (RGIs)}} + \underbrace{\begin{bmatrix} F_{\mathrm{D}} \\ F_{\mathrm{B}} \\ B_{\mathrm{B}} \\ \xi \\ \ldots \end{bmatrix}}_{\text{Predictions}}$$

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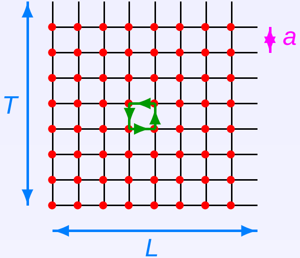
$$\underbrace{\begin{bmatrix} F_\pi \\ m_\pi \\ m_{\rm K} \\ m_{\rm D} \\ m_{\rm B}\end{bmatrix}}_{\text{Experiment}} \xRightarrow{\mathcal{L}_{\rm QCD}[g_0,m_f]} \underbrace{\begin{bmatrix} \Lambda_{\rm QCD} \\ \frac{1}{2}(M_{\rm u}+M_{\rm d}) \\ M_{\rm s} \\ M_{\rm c} \\ M_{\rm b}\end{bmatrix}}_{\text{QCD parameters (RGIs)}} + \underbrace{\begin{bmatrix} F_{\rm D} \\ F_{\rm B} \\ B_{\rm B} \\ \xi \\ \ldots\end{bmatrix}}_{\text{Predictions}}$$

$\xRightarrow{\mathcal{L}_{\rm QCD}[g_0,m_f]}$ means discretization with:

- Gauge invariance
- Locality
- Unitarity

$U_\mu(x) = {\rm e}^{iag_0A_\mu(x)}$ $\quad \psi(x)$

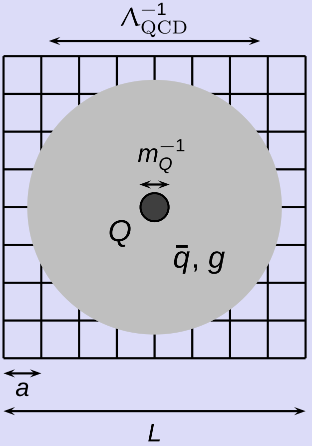


Issues/Obstacles:

- Renormalization
- Continuum limit (CL)
- ...
- $\mathrm{O}\left(1/\sqrt{t_{\rm CPU}}\right)$ errors

Typical momentum scales in heavy-light and heavy-heavy mesons:

Heavy-light ($Q\overline{q}$) $\longrightarrow$ **HQET**



- Q almost at rest at bound state's center, surrounded by light DOFs
- Motion of the heavy quark is suppressed by Λ_{QCD}/m_Q

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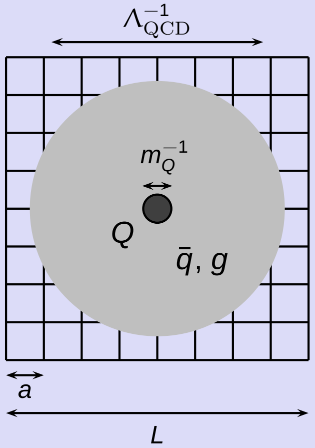

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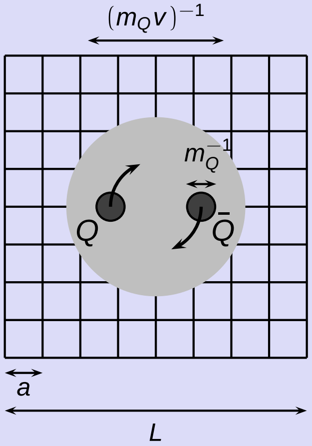
Heavy-heavy ($\mathrm{Q}\overline{\mathrm{Q}}$) $\longrightarrow$ NRQCD



- Non-relativistic kinetic and the potential energy to be balanced
- Separate: m_{Q}, $\langle p \rangle \simeq m_{\mathrm{Q}} v$ and binding energy $\langle p^2 \rangle / m_{\mathrm{Q}} \simeq m_{\mathrm{Q}} v^2$

Problems with lattice regularized HQET

In the past: Difficulties/Limitations on the

theoretical side

At each order in $\frac{1}{m}$, new parameters arise in the effective theory, which (due to mixings among operators of different dimensions) leave power divergences in the lattice spacing if only estimated *perturbatively*

$\Rightarrow$ *Continuum limit does not exist*

technical side

Rapid growth of statistical errors as the time separation of B-meson correlation functions increases:

$$S_{\mathrm{h}}^{\mathrm{Eichten-Hill}} = a^4 \sum_x \overline{\psi}_{\mathrm{h}}(x) D_0 \psi_{\mathrm{h}}(x)$$

$$\frac{\mathrm{noise}}{\mathrm{signal}} \propto \exp(x_0 \Delta) \qquad \begin{aligned} \Delta &= E_{\mathrm{stat}} - m_\pi \\ E_{\mathrm{stat}} &\sim e_1 \times g_0^2/a \end{aligned}$$

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Progress by two recent developments:

Non-perturbative renormalization of HQET through its *non-perturbative matching to QCD in finite volume*

[H. & Sommer, 2004]

Alternative discretizations of HQET, leading to a substantial reduction of statistical fluctuations in correlators

[ALPHA Collaboration , Della Morte et al., 2003 & 2005]

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2g_0^2}\,{\rm Tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \overline{\psi}_f\,\{\gamma_\mu\,(\partial_\mu + g_0 A_\mu) + m_f\}\,\psi_f$$

Consider:
Energies & matrix elements of states containing a single b-quark at rest

HQET Lagrangian by formal $1/m_{\rm b}$–expansion of continuum QCD

$$\overline{\psi}_{\rm b}\,\{\gamma_\mu D_\mu + m_{\rm b}\}\,\psi_{\rm b} \quad \rightarrow \quad \mathcal{L}_{\rm stat} + \mathcal{L}^{(1)} + \ldots$$

$$\mathcal{L}_{\rm stat}(x) = \overline{\psi}_{\rm h}(x)\,\{D_0 + \delta m\}\,\psi_{\rm h}(x)$$

- 4–component effective heavy quark field $\psi_{\rm h}$ with constraint

  $$P_+\psi_{\rm h} = \psi_{\rm h} \qquad \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h} \qquad P_+ = \tfrac{1}{2}\,(1+\gamma_0) \qquad \Rightarrow \qquad 2 \text{ d.o.f.}$$

- Composite fields involving b-quarks translate to the effective theory:

  $$A_0(x) = Z_{\rm A}\overline{\psi}_{\rm l}(x)\gamma_0\gamma_5\psi_{\rm b}(x) \quad \rightarrow \quad A_0^{\rm stat} = Z_{\rm A}^{\rm stat}\overline{\psi}_{\rm l}(x)\gamma_0\gamma_5\psi_{\rm h}(x)$$

  $Z_{\rm A}$, $Z_{\rm A}^{\rm stat}$: renormalization constants of the axial currents

- Expansion is accurate for heavy quark masses $m \equiv m_{\rm h} \gg \Lambda_{\rm QCD}$, yields valid description for low-lying energy levels & matrix elements

Example

$$\Phi^{\mathrm{QCD}} \equiv F_{\mathrm{B}}\sqrt{m_{\mathrm{B}}} = Z_{\mathrm{A}}\langle\, \mathrm{B}\,|\, A_0\,|\, 0\,\rangle$$

- Scale independent due to the chiral symmetry of (massless) QCD
- In HQET: chiral symmetry absent $\Rightarrow$ $Z_{\mathrm{A}}^{\mathrm{stat}} = Z_{\mathrm{A}}^{\mathrm{stat}}(\mu)$

Rather than $\Phi^{\mathrm{stat}}(\mu) \equiv Z_{\mathrm{A}}^{\mathrm{stat}}(\mu)\langle\, \mathrm{B}\,|\, A_0^{\mathrm{stat}}\,|\, 0\,\rangle$, focus on the μ & scheme independent renormalization group invariant (RGI) matrix element

$$\Phi_{\mathrm{RGI}} = \lim_{\mu\to\infty} \left[\, 2b_0\bar{g}^2(\mu)\,\right]^{-\gamma_0/(2b_0)} \times \Phi^{\mathrm{stat}}(\mu)$$

Example

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$\Rightarrow$ Generic form of the HQET-expansion of the QCD matrix elements:

$$\begin{aligned}
\Phi^{\rm QCD} &= C_{\rm PS}\,(M_{\rm b}/\Lambda_{\overline{\rm MS}}) \times \Phi_{\rm RGI} + {\rm O}\,(1/M_{\rm b}) \\
M_{\rm b} &= \lim_{\mu\to\infty} \left[\, 2b_0\bar{g}^2(\mu)\,\right]^{-d_0/(2b_0)} \times \overline{m}_{\rm b}(\mu) \\
\Lambda_{\overline{\rm MS}} &= \lim_{\mu\to\infty} \mu \left[\, b_0\bar{g}^2_{\overline{\rm MS}}(\mu)\,\right]^{-b_1/(2b_0^2)} {\rm e}^{-1/(2b_0\bar{g}^2_{\overline{\rm MS}}(\mu))}
\end{aligned}$$

with $\beta(\bar{g}) = \mu\,(\partial\bar{g}/\partial\mu) = -b_0\bar{g}^3 + {\rm O}(\bar{g}^5)$ and associated anomalous dimensions

$$\tau(\bar{g}) = \frac{\mu}{\overline{m}}\,\frac{\partial\overline{m}}{\partial\mu} = -d_0\bar{g}^2 + {\rm O}(\bar{g}^4) \qquad \gamma(\bar{g}) = \frac{\mu}{Z_{\rm A}^{\rm stat}}\,\frac{\partial Z_{\rm A}^{\rm stat}}{\partial\mu} = -\gamma_0\bar{g}^2 + {\rm O}(\bar{g}^4)$$

What is the meaning of $C_{\mathrm{PS}}(M_{\mathrm{b}}/\Lambda_{\overline{\mathrm{MS}}})$?

Conversion to the *matching* scheme

To extract QCD predictions from results obtained in the (static) effective theory, its RGIs must be related to QCD observables at finite quark mass

$\Leftrightarrow$ Translation to another renormalization scheme:
The *matching* scheme — defined by the condition that for arbitrary renormalized matrix elements Φ in QCD and in the effective theory

$$\Phi^{\mathrm{QCD}} = \Phi^{\mathrm{HQET}}(\mu)\big|_{\mu=m} + \mathrm{O}(1/m)$$

(in PT, one typically identifies $m = m_{\mathrm{Q}} = \text{pole mass}$ or $m = \bar{m}_{*} = \overline{\mathrm{MS}}\ \text{mass}$)

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In case of the static axial current:

$$\Phi^{\mathrm{QCD}} = C_{\mathrm{match}}(m_{\mathrm{b}}/\mu) \times \Phi_{\overline{\mathrm{MS}}}(\mu) + \mathrm{O}(1/m_{\mathrm{b}}) \qquad (\star)$$

- $\Phi_{\overline{\mathrm{MS}}}(\mu)$: renormalized in HQET in the $\overline{\mathrm{MS}}$ scheme
- $C_{\mathrm{match}}(m_{\mathrm{b}}/\mu)$:
  Matching coefficient depending on m_{b}, defined by $\overline{m}_{\overline{\mathrm{MS}}}(m_{\mathrm{b}}) = m_{\mathrm{b}}$
- Once C_{match} is determined (usually in PT) such that $(\star)$ holds for some particular current matrix element, it applies to *all* of them

Change to a more convenient argument of the conversion function via

$$\frac{\Phi_{\mathrm{RGI}}}{\Phi_{\overline{\mathrm{MS}}}(\mu)} = \left[2b_0\bar{g}^2(\mu)\right]^{-\gamma_0/(2b_0)} \exp\left\{\int_0^{\bar{g}(\mu)} \mathrm{d}g \left[\frac{\gamma_{\overline{\mathrm{MS}}}(g)}{\beta_{\overline{\mathrm{MS}}}(g)} - \frac{\gamma_0}{b_0 g}\right]\right\} \quad [\bar{g} = \bar{g}_{\overline{\mathrm{MS}}}]$$

and choosing the arbitrary renormalization point as $\mu = m_{\mathrm{b}}$

$$\Rightarrow \qquad C_{\mathrm{PS}}\left(M_{\mathrm{b}}/\Lambda_{\overline{\mathrm{MS}}}\right) = C_{\mathrm{match}}(1) \times \frac{\Phi_{\overline{\mathrm{MS}}}(\mu)}{\Phi_{\mathrm{RGI}}} =$$

$$\left[2b_0\bar{g}^2(m_{\mathrm{b}})\right]^{\gamma_0/(2b_0)} \exp\left\{\int_0^{\bar{g}(m_{\mathrm{b}})} \mathrm{d}g \left[\frac{\gamma^{\mathrm{match}}(g)}{\beta_{\overline{\mathrm{MS}}}(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

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- $C_{\rm PS}$ 'defines' the anomalous dimension $\gamma^{\rm match}$ in the matching scheme:

  $$\gamma^{\rm match}(\bar{g}) = \gamma^{\overline{\rm MS}}(\bar{g}) + \rho(\bar{g})$$

  with a contribution $\rho(\bar{g})$ from $C_{\rm match}$

- advantages of the *ratio* of RGIs M/Λ:
  - can be fixed in lattice calculations without perturbative uncertainties
  - $C_{\rm PS}$ independent of the choice of scheme for the effective operators

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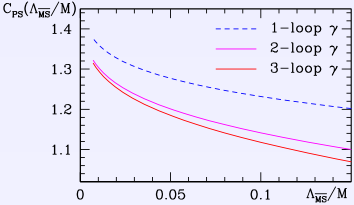

✓ weak logarithmic mass dependence

✓ PT under control ⇐ 3-loop AD [Chetyrkin & Grozin, 2003]

✓ remaining $O(\bar{g}^6(m_{\rm b}))$ errors small

Non-perturbative formulation of HQET

Let the effective theory be regularized on a space-time lattice

$$S_{\rm HQET} = a^4 \sum_x \left\{ \mathcal{L}_{\rm stat}(x) + \sum_{\nu=1}^{n} \mathcal{L}^{(\nu)}(x) \right\} \qquad \mathcal{L}^{(\nu)}(x) = \sum_i \omega_i^{(\nu)} \mathcal{L}_i^{(\nu)}(x)$$

with static action $\mathcal{L}_{\rm stat}(x) = \overline{\psi}_{\rm h}(x)\,[\,\nabla_0^* + \delta m\,]\,\psi_{\rm h}(x)$ and the $1/m$ –parts

$$\mathcal{L}_1^{(1)} = \overline{\psi}_{\rm h}\Big(-\tfrac{1}{2}\,\boldsymbol{\sigma}\cdot\mathbf{B}\Big)\psi_{\rm h}$$ $\longrightarrow$ chromomagnetic interaction with the gluon field

$$\mathcal{L}_2^{(1)} = \overline{\psi}_{\rm h}\Big(-\tfrac{1}{2}\,\mathbf{D}^2\Big)\psi_{\rm h}$$ $\longrightarrow$ kinetic energy from the heavy quark's residual motion

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coefficients $\omega = \omega(g_0, m)$

- must be determined such that HQET matches QCD
- at the classical level this fixes

$$\omega_1^{(1)} = \omega_2^{(1)} = 1/m + \mathrm{O}\left(g_0^2\right) \qquad \delta m = 0 + \mathrm{O}\left(g_0^2\right)$$

(Removal of $m\,\overline{\psi}_{\rm h}\psi_{\rm h}$ from the action, corresponding to a universal energy shift, reflects the heavy-light dynamics' independence of the scale m at lowest order)

Insert: Derivation of the HQET Lagrangian

Start from the Euclidean Dirac-Lagrangian in the continuum

$$\begin{aligned} \mathcal{L} &= \overline{\psi}(D_\mu \gamma_\mu + m)\psi = \psi^\dagger \mathcal{D} \psi \\ \mathcal{D} &\equiv m\gamma_0 + D_0 + \gamma_0 D_k \gamma_k \end{aligned}$$

and perform a field rotation (i.e. a Foldy-Wouthuysen-Tani transformation) to decouple 'large' and 'small' components:

$$\psi \to \phi = \mathrm{e}^{S}\psi \qquad \psi^\dagger \to \phi^\dagger = \psi^\dagger \mathrm{e}^{-S}$$

$\Rightarrow$ $$\mathcal{L} = \phi^\dagger \mathcal{D}' \phi$$

with $$\mathcal{D}' = \mathrm{e}^{S}\mathcal{D}\mathrm{e}^{-S}$$

and $$S \equiv \frac{1}{2m} D_k \gamma_k = -S^\dagger = \mathrm{O}\left(\frac{1}{m}\right) \qquad \left[\, \mathcal{D} = \mathrm{O}\,(m) \,\right]$$

In this way the $D_k\gamma_k$–term is rotated away

Classical theory:
One has smooth fields and thus can count

$$D_\mu = \mathrm{O}\left(\left[\tfrac{1}{m}\right]^0\right)$$

so that it makes sense to expand in $1/m$

$$\begin{aligned}
\mathcal{D}' &= \mathcal{D} + \frac{1}{2m}\left[D_k\gamma_k, \mathcal{D}\right] + \frac{1}{8m^2}\left[D_l\gamma_l, [D_k\gamma_k, \mathcal{D}]\right] + \mathrm{O}\left(1/m^2\right) \\
&= \mathcal{D} + \frac{1}{2m}\left[D_k\gamma_k, \mathcal{D}\right] - \frac{1}{4m}\left[D_l\gamma_l, \gamma_0 D_k\gamma_k\right] + \mathrm{O}\left(1/m^2\right) \\
&= \gamma_0\left\{\gamma_0 D_0 + m + \frac{1}{2m}\left(-D_k D_k - \frac{1}{2i}F_{kl}\sigma_{kl}\right) + \frac{1}{2m}F_{k0}\gamma_0\gamma_k\right\} \\
&\quad + \mathrm{O}\left(1/m^2\right)
\end{aligned}$$

$$\mathcal{L} = \mathcal{L}_{\mathrm{h}}^{\mathrm{stat}} + \mathcal{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} + \frac{1}{2m}\left\{\mathcal{L}_{\mathrm{h}}^{(1)} + \mathcal{L}_{\bar{\mathrm{h}}}^{(1)} + \mathcal{L}_{\mathrm{h}\bar{\mathrm{h}}}^{(1)}\right\} + \mathrm{O}\left(1/m^2\right)$$

Here we have introduced

$$\mathcal{L}_{\mathrm{h}}^{\mathrm{stat}} = \overline{\psi}_{\mathrm{h}}(D_0 + m)\psi_{\mathrm{h}}$$
$$\mathcal{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} = \overline{\psi}_{\bar{\mathrm{h}}}(D_0 - m)\psi_{\bar{\mathrm{h}}}$$
$$\mathcal{L}_{\mathrm{h}}^{(1)} = \overline{\psi}_{\mathrm{h}}\left(-D_k D_k - \frac{1}{2i}F_{kl}\,\sigma_{kl}\right)\psi_{\mathrm{h}} = \overline{\psi}_{\mathrm{h}}\left(-\mathbf{D}^2 - \mathbf{B}\boldsymbol{\sigma}\right)\psi_{\mathrm{h}}$$

with

$$P_+\psi_{\mathrm{h}} = \psi_{\mathrm{h}} \qquad \overline{\psi}_{\mathrm{h}}P_+ = \overline{\psi}_{\mathrm{h}} \qquad P_\pm = \frac{1 \pm \gamma_0}{2}$$
$$P_-\psi_{\bar{\mathrm{h}}} = \psi_{\bar{\mathrm{h}}} \qquad \overline{\psi}_{\bar{\mathrm{h}}}P_- = \overline{\psi}_{\bar{\mathrm{h}}}$$
$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \qquad F_{kl} = [D_k, D_l]$$

- $\mathcal{L}_{\mathrm{h}\bar{\mathrm{h}}}^{(1)}$ – terms may be dropped in $\mathcal{L}$ at the order considered
- The expressions are discretized in a straightforward way:

  $D_0 \to \nabla_0^*$: backward lattice derivative $\qquad D_k D_k \to \nabla_k^* \nabla_k \qquad F_{kl} \to \widehat{F}_{ij}$
- The prefactors of the various operators are to be determined by a non-trivial matching of HQET and QCD in the quantum theory

Eichten-Hill action for static quarks on the lattice:

$$S_{\rm h}[U,\overline{\psi}_{\rm h},\psi_{\rm h}] = a^4\,\frac{1}{1+a\,\delta m}\,\sum_x \overline{\psi}_{\rm h}(x)\,(\nabla_0^* + \delta m)\,\psi_{\rm h}(x)$$

$$\nabla_0^*\psi_{\rm h}(x) = \frac{1}{a}\left[\,\psi_{\rm h}(x) - U^\dagger(x-a\hat{0},0)\psi_{\rm h}(x-a\hat{0})\,\right]$$

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- Static quarks propagate only forward in time
  $\Rightarrow$ Associated quark propagator reads

$$\begin{aligned} S_{\rm h}(x,y) &= U(x-a\hat{0},0)^{-1}\,U(x-2a\hat{0},0)^{-1}\cdots U(y,0)^{-1} \\ &\quad\times\theta(x_0-y_0)\delta(\mathbf{x}-\mathbf{y})(1+a\,\delta m)^{-(x_0-y_0)/a}\,P_+ \end{aligned}$$

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  (timelike Wilson line, δm cancels divergence in static quark's self-energy)

- $\mathrm{O}(a)$ improvement:
  Preserving on the lattice the symmetries of the static theory
  - heavy quark spin-symmetry,
  - local conservation of heavy quark flavour number
  - plus gauge invariance, parity and cubic symmetry

  guarantees that both universality class and $\mathrm{O}(a)$ improvement are unchanged w.r.t. the Eichten-Hill action, i.e. the static-light action is already improved if the light quark sector is
  [Kurth & Sommer, 2001]

Correlation functions of composite fields ...

... are of interest for applications involving transition matrix elements

Example

Expansion of the (time component of the) axial current in HQET:

$$A_0^{\mathrm{HQET}}(x) = \sum_{\nu=0}^{n} \mathcal{A}^{(\nu)}(x)$$

$$\mathcal{A}^{(0)}(x) = \alpha_0^{(0)} A_0^{\mathrm{stat}}(x) \qquad A_0^{\mathrm{stat}}(x) = \overline{\psi}_{\mathrm{l}}(x)\gamma_0\gamma_5\psi_{\mathrm{h}}(x)$$

$$\mathcal{A}^{(\nu)}(x) = \sum_i \alpha_i^{(\nu)} \mathcal{A}_i^{(\nu)}(x) \qquad \nu > 0$$

where ψ_{l} denotes a light quark field and $\mathcal{A}_i^{(\nu)}$ is of mass dimension $3+\nu$

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\end{aligned}
$$

where ψ_{l} denotes a light quark field and $\mathcal{A}_i^{(\nu)}$ is of mass dimension $3+\nu$

Then, for the correlator [with $(\overline{\psi}_i \Gamma \psi_j)^\dagger \equiv \overline{\psi}_j \gamma_0 \Gamma^\dagger \gamma_0 \psi_i$]

$$
C_{\mathrm{AA}}^{\mathrm{HQET}}(x_0) = a^3 \sum_{\mathbf{x}} \left\langle A_0^{\mathrm{HQET}}(x) \left(A_0^{\mathrm{HQET}}\right)^\dagger(0) \right\rangle
$$

the leading and subleading terms at the classical level are given by

$$
\alpha_0^{(0)} = 1 \qquad \mathcal{A}_1^{(1)} = \overline{\psi}_{\mathrm{l}}\gamma_j\gamma_5 \overleftarrow{D}_j \psi_{\mathrm{h}} \qquad \alpha_1^{(1)} = 1/m
$$

Expectation values

At the quantum level:
Expectation values are defined via the path integral representation

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, O[\varphi]\, \mathrm{e}^{-(S_{\mathrm{rel}}+S_{\mathrm{HQET}})} \qquad \mathcal{Z} = \int \mathcal{D}[\varphi]\, \mathrm{e}^{-(S_{\mathrm{rel}}+S_{\mathrm{HQET}})}$$

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Important element in the definition of the effective field theory

It is understood that the *integrand* of the path integral is expanded in a *power series* in $1/m$ with power counting according to

$$\omega_i^{(\nu)} = \mathrm{O}\left(1/m^{\nu}\right) \qquad \alpha_i^{(\nu)} = \mathrm{O}\left(1/m^{\nu}\right)$$

$\Rightarrow$ Replace

$$\begin{aligned}
&\exp\{-(S_{\rm rel} + S_{\rm HQET})\} = \\
&\qquad \exp\left\{-\left(S_{\rm rel} + a^4 \textstyle\sum_x \mathcal{L}_{\rm stat}(x)\right)\right\} \\
&\qquad \times \left\{1 - a^4 \textstyle\sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2}\left[a^4 \sum_x \mathcal{L}^{(1)}(x)\right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots\right\}
\end{aligned}$$

⇒ $1/m$ –terms appear only as insertions of local operators $\mathcal{O}_i^{(\nu)}(x)$ and $\mathcal{A}_i^{(\nu)}(x)$ into correlators, while the true PI average is taken w.r.t. the action in the static approximation for the heavy quark

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Discussion of the renormalization properties of lattice HQET

Power counting arguments:

- Static effective theory expected to be *renormalizable*, requiring a finite number of parameters to be fixed to obtain a continuum limit
  (Note: With one of the $1/m$–terms kept in the exponent, as in NRQCD, renormalizability would be lost!)
- Consequences for renormalization of EVs $\langle O \rangle$ after inserting the expanded form of $\exp\{-(S_{\rm rel} + S_{\rm HQET})\}$:
  - ✓ Problem of renormalizing correlation functions of local composite operators in the static effective theory

⇒ Conclusion:
Upon inclusion of all local operators with proper symmetries and dimensions up to that of the highest-dimensional one ($\nu \leqslant n$), their coefficients may be chosen so that all EVs have a continuum limit

$\Rightarrow$ *HQET truncated at any finite order in $1/m$ is renormalizable*

Crucial for the lattice theory, because this means that the CL exists and is independent of the details of the lattice formulation (universality)

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Formally:
The effective field theory is now defined in terms of the parameter set

$$\mathcal{C}_{\mathrm{HQET}} \equiv \{c_k\} = \mathcal{C}_{N_\mathrm{f}-1} \cup \left\{\delta m\right\} \cup \left\{\omega_i^{(\nu)}\right\} \cup \left\{\alpha_j^{(\nu)}\right\} \cup \ldots \qquad c_1 \equiv g_0^2$$

which for $k > 1$ must be adjusted as function of g_0^2 to get a decent CL
(i.e. renormalizations of composite fields are among $\mathcal{C}_{\mathrm{HQET}}$, e.g. $\alpha_0^{(0)} \equiv Z_{\mathrm{A}}^{\mathrm{stat}}$)

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- Since the terms in S_HQET are organized just by their mass dimension, the existence of a CL (non-perturbative renormalizability) is equivalent to expect that composite operators mix only with same- and lower-dimensional ones
- Generally, as the $1/m$ – and a –expansion aren't independent but regarded as one expansion in the dimension of $\mathcal{L}_i^{(\nu)}, \mathcal{A}_i^{(\nu)}$, count $a = \mathrm{O}(1/m)$ and start with all $\mathcal{O}_i^{(\nu)}$ of given dimension, restricted only by lattice symmetries
- In particular: S_rel has to be $\mathrm{O}(a)$ improved to go to order $1/m$

Caveat: Operator mixing induces power divergences

Mixings are allowed between operators of different dimensions, e.g.

$$\mathcal{O}_{\mathrm{R}}^{\mathrm{d}=5} = \sum_k z_k \, \mathcal{O}_k^{\mathrm{d}=5} + \sum_k c_k \, \mathcal{O}_k^{\mathrm{d}=4} \qquad c_k = a^{-1} \times \left\{ c_k^{(0)} + c_k^{(1)} g_0^2 + \dots \right\}$$

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Perturbative precision insufficient to determine the coefficients $\{c_k\}$

$\Rightarrow$ Power-law divergences, remainders $\sim a^{-p}$, i.e. *no* continuum limit

example: at the static level a linearly divergent, additive mass counterterm

$$\delta m = (c_1 g_0^2 + \ldots)/a$$

originates from the mixing of $\overline{\psi}_{\mathrm{h}} D_0 \psi_{\mathrm{h}}$ with $\overline{\psi}_{\mathrm{h}} \psi_{\mathrm{h}}$

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in general: since the lattice spacing decreases as

$$a, \sim \exp\left\{-1/(2b_0 g_0^2)\right\} \quad \text{for small bare gauge coupling } g_0$$

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$\Rightarrow$ *Non-perturbative method needed to determine (at least some) $\{c_k\}$*

Matching of HQET and QCD

Implication: Non-perturbative renormalization of the theory required

From the discussion so far we infer:
HQET is an approximation to QCD when the coefficients $\{c_k\}$ are chosen correctly such that

$$\Phi^{\mathrm{HQET}}(M) = \Phi^{\mathrm{QCD}}(M) + \mathrm{O}\left(1/[r_0 M]^{n+1}\right)$$

M = RGI (heavy) quark mass to be free of any renormalization scheme dependence

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Example

for a quantity $\Phi^{\rm QCD}$: Correlation function of the heavy-light axial current

$$C_{\rm AA}(x_0) = Z_{\rm A}^2 a^3 \sum_{\mathbf{x}} \langle A_0(x)(A_0)^{\dagger}(0)\rangle \qquad A_\mu \equiv A_\mu|^{\rm QCD} = \overline{\psi}_{\rm l}\gamma_\mu\gamma_5\psi_{\rm b}$$

($Z_{\rm A}$ ensures natural normalization of A_μ consistent with current algebra)

Then: $\Phi^{\rm HQET} = {\rm e}^{-mx_0} C_{\rm AA}^{\rm HQET}(x_0)$ in the region $1/x_0 \ll M$

Obvious strategy:

Determine the $\{c_k, k = 1, \ldots, N_n\}$ by imposing *matching conditions*

$$\Phi_k^{\mathrm{HQET}}(M) = \Phi_k^{\mathrm{QCD}}(M) \qquad k = 1, \ldots, N_n \qquad (\star)$$

for this equivalence between the effective theory and QCD to hold

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$\Rightarrow$ These conditions define the set $\{c_k\}$ for any value of the lattice spacing (or bare coupling)

- Observables used originally to fix the parameters of QCD (e.g. via requiring hadron masses to agree with experiment) may be amongst the Φ_k^{QCD}
- To preserve the predictability of HQET, Φ_k^{QCD} should not be experimentally accessible observables but certain quantities calculable in the continuum limit of lattice QCD

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⇒ However:
Demands to treat the heavy quark as *relativistic particle on the lattice*, though small enough a to do this are very difficult to reach

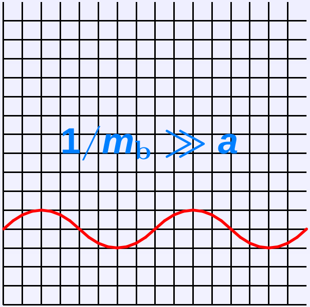
⇒ Impose the matching conditions in *small* volume

Non-perturbative HQET ...

... and the basic idea of exploiting *finite*-volume physics

Goal: *non*-perturbative matching of HQET & QCD

QCD



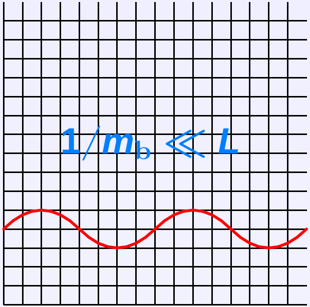
matching condition

$$\Phi^{\mathrm{QCD}} = \Phi^{\mathrm{HQET}}$$

for observables Φ

(e.g. matrix elements)

HQET



Non-perturbative HQET ...

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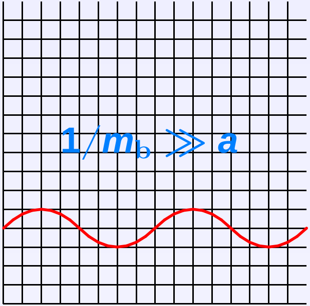
Objection: how do you simulate the b-quark as a relativistic fermion ?

QCD



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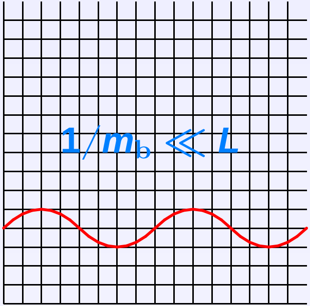
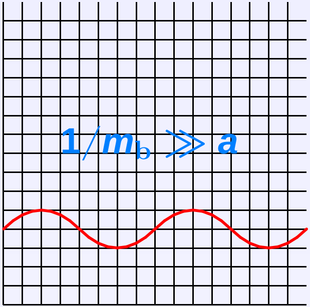


Goal: *non*-perturbative matching of HQET & QCD

Objection: how do you simulate the b-quark as a relativistic fermion ?

$\Rightarrow$ Trick: start with QCD in small volume, $L \equiv L_0 \simeq 0.2\,\mathrm{fm}$

QCD



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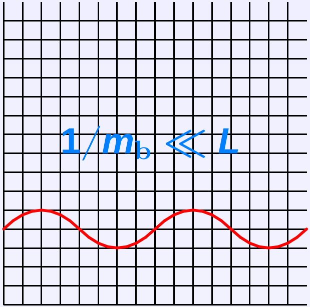

- ✓ Fix parameters of the effective theory through its relation to QCD observables in small volume
- ✓ Legitimate:
  The underlying Lagrangian does not know about the finite volume !

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The underlying Lagrangian does not know about the finite volume !

Further remarks

- Observables Φ are assumed to be renormalized
- New Φ_k have to be added when increasing the order n in the expansion, while the parameters $\{c_i, i \leqslant N_{n-1}\}$ of the lower-order $\mathcal{L}_{\rm HQET}$ might change due to operator mixing
- Most convenient to take the continuum limit of $\Phi_k^{\rm QCD}$ *before* imposing the matching conditions
- Interpreting some of the conditions as improvement conditions, Symanzik $\mathrm{O}(a)$ improvement is accounted for automatically

$$\text{errors} = \mathrm{O}\left([\,1/m\,]^{n+1}\right) = \mathrm{O}\left(M^{-(n+1)}[\,aM\,]^{k}\right) \qquad k = 0, 1, \ldots, n+1$$

E.g. treating the theory including the next-to-leading operators
$\rightarrow$ $(1/M)^0$–terms with $\mathrm{O}(a^2)$ errors
$\rightarrow$ linear $1/M$ –corrections with $\mathrm{O}(a)$ uncertainties

Matching in finite volume and finite-size scaling

Assuming both QCD & HQET to be applicable in finite volume and the parameters in $\mathcal{L}_{\mathrm{QCD/HQET}}$ to be independent of it, we evaluate $(\star)$ as

$$\Phi_k^{\mathrm{HQET}}(L,M) = \Phi_k^{\mathrm{QCD}}(L,M) \qquad k = 1,\ldots,N_n$$

- Allows much smaller a on the r.h.s. to eventually approach the CL
- Typical choice: $L = L_0 \simeq 0.2 - 0.4\,\mathrm{fm}$
- HQET parameters determined at small spacings $a = 0.01 - 0.4\,\mathrm{fm}$ so that large volumes, needed to extract the physical mass spectrum or matrix elements, require very large lattices of $L/a > 50$
  $\longrightarrow$ How can we bridge the gap to practicable lattice spacings ?

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A well-defined procedure: Finite-size scaling

Define step scaling functions σ_k by

$$\Phi_k^{\rm HQET}(sL,M) = \sigma_k\left(\left\{\Phi_j^{\rm HQET}(L,M)\,,\, j = 1,\ldots,N_n\right\}\right) \quad k = 1,\ldots,N_n$$

- σ_k describe the change of the complete set $\{\Phi_k^{\rm HQET}\}$ under $L \longrightarrow s\,L$
- Ends up with a's appropriate for infinite volume computations ($L_K = s^K L_0 \simeq 1\,\mathrm{fm}$ where typically $s = 2$ and $k = 2,3$)

QCD Schrödinger functional (SF)

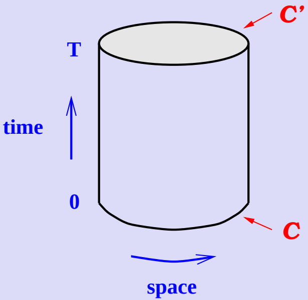
A finite-volume renormalization scheme

Definition [Lüscher et al.]

- SF $\equiv$ QCD partition function on a Euclidean $T \times L^3$ cylinder:

$$\int_{T\times L^3} \mathcal{D}[U,\psi,\overline{\psi}]\, \mathrm{e}^{-S[U,\psi,\overline{\psi}]} = \mathrm{e}^{-\Gamma}$$

- Gauge & quark fields satisfy Dirichlet BCs in time and are periodic in space, e.g. $U(x,k)|_{x_0=0} = \mathrm{e}^{aC_k(\mathbf{x})}$ $\quad U(x,k)|_{x_0=T} = \mathrm{e}^{aC'_k(\mathbf{x})}$

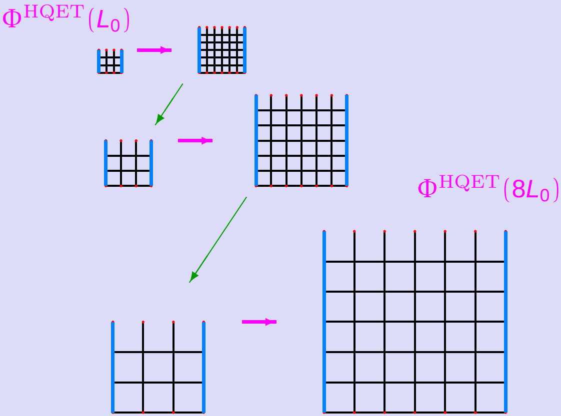


(Lattice) Correlation functions are constructed according to

$$f_{\mathrm{A}}(x_0) = -\frac{1}{2}\langle A_0(x)\mathcal{O}\rangle \qquad \mathcal{O} = \sum_{\mathbf{y},\mathbf{z}} \bar{\zeta}(\mathbf{y})\gamma_5\,\zeta(\mathbf{z}) : \text{ (PS) Boundary source}$$

Similar: f_{P}, k_{V}, ..., and $f_1 = -\frac{1}{2}\langle\mathcal{O}'\mathcal{O}\rangle$ = boundary-to-boundary correlator

Recursive finite-size scaling



- Connects small and large volumes
  (resp. low-energy scales to *perturbative* ones $\sim 1/L_0$)
- Continuum limit can be taken
- Fully non-perturbative

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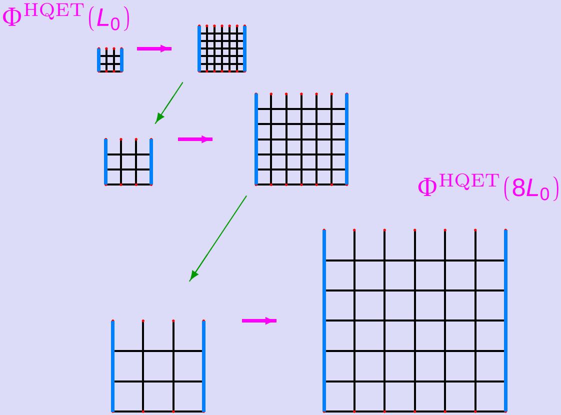
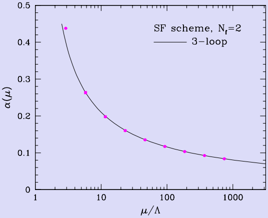


- Connects small and large volumes
  (resp. low-energy scales to *perturbative* ones $\sim 1/L_0$)
- Continuum limit can be taken
- Fully non-perturbative

Benefits

- Identifies $\mu = 1/L$
- L^{-1} runs, separated from a^{-1}
- Spans large range in energy $\mu = 1/L$

$\Rightarrow$ Framework to solve scale dependent renorm. problems



[ALPHA Collaboration 2004]

LECTURE 2

Applications

Non-perturbative tests of HQET

[H., Jüttner, Sommer & Wennekers, 2004]

Motivation:

Though HQET is commonly accepted as effective theory of QCD, explicit demonstrations of this are rare or based on phenomenological analyses

Requirement for a pure, non-perturbative theory test

QCD including a heavy enough quark must be simulated on the lattice at lattice spacings small enough to be able to take the continuum limit

$\Rightarrow$ Perform such tests in a *finite* volume

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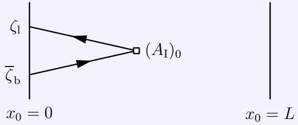
$\Rightarrow$ Perform such tests in a *finite* volume

Realization:

- Put the theory in a Schrödinger functional box with moderate $T = L$ such that $am_{\rm b} \ll 1$
- Equivalent boundary conditions can be imposed on the HQET side
- Build correlators of boundary quark fields ζ and composite fields

$$f_{\rm A}(x_0) = -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}} \left\langle (A_{\rm I})_0(x)\,\overline{\zeta}_{\rm b}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z})\right\rangle$$

$$(A_{\rm I})_0(x) = A_0(x) + a\,c_{\rm A}\,\tfrac{1}{2}(\partial_\mu + \partial^*_\mu)P(x)$$



Form a proper ratio where the boundary renormalization factors drop out:

$$Y_{\rm PS}(L,M) \;\equiv\; Z_{\rm A}\,\frac{f_{\rm A}(L/2)}{\sqrt{f_1}} \;=\; \frac{\langle\,\Omega(L)\,|\,\mathbb{A}_0\,|\,B(L)\,\rangle}{\|\,|\Omega(L)\rangle\,\|\;\|\,|B(L)\rangle\,\|}$$

$$|B(L)\rangle = {\rm e}^{-L\mathbb{H}/2}\,|\varphi_{\rm B}(L)\rangle \qquad |\Omega(L)\rangle = {\rm e}^{-L\mathbb{H}/2}\,|\varphi_0(L)\rangle$$

$|\varphi_0(L)\rangle$, $|\varphi_{\rm B}(L)\rangle$: SF (vacuum and pseudoscalar) boundary states
$|\Omega(L)\rangle$, $|B(L)\rangle$: states with vacuum and B-meson quantum numbers

- Time evolution ensures dominance by contributions with $\Delta E \lesssim 2/L$
- Conclusion: HQET is applicable if $1/L \ll M$ (and $\Lambda \ll M$)

$\Rightarrow$ One expects (for fixed ΛL) the large-mass asymptotics of $Y_{\rm PS}$ to obey

$$Y_{\rm PS}(L,M) \overset{M\to\infty}{\sim} C_{\rm PS}\,(M/\Lambda) \times X_{\rm RGI}(L) \;+\; {\rm O}(1/z) \qquad z = ML$$

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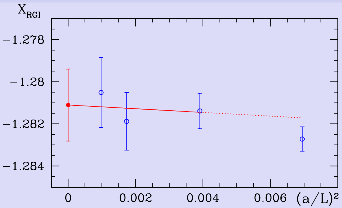
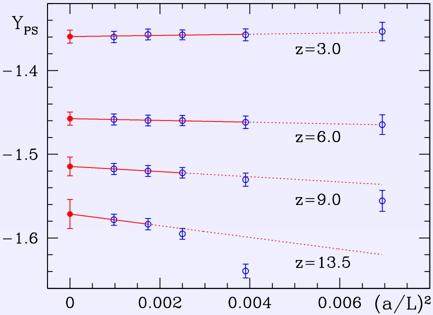
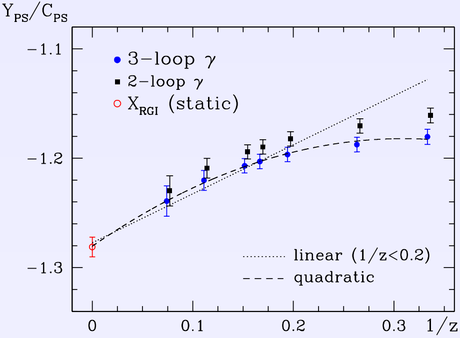
$X_{\rm RGI}$ = static-limit analogue of $Y_{\rm PS}$

- $X_{\rm RGI}(L) = Z_{\rm RGI}\,X(L) \propto \frac{\Phi_{\rm RGI}}{\Phi_{\rm SF}(\mu=1/L)}$
- demands a lattice computation in static approximation ($L \simeq 0.2\,{\rm fm}$) and to extrapolate to the continuum



Quenched study of the large $-z$ behaviour of $Y_{\rm PS}$ in small-volume QCD:





- The finite-mass observable turns smoothly into the HQET prediction (Note: $C_{\rm PS}$ reduces the mass dependence of $Y_{\rm PS}(L,M)$ by a factor > 2)
- More such successful test (e.g. for the spin splitting) are available, outcome: magnitude of z^{-n}–coefficients reasonably small, $\sim \mathrm{O}(1)$
- Power-corrections larger than perturbative ones at $z^{-1} = 0.1 - 0.2$, but a theoretically consistent evaluation of the former requires a fully non-perturbative formulation of HQET including its matching to QCD

Non-perturbative matching in a concrete example

Computation of M_{b} in lowest-order of HQET (static approximation) [H. & Sommer, 2004]

Recall:

- Matching conditions $\Phi_k^{\mathrm{HQET}}(L, M) = \Phi_k^{\mathrm{QCD}}(L, M)$ in $L \simeq 0.2\,\mathrm{fm}$
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Realization to determine the mass of the b-quark

$$\left.\begin{array}{l}\Gamma(L,M)\\ \Gamma_{\mathrm{stat}}(L)\end{array}\right\} = \left\{\begin{array}{l}\text{B-meson mass in a \textit{finite volume} of extent } L^4\\ \text{energy of a state with B-meson quantum numbers in } L^4\end{array}\right.$$

Now implicitly replace

$$m_{\mathrm{bare}} = m + \frac{1}{a}\ln(1 + a\,\delta m) \qquad \text{in} \qquad m_{\mathrm{B}} = E_{\mathrm{stat}} + m_{\mathrm{bare}}$$

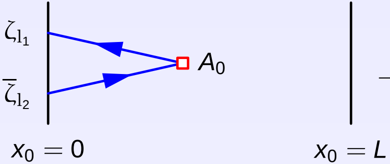
via the set of conditions

$$\begin{array}{rcccccccc} & & \Phi_1^{\mathrm{HQET}} & = & \Phi_1^{\mathrm{QCD}} & \equiv & \bar{g}^2(L_0) & = & \text{constant}\\ & & \Phi_2^{\mathrm{HQET}} & = & \Phi_2^{\mathrm{QCD}} & \equiv & m_\mathrm{l} & = & 0\\ \Gamma_{\mathrm{stat}}(L_0) + m_{\mathrm{bare}} & \equiv & \Phi_3^{\mathrm{HQET}} & = & \Phi_3^{\mathrm{QCD}} & \equiv & \Gamma(L_0, M_\mathrm{b}) & & \end{array}$$

As anticipated before:

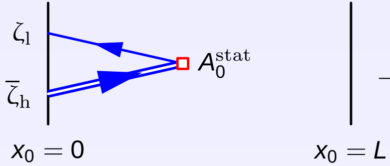
Use L–dependent energies from SF-correlators in the B-meson channel

$C(x_0, M)$:



$$\to \Gamma(L, M) \equiv -\frac{\mathrm{d}}{\mathrm{d}x_0} \ln \left[C(x_0, M)\right] \Big|_{x_0=\frac{L}{2}}$$

$C_{\mathrm{stat}}(x_0)$:



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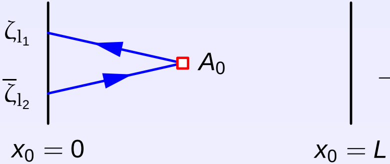

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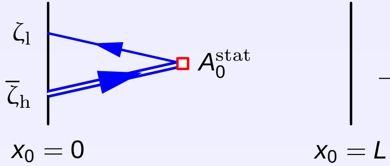

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$\Rightarrow$ Matching condition by equating in small volume with linear extent L_0 :

$$\Gamma_{\rm stat}(L_0) + m_{\rm bare} = \Gamma(L_0, M_{\rm b})$$

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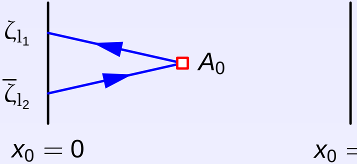

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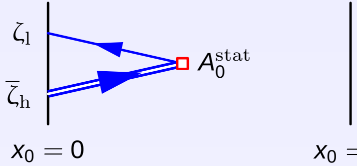

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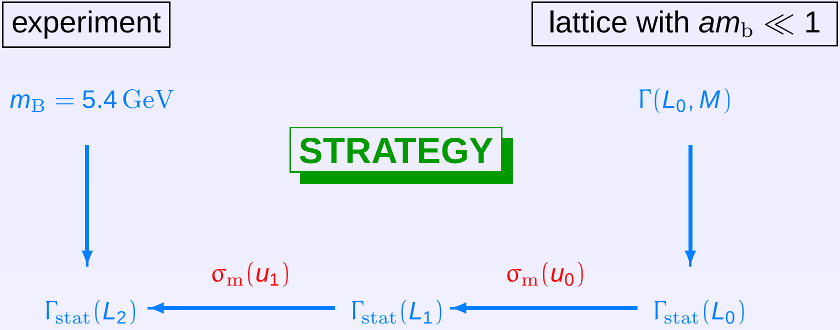
$$\Gamma_{\mathrm{stat}}(L_0) + m_{\mathrm{bare}} = \Gamma(L_0, M_{\mathrm{b}})$$

As $C(x_0) \overset{x_0\to\infty}{\sim} \mathrm{e}^{-m_{\mathrm{B}}x_0}$ and $C_{\mathrm{stat}}(x_0) \overset{x_0\to\infty}{\sim} \mathrm{e}^{-E_{\mathrm{stat}}x_0}$ in the large– L limit, we have to connect this condition (by finite-size scaling) to

$$E_{\mathrm{stat}} + m_{\mathrm{bare}} = m_{\mathrm{B}}$$

To bridge between the matching in small volume and a physical situation (i.e. $L \geqslant 1.5\,\mathrm{fm}$ & $a \gtrsim 0.05\,\mathrm{fm}$ to accommodate a B-meson), adopt a few *recursive finite-size scaling* steps in an intermediate SF scheme:



- Γ , Γ_{stat} : suitable quantities for matching
- Introduce a step scaling function $\sigma_{\mathrm{m}}(u) \equiv 2L\,[\,\Gamma_{\mathrm{stat}}(2L) - \Gamma_{\mathrm{stat}}(L)\,]$ to evolve $L_0 \to L_2 = 2^2 L_0 \simeq 1\,\mathrm{fm}$
- For $L \simeq 2\,\mathrm{fm}$ @ same resolution: calculate physical observables

⇒ Equation to solve for the b-quark mass

$$m_{\rm B} = \underbrace{E_{\rm stat} - \Gamma_{\rm stat}(L_2)}_{a \to 0 \text{ in HQET}}$$
$$+ \underbrace{\Gamma_{\rm stat}(L_2) - \Gamma_{\rm stat}(L_0)}_{a \to 0 \text{ in HQET}}$$
$$+ \underbrace{\Gamma(L_0, M_{\rm b})}_{a \to 0 \text{ in QCD } (L_0^4)}$$

- Linearly divergent static quark's self-energy δm cancels in differences !
- $\Gamma(L, M)$ is defined in small-volume QCD with a *relativistic b-quark*:

$$z \equiv L_0 M \gg 1 \qquad L_0 \simeq 0.2\,\mathrm{fm}$$

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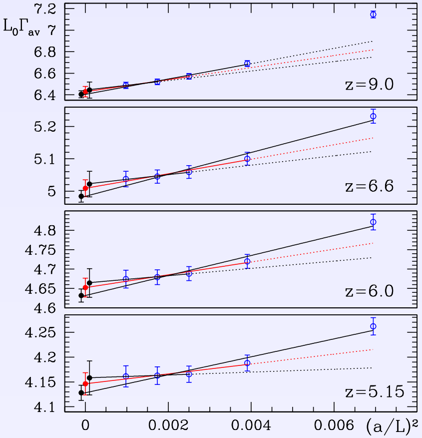


In practice: choose fixed SF coupling $\bar{g}^2(L_0/2)$ with $L_0 = L_{\rm max}/2 = 0.36 r_0$ ⇒ $(L/a, \beta, \kappa_{\rm l})$ from previous work

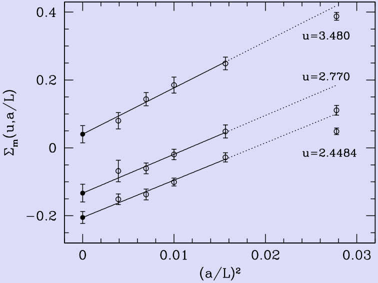
Desired quark mass values $z = L_0 M$ traded for $\kappa_{\rm h}$ used in the simulations:
$z = L_0 \frac{M}{\overline{m}_{\rm h}(\mu_0)} Z_{\rm m} m_{\rm q,h}(1 + b_{\rm m} a m_{\rm q,h})$

[H. & Wennekers, 2004]

$L_0 \times [\,\Gamma_{\rm stat}(L_2) - \Gamma_{\rm stat}(L_0)\,]$

- The SSF connects the small *'matching'* volume $L_0 \simeq 0.2\,\mathrm{fm}$ to $L_2 \simeq 1\,\mathrm{fm}$

⇒ Contact with physical quantities: $E_{\rm stat}$, $m_{\rm B}$ in large volume



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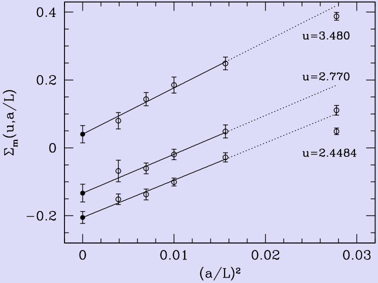
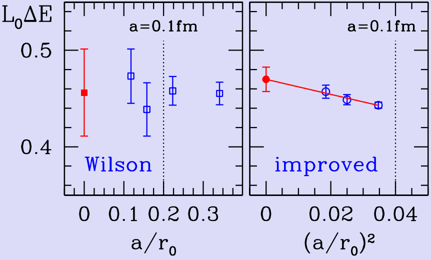


$L_0\Delta E \equiv L_0 \times [\,E_{\rm stat} - \Gamma_{\rm stat}(L_2)\,]$

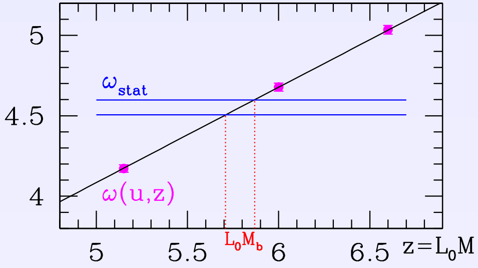
- Left: Wilson fermions, $E_{\rm stat}$ from the Fermilab group [Duncan et al., PRD51(1995)5101]
- Right: [non-] perturbatively $\mathrm{O}(a)$ improved ($+$ enhanced signal/noise-ratios by change of discretization of $S_{\rm HQET}$)



The RGI b-quark mass $M_{\rm b}$ is finally obtained from intercept of

$$\omega(z,u) \equiv \lim_{a\to 0} L_0\Gamma \quad \text{with} \quad \omega_{\rm stat} \equiv L_0 m_{\rm B} - \left\{\tfrac{1}{2}\sigma_{\rm m}(u_0) + \tfrac{1}{4}\sigma_{\rm m}(u_1)\right\} - L_0\Delta E$$

- $\Gamma = \Gamma_{\rm av} \equiv \frac{1}{4}\Gamma_{\rm PS} + \frac{3}{4}\Gamma_{\rm V}$: spin-averaged combination to minimize the size of $1/M$ – effects
- Continuum limit in all steps
- Non-perturbative renormalization



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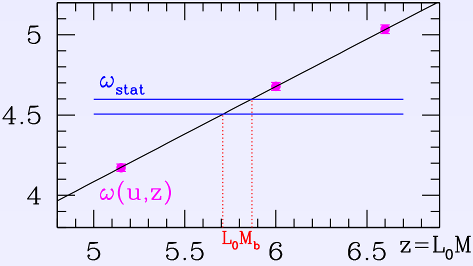

Result [H. & Sommer, 2004]

$$r_0 M_{\rm b} = 16.12(29) \quad \rightarrow \quad \overline{m}_{\rm b}^{\overline{\rm MS}}(\overline{m}_{\rm b}^{\overline{\rm MS}}) = 4.12(8)\,{\rm GeV}$$

Uncertainties and expected improvements:

✓ Valid up to $\mathrm{O}(\frac{\Lambda}{L_0 M_{\rm b}}) \sim \mathrm{O}(\frac{\Lambda^2}{M_{\rm b}})$ corrections, quenched approximation

✓ Computation of $aE_{\rm stat}$ including the improvements just mentioned will yield a continuum limit of $L_0\Delta E$ with a much smaller error [ALPHA Collaboration , to come soon]

Towards a precision determination of $F_{\mathrm{B_s}}$

Two-step strategy

1. Calculation of $F_{\mathrm{B_s}}$ in lowest order of HQET (= static approximation)

$$F_{\mathrm{PS}}\sqrt{m_{\mathrm{PS}}} = C_{\mathrm{PS}}\left(M/\Lambda_{\overline{\mathrm{MS}}}\right) \times \Phi_{\mathrm{RGI}} + \mathrm{O}\left(1/M\right)$$

$\Phi_{\mathrm{RGI}} =$ RGI matrix element of the static axial current

$$\Phi_{\mathrm{RGI}} = Z_{\mathrm{RGI}}\langle\,\mathrm{PS}\,|\,A_0^{\mathrm{stat}}\,|\,0\,\rangle \quad A_0^{\mathrm{stat}} = \overline{\psi}_{\mathrm{s}}\gamma_0\gamma_5\psi_{\mathrm{b}}^{\mathrm{stat}} \quad \text{for} \quad \mathrm{PS} = \mathrm{B}$$

$$\left[\ \Phi_{\mathrm{RGI}}(x_0) \ \propto \ Z_{\mathrm{RGI}} \times \frac{f_{\mathrm{A}}^{\mathrm{stat}}(x_0)}{\sqrt{f_1}}\ \mathrm{e}^{(x_0-T/2)E_{\mathrm{stat}}(x_0)} \quad \text{in the SF}\ \right]$$

2. Combine with results for $F_{\mathrm{PS}}(m_{\mathrm{PS}})$ around the charm quark region by (linear) interpolation in $1/m_{\mathrm{PS}}$

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- Non-perturbative renormalization: Z_{RGI} known [H., Kurth & Sommer, 2003]
- Calculation employs further (mainly new) ingredients, namely ...

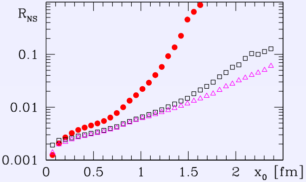
- Linear $a-$effects removed

- Linear a – effects removed
- Modified (static) action with reduced statistical errors by change of parallel transporters in covariant derivative:

$$D_0\psi_{\mathrm{h}}(x) = a^{-1}[\,\psi_{\mathrm{h}}(x) - W^{\dagger}(x - a\hat{0}, 0)\psi_{\mathrm{h}}(x - a\hat{0})\,]$$

✓ $W(x, 0) =$ function of gauge fields in the neighbourhood of $x, x + a\hat{0}$

✓ Quite the same small lattice artifacts



Best version: ‘HYP-smearing’

[Hasenfratz & Knechtli, 2001]

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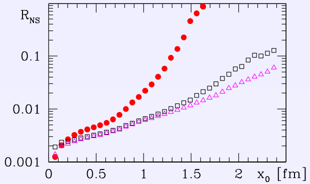
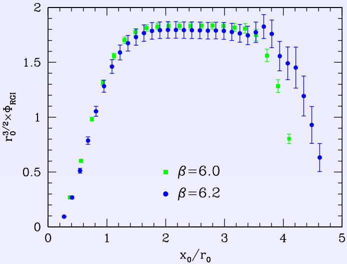


Best version: 'HYP-smearing'

[Hasenfratz & Knechtli, 2001]

- Wave functions at boundaries of the SF-cylinder to suppress excited B-meson state contributions to correlators

- Linear $a-$effects removed



- Modified (static) action with reduced statistical errors by change of parallel transporters in covariant derivative:

$$D_0\psi_h(x) = a^{-1}[\psi_h(x) - W^\dagger(x-a\hat{0},0)\psi_h(x-a\hat{0})]$$

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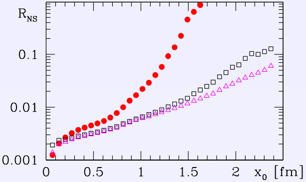

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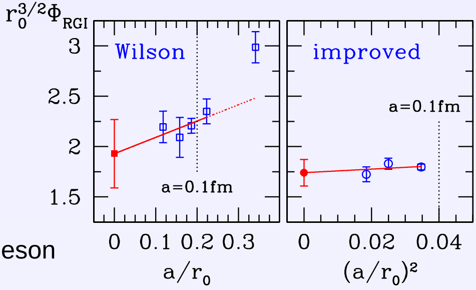

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Interpolation between leading-order HQET and $F_{\mathrm{D_s}}$

Extrapolation of $r_0^{3/2} F_{\mathrm{PS}} \sqrt{m_{\mathrm{PS}}} / C_{\mathrm{PS}}$ from the charm region to the static estimate $r_0^{3/2} \Phi_{\mathrm{RGI}}$ using results on $F_{\mathrm{PS}}(m_{\mathrm{PS}}) \big|_{m \simeq m_c}$ [Rolf & Jüttner, 2003]

- Linear interpolation in $1/(m_{\mathrm{PS}} r_0)$:
  - ▶ motivated by HQET
  - ▶ justified by the data
- mass dependent discretization errors near m_{c}
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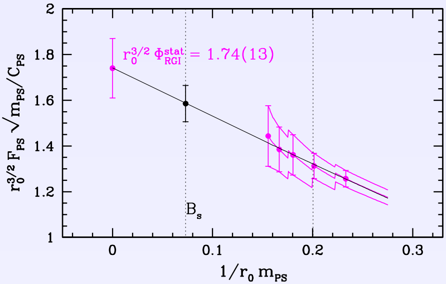

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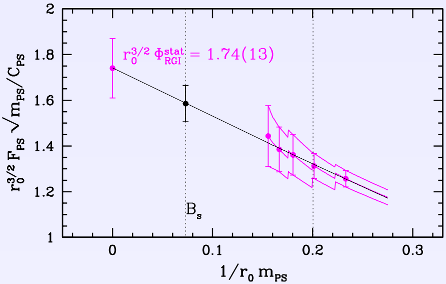

Preliminary result [ALPHA Collaboration 2003]

$$\Lambda_{\overline{\mathrm{MS}}} = 238(19)\,\mathrm{MeV}\ ,\ r_0 = 0.5\,\mathrm{fm} \quad \rightarrow \quad F_{\mathrm{B_s}} = 205(12)\,\mathrm{MeV}$$

✓ Includes all errors except for quenching (scale ambiguity is $\simeq 12\%$)

✓ Extrapolation without the static constraint looks similar but depends significantly on functional form assumed $\Rightarrow$ *interpolation* much safer

Alternative to determine the B-meson decay constant

Further application of the non-perturbative matching strategy

To lowest order in $1/m$ we have

$$\mathcal{M}(g_0) \equiv \langle\, \mathrm{B}(\mathbf{p}=\mathbf{0})\,|\, A_0^{\mathrm{stat}}(0)\,|\,0\,\rangle \qquad F_{\mathrm{B}}\sqrt{m_{\mathrm{b}}} = \lim_{a\to 0} Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, aM_{\mathrm{b}})\,\mathcal{M}(g_0)$$

- $Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, aM_{\mathrm{b}})$ computed in quenched approximation via a *matching through the RGI operator* with finite-size scaling techniques
  ($N_{\mathrm{f}}=2$ also in progress $\rightarrow$ P. Fritzsch's talk)
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In the spirit of our non-perturbative matching of HQET and QCD in finite volume, the master formula valid up to corrections of $\mathrm{O}(1/m)$ is

$$F_{\mathrm{B}}\sqrt{m_{\mathrm{b}}} = \frac{F_{\mathrm{B}}\sqrt{m_{\mathrm{b}}}\,|^{\mathrm{HQET}}}{\Phi^{\mathrm{HQET}}(L_2, M_{\mathrm{b}})} \times \frac{\Phi^{\mathrm{HQET}}(L_2, M_{\mathrm{b}})}{\Phi^{\mathrm{HQET}}(L_1, M_{\mathrm{b}})} \times \frac{\Phi^{\mathrm{HQET}}(L_1, M_{\mathrm{b}})}{\Phi^{\mathrm{HQET}}(L_0, M_{\mathrm{b}})} \times \Phi^{\mathrm{QCD}}(L_0, M_{\mathrm{b}})$$

- applying to multiplicative, scale dependent renormalizations and
- provided that the b-quark mass is already known

Ingredients:

- Matching equation to be imposed in the small volume

$$\Phi^{\rm HQET}(L_0, M_{\rm b}) = \Phi^{\rm QCD}(L_0, M_{\rm b}) \quad \text{with} \quad \bar{g}^2(L_0) = u_0 = \text{fixed}$$

- Finite-size scaling in terms of step scaling functions built as

$$\Phi^{\rm HQET}(2L, M_{\rm b})\big|_{a=0} = \sigma_{\rm X}\left(\bar{g}^2(L)\right) \times \Phi^{\rm HQET}(L, M_{\rm b})\big|_{a=0}$$

- Then the previous formula finally combines to

$$F_{\rm B}\sqrt{m_{\rm b}} = \rho(u_2) \times \sigma_{\rm X}(u_1) \times \sigma_{\rm X}(u_0) \times \Phi^{\rm QCD}(L_0, M_{\rm b})$$

$$\rho(u) \equiv \lim_{a/L\to 0} \frac{\mathcal{M}(g_0)}{X(g_0, L/a)}\bigg|_{\bar{g}^2(L)=u} \qquad X(g_0, L/a) \equiv \frac{f_{\rm A}^{\rm stat}(L/2)}{\sqrt{f_1^{\rm stat}}}$$

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Key difference to obtaining RGIs & conversion to the matching scheme

- is not the absence of perturbative errors in $C_{\rm PS}\,(M_{\rm b}/\Lambda_{\overline{\rm MS}})$
- but the tempting possibility to *include* $1/m$ *–corrections*

Conclusions & Perspectives

- New quality of the computations employing lattice HQET:
  - ✓ Non-perturbative renormalization
  - ✓ Continuum limit at *large* quark masses (small-volume setup !)

- Discretizations for static quarks entailing exponentially improved statistical precision

- Physics results are still quenched, but an extension of the methods to dynamical fermions is straightforward
  (‘only’ the usual problems with light quarks to be solved)

- Even more interesting:
  Systematic improvement by implementing the effective theory beyond the leading order in $1/m$ to reach an acceptable precision
  - ✓ First tests and ideas seem to be promising
  - ✓ To do this consistently, conversion functions such as $C_{\rm PS}$ have to be known non-perturbatively

Towards an inclusion of $1/m$ – corrections

The $1/m$ – expansion of the correlator f_A receives new contributions:

$$f_A \;\propto\; f_A^{stat} \left\{ 1 + \frac{\alpha^{(1)}\,\delta f_A^{stat}}{\alpha^{(0)}\,f_A^{stat}} + \omega_{kin}\,\frac{f_A^{kin}}{f_A^{stat}} + \omega_{spin}\,\frac{f_A^{spin}}{f_A^{stat}} \right\}$$

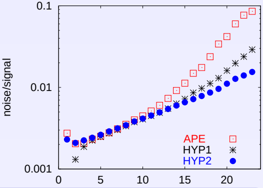
with bulk insertions

$$X^{kin} = \overline{\psi}_h \mathbf{D}^2 \psi_h \qquad X^{spin} = \overline{\psi}_h \boldsymbol{\sigma}\mathbf{B} \psi_h$$

in

$$f_A^{kin/spin}(x_0) = -\tfrac{1}{2}\left\langle (A_I^{stat})_0(x) \sum_u X^{kin/spin}(u)\, \mathcal{O} \right\rangle$$

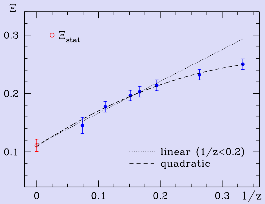
first numerical exploration encouraging [Dürr et al., 2004]



How can one match the ω_{kin} – term in a computation of M_b ?

Proposal: use a combination of energies

$$\begin{aligned} \Xi(L,M) &= L\,[\,\Gamma_{av}(L/2,M) - \Gamma_{av}(L/4,M)\,] \\ &= \Xi_{stat}(L) + \frac{1}{2z}\,\Xi_{kin}(L) + \mathrm{O}(1/z^2) \end{aligned}$$



- Ξ_{kin} encodes matrix elements of $\overline{\psi}_h \mathbf{D}^2 \psi_h$
- Reparametrization invariance restricts Ξ_{kin} to be free of logarithmic modifications