Ward Identities and Radiative Rare Semileptonic B-decays

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Standard Model process

$$B \to l\nu$$

- *Direct measurement of* f_B
- *CKM matrix element –* V_{ub}
- *New Physics beyond S.M.*

(at tree level)

The decay width

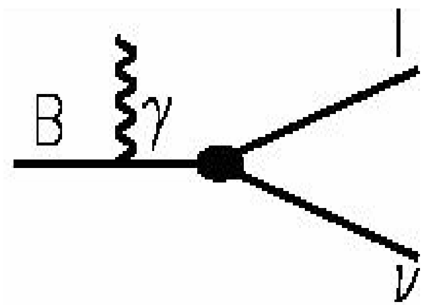
$$\Gamma\left(B \to l\nu\right) = \frac{G_F^2}{8\pi}\left|V_{ub}\right|^2 f_B^2 \frac{m_l^2}{M_B^2} M_B^3 \left(1 - \frac{m_l^2}{M_B^2}\right)^2$$

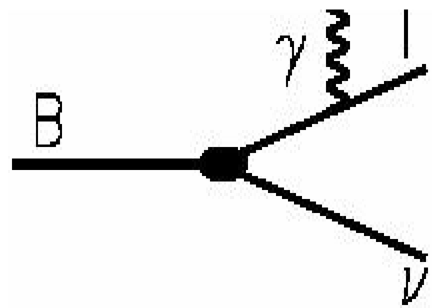
$$Br\left(B \to l\nu\right) \approx \begin{cases} 5.8\times 10^{-12} & \text{for } e^- \\ 2.2\times 10^{-7} & \text{for } \mu^- \end{cases}$$

The Radiative Partner $B \to \gamma l \nu$

In Radiative B-decay Process, there are two major contributions to the amplitude:

- ***Inner Bremsstrahlung (IB)***



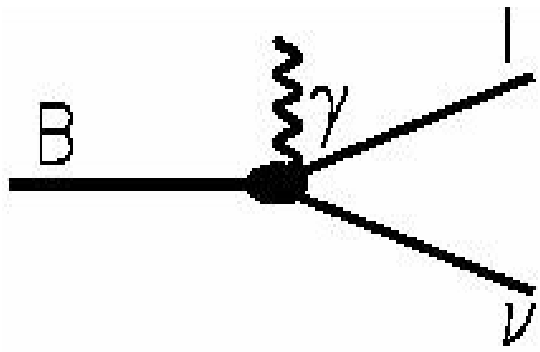


$$M_{IB} = ie\frac{G_F}{\sqrt{2}} V_{ub} f_B m_l \epsilon_\mu^* L^\mu$$

with

$$L^\mu = m_l \bar{u}(p_\nu)(1+\gamma_5)\left(\frac{2p^\mu}{2p\cdot k} - \frac{2p_l^\mu + \not{k}\gamma^\mu}{2p_l\cdot k}\right) v(p_l, s_l)$$

- ***Structure Dependent (SD)***



$$M_{SD} = -i\frac{G_F}{\sqrt{2}} V_{ub} f_B m_l \epsilon_\mu^* \tilde{H}^{\mu\nu} l_\nu$$

where

$$\tilde{H}^{\mu\nu} = iF_V(q^2)\epsilon^{\mu\nu\alpha\beta} k_\alpha p_\beta - F_A(q^2)(p \bullet k g^{\mu\nu} - p^\mu k^\nu)$$

$$l^\mu = \bar{u}(p_\nu)\gamma^\mu(1+\gamma_5)v(p_l, s_l)$$

$$q^\mu = (p-k)^\mu = (p_l + p_\nu)^\mu$$

It depends on vector and axial vector form factors.

The decay constant and form factors are defined as

$$\langle 0|\bar{u}\gamma^{\mu}\gamma_5 b|B(p)\rangle = -if_B p^{\mu}$$

$$\langle\gamma(k)|\bar{u}\gamma^{\mu}\gamma_5 b|B(p)\rangle = -[(\epsilon^* \bullet p)k^{\mu} - \epsilon^{*\mu}(p \bullet k)]F_A(q^2)$$

$$\langle\gamma(k)|\bar{u}\gamma^{\mu}b|B(p)\rangle = -i\epsilon^{\mu\nu\alpha\beta}\epsilon^*_{\nu}p_{\alpha}k_{\beta}F_V(q^2)$$

The Structure Dependent part is given by

$$iH^{\mu\nu} = i\int d^4x\, e^{ik\cdot x}\langle 0|T(j^{\mu}_{em}(x)J^{\nu}_{2}(0))|B(p)\rangle$$

For real photon we can write

$$H^{\mu\nu} = \tilde{H}^{\mu\nu} + f_B\frac{p^{\mu}p^{\nu}}{p\cdot k}$$

with $k_{\mu}\tilde{H}^{\mu\nu} = 0$

The absorptive part is given by

$$Abs[iH^{\mu\nu}] = \frac{1}{2}\int d^4x\, e^{ik\cdot x}\langle 0|[j^{\mu}_{em}(x), J^{\nu}_{2}(0)]|B(p)\rangle$$

$$= \frac{1}{2}(2\pi)^4\ [\sum_n \langle 0|j^{\mu}_{em}(0)|n\rangle\langle n|J^{\nu}_{2}(0)|B(p)\rangle\delta^4(k-p_n)$$

$$-\sum_n \langle 0|J^{\nu}_{2}(0)|n\rangle\langle n|j^{\mu}_{em}(0)|B(p)\rangle\delta^4(k+p_n-p)]$$

The contribution to absorptive part are all possible intermediate states that couple to $B\gamma$ and are annihilated by the weak vertex $\langle 0|J_2^\nu(0)|n\rangle$
These include the multiparticle contrinum as well resonances with quantum numbers 1⁻ and 1⁺.

$$F_V(t) = \frac{g_{BB^*\gamma}}{M_{B^*}^2 - t} f_{B^*} + \cdots$$

$$F_A(t) = \frac{f_{B_A^* B\gamma}}{M_{B_A^*}^2 - t} f_{B_A^*} + \cdots$$

We assume that the contributions from the radial excitations of B^* and B_A^* dominate the higher state contribution.

$$F_V(t) = \frac{R_V}{1 - t/M_{B^*}^2} + \sum_i \frac{R_{V_i}}{1 - t/M_{B_i^*}^2} + \frac{1}{\pi}\int_{S_0}^{M^2} \frac{\mathrm{Im}\, F_V^{\mathrm{Cont}}(s)}{s - t - i\varepsilon} ds$$

$$F_A(t) = \frac{R_A}{1 - t/M_{B_A^*}^2} + \sum_i \frac{R_{A_i}}{1 - t/M_{B_{A\,i}^*}^2} + \frac{1}{\pi}\int_{S_0}^{M^2} \frac{\mathrm{Im}\, F_A^{\mathrm{Cont}}(s)}{s - t - i\varepsilon} ds$$

$$S_0 = M_B + m_\pi$$

where

$$R_V = \frac{g_{BB^*\gamma}}{M_{B^*}^2} f_{B^*}$$

$$R_A = \frac{f_{B_A^* B\gamma}}{M_{B_A^*}^2} f_{B_A^*}$$

If we model the continum contributions by quark triangle graph, we have

$$F_V^{\text{Cont}} = F_A^{\text{Cont}} = \frac{f_B}{M_B}\left\{\frac{Q_u}{\bar{\Lambda}} - \frac{Q_b}{M_B}\left(1 + \frac{\bar{\Lambda}}{M_B}\right)\right\}\frac{1}{1 - q^2/M_B^2}$$

where $\bar{\Lambda} = M_B - m_b$, together with the term

$$(Q_u - Q_b) f_B \frac{p^\mu p^\nu}{k \cdot p} = f_B \frac{p^\mu p^\nu}{k \cdot p}$$

Calculation of Vector and Axial Vector Form Factors

- Ward Identities
- Gauge Invariance
- Pole Contributions
- Coupling Constants
- Branching Ratio

Ward Identities and Gauge Invariance

Define

$$\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu} q_\nu b|B(p)\rangle = -i\varepsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu k_\alpha p_\beta F_1(q^2)$$

$$\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu}\gamma_5 q_\nu b|B(p)\rangle = [(q\cdot k)\epsilon^{*\mu} - (\epsilon^*\cdot q)k^\mu]F_3(q^2)$$

Ward Identities used to relate different form factors appearing in our calculation are

$$\begin{aligned}\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu} q_\nu b|B(p)\rangle &= -(m_b+m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu b|B(p)\rangle \\ &\quad +(p^\mu+k^\mu)\langle \gamma(k,\epsilon)|\bar{u} b|B(p)\rangle \\ &= -(m_b+m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu b|B(p)\rangle\end{aligned}$$

$$\begin{aligned}\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu}\gamma_5 q_\nu b|B(p)\rangle &= (m_b-m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu\gamma_5 b|B(p)\rangle \\ &\quad +(p^\mu+k^\mu)\langle \gamma(k,\epsilon)|\bar{u}\gamma_5 b|B(p)\rangle \\ &= (m_b-m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu\gamma_5 b|B(p)\rangle\end{aligned}$$

Using gauge invariance we have

$$F_V(q^2) = \frac{1}{m_b+m_q} F_1(q^2)$$

$$F_A(q^2) = \frac{1}{m_b-m_q} F_3(q^2)$$

To make use of Ward Identities to relate different form factors, define

$$\langle \gamma(k,\epsilon)| i\bar{u}\sigma_{\alpha\beta} b |B(p)\rangle = -i\varepsilon_{\alpha\beta\rho\sigma}\epsilon^{*\rho}(k)[(p+k)^\sigma g_+ + q^\sigma g_-]$$
$$-iq\cdot\epsilon^*(k)\varepsilon_{\alpha\beta\rho\sigma}(p+k)^\rho q^\sigma h$$
$$-i[q_\alpha\varepsilon_{\beta\rho\sigma\tau}\epsilon^{*\rho}(k)(p+k)^\sigma q^\tau - \alpha\leftrightarrow\beta]h_1$$
$$-i[(p+k)_\alpha\varepsilon_{\beta\rho\sigma\tau}\epsilon^{*\rho}(k)(p+k)^\sigma q^\tau - \alpha\leftrightarrow\beta]h_2$$

And using Dirac algebra we can write

$$\langle \gamma(k,\epsilon)| i\bar{u}\sigma^{\mu\nu}\gamma_5 b |B(p)\rangle = -\frac{i}{2}\varepsilon^{\mu\nu\alpha\beta}\langle \gamma(k,\epsilon)| i\bar{u}\sigma_{\alpha\beta} b |B(p)\rangle$$

Using Gauge Invariance we can write

$$F_1(q^2) = 2[g_+ - q^2 h_1 - M_B^2 h_2]$$

$$F_3(q^2) = 2\left[-g_+ - q^2 h - \left(M_B^2 - q^2\right) h_2\right]$$

Finally

$$F_V = \frac{2}{m_b + m_q}\left(g_+ - q^2 h_1 - M_B^2 h_2\right)$$

$$F_A = \frac{2}{m_b - m_q}\left(g_+ - q^2 h - \left(M_B^2 - q^2\right) h_2\right)$$

The normalization of these form factors at $q^2 = 0$ is determined by the universal from factor $g_+(0)$.

Pole Contributions

The parent B-meson can go into a vector meson state or an axial vector meson state after emitting a real photon. There appear a pole term if momentum transfer become equal to the mass of the intermediate state. In context of *HQS*, the axial vector meson has *L=1*, and belongs to two separate spin doublets. This give rise to *S* wave and *D* wave contributions to the axial vector meson.

Only h_1, g_- and h get pole contribution from $B^*(1^-)$ and $B_A^*(1^+)$ mesons

$$h_1 \Big|_{pole} = -\frac{1}{2}\frac{g_{B^*B\gamma}}{M_{B^*}^2}\frac{f_T^{B^*}}{1-q^2/M_{B^*}^2} = -\frac{1}{2}(m_b + m_q)\frac{R_V}{M_{B^*}^2}\frac{1}{1-q^2/M_{B^*}^2}$$

$$g_- \Big|_{pole} = -\frac{g_{B_A^*B\gamma}}{M_{B_A^*}^2}\frac{f_T^{B_A^*}}{1-q^2/M_{B_A^*}^2} = (m_b - m_q)\frac{R_A^S}{M_{B_A^*}^2}\frac{1}{1-q^2/M_{B_A^*}^2}$$

$$h \Big|_{pole} = \frac{1}{2}\frac{f_{B_A^*B\gamma}}{M_{B_A^*}^2}\frac{f_T^{B_A^*}}{1-q^2/M_{B_A^*}^2} = -\frac{1}{2}(m_b - m_q)\frac{R_A^D}{M_{B_A^*}^2}\frac{1}{1-q^2/M_{B_A^*}^2}$$

On the other hand g_+ get contribution from triangle graph

$$g_+ = f_B \left\{ \frac{Q_u}{2\bar{\Lambda}} - \frac{Q_b}{2M_B}\left(1 - \frac{m_q}{M_B}\right) \right\} \frac{1}{1-q^2/M_B^2} \qquad \text{(A)}$$

g_+, g_- and h are related through the equation

$$g_+ + g_- + 2(q \cdot k)h = 0$$

and the coupling constants $g_{B_A^* B\gamma}, f_{B_A^* B\gamma}$ are defined as follows

$$\langle B^{*-}(q,\eta)\gamma(k,\epsilon) \mid B^-(P)\rangle = i g_{B^* B\gamma} \varepsilon_{\alpha\rho\mu\sigma} \epsilon^{*\alpha} q^\rho \eta^{*\mu} p^\sigma$$

$$\langle 0|i\bar{u}\sigma_{\mu\nu} b|B^{*-}(q,\eta)\rangle = f_T^{B^*}(q_\mu \eta_\nu - q_\nu \eta_\mu)$$

$$\langle B_A^{*-}(q,\eta)\gamma(k,\epsilon) \mid B^-(P)\rangle = i g_{B_A^* B\gamma}(\epsilon^*.\eta^*) - i f_{B_A^* B\gamma}(q.\epsilon^*)(k.\eta^*)$$

$$\langle 0|i\bar{u}\sigma_{\mu\nu} b|B_A^{*-}(q,\eta)\rangle = f_T^{B_A^*} \varepsilon_{\mu\nu\alpha\beta} \eta^\alpha q^\beta$$

Using Ward Identity we take the matrix element between $\langle 0|$ and $|B^*\rangle$, we obtain

$$\langle 0|i\bar{u}\sigma^{\mu\nu}q_\nu b|B^*(q,\eta)\rangle = -(m_b + m_q)f_{B^*}\eta^\mu$$

where $\langle 0|i\bar{u}\gamma^\mu b|B^*(q,\eta)\rangle = f_{B^*}\eta^\mu$, so we can write

$$f_T^{B^*} = \frac{(m_b+m_q)}{M_{B^*}^2}f_{B^*} = \frac{(m_b+m_q)}{M_{B^*}}f_B = \frac{M_B}{M_{B^*}}f_B = f_B$$

Working on the same line we can write

$$\langle 0|i\bar{u}\sigma^{\mu\nu}q_\nu\gamma_5 b|B_A^*(q,\eta)\rangle = (m_b - m_q)f_{B_A^*}\eta^\mu$$

and

$$f_T^{B_A^*} = -\frac{(m_b-m_q)}{M_{B_A^*}^2}f_{B_A^*}$$

Using the gauge invariance the ratio of *S*-wave and *D*-wave couplings is given as

$$\frac{R_A^S}{R_A^D} = -\frac{2g_{B_A^* B\gamma}}{f_{B_A^* B\gamma}} = -(M_B^2 - q^2)$$

We will use this ratio to predict the coupling of γ with B and B_A^* vertex. We will also predict the coupling $g_{B^* B\gamma}$ for B^* taken as an intermediate state.

Form Factors and determination of Coupling constants

Using the pole contributions calculated above the form factors can be written as

$$F_V(q^2) = \left\{ \frac{2}{m_b+m_q} g_+(q^2) + R_V \frac{q^2}{M^2_{B^*}} \frac{1}{1-q^2/M^2_{B^*}} + \sum_i \frac{q^2}{M^2_{B^*_i}} \frac{R_{V_i}}{1-q^2/M^2_{B^*_i}} \right\}$$

$$F_A(q^2) = \left\{ \frac{2}{m_b-m_q} g_+(q^2) + R^D_A \frac{q^2}{M^2_{B^*_A}} \frac{1}{1-q^2/M^2_{B^*_A}} + \sum_i \frac{q^2}{M^2_{B^*_i}} \frac{R^D_{A_i}}{1-q^2/M^2_{B^*_i}} \right\}$$

The constraint

$$R + \sum_i R_i = 0$$

gives restriction to the first radial excitation,

$$F_V(q^2) = \frac{2}{m_b+m_q} g_+(q^2) + R_V q^2 \frac{(M_{B_1^*}^2 - M_{B^*}^2)}{(M_{B^*}^2 - q^2)(M_{B_1^*}^2 - q^2)}$$

$$F_A(q^2) = \frac{2}{m_b-m_q} g_+(q^2) + R_A^D q^2 \frac{(M_{B_{A_1}^*}^2 - M_{B_A^*}^2)}{(M_{B_A^*}^2 - q^2)(M_{B_{A_1}^*}^2 - q^2)}$$

The pole behavior is softened by an effective suppression factor $(M_{B_1^*}^2 - M_{B^*}^2)$ which takes care of the off-shell-ness of the couplings of B^* or B_A^* with $B\gamma$ channel. We can not expect the above relations obtained from Ward identities, to hold for all q^2 for which we use the parameterization

$$F(q^2) = \frac{F(0)}{1 + a q^2 + b q^4}$$

In this way we obtain

$$F(q^2) = \frac{F(0)}{1 - \frac{q^2}{M^2} - \frac{R}{F(0)} \frac{q^2}{M_1^2} \left(\frac{M_1^2 - M^2}{M^2} \right) \left(1 - \frac{q^2}{M^2} \frac{M_1^2 - M^2}{M_1^2} \left(1 + \frac{R}{F(0)} \right) \right)}$$

Now it is tempting to factor out $\frac{1}{1-q^2/M^2}$ pole behavior, which gives

$$R = \left(\frac{1}{\frac{M_1^2}{M^2} - 1} \right) \frac{2g_+}{M_B}$$

$$F(q^2) = \frac{F(0)}{\left(1 - \frac{q^2}{M^2}\right)\left(1 - \frac{q^2}{M_1^2}\right)}$$

The couplings can be obtained as

$$g_{B^*B\gamma} \simeq \frac{2g_+(0)}{f_B \left(\frac{M_{B_1^*}^2}{M_{B^*}^2} - 1 \right)}$$

$$f_{B_A^*B\gamma} = \frac{M_{B_A^*}^2}{M_B} \frac{2g_+(0)}{f_{B_A^*} \left(\frac{M_{B_{A1}^*}^2}{M_{B_A^*}^2} - 1 \right)}$$

From Eq. (A) for $\bar{\Lambda} = 0.4\ \mathrm{GeV}^{-1}$ we have

$$g_+(0) = \frac{2}{3}\frac{f_B}{2\bar{\Lambda}} = 0.15$$

and the same value gives us the coupling constants

$$g_{B^*B\gamma} = \frac{2.2}{\bar{\Lambda}} = 5.6\ \mathrm{GeV}^{-1}$$

$$f_{B_A^*B\gamma} = 6.5\frac{f_B M_{B_A^*}}{f_{B_A^*}}\ \mathrm{GeV}^{-1}$$

The relation between *S*-wave and *D*-wave couplings near the pole at $q^2 = M_{B_A^*}^2$ is

$$g_{B_A^*B\gamma} = \frac{M_B^2 - M_{B_A^*}^2}{2} f_{B_A^*B\gamma}$$

$$= -2.36 \times f_{B_A^*B\gamma}$$

The final expression for form factors becomes

$$F_V(q^2) = \frac{F_V(0)}{\left(1-q^2/M^2_{B^*}\right)\left(1-q^2/M^2_{B^*_1}\right)}$$

$$F_A(q^2) = \frac{F_A(0)}{\left(1-q^2/M^2_{B^*_A}\right)\left(1-q^2/M^2_{B^*_{A_1}}\right)}$$

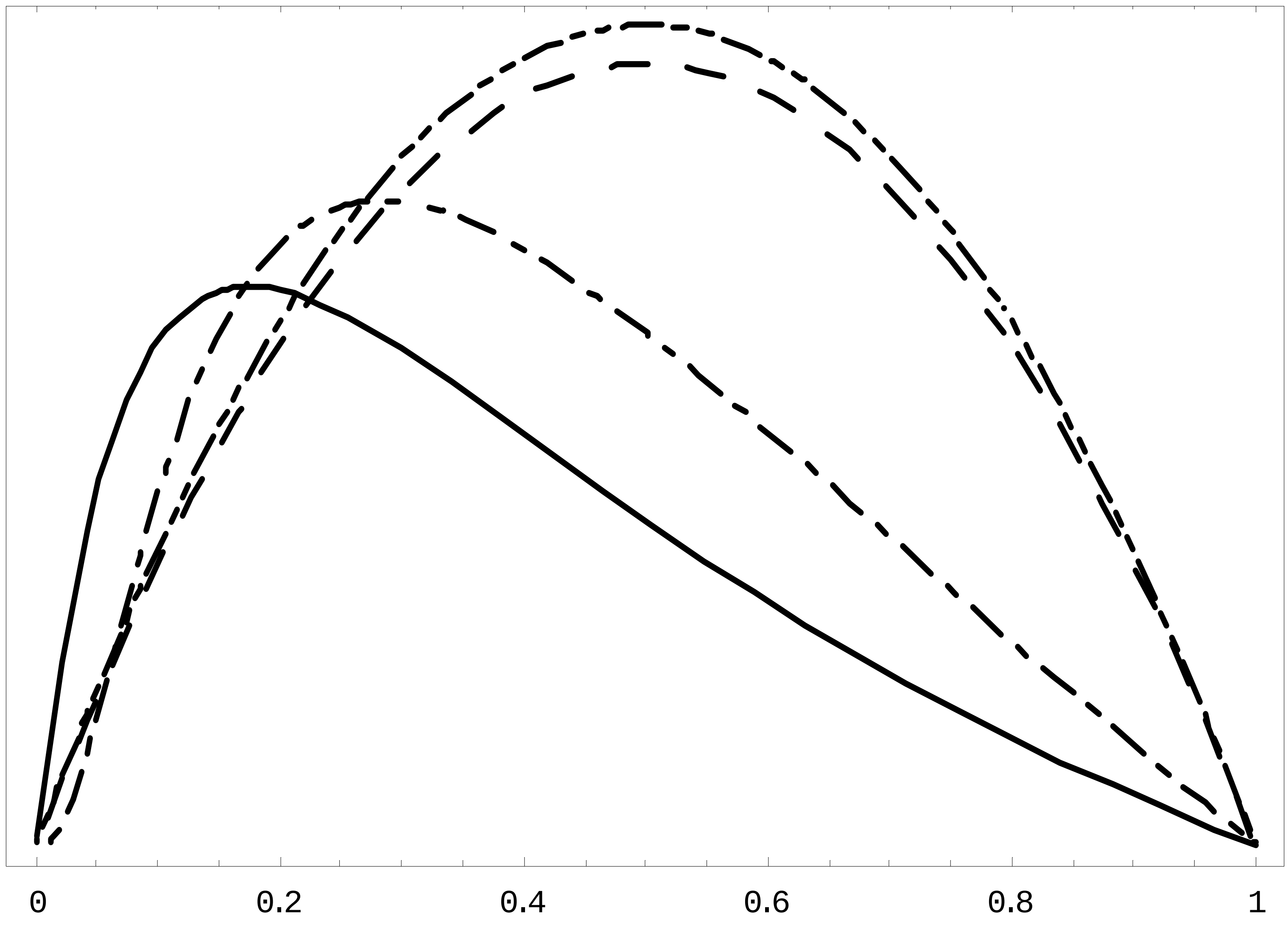
$$F_{V,A}(0) = \frac{2g_+(0)}{M_B}$$

Branching Ratio

Using the form factors calculated above we have

$$\mathcal{B}(B \to \gamma l v_l) = 0.5 \times 10^{-6} \qquad for\ \ l = \mu$$

- CLEO $2\mathrm{x}10^{-6}$
- Bethe-Salpeter approach $0.9\ \mathrm{x}10^{-6}$
- Light-Cone QCD (2-5) $\mathrm{x}10^{-6}$
- Monte-Carlo Simulation $5.2\mathrm{x}10^{-5}$


0
0.2
0.4
0.6
0.8
1

Conclusion

- We have studied $B \to \gamma l \nu_l$ decay using Ward Identities.
- The form factors $F_V(q^2)$ and $F_A(q^2)$ have been calculated and it is found that their normalization is essentially determined by a single constant $g_+(0)$.
- We use parameterization which takes into account potential corrections to single pole dominance arising from radial excitation of *M*.
- We have calculated the value of $g_+(0)$ and using this we have found the ratio of S-wave to D-wave coupling.
- Branching ratio is calculated and compared it with different approaches.
- Finally the partial decay width vs. the photon energy spectrum is plotted and it is found that our peak shifts towards the lower value of x.

Thanks!