B-meson distribution amplitudes

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HQET

Quark-antiquark distribution amplitudes

Quark-antiquark-gluon distribution amplitudes

Evolution

Sum rules

Introduction

Amplitudes of many exclusive B-decays can be factorized into a hard kernel (perturbatively calculable) and light-cone distribution amplitudes of the initial B-meson and final hadron(s). They describe large-distance (soft) structure of these hadrons, and cannot be calculated in perturbation theory.

References

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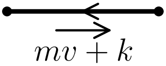
HQET

Heavy antiquark with momentum $p = mv + k$



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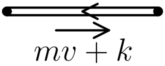

$$+ \mathcal{O}\left(\frac{1}{m}\right)$$

$$\frac{m - m\not{v} - \not{k}}{(mv + k)^2 - m^2 + i0} = \frac{1 - \not{v}}{2} \frac{1}{k \cdot v + i0} + \mathcal{O}\left(\frac{1}{m}\right)$$

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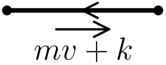

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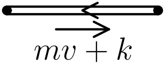

$+ \mathcal{O}\left(\frac{1}{m}\right)$

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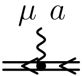
$$\frac{1-\not{v}}{2}\gamma^\mu\frac{1-\not{v}}{2} = \frac{1-\not{v}}{2}(-v^\mu)\frac{1-\not{v}}{2}$$

Projector can also be inserted near external legs



$= ig_0 t^a (-v^\mu)$

Lagrangian

$$L = \bar{Q}_v i v \cdot \overleftarrow{D} Q_v + \text{light fields}$$
$$\bar{Q}_v \not{v} = -\bar{Q}_v$$

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QCD tree diagrams are reproduced up to $\mathcal{O}(k_i/m)$

Heavy quark symmetry

The heavy quark chromomagnetic moment is $\sim 1/m$ by dimensionality. At the leading order in $1/m$, the heavy-quark spin does not interact with the gluon field. Therefore, it may be rotated at will, without changing the physics (heavy-quark spin symmetry). It may even be switched off (superflavour symmetry).

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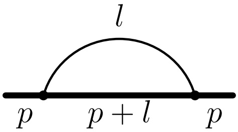
$$L = Q_v^* iv \cdot \overleftarrow{D} Q_v + \text{light fields}$$

Heavy quark near mass shell



$$p = mv + k$$

$$k = \omega v$$

$$I = \frac{-im^2}{\pi^{d/2}} \int \frac{d^d l}{\left[m^2 - (mv + k + l)^2\right]^2 (-l^2)}$$

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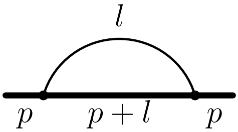

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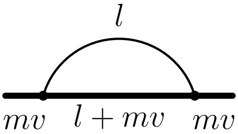
- Hard region $l \sim m$

$$\frac{1}{\left[m^2 - (mv + l)^2\right]^2 (-l^2)} + 4 \frac{(m + l \cdot v)\omega}{\left[m^2 - (mv + l)^2\right]^3 (-l^2)} + \cdots$$

- Soft region $l \sim \omega$

$$\frac{1}{\left[-2m(k + l) \cdot v\right]^2 (-l^2)} + 2 \frac{(k + l)^2}{\left[-2m(k + l) \cdot v\right]^3 (-l^2)} + \cdots$$

Hard region

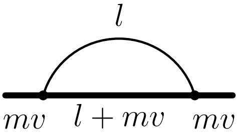


$$\int \frac{d^d l}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} m^{d-2(n_1+n_2)} M(n_1, n_2)$$

$$D_1 = m^2 - (l + mv)^2 - i0 \qquad D_2 = -l^2 - i0$$

$$M(n_1, n_2) = \frac{\Gamma(d - n_1 - 2n2)\Gamma(-d/2 + n_1 + n_2)}{\Gamma(n_1)\Gamma(d - n_1 - n_2)}$$

Hard region



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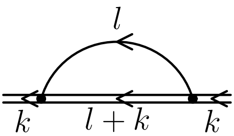
$$I_h = m^{-2\varepsilon} \left[M(2,1) + 2\frac{\omega}{m} \left[M(3,0) - M(2,1) + 2M(3,1) \right] + \cdots \right]$$

$$\frac{\mu^{2\varepsilon} I_h}{\Gamma(1+\varepsilon)} = -\frac{1}{2\varepsilon} + \log\frac{m}{\mu} + \left(\frac{1}{\varepsilon} - 2\log\frac{m}{\mu} - 1 \right) \frac{\omega}{m} + \cdots$$

Soft region



$$\omega = k \cdot v$$

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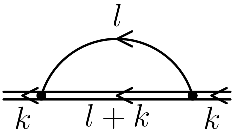

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$$I_s = (-2\omega)^{-2\varepsilon}\left[I(2,1) + \frac{\omega}{m}\left[I(3,1) - 2I(2,1)\right] + \cdots\right]$$

$$\frac{\mu^{2\varepsilon} I_s}{\Gamma(1+\varepsilon)} = \frac{1}{2\varepsilon} - \log\frac{-2\omega}{\mu} - \left(\frac{1}{\varepsilon} - 2\log\frac{-2\omega}{\mu} - \frac{1}{2}\right)\frac{\omega}{m} + \cdots$$

Result

IR divergence in I_h cancels UV divergence in I_s

$$I = -\log\frac{-2\omega}{m} + \left(2\log\frac{-2\omega}{m} - \frac{1}{2}\right)\frac{\omega}{m} + \cdots$$

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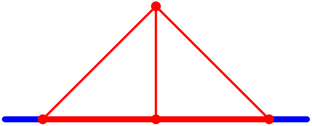
2 loops: hard and soft



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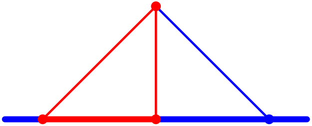

- Hard loop: scale m, expand in soft momenta
- Soft loop: scale ω, hard loop is a local vertex

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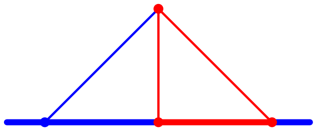

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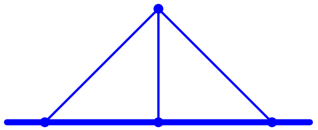

Multiloop diagrams

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- Each of them must contain at least one heavy line
- From the point of view of the soft diagram, each hard part is a local vertex
- If some soft subdiagram is only connected to one such vertex, it vanishes

Usual HQET formalism

- Lagrangian – local operators with matching coefficients

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- Lagrangian – local operators with matching coefficients
- QCD currents – similar
- Matching coefficients depend on the hard scale m, come from hard subdiagrams
- Local operators produce vertices in the (soft) HQET diagram

Spin 0 heavy antiquark

$\bar{Q}q$ mesons: S-wave $\frac{1}{2}^+$; P-wave $\frac{1}{2}^-$, $\frac{3}{2}^-$

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Currents with parity $P = \pm 1$

$$j_P = \frac{1 + P\gamma^0}{2} j$$

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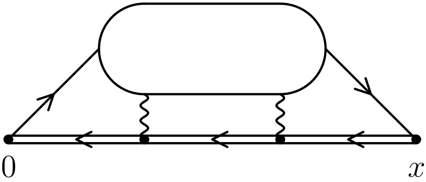
Ground-state meson M: $\gamma^0 u = u$, $\bar{u}u = 1$

$$<0|j|M> = Fu$$

Non-relativistic normalization

$$<M, \vec{p}\,'|M, \vec{p}> = (2\pi)^3 \delta(\vec{p}\,' - \vec{p})$$

Correlator



$$i\langle Tj(x)\bar{j}(0)\rangle = \delta(\vec{x}\,)\Pi(x^0)$$

$$\Pi(x^0) = A + B\not{v}$$

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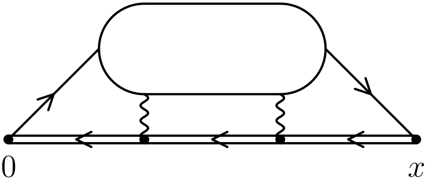

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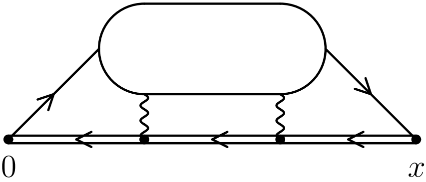

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Spectral density

$$\rho_+(\varepsilon) = F^2 \delta(\varepsilon - \bar{\Lambda}) + \cdots$$

Switching the heavy-antiquark spin on

Doublets: S-wave 0^-, 1^-; P-wave 0^+, 1^+; 1^+, 2^+

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Doublets: S-wave 0^-, 1^-; P-wave 0^+, 1^+; 1^+, 2^+
Currents $\bar{Q}_v \Gamma q$ ($\bar{Q}_v \gamma^0 = -\bar{Q}_v$):

$$\Gamma = \gamma_5\,, \quad \vec{\gamma} \qquad \text{and} \qquad \Gamma = 1\,, \quad \vec{\gamma}\gamma_5$$

Currents with $\Gamma = \gamma_5$, $\vec{\gamma}$ (anticommution with γ^0) – ground state 0^-, 1^- mesons; with $\Gamma = 1$, $\vec{\gamma}\gamma_5$ (commution with γ^0) – P-wave 0^+, 1^+ mesons

Correlators

$$i\langle T j_2(x) j_1^+(0)\rangle = \delta(\vec{x}\,)\Pi_{12}(x^0)$$

$$\Pi_{12} = \operatorname{Tr}\overline{\Gamma}_1 \frac{1-\gamma^0}{2}\Gamma_2 \Pi \to \Pi_P \operatorname{Tr}\overline{\Gamma}_1 \frac{1-\gamma^0}{2}\Gamma_2$$

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For $\Gamma = \gamma_5$ and γ^i, the correlators are $2\Pi_+$ and $2\Pi_+\delta^{ij}$

$$<0|\bar{Q}_v \gamma_5 q|B> = F_B \qquad <0|\bar{Q}_v \vec{\gamma} q|B^*> = F_{B^*}\vec{e}$$

Spectral densities $F_B^2 \delta(\varepsilon - \bar{\Lambda})$ and $F_{B^*}^2 \delta(\varepsilon - \bar{\Lambda})\delta^{ij}$

$$F_B = F_{B^*} = \sqrt{2} F$$

QCD decay constants

Relativistic normalization

$$ {}_r\!<B, p'|B, p>_r = (2\pi)^3 2p^0 \delta(\vec{p}\,' - \vec{p}) $$

Definitions

$$ <0|\bar{Q}_v \gamma^\mu \gamma_5 q|B>_r = i f_B p^\mu \qquad <0|\bar{Q}_v \gamma^\mu q|B^*>_r = i m f_{B^*} e^\mu $$

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$$ f_B = f_{B^*} = \frac{2F}{\sqrt{m}} $$

Trace formalism

$$<0|j_2|M><M|j_1^+|0> = F^2 \,\mathrm{Tr}\, \Gamma_2 \frac{1+\gamma^0}{2} \overline{\Gamma}_1$$

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$$\Gamma_M = \begin{cases} -i\gamma_5 & \text{for } B \\ i\not{\epsilon} & \text{for } B^* \end{cases}$$

Trace formalism

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$$<0|\bar{Q}_v \Gamma q|M>_r = \sqrt{m} F \,\mathrm{Tr}\, \Gamma \mathcal{M}$$

$$\mathcal{M} = \frac{1+\not{v}}{2} \begin{cases} -i\gamma_5 & \text{for } B \\ i\not{e} & \text{for } B^* \end{cases}$$

Quark-antiquark distribution amplitudes

$$\tilde{O}(t) = Q_v^*(0)[0, z]q(z) \qquad z^2 = 0 \qquad t = v \cdot z$$

$$[x, y] = P \exp\left[-i \int_x^y A_\mu(z) dz^\mu\right]$$

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$$<0|\tilde{O}(t)|M> = F\left[\tilde{\varphi}_+(t) + \frac{\tilde{\varphi}_-(t) - \tilde{\varphi}_+(t)}{2t}\not{z}\right]u$$

Light-front components

$$n_\pm^\mu = (1, \mp 1, \vec{0})$$

$$n_+^2 = n_-^2 = 0 \qquad n_+ \cdot n_- = 2$$

$$a_\pm = a \cdot n_\pm = a^0 \pm a^1$$

$$a^\mu = \frac{1}{2}\left(a_+ n_-^\mu + a_- n_+^\mu\right) + a_\perp^\mu$$

$$a \cdot b = \frac{1}{2}\left(a_+ b_- + a_- b_+\right) - \vec{a}_\perp \cdot \vec{b}_\perp$$

$$v^\mu = \frac{1}{2}\left(n_+^\mu + n_-^\mu\right) \qquad v_+ = v_- = 1 \qquad \vec{v}_\perp = \vec{0}$$

$$\gamma_\pm = \gamma \cdot n_\pm = \not{n}_\pm$$

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$$\varphi_\pm(\omega) = \frac{1}{2\pi} \int \tilde{\varphi}_\pm(t) e^{i\omega t} dt \qquad \tilde{\varphi}_\pm(t) = \int \varphi_\pm(\omega) e^{-i\omega t} d\omega$$

$$<0|\tilde{O}(t)|M> = \frac{1}{2}F\left[\tilde{\varphi}_+(t)\gamma_- + \tilde{\varphi}_-(t)\gamma_+\right]u$$

$$\tilde{O}_\pm(t) = \gamma_\pm\tilde{O}(t) \qquad <0|\tilde{O}_\pm(t)|M> = F\tilde{\varphi}_\pm(t)\gamma_\pm u$$

$$\varphi_\pm(\omega) = \frac{1}{2\pi}\int\tilde{\varphi}_\pm(t)e^{i\omega t}dt \qquad \tilde{\varphi}_\pm(t) = \int\varphi_\pm(\omega)e^{-i\omega t}d\omega$$

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$$O_\pm(\omega) = \frac{1}{2\pi}\int \tilde{O}_\pm(t) e^{i\omega t} dt = Q_v^*(0)\gamma_\pm \delta(iD_+ - \omega) q(0)$$

$$\tilde{O}_\pm(t) = \int O_\pm(\omega) e^{-i\omega t} d\omega$$

$$<0|O_\pm(\omega)|M> = F\varphi_\pm(\omega)\gamma_\pm u$$

Distribution in p_+ of the light quark

Moments

$$\tilde{O}_\pm(t) = \sum_{n=0}^{\infty} O_\pm^{(n)} \frac{(-it)^n}{n!}$$

$$O_\pm^{(n)} = \int O_\pm(\omega)\omega^n d\omega = Q_v^* \gamma_\pm (iD_+)^n q$$

$$O_\pm(\omega) = \int_{-i\infty}^{+i\infty} O_\pm^{(n)} \omega^{-n-1} \frac{dn}{2\pi i}$$

Moments

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$$\tilde{\varphi}_\pm(t) = \sum_{n=0}^{\infty} {<}\omega^n{>}_\pm \frac{(-it)^n}{n!}$$

$$<\omega^n>_\pm = \int_0^\infty \varphi_\pm(\omega)\omega^n d\omega$$

$$<0|O_\pm^{(n)}|M> = F{<}\omega^n{>}_\pm \gamma_\pm u$$

$$\varphi_\pm(\omega) = \int_{-i\infty}^{+i\infty} <\omega^n>_\pm \omega^{-n-1} \frac{dn}{2\pi i}$$

First moments

$$<0|Q_v^* D_0 q|M> = <0|\partial_0 (Q_v^* q)|M> = -iF\bar{\Lambda}u$$

because $Q_v^* \overleftarrow{D}_0 = 0$

First moments

$<0|Q_v^* D_0 q|M> = <0|\partial_0(Q_v^* q)|M> = -iF\bar{\Lambda}u$
because $Q_v^* \overleftarrow{D}_0 = 0$
$<0|Q_v^* \vec{D} q|M> = aF\vec{\gamma}\,u$
Equation of motion $<0|Q_v^* \not{D} q|M> = 0$:

$$<0|Q_v^* \vec{D} q|M> = \frac{i}{3} F\bar{\Lambda}\vec{\gamma}u$$

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Equation of motion $<0|Q_v^* \not{D} q|M> = 0$:

$$<0|Q_v^* \vec{D} q|M> = \frac{i}{3} F\bar{\Lambda}\vec{\gamma}u$$

$$<\omega>_+ = \frac{4}{3}\bar{\Lambda} \qquad <\omega>_- = \frac{2}{3}\bar{\Lambda}$$

Second moments

$$<0|Q_v^* D_0^2 q|M> = -F\bar{\Lambda}^2 u \qquad <0|Q_v^* D_0 \vec{D} q|M> = \frac{1}{3} F\bar{\Lambda}^2 \vec{\gamma} u$$

Second moments

$$<0|Q_v^* D_0^2 q|M> = -F\bar{\Lambda}^2 u \qquad <0|Q_v^* D_0 \vec{D} q|M> = \frac{1}{3} F\bar{\Lambda}^2 \vec{\gamma} u$$

$$<0|Q_v^* \vec{E}\cdot\vec{\alpha} q|M> = -iF\lambda_E^2 u \qquad <0|Q_v^* \vec{H}\cdot\vec{\sigma} q|M> = -F\lambda_H^2 u$$

$$\vec{E} = i[D_0, \vec{D}] \qquad \vec{H} = i\vec{D}\times\vec{D} \qquad \vec{\alpha} = \gamma^0\vec{\gamma} \qquad \vec{\sigma} = -\vec{\gamma}\gamma_5\gamma^0$$

Second moments

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$$<0|Q_v^* \vec{D} D_0 q|M> = \frac{1}{3} F\left(\bar{\Lambda}^2 + \lambda_E^2\right)\vec{\gamma} u$$

Second moments

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$$<0|Q_v^* \vec{D} D_0 q|M> = \frac{1}{3} F\left(\bar{\Lambda}^2 + \lambda_E^2\right)\vec{\gamma} u$$

$$<0|Q_v^* D^i D^j q|M> = F\left(b\delta^{ij} - \tfrac{i}{6}\lambda_H^2 \varepsilon^{ijk}\sigma^k\right) u$$

Equation of motion

$$<0|Q_v^* D^i D^j q|M> = -\frac{1}{3} F\left[\left(\bar{\Lambda}^2 + \lambda_E^2 + \lambda_H^2\right)\delta^{ij} + \frac{i}{2}\lambda_H^2 \varepsilon^{ijk}\sigma^k\right] u$$

Second moments

$$<0|Q_v^* D_0^2 q|M> = -F\bar{\Lambda}^2 u \qquad <0|Q_v^* D_0 \vec{D} q|M> = \frac{1}{3} F\bar{\Lambda}^2 \vec{\gamma} u$$

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Equation of motion

$$<0|Q_v^* D^i D^j q|M> = -\frac{1}{3} F\left[\left(\bar{\Lambda}^2 + \lambda_E^2 + \lambda_H^2\right)\delta^{ij} + \frac{i}{2}\lambda_H^2 \varepsilon^{ijk}\sigma^k\right] u$$

$$<\omega^2>_+ = 2\bar{\Lambda}^2 + \frac{2}{3}\lambda_E^2 + \frac{1}{3}\lambda_H^2 \qquad <\omega^2>_- = \frac{2}{3}\bar{\Lambda}^2 + \frac{1}{3}\lambda_H^2$$

B distribution amplitudes

$$<0|Q_v^*(0)[0,z]\gamma_5 q(z)|B>_r = -if_B m\tilde{\varphi}_P$$
$$<0|Q_v^*(0)[0,z]\gamma^\mu\gamma_5 q(z)|B>_r = f_B\left[i\tilde{\varphi}_{A1}p^\mu - m\tilde{\varphi}_{A2}z^\mu\right]$$
$$<0|Q_v^*(0)[0,z]\sigma^{\mu\nu}\gamma_5 q(z)|B>_r = if_B\tilde{\varphi}_T(p^\mu z^\nu - p^\nu z^\mu)$$

B distribution amplitudes

$$
\begin{aligned}
<0|Q_v^*(0)[0,z]\gamma_5 q(z)|B>_r &= -i f_B m \tilde{\varphi}_P \\
<0|Q_v^*(0)[0,z]\gamma^\mu\gamma_5 q(z)|B>_r &= f_B \left[i\tilde{\varphi}_{A1} p^\mu - m\tilde{\varphi}_{A2} z^\mu \right] \\
<0|Q_v^*(0)[0,z]\sigma^{\mu\nu}\gamma_5 q(z)|B>_r &= i f_B \tilde{\varphi}_T (p^\mu z^\nu - p^\nu z^\mu)
\end{aligned}
$$

$$
<0|Q_v^*(0)[0,z]\Gamma q(z)|M>_r = F\,\mathrm{Tr}\,\Gamma \left[\tilde{\varphi}_+ + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{2t} \not{z} \right] \mathcal{M}
$$

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$$\tilde{\varphi}_P = \frac{\tilde{\varphi}_+(t) + \tilde{\varphi}_-(t)}{2} \qquad \tilde{\varphi}_{A1} = \tilde{\varphi}_+(t)$$

$$\tilde{\varphi}_{A2} = \tilde{\varphi}_T = \frac{i}{2}\frac{\tilde{\varphi}_+(t) - \tilde{\varphi}_-(t)}{t}$$

$x = \omega/m \to 0$: $\varphi_{A1}(x) \sim x$, $\varphi_P(x) \sim 1 \Rightarrow$
$\varphi_+(\omega) \sim \omega$, $\varphi_-(\omega) \sim 1$

B distribution amplitudes

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$x = \omega/m \to 0$: $\varphi_{A1}(x) \sim x$, $\varphi_P(x) \sim 1 \Rightarrow$
$\varphi_+(\omega) \sim \omega$, $\varphi_-(\omega) \sim 1$

$$\tilde{\varphi}_P(0) = \tilde{\varphi}_{A1}(0) = 1 \qquad \tilde{\varphi}_{A2}(0) = \tilde{\varphi}_T(0) = \frac{\bar{\Lambda}}{3}$$

Quark-antiquark-gluon distribution amplitudes

Fixed-point gauge $x_\mu A^\mu(x) = 0$

$$<0|Q_v^*(0)q(z)|M> = F\left[\tilde{\varphi}_+(t,z^2) + \frac{\tilde{\varphi}_-(t,z^2) - \tilde{\varphi}_+(t,z^2)}{2t}\not{z}\right]u$$

Quark-antiquark-gluon distribution amplitudes

Fixed-point gauge $x_\mu A^\mu(x) = 0$

$$<0|Q_v^*(0)q(z)|M> = F\left[\tilde{\varphi}_+(t,z^2) + \frac{\tilde{\varphi}_-(t,z^2) - \tilde{\varphi}_+(t,z^2)}{2t}\not{z}\right]u$$

Differentiate $\gamma^\mu \dfrac{\partial}{\partial z^\mu}$

Left-hand side

$$<0|Q_v^*(0)\gamma^\mu(\partial_\mu - iA_\mu(z) + iA_\mu(z))q(z)|M>$$

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Fixed-point gauge ($G_{\mu\nu} = gG^a_{\mu\nu}t^a$)

$$A_\mu(z) = \int_0^1 G_{\nu\mu}(uz)uz^\nu du$$

Left-hand side

$$<0|Q_v^*(0)\gamma^\mu(\partial_\mu - iA_\mu(z) + iA_\mu(z))q(z)|M>$$

Fixed-point gauge ($G_{\mu\nu} = gG^a_{\mu\nu}t^a$)

$$A_\mu(z) = \int_0^1 G_{\nu\mu}(uz)uz^\nu du$$

Definition

$$<0|Q_v^*(0)[0,uz]iG_{\nu\mu}(uz)z^\nu[uz,z]q(z)|M> =$$
$$-F\left[(v_\mu \not{z} - t\gamma_\mu)(\tilde{\psi}_A - \tilde{\psi}_V) + i\sigma_{\mu\nu}z^\nu\tilde{\psi}_V - z_\mu\tilde{\psi}_X + \frac{z_\mu}{t}\not{z}\tilde{\psi}_Y\right]u$$

Left-hand side

$$<0|Q_v^*(0)\gamma^\mu(\partial_\mu - iA_\mu(z) + iA_\mu(z))q(z)|M>$$

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$$\tilde{\varphi}'_- + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{t} = 2t\int_0^1 (\tilde{\psi}_A - \tilde{\psi}_V)u\,du$$
$$\tilde{\varphi}'_+ - \tilde{\varphi}'_- + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{t} + 4t\frac{\partial\tilde{\varphi}_+}{\partial z^2} = 2t\int_0^1 (2\tilde{\psi}_V + \tilde{\psi}_A + \tilde{\psi}_X)u\,du$$

Heavy-quark equation of motion

$$Q_v^*(0)v^\mu \frac{\partial}{\partial z^\mu} q(z) = v^\mu \partial_\mu (Q_v^*(0) q(z)) \\ - v^\mu Q_v^*(0)(\overleftarrow{\partial}_\mu + iA_\mu(0) - iA_\mu(0)) q(z)$$

Heavy-quark equation of motion

$$Q_v^*(0) v^\mu \frac{\partial}{\partial z^\mu} q(z) = v^\mu \partial_\mu (Q_v^*(0) q(z)) - v^\mu Q_v^*(0)(\overleftarrow{\partial}_\mu + iA_\mu(0) - iA_\mu(0)) q(z)$$

Gauge $(x-z)^\mu A_\mu(x) = 0$

$$A_\mu(0) = -\int_0^1 G_{\nu\mu}(uz)(1-u) z^\nu du$$

Heavy-quark equation of motion

$$Q_v^*(0)v^\mu \frac{\partial}{\partial z^\mu} q(z) = v^\mu \partial_\mu (Q_v^*(0)q(z))$$
$$- v^\mu Q_v^*(0)(\overleftarrow{\partial}_\mu + iA_\mu(0) - iA_\mu(0))q(z)$$

Gauge $(x-z)^\mu A_\mu(x) = 0$

$$A_\mu(0) = -\int_0^1 G_{\nu\mu}(uz)(1-u)z^\nu du$$

$$\tilde{\varphi}_+' + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{2t} + i\bar{\Lambda}\tilde{\varphi}_+ + 2t\frac{\partial \tilde{\varphi}_+}{\partial z^2} = -t\int_0^1 (\tilde{\psi}_A + \tilde{\psi}_X)(1-u)du$$

$$\tilde{\varphi}_-' - \tilde{\varphi}_+' + \frac{\tilde{\varphi}_+ - \tilde{\varphi}_-}{t} + i\bar{\Lambda}(\tilde{\varphi}_- - \tilde{\varphi}_+) + 2t\left(\frac{\partial \tilde{\varphi}_-}{\partial z^2} - \frac{\partial \tilde{\varphi}_+}{\partial z^2}\right)$$
$$= 2t\int_0^1 (\tilde{\psi}_A + \tilde{\psi}_Y)(1-u)du$$

Equations for $\tilde{\varphi}_\pm(t)$

$$\tilde{\varphi}'_- + \frac{\tilde{\varphi}_- - \tilde{\varphi}_+}{t} = 2t\int_0^1 (\tilde{\psi}_A - \tilde{\psi}_V)u\,du$$

$$\tilde{\varphi}'_+ + \tilde{\varphi}'_- + 2i\bar{\Lambda}\tilde{\varphi}_+ = -2t\int_0^1 (\tilde{\psi}_A + \tilde{\psi}_X + 2\tilde{\psi}_V u)du$$

Momentum space

$$\tilde{\psi}_i(t, u) = \int \psi_i(\omega, \xi) e^{-i(\omega + \xi u)t} d\omega \, d\xi$$

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$$\omega \frac{d\varphi_-(\omega)}{d\omega} + \varphi_+(\omega) = I(\omega)$$

$$(\omega - 2\bar{\Lambda})\varphi_+(\omega) + \omega\varphi_-(\omega) = J(\omega)$$

Momentum space

$$\tilde{\psi}_i(t,u) = \int \psi_i(\omega,\xi) e^{-i(\omega+\xi u)t} d\omega\, d\xi$$

$$\omega \frac{d\varphi_-(\omega)}{d\omega} + \varphi_+(\omega) = I(\omega)$$

$$(\omega - 2\bar{\Lambda})\varphi_+(\omega) + \omega\varphi_-(\omega) = J(\omega)$$

$$I(\omega) = 2\frac{d}{d\omega}\int_0^\omega d\rho \int_{\omega-\rho}^\infty \frac{d\xi}{\xi}\frac{\partial}{\partial\xi}\left[\psi_A(\rho,\xi) - \psi_V(\rho,\xi)\right]$$

$$J(\omega) = -2\frac{d}{d\omega}\int_0^\omega d\rho \int_{\omega-\rho}^\infty \frac{d\xi}{\xi}\left[\psi_A(\rho,\xi) + \psi_X(\rho,\xi)\right]$$

$$-4\int_0^\omega d\rho \int_{\omega-\rho}^\infty \frac{d\xi}{\xi}\frac{\partial}{\partial\xi}\psi_V(\rho,\xi)$$

Momentum space

$$\tilde{\psi}_i(t,u) = \int \psi_i(\omega,\xi) e^{-i(\omega+\xi u)t} d\omega\, d\xi$$

$$\omega \frac{d\varphi_-(\omega)}{d\omega} + \varphi_+(\omega) = I(\omega)$$

$$(\omega - 2\bar{\Lambda})\varphi_+(\omega) + \omega\varphi_-(\omega) = J(\omega)$$

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$$-4\int_0^\omega d\rho \int_{\omega-\rho}^\infty \frac{d\xi}{\xi}\frac{\partial}{\partial\xi}\psi_V(\rho,\xi)$$

$$J(0) = -2\int_0^\infty \frac{d\xi}{\xi}\left[\psi_A(0,\xi) + \psi_X(0,\xi)\right] = 0$$

Solution

$$\varphi_{\pm}(\omega) = \varphi_{\pm}^{(WW)}(\omega) + \varphi_{\pm}^{(g)}(\omega)$$

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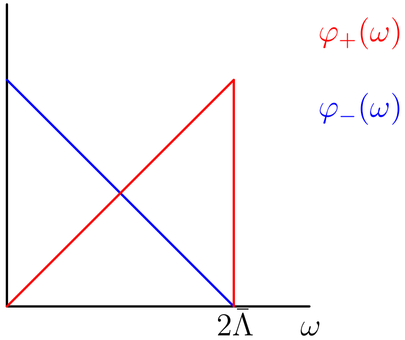
$$\varphi_\pm(\omega) = \varphi_\pm^{(WW)}(\omega) + \varphi_\pm^{(g)}(\omega)$$

Wandzura–Wilczek part

$$\varphi_+^{(WW)}(\omega) = \frac{\omega}{2\bar\Lambda^2}\theta(2\bar\Lambda - \omega)$$

$$\varphi_-^{(WW)}(\omega) = \frac{2\bar\Lambda - \omega}{2\bar\Lambda^2}\theta(2\bar\Lambda - \omega)$$



Moments

$$<\omega^n>_+^{(WW)} = \frac{2(2\bar{\Lambda})^n}{n+2}$$
$$<\omega^n>_-^{(WW)} = \frac{2(2\bar{\Lambda})^n}{(n+1)(n+2)}$$

$<\omega>_\pm$

Moments

$$<\omega^n>_+^{(WW)} = \frac{2(2\bar{\Lambda})^n}{n+2}$$

$$<\omega^n>_-^{(WW)} = \frac{2(2\bar{\Lambda})^n}{(n+1)(n+2)}$$

$<\omega>_\pm$

$$\int \psi_A(\omega,\xi) d\omega\, d\xi = \frac{1}{3}\lambda_E^2$$

$$\int \psi_V(\omega,\xi) d\omega\, d\xi = \frac{1}{3}\lambda_H^2$$

$$\int \psi_X(\omega,\xi) d\omega\, d\xi = 0$$

$<\omega^2>_\pm$

Bare operators

$$\tilde{O}_+(t) = Q_v^*(0)\gamma_+[0, z]q(z)$$
$$O_+(\omega) = Q_v^*(0)\gamma_+\delta(iD_+ - \omega)q(0)$$
$$O_+^{(n)} = Q_v^*(0)\gamma_+(iD_+)^n q(0)$$

Bare operators

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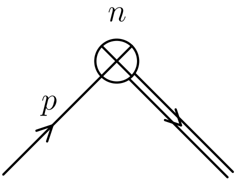
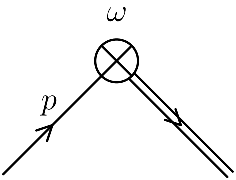
$$O_+(\omega) = Q_v^*(0)\gamma_+\delta(iD_+ - \omega)q(0)$$
$$O_+^{(n)} = Q_v^*(0)\gamma_+(iD_+)^n q(0)$$

$$\tilde{O}_+(t) = \int O_+(\omega)e^{-i\omega t}d\omega = \sum_{n=0}^{\infty} O_+^{(n)}\frac{(-it)^n}{n!}$$
$$O_+(\omega) = \int \tilde{O}_+(t)e^{i\omega t}\frac{dt}{2\pi} = \int_{-i\infty}^{+i\infty} O_+^{(n)}\omega^{-n-1}\frac{dn}{2\pi i}$$
$$O_+^{(n)} = \left(i\frac{d}{dt}\right)^n O_+(t)\Big|_{t=0} = \int_0^{\infty} O_+(\omega)\omega^n d\omega$$

Feynman rules



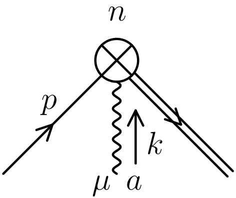
$$= p_+^n \gamma_+$$



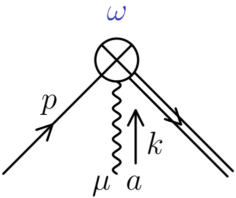
$$= \delta(p_+ - \omega)\gamma_+$$

$$(i\partial_+ + A_+)^n \to \sum_{m=1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} \left[(i\partial_+)^{m-1} A_+\right] (i\partial_+)^{n-m}$$

$$\sum_{m=1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} k_+^{m-1} p_+^{n-m} = \frac{(p_+ + k_+)^n - p_+^n}{k_+}$$



$$= \frac{(p_+ + k_+)^n - p_+^n}{k_+} g_0 t^a n_+^\mu \gamma_+$$



$$= \frac{\delta(p_+ + k_+ - \omega) - \delta(p_+ - \omega)}{k_+} g_0 t^a n_+^\mu \gamma_+$$

Renormalization

$$O_+(\omega) = \int Z_+(\omega, \omega'; \mu) O_+(\omega'; \mu) d\omega'$$

$$O_+(\omega; \mu) = \int Z_+^{-1}(\omega, \omega'; \mu) O_+(\omega') d\omega'$$

$$\int Z_+(\omega, \omega''; \mu) Z_+^{-1}(\omega'', \omega'; \mu) d\omega'' = \delta(\omega - \omega')$$

$$\int Z_+^{-1}(\omega, \omega''; \mu) Z_+(\omega'', \omega'; \mu) d\omega'' = \delta(\omega - \omega')$$

Evolution equation

$$\frac{\partial O_+(\omega;\mu)}{\partial \log \mu} + \int \Gamma_+(\omega,\omega';\mu) O_+(\omega';\mu) d\omega' = 0$$

$$\Gamma_+(\omega,\omega';\mu) = \int Z_+^{-1}(\omega,\omega'';\mu) \frac{\partial Z_+(\omega'',\omega';\mu)}{\partial \log \mu} d\omega''$$

$$= -\int \frac{\partial Z_+^{-1}(\omega,\omega'';\mu)}{\partial \log \mu} Z_+(\omega'',\omega';\mu) d\omega''$$

1 loop

$$Z_+(\omega,\omega';\mu) = \delta(\omega-\omega') + z_+^{(1)}(\omega,\omega';\mu)a_s + \cdots$$
$$Z_+^{-1}(\omega,\omega';\mu) = \delta(\omega-\omega') - z_+^{(1)}(\omega,\omega';\mu)a_s + \cdots$$
$$\Gamma_+(\omega,\omega';\mu) = \Gamma_+^{(1)}(\omega,\omega';\mu)a_s + \cdots$$

$$a_s = \frac{\alpha_s(\mu)}{4\pi}$$

1 loop

$$Z_+(\omega,\omega';\mu) = \delta(\omega-\omega') + z_+^{(1)}(\omega,\omega';\mu)a_s + \cdots$$
$$Z_+^{-1}(\omega,\omega';\mu) = \delta(\omega-\omega') - z_+^{(1)}(\omega,\omega';\mu)a_s + \cdots$$
$$\Gamma_+(\omega,\omega';\mu) = \Gamma_+^{(1)}(\omega,\omega';\mu)a_s + \cdots$$

$$a_s = \frac{\alpha_s(\mu)}{4\pi}$$

$$\Gamma_+^{(1)}(\omega,\omega';\mu) = \frac{\partial z_+^{(1)}(\omega,\omega';\mu)}{\partial \log\mu} - 2\varepsilon z_+^{(1)}(\omega,\omega';\mu)$$

Calculation

$$
\begin{aligned}
M &= \langle 0|O_+(\omega)|q(p), Q_v^*(p')\rangle \\
&= Z_q^{1/2}\tilde{Z}_Q^{1/2}\left[\delta(p_+ - \omega)\gamma_+ + M_1 + M_2 + M_3\right] \\
&= \langle 0|O_+(\omega;\mu)|q(p), Q_v^*(p')\rangle \\
&\quad + a_s \int z_+^{(1)}(\omega,\omega';\mu)\delta(p_+ - \omega')\gamma_+ d\omega'
\end{aligned}
$$

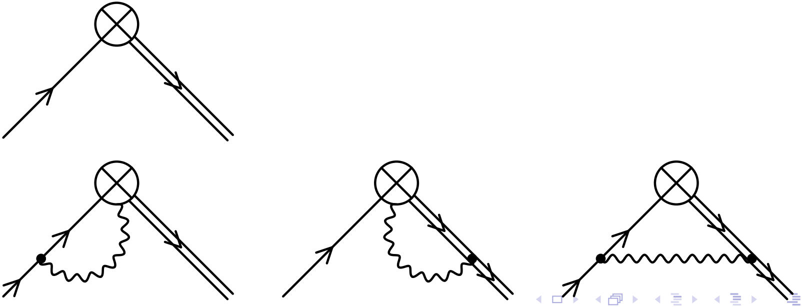
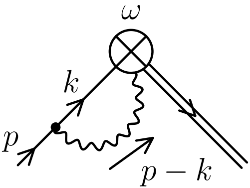


Diagram 1



$$p_+ = \omega'$$

$$p_\perp = 0$$

$$p^2 = p_+ p_- < 0$$

$$M_1 = iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\delta(p_+ - \omega) - \delta(k_+ - \omega)}{p_+ - k_+} \times \frac{\gamma_+ \not{k} \gamma_+}{\left[-(p-k)^2 - i0\right]\left[-k^2 - i0\right]}$$

$$\gamma_+ \not{k} \gamma_+ = 2k_+ \gamma_+$$

$$M_1 = 2C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+$$

$$\times \left[f_1(\omega, \omega') - \delta(\omega - \omega') \int f_1(\omega'', \omega') d\omega'' \right]$$

$$\pi^{d/2}(-p^2)^{-\varepsilon}\Gamma(\varepsilon) f_1(\omega, \omega')$$

$$= -i \frac{\omega}{\omega' - \omega} \int d^d k \frac{\delta(k_+ - \omega)}{[-(p-k)^2 - i0] [-k^2 - i0]}$$

$$M_1 = 2C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+$$

$$\times \left[f_1(\omega,\omega') - \delta(\omega-\omega') \int f_1(\omega'',\omega') d\omega'' \right]$$

$$\pi^{d/2}(-p^2)^{-\varepsilon}\Gamma(\varepsilon) f_1(\omega,\omega')$$

$$= -i \frac{\omega}{\omega'-\omega} \int d^d k \frac{\delta(k_+ - \omega)}{[-(p-k)^2 - i0]\,[-k^2 - i0]}$$

α parametrization

$$\frac{1}{-k^2 - i0} = \int_0^\infty e^{(k^2+i0)\alpha} d\alpha$$

$$M_1 = 2C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+$$
$$\times \left[f_1(\omega,\omega') - \delta(\omega-\omega') \int f_1(\omega'',\omega') d\omega'' \right]$$

$$\pi^{d/2}(-p^2)^{-\varepsilon}\Gamma(\varepsilon) f_1(\omega,\omega')$$
$$= -i \frac{\omega}{\omega'-\omega} \int d^d k \frac{\delta(k_+ - \omega)}{[-(p-k)^2 - i0]\,[-k^2 - i0]}$$

α parametrization

$$\frac{1}{-k^2 - i0} = \int_0^\infty e^{(k^2+i0)\alpha} d\alpha$$

$$= -2i \frac{\omega}{\omega'-\omega} \int d^d k \, d\alpha_1 \, d\alpha_2 \, \frac{d\nu}{2\pi}$$
$$\times \exp\left[\alpha_1(k-p)^2 + \alpha_2 k^2 + 2i\nu(k\cdot n_+ - \omega)\right]$$

Shift $k' = k - \dfrac{\alpha_1 p - i\nu n_+}{\alpha_1 + \alpha_2}$

$$
\begin{aligned}
= & -2i\frac{\omega}{\omega' - \omega}\int d\alpha_1\, d\alpha_2 \exp\left[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2\right] \\
& \times \int \frac{d\nu}{2\pi} \exp\left[2i\nu\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\omega' - \omega\right)\right] \\
& \qquad \times \int d^dk' \exp\left[(\alpha_1+\alpha_2)(k'^2 + i0)\right]
\end{aligned}
$$

Shift $k' = k - \dfrac{\alpha_1 p - i\nu n_+}{\alpha_1 + \alpha_2}$

$$\begin{aligned} = & -2i\frac{\omega}{\omega' - \omega} \int d\alpha_1\, d\alpha_2 \exp\left[\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2\right] \\ & \times \int \frac{d\nu}{2\pi} \exp\left[2i\nu \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\omega' - \omega\right)\right] \\ & \quad \times \int d^d k' \exp\left[(\alpha_1 + \alpha_2)(k'^2 + i0)\right] \end{aligned}$$

Wick rotation $k_0 = i k_{E0}$

$$\int d^d k\, e^{\alpha(k^2 + i0)} = i \int d^d k_E\, e^{-\alpha k_E^2} = i \left(\frac{\pi}{\alpha}\right)^{d/2}$$

$$
\begin{aligned}
&(-p^2)^{-\varepsilon}\Gamma(\varepsilon) f_1(\omega,\omega') \\
&= \frac{\omega}{\omega'-\omega}\int d\alpha_1\, d\alpha_2\, (\alpha_1+\alpha_2)^{-d/2} \exp\left[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} p^2\right] \\
&\qquad \times \delta\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\omega' - \omega\right)
\end{aligned}
$$

$$
\begin{aligned}
&(-p^2)^{-\varepsilon}\Gamma(\varepsilon) f_1(\omega,\omega') \\
&= \frac{\omega}{\omega'-\omega}\int d\alpha_1\, d\alpha_2\, (\alpha_1+\alpha_2)^{-d/2}\exp\left[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2\right] \\
&\qquad \times \delta\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\omega'-\omega\right) \ \times \delta(\alpha_1+\alpha_2-\eta)d\eta
\end{aligned}
$$

$$
\begin{aligned}
&(-p^2)^{-\varepsilon}\Gamma(\varepsilon) f_1(\omega,\omega') \\
&= \frac{\omega}{\omega'-\omega}\int d\alpha_1\, d\alpha_2\, (\alpha_1+\alpha_2)^{-d/2}\exp\left[\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2\right] \\
&\qquad \times \delta\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\omega'-\omega\right) \times \delta(\alpha_1+\alpha_2-\eta)d\eta
\end{aligned}
$$

Substitution $\alpha_i = \eta x_i$

$$
\begin{aligned}
&= \frac{\omega}{\omega'-\omega}\int dx_1\, dx_2\, \delta(x_1+x_2-1)\delta(x_1\omega'-\omega) \\
&\quad \times \int d\eta\eta^{-1+\varepsilon} e^{-(-p^2)x_1x_2\eta}
\end{aligned}
$$

$$f_1(\omega, \omega') = \frac{\theta(\omega' - \omega)}{(\omega' - \omega)^{1+\varepsilon}} \frac{\omega^{1-\varepsilon}}{(\omega')^{1-2\varepsilon}}$$

Distribution

$$\int \left[F(\omega,\omega')\right]_+ \varphi(\omega')d\omega' = \int F(\omega,\omega')\left(\varphi(\omega') - \varphi(\omega)\right)d\omega'$$

Formally

$$F(\omega,\omega') = \left[F(\omega,\omega')\right]_+ + \delta(\omega-\omega')\int F(\omega,\omega'')d\omega''$$

Distribution

$$\int [F(\omega,\omega')]_+ \varphi(\omega')d\omega' = \int F(\omega,\omega')\left(\varphi(\omega') - \varphi(\omega)\right) d\omega'$$

Formally

$$F(\omega,\omega') = [F(\omega,\omega')]_+ + \delta(\omega-\omega') \int F(\omega,\omega'')d\omega''$$

$$M_1 = 2C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+ \Bigg[[f_1(\omega,\omega')]_+ + \delta(\omega-\omega')\left(\int f_1(\omega,\omega'')d\omega'' - \int f_1(\omega'',\omega)d\omega'' \right) \Bigg]$$

Coefficient of $\delta(\omega - \omega')$: substitution $x = \omega''/\omega$

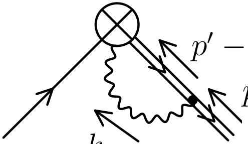
$$\int_1^\infty x^{-1+2\varepsilon}(1-x)^{-1-\varepsilon}dx - \int_0^1 x^{1-\varepsilon}(1-x)^{-1-\varepsilon}dx$$
$$= \int_0^1 \left(x^{-\varepsilon} - x^{1-\varepsilon}\right)(1-x)^{-1-\varepsilon}dx = \int_0^1 x^{-\varepsilon}(1-x)^{-\varepsilon}dx$$

Coefficient of $\delta(\omega-\omega')$: substitution $x=\omega''/\omega$

$$\int_1^\infty x^{-1+2\varepsilon}(1-x)^{-1-\varepsilon}dx-\int_0^1 x^{1-\varepsilon}(1-x)^{-1-\varepsilon}dx$$
$$=\int_0^1\left(x^{-\varepsilon}-x^{1-\varepsilon}\right)(1-x)^{-1-\varepsilon}dx=\int_0^1 x^{-\varepsilon}(1-x)^{-\varepsilon}dx$$

$$M_1=2C_F\frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\gamma_+$$
$$\times\left[\left(\frac{\theta(\omega'-\omega)}{(\omega'-\omega)^{1+\varepsilon}}\frac{\omega^{1-\varepsilon}}{(\omega')^{1-2\varepsilon}}\right)_++\frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\delta(\omega-\omega')\right]$$

Diagram 2



$$p' \cdot v = \omega_1 < 0$$

$$M_2 = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\delta(p_+ + k_+ - \omega) - \delta(p_+ - \omega)}{k_+} \times \frac{v_+ \gamma_+}{[-k^2 - i0]\,[-(p' - k) \cdot v - i0]}$$

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+ \times \left[f_2(\omega - \omega') - \delta(\omega - \omega') \int f_2(\omega'') d\omega'' \right]$$

$$\pi^{d/2}\Gamma(\varepsilon) f_2(\omega'') = -\frac{i}{2\omega''} \int d^d k \frac{\delta(k_+ - \omega'')}{[-k^2 - i0]\,[-(p' - k)\cdot v - i0]}$$

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+$$
$$\times \left[f_2(\omega - \omega') - \delta(\omega - \omega') \int f_2(\omega'') d\omega'' \right]$$

$$\pi^{d/2}\Gamma(\varepsilon) f_2(\omega'')$$
$$= -\frac{i}{2\omega''} \int d^d k \frac{\delta(k_+ - \omega'')}{\left[-k^2 - i0\right]\left[-(p' - k)\cdot v - i0\right]}$$

$$= -\frac{2i}{\omega''} \int d^d k \, d\alpha_1 \, d\alpha_2 \, \frac{d\nu}{2\pi}$$
$$\times \exp\left[\alpha_1 k^2 + 2\alpha_2 (p' - k)\cdot v + 2i\nu(k \cdot n_+ - \omega'')\right]$$

Shift $k' = k - \dfrac{\alpha_2 v - i\nu n_+}{\alpha_1}$

$$= -\frac{2i}{\omega''}\int d\alpha_1\, d\alpha_2 \exp\left[-\frac{\alpha_2^2}{\alpha_1} + 2\omega_1\alpha_2\right] \times \int \frac{d\nu}{2\pi}\exp\left[2i\nu\left(\frac{\alpha_2}{\alpha_1} - \omega''\right)\right] \times \int d^d k' \exp\left[\alpha_1(k'^2 + i0)\right]$$

Shift $k' = k - \dfrac{\alpha_2 v - i\nu n_+}{\alpha_1}$

$$= -\frac{2i}{\omega''}\int d\alpha_1\, d\alpha_2 \exp\left[-\frac{\alpha_2^2}{\alpha_1} + 2\omega_1\alpha_2\right]$$
$$\times \int \frac{d\nu}{2\pi} \exp\left[2i\nu\left(\frac{\alpha_2}{\alpha_1} - \omega''\right)\right]$$
$$\times \int d^d k' \exp\left[\alpha_1(k'^2 + i0)\right]$$

$$\Gamma(\varepsilon) f_2(\omega'') = \frac{1}{\omega''}\int d\alpha_1\, d\alpha_2\, \alpha_1^{-d/2} \exp\left[-\frac{\alpha_2^2}{\alpha_1} + 2\omega_1\alpha_2\right]$$
$$\times\, \delta\left(\frac{\alpha_2}{\alpha_1} - \omega''\right)$$

Shift $k' = k - \dfrac{\alpha_2 v - i\nu n_+}{\alpha_1}$

$$= -\frac{2i}{\omega''}\int d\alpha_1\, d\alpha_2 \exp\left[-\frac{\alpha_2^2}{\alpha_1} + 2\omega_1\alpha_2\right]$$

$$\times \int \frac{d\nu}{2\pi} \exp\left[2i\nu\left(\frac{\alpha_2}{\alpha_1} - \omega''\right)\right]$$

$$\times \int d^d k' \exp\left[\alpha_1(k'^2 + i0)\right]$$

$$\Gamma(\varepsilon) f_2(\omega'') = \frac{1}{\omega''}\int d\alpha_1\, d\alpha_2\, \alpha_1^{-d/2} \exp\left[-\frac{\alpha_2^2}{\alpha_1} + 2\omega_1\alpha_2\right]$$

$$\times\, \delta\left(\frac{\alpha_2}{\alpha_1} - \omega''\right)$$

Substitution $\alpha_2 = \alpha_1 y$

$$= \frac{1}{\omega''}\int dy\, \delta(y - \omega'') \int d\alpha_1 \alpha_1^{-1+\varepsilon} e^{-y(y-2\omega_1)\alpha_1}$$

$$f_2(\omega'') = \frac{\theta(\omega'')}{(\omega'')^{1+\varepsilon}(\omega'' - 2\omega_1)^{\varepsilon}}$$

$$f_2(\omega'') = \frac{\theta(\omega'')}{(\omega'')^{1+\varepsilon}(\omega'' - 2\omega_1)^{\varepsilon}}$$

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+ \Bigg[[f_2(\omega - \omega')]_+ + \delta(\omega - \omega') \left(\int_0^{\omega} f_2(\omega - \omega'') d\omega'' - \int_0^{\infty} f_2(\omega'') d\omega'' \right) \Bigg]$$

$$f_2(\omega'') = \frac{\theta(\omega'')}{(\omega'')^{1+\varepsilon}(\omega'' - 2\omega_1)^{\varepsilon}}$$

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+ \Bigg[[f_2(\omega - \omega')]_+ + \delta(\omega - \omega') \left(\int_0^{\omega} f_2(\omega - \omega'') d\omega'' - \int_0^{\infty} f_2(\omega'') d\omega'' \right) \Bigg]$$

Coefficient of $\delta(\omega - \omega')$ at $|\omega_1| \ll \omega$

$$-\int_{\omega}^{\infty} f_2(\omega'') d\omega'' = -\int_{\omega}^{\infty} (\omega'')^{-1-2\varepsilon} d\omega'' = -\frac{\omega^{-2\varepsilon}}{2\varepsilon}$$

$$f_2(\omega'') = \frac{\theta(\omega'')}{(\omega'')^{1+\varepsilon}(\omega'' - 2\omega_1)^{\varepsilon}}$$

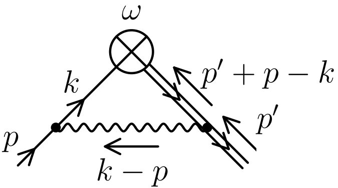
$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+ \Bigg[[f_2(\omega - \omega')]_+ + \delta(\omega - \omega') \left(\int_0^{\omega} f_2(\omega - \omega'') d\omega'' - \int_0^{\infty} f_2(\omega'') d\omega'' \right) \Bigg]$$

Coefficient of $\delta(\omega - \omega')$ at $|\omega_1| \ll \omega$

$$-\int_{\omega}^{\infty} f_2(\omega'') d\omega'' = -\int_{\omega}^{\infty} (\omega'')^{-1-2\varepsilon} d\omega'' = -\frac{\omega^{-2\varepsilon}}{2\varepsilon}$$

$$M_2 = 2C_F \frac{g_0^2}{(4\pi)^{d/2}} \Gamma(\varepsilon)\gamma_+ \left[\left(\frac{\theta(\omega - \omega')}{(\omega - \omega')^{1+2\varepsilon}} \right)_+ - \frac{\omega^{-2\varepsilon}}{2\varepsilon} \delta(\omega - \omega') \right]$$

Diagram 3



$$M_3 = -iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d}$$
$$\times \frac{\delta(k_+ - \omega)\,\gamma_+ \not{k} \not{v}}{[-(k-p)^2 - i0]\,[-k^2 - i0]\,[-(p'+p-k)\cdot v - i0]}$$

$\gamma_+ \not{k} \not{v} = k_+ \gamma_+$

UV finite

Result

Re-expressing $\dfrac{g_0^2}{(4\pi)^{d/2}} = a_s \mu^{2\varepsilon} e^{\gamma_E \varepsilon}$

$$M = \gamma_+ \left(Z_q \tilde{Z}_Q \right)^{1/2} \Bigg[\delta(\omega - \omega') + 2C_F \frac{\alpha_s(\mu)}{4\pi\varepsilon} \times \left(\left(\frac{\theta(\omega' - \omega)}{\omega' - \omega} \frac{\omega}{\omega'} \right)_+ + \delta(\omega - \omega') + \left(\frac{\theta(\omega - \omega')}{\omega - \omega'} \right)_+ - \left(\frac{1}{2\varepsilon} - \log \frac{\omega}{\mu} \right) \delta(\omega - \omega') + \mathcal{O}(\varepsilon) \right) \Bigg]$$

Result

Re-expressing $\dfrac{g_0^2}{(4\pi)^{d/2}} = a_s \mu^{2\varepsilon} e^{\gamma_E \varepsilon}$

$$M = \gamma_+ \left(Z_q \tilde{Z}_Q \right)^{1/2} \left[\delta(\omega - \omega') + 2C_F \frac{\alpha_s(\mu)}{4\pi\varepsilon} \right.$$
$$\times \left(\left(\frac{\theta(\omega' - \omega)}{\omega' - \omega} \frac{\omega}{\omega'} \right)_+ + \delta(\omega - \omega') \right.$$
$$\left. \left. + \left(\frac{\theta(\omega - \omega')}{\omega - \omega'} \right)_+ - \left(\frac{1}{2\varepsilon} - \log \frac{\omega}{\mu} \right) \delta(\omega - \omega') + \mathcal{O}(\varepsilon) \right) \right]$$

Feynman gauge

$$Z_q = 1 - C_F \frac{\alpha_s}{4\pi\varepsilon} \qquad \tilde{Z}_Q = 1 + 2C_F \frac{\alpha_s}{4\pi\varepsilon}$$

$$
\begin{aligned}
Z_+(\omega, \omega'; \mu) = \delta(\omega - \omega') + 2C_F \frac{\alpha_s(\mu)}{4\pi\varepsilon} \\
\times \left[\left(\frac{\theta(\omega' - \omega)}{\omega' - \omega} \frac{\omega}{\omega'} + \frac{\theta(\omega - \omega')}{\omega - \omega'} \right)_+ \right. \\
\left. + \left(-\frac{1}{2\varepsilon} + \log \frac{\omega}{\mu} + \frac{5}{4} \right) \delta(\omega - \omega') \right]
\end{aligned}
$$

Evolution kernel

$$\Gamma_+(\omega, \omega'; \mu) = \Gamma(\omega, \omega'; a_s) + \left[-\Gamma(\omega, \omega'; a_s) \log \frac{\omega}{\mu} + \tilde{\gamma}_j(a_s) + \gamma(a_s)\right] \delta(\omega - \omega')$$

Evolution kernel

$$\Gamma_+(\omega,\omega';\mu) = \Gamma(\omega,\omega';a_s) + \left[-\Gamma(\omega,\omega';a_s)\log\frac{\omega}{\mu} + \tilde{\gamma}_j(a_s) + \gamma(a_s)\right]\delta(\omega-\omega')$$

Cusp anomalous dimension

$$\Gamma(a_s) = \Gamma_0 a_s + \Gamma_1 a_s^2 + \cdots \qquad \Gamma_0 = 4C_F$$

Evolution kernel

$$\Gamma_+(\omega,\omega';\mu) = \Gamma(\omega,\omega';a_s) + \left[-\Gamma(\omega,\omega';a_s)\log\frac{\omega}{\mu} + \tilde{\gamma}_j(a_s) + \gamma(a_s)\right]\delta(\omega-\omega')$$

Cusp anomalous dimension

$$\Gamma(a_s) = \Gamma_0 a_s + \Gamma_1 a_s^2 + \cdots \qquad \Gamma_0 = 4C_F$$

Anomalous dimension of $F(\mu)$

$$\tilde{\gamma}_j(a_s) = \tilde{\gamma}_{j0} a_s + \tilde{\gamma}_{j1} a_s^2 + \cdots \qquad \tilde{\gamma}_{j0} = -3C_F$$

Evolution kernel

$$\Gamma_+(\omega, \omega'; \mu) = \Gamma(\omega, \omega'; a_s) + \left[-\Gamma(\omega, \omega'; a_s) \log \frac{\omega}{\mu} + \tilde{\gamma}_j(a_s) + \gamma(a_s) \right] \delta(\omega - \omega')$$

Cusp anomalous dimension

$$\Gamma(a_s) = \Gamma_0 a_s + \Gamma_1 a_s^2 + \cdots \qquad \Gamma_0 = 4C_F$$

Anomalous dimension of $F(\mu)$

$$\tilde{\gamma}_j(a_s) = \tilde{\gamma}_{j0} a_s + \tilde{\gamma}_{j1} a_s^2 + \cdots \qquad \tilde{\gamma}_{j0} = -3C_F$$

$$\gamma(a_s) = \gamma_0 a_s + \gamma_1 a_s^2 + \cdots \qquad \gamma_0 = -2C_F$$

Evolution kernel

$$\Gamma_+(\omega,\omega';\mu) = \Gamma(\omega,\omega';a_s) + \left[-\Gamma(\omega,\omega';a_s)\log\frac{\omega}{\mu} + \tilde{\gamma}_j(a_s) + \gamma(a_s)\right]\delta(\omega-\omega')$$

Cusp anomalous dimension

$$\Gamma(a_s) = \Gamma_0 a_s + \Gamma_1 a_s^2 + \cdots \qquad \Gamma_0 = 4C_F$$

Anomalous dimension of $F(\mu)$

$$\tilde{\gamma}_j(a_s) = \tilde{\gamma}_{j0} a_s + \tilde{\gamma}_{j1} a_s^2 + \cdots \qquad \tilde{\gamma}_{j0} = -3C_F$$

$$\gamma(a_s) = \gamma_0 a_s + \gamma_1 a_s^2 + \cdots \qquad \gamma_0 = -2C_F$$

$$\Gamma(\omega,\omega',a_s) = \Gamma_0(\omega,\omega')a_s + \cdots$$

$$\Gamma_0(\omega,\omega') = \left(\frac{\theta(\omega'-\omega)}{\omega'-\omega}\frac{\omega}{\omega'} + \frac{\theta(\omega-\omega')}{\omega-\omega'}\right)_+$$

Evolution equation

$$\frac{\partial \varphi_+(\omega;\mu)}{\partial \log \mu} + \left[-\Gamma(a_s) \log \frac{\omega}{\mu} + \gamma(a_s) \right] \varphi_+(\omega;\mu) \\ + \int \Gamma(\omega,\omega';a_s) \varphi_+(\omega';\mu) d\omega' = 0$$

Evolution equation

$$\frac{\partial \varphi_+(\omega;\mu)}{\partial \log \mu} + \left[-\Gamma(a_s)\log\frac{\omega}{\mu} + \gamma(a_s)\right]\varphi_+(\omega;\mu) + \int \Gamma(\omega,\omega';a_s)\varphi_+(\omega';\mu)d\omega' = 0$$

$$\int \Gamma(\omega,\omega';a_s)\omega'^n d\omega' = \tilde{\Gamma}(n,a_s)\omega^n$$

$$\tilde{\Gamma}(n,a_s) = \tilde{\Gamma}_0 a_s + \cdots$$

$$\tilde{\Gamma}_0(n) = 4C_F\left[\psi(1+n) + \psi(1-n) + 2\gamma_E\right]$$

Solution

$$\frac{\partial}{\partial \log \mu} \left(\frac{\omega}{\mu}\right)^{n+\xi(\mu)} = \left(\frac{\omega}{\mu}\right)^{n+\xi(\mu)} \left[\frac{d\xi}{d \log \mu} \log \frac{\omega}{\mu} - n - \xi\right]$$

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$$\frac{d\xi}{d \log \mu} = \Gamma(a_s)$$

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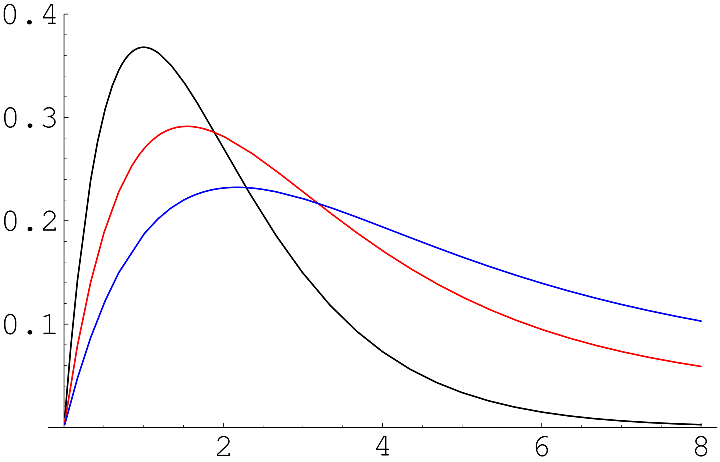
$$\frac{d\xi}{d \log \mu} = \Gamma(a_s)$$

$$\begin{aligned}
\varphi_+(\omega; \mu) = & \left(\frac{\mu_0}{\mu} \right)^{\frac{\Gamma_0}{2\beta_0}} \\
& \times \exp \left[-\frac{\beta_1}{2\beta_0 \Gamma_0} \xi^2 + \left(\frac{\beta_1}{\beta_0^2} - \frac{\Gamma_1}{2\beta_0 \Gamma_0} - \frac{\gamma_0}{\Gamma_0} - 2\gamma_E \right) \xi \right] \left(\frac{\omega}{\Lambda_{\overline{\mathrm{MS}}}} \right)^{\xi} \\
& \times \int_{-\infty}^{+\infty} <\omega^{-1-in}>_+^{(\mu_0)} \omega^{in} \frac{\Gamma(1-\xi-in)\Gamma(1+in)}{\Gamma(1+\xi+in)\Gamma(1-in)} \\
& \quad \times \exp \left(-i \frac{\beta_1}{\beta_0 \Gamma_0} \xi n \right) \frac{dn}{2\pi}
\end{aligned}$$



0.4
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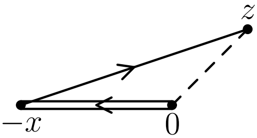
Sum rules



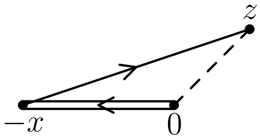
$$i<TO_\pm(t)\bar{\jmath}_+(-x)> =$$

$$\gamma_\pm \frac{1+\gamma^0}{2}\delta(\vec{x}\,)\theta(x^0)\tilde{\Pi}_\pm(x^0,t)$$

Fourier transform

$$\Pi_\pm(x^0,\omega) = \int \tilde{\Pi}_\pm(x^0,t)e^{i\omega t}\frac{dt}{2\pi}$$

Sum rules



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Analytically continuing from $x^0 > 0$ to $x^0 = -i\tau$

$$\Pi_\pm(\tau,\omega) = \int \rho_\pm(\varepsilon,\omega)e^{-\varepsilon\tau}d\varepsilon = F^2\tilde{\varphi}_\pm(\omega)e^{-\bar{\Lambda}\tau} + \Pi^c_\pm(\tau,\omega)$$

Perturbative contribution

Fixed-point gauge $x^\mu A_\mu(x) = 0$

$$\tilde{\Pi}_+^{(1)}(\tau, t) = \frac{N_c}{2\pi^2\tau(\tau + 2it)^2} \qquad \tilde{\Pi}_-^{(1)}(\tau, t) = \frac{N_c}{2\pi^2\tau^2(\tau + 2it)}$$

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Spectral densities (contour $a > 0$)

$$\rho_\pm(\varepsilon,\omega) = \int_{a-i\infty}^{a+i\infty} \Pi_\pm(\tau,\omega)e^{\varepsilon\tau}\frac{d\tau}{2\pi i}$$

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Quark condensate

$$\tilde{\Pi}^{(2)}_{\pm}(x^0, t) = -\frac{1}{4} {<}\bar{q}q{>} f_S((x+z)^2)$$

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Bilocal quark condensate

$$f_S(x^2) = \frac{< \bar{q}(0)[0,x]q(x) >}{< \bar{q}q >} = 1 + \frac{m_0^2}{16} x^2 + \cdots$$

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$$\int \tilde{f}_S(\nu) d\nu = 1 \qquad \int \tilde{f}_S(\nu)\nu d\nu = \frac{m_0^2}{16} \qquad \ldots$$

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$$\Pi_{\pm}^{(2)}(\tau, \omega) = -\frac{<\bar{q}q>}{8\tau} \tilde{f}_S \left(\frac{\omega}{2\tau}\right) e^{-\omega\tau/2}$$

Bilocal quark condensate

Local OPE

$$\tilde{f}_S(\nu) = \delta(\nu) - \frac{m_0^2}{16}\delta'(\nu) + \cdots$$

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Asymptotics at large $-x^2$

$$f_S(x^2) \sim e^{-\bar{\Lambda}\sqrt{-x^2}}$$

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$$f_S(x^2) \sim e^{-\bar{\Lambda}\sqrt{-x^2}}$$

Bakulev–Mikhailov anzatz

$$\tilde{f}_S(\nu) = N \exp\left(-\frac{\bar{\Lambda}^2}{4\nu} - \sigma\nu\right)$$

Sum rules

Model of spectral densities

$$\rho_{\pm}(\varepsilon,\omega) = F^2\varphi_{\pm}(\omega)\delta\left(\varepsilon - \bar{\Lambda}\right) + \rho_{\pm}^{(1)}(\varepsilon,\omega)\theta(\varepsilon-\varepsilon_c)$$

Sum rules

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Sum rules

$$\begin{aligned}
F^2\varphi_{+}(\omega)e^{-\bar{\Lambda}\tau} &= \frac{N_c}{8\pi^2\tau}\omega e^{-\omega\tau/2}\delta_0\left(\left(\varepsilon_c-\frac{\omega}{2}\right)\tau\right) \\
&\quad - \frac{<\bar{q}q>}{8\tau}\tilde{f}_S\left(\frac{\omega}{2\tau}\right)e^{-\omega\tau/2} \\
F^2\varphi_{-}(\omega)e^{-\bar{\Lambda}\tau} &= \frac{N_c}{4\pi^2\tau^2}e^{-\omega\tau/2}\delta_1\left(\left(\varepsilon_c-\frac{\omega}{2}\right)\tau\right) \\
&\quad - \frac{<\bar{q}q>}{8\tau}\tilde{f}_S\left(\frac{\omega}{2\tau}\right)e^{-\omega\tau/2}
\end{aligned}$$

Sum rules

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$$\rho_\pm(\varepsilon,\omega) = F^2\varphi_\pm(\omega)\delta\left(\varepsilon - \bar\Lambda\right) + \rho_\pm^{(1)}(\varepsilon,\omega)\theta(\varepsilon-\varepsilon_c)$$

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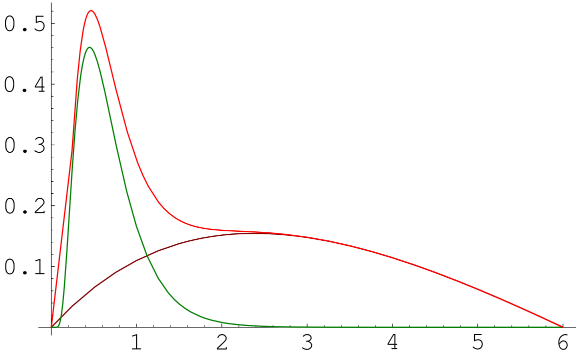
&\quad - \frac{<\bar q q>}{8\tau}\tilde f_S\left(\frac{\omega}{2\tau}\right)e^{-\omega\tau/2}\\
F^2\varphi_-(\omega)e^{-\bar\Lambda\tau} &= \frac{N_c}{4\pi^2\tau^2}e^{-\omega\tau/2}\delta_1\left(\left(\varepsilon_c - \frac{\omega}{2}\right)\tau\right)\\
&\quad - \frac{<\bar q q>}{8\tau}\tilde f_S\left(\frac{\omega}{2\tau}\right)e^{-\omega\tau/2}
\end{aligned}$$

$$\delta_n(x) = \theta(x)\left(1 - e^{-x}\sum_{m=0}^{n}\frac{x^m}{m!}\right)$$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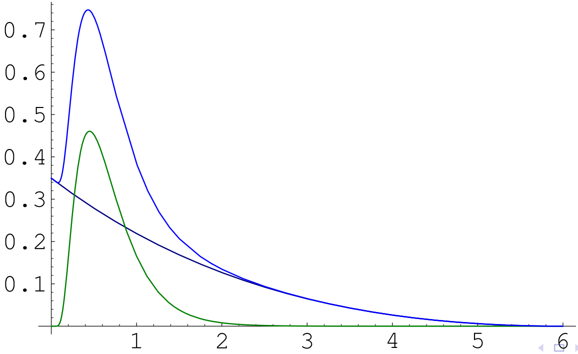
0.7
0.6
0.5
0.4
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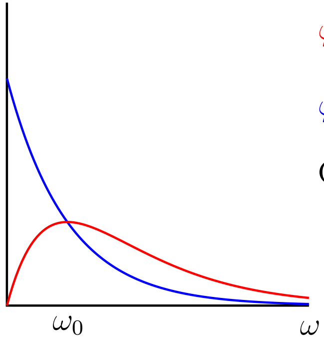
0.2
0.1
1
2
3
4
5
6

Grozin–Neubert model

$$\varphi_+(\omega) = \frac{\omega}{\omega_0^2} e^{-\omega/\omega_0} \qquad \varphi_-(\omega) = \frac{1}{\omega_0} e^{-\omega/\omega_0}$$

From $<\omega>_\pm$

$$\omega_0 = \frac{2}{3}\bar{\Lambda}$$



$\varphi_+(\omega)$

$\varphi_-(\omega)$

Compare with Wandzura–Wilczek

Radiative correction

$$\rho_+(\varepsilon,\omega) = \frac{N_c}{8\pi^2}\omega$$

$$\times \begin{cases} x>1: & 1 + C_F\frac{\alpha_s}{4\pi}\Big[-2\log^2\frac{\omega}{\mu} - 4\big(\log(x-1)+1\big)\log\frac{\omega}{\mu} \\ & \quad + 2\,\mathrm{Li}_2\left(\frac{1}{1-x}\right) - \log^2(x-1) \\ & \quad - (2x+3)\log(x-1) + 2x\log x + \frac{7}{12}\pi^2 + 7\Big] \\ x<1: & 2C_F\frac{\alpha_s}{4\pi}\Big[2\big(\log(1-x)+x\big)\log\frac{\omega}{\mu} + 2\log^2(1-x) \\ & \quad + (2x-1)\log(1-x) - x\Big] \end{cases}$$

$x = \frac{2\varepsilon}{\omega}$ Perturbative $1/\omega$ tail with $\alpha_s/(4\pi)$