Transverse Momentum Distribution of Hadrons in the Tsallis Statistics

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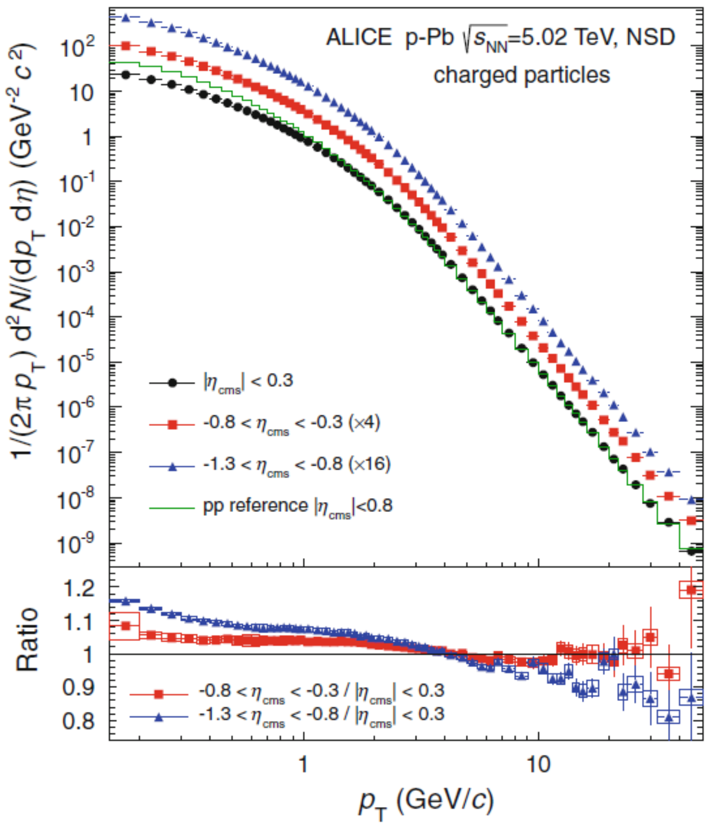
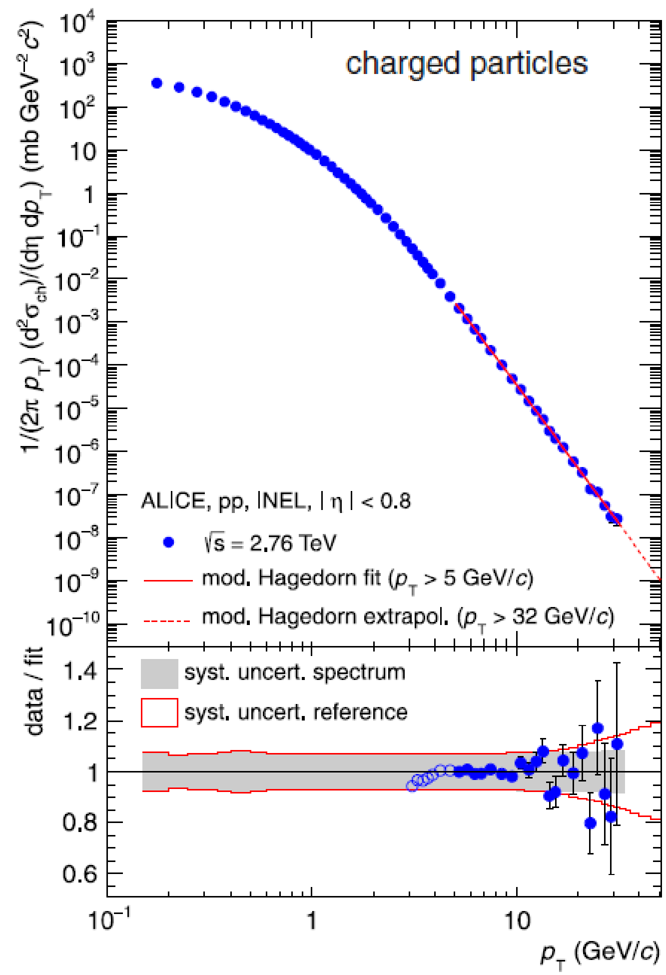
Transverse momentum distributions of hadrons at high energies

❑ Experiment:

Eur. Phys. J. C (2014) 74:3054



Eur. Phys. J. C (2013) 73:2662



❑ Statistical Theory:

Tsallis-factorized distribution **Boltzmann-Gibbs distribution**

$$\left[1+(q_c-1)\frac{\varepsilon_{\vec{p}}-\mu}{T}\right]^{\frac{q_c}{1-q_c}} \qquad q_c \to 1 \qquad e^{-\frac{\varepsilon_{\vec{p}}-\mu}{T}}$$

J. Cleymans, D. Worku,
Eur. Phys. J. A 48, 160 (2012)

$$\varepsilon_{\vec{p}} = m_T \cosh y, \quad m_T = \sqrt{p_T^2 + m^2}$$

Is the Tsallis-factorized distribution related to the Tsallis statistics?

What is the Tsallis statistics?

1.) Definitions:

Boltzmann-Gibbs Statistics	Tsallis-1 Statistics	Tsallis-2 Statistics
$S = -\sum_i p_i \ln p_i, \quad q = 1$	$S = -\sum_i \frac{p_i - p_i^q}{1-q} \quad 0 < q < \infty$	entropy $S = -\sum_i \frac{p_i - p_i^{q_c}}{1-q_c} \quad 0 < q_c < \infty$
$\sum_i p_i = 1$	$\sum_i p_i = 1$	$\sum_i p_i = 1$ -norm equation
$E = \sum_i p_i E_i$	$E = \sum_i p_i E_i$	$E = \sum_i p_i^{q_c} E_i$ -energy
$\langle N \rangle = \sum_i p_i N_i$	$\langle N \rangle = \sum_i p_i N_i$	$\langle N \rangle = \sum_i p_i^{q_c} N_i$ -number of particles

$p_i -$ probability of *i*-th microstate of the system

2.) Legendre Transform:

$$\Omega = E - TS - \mu \langle N \rangle$$

3.) Thermodynamic potentials:

B-G

$$\Omega = T \sum_i p_i \left[\ln p_i + \frac{E_i - \mu N_i}{T} \right]$$

T-1

$$\Omega = T \sum_i p_i \left[\frac{1 - p_i^{q-1}}{1-q} + \frac{E_i - \mu N_i}{T} \right]$$

T-2

$$\Omega = T \sum_i p_i^{q_c} \left[\frac{p_i^{1-q_c} - 1}{1-q_c} + \frac{E_i - \mu N_i}{T} \right]$$

What is the Tsallis statistics?

4.) Constrained Local Extrema of the Thermodynamic Potential (Method of Lagrange Multipliers):

$$\Phi = \Omega - \lambda\phi, \qquad \phi = \sum_i p_i - 1 = 0$$

- Lagrange function
- constrained equation

$$\frac{\partial \Phi}{\partial p_i} = 0$$

- extremization

5.) Many-body distribution functions (Probabilities of Microstates of the System) and the norm functions:

Boltzmann-Gibbs Statistics	Tsallis-1 Statistics	Tsallis-2 Statistics
	- many-body distribution function	
$p_i = \frac{1}{Z}\exp\left(-\frac{E_i - \mu N_i}{T}\right)$	$p_i = \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}$	$p_i = \frac{1}{Z}\left[1 - (1-q_c)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q_c}}$
	- norm function (partition function)	
$Z = \sum_i \exp\left(-\frac{E_i - \mu N_i}{T}\right)$	$\sum_i \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}} = 1$	$Z = \sum_i \left[1 - (1-q_c)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q_c}}$
$\Omega = -T\ln Z = \lambda - T$	$\Lambda = \lambda - T$	$-Tq_c\frac{Z^{1-q_c} - 1}{1-q_c} = \lambda - T$

Equilibrium Thermodynamics

- **Thermodynamic potential and its partial derivatives:**

$f(x) = f(x_1,\dots,x_n)$ (Thermodynamic potential)

Variables of state

$$df = \sum_{i=1}^{n} u_i dx_i, \quad u_i = \frac{\partial f}{\partial x_i}, \qquad d^2 f = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} dx_i dx_j, \quad a_{ij} = a_{ji} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad \frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j} \qquad (i, j = 1,\dots,n)$$

-Maxwell relations,

- **Legendre transform:**

$$g = g(u_1,\dots,u_m, x_{m+1},\dots,x_n) = f - \sum_{i=1}^{m} u_i x_i, \quad \frac{\partial f}{\partial x_i} = u_i, \quad x_i = x_i(u_1,\dots,u_m, x_{m+1},\dots,x_n) \quad (i = 1,\dots,m).$$

Equilibrium thermodynamics is defined only on the class of homogeneuos functions:

$$system(1+2) = system(1) + system(2)$$

$$x_i^{1+2} = x_i^1 + x_i^2, \qquad i = 1,\dots,m \qquad \text{(extensive variables of state)}$$

$$x_i^{1+2} = x_i^1 = x_i^2, \qquad i = m+1,\dots,n \qquad \text{(intensive variables of state)}$$

- **The homogeneous function of the first order (extensive):**

$$f(\lambda x_1,\dots,\lambda x_m, x_{m+1},\dots,x_n) = \lambda f(x_1,\dots,x_m, x_{m+1},\dots,x_n),$$

$$\sum_{i=1}^{m} \frac{\partial f}{\partial x_i} x_i = f. \qquad \text{(Euler theorem)}$$

$$f(x_1^{1+2},\dots,x_m^{1+2}, x_{m+1}^{1+2},\dots x_n^{1+2}) = f^1(x_1^1,\dots,x_m^1, x_{m+1}^1,\dots x_n^1) + f^2(x_1^2,\dots,x_m^2, x_{m+1}^2,\dots x_n^2)$$

- **The homogeneous function of the zero order (intensive):**

$$\phi(\lambda x_1,\dots,\lambda x_m, x_{m+1},\dots,x_n) = \phi(x_1,\dots,x_m, x_{m+1},\dots,x_n),$$

$$\sum_{i=1}^{m} \frac{\partial \phi}{\partial x_i} x_i = 0. \qquad \text{(Euler theorem)}$$

$$\varphi(x_1^{1+2},\dots,x_m^{1+2}, x_{m+1}^{1+2},\dots x_n^{1+2}) = \varphi^1(x_1^1,\dots,x_m^1, x_{m+1}^1,\dots x_n^1) = \varphi^2(x_1^2,\dots,x_m^2, x_{m+1}^2,\dots x_n^2)$$

A.S.P., In Recent Advances in Thermo and Fluid Dynamics, ed. Mod Gorji-Bandpy, InTech, Chapter 11, 2015, pp.303-331,

What is the Tsallis-factorized statistics?

Boltzmann-Gibbs Statistics

- **Ideal Gas (Maxwell-Boltzmann):**

$$\langle n_{\vec{p}\sigma}\rangle = e^{-\frac{\varepsilon_{\vec{p}}-\mu}{T}}$$

$$S = -\sum_{\vec{p}\sigma}\left[\langle n_{\vec{p}\sigma}\rangle \ln\langle n_{\vec{p}\sigma}\rangle - \langle n_{\vec{p}\sigma}\rangle\right]$$

generalization →

$$\langle N\rangle = \sum_{\vec{p}\sigma}\langle n_{\vec{p}\sigma}\rangle$$

$$E = \sum_{\vec{p}\sigma}\langle n_{\vec{p}\sigma}\rangle \varepsilon_{\vec{p}}$$

$$\Omega = E - TS - \mu\langle N\rangle$$

$$= T\sum_{\vec{p}\sigma}\langle n_{\vec{p}\sigma}\rangle\left[\ln\langle n_{\vec{p}\sigma}\rangle - 1 + \frac{\varepsilon_{\vec{p}}-\mu}{T}\right]$$

$$\frac{\partial\Omega}{\partial\langle n_{\vec{p}\sigma}\rangle} = 0, \rightarrow \langle n_{\vec{p}\sigma}\rangle = e^{-\frac{\varepsilon_{\vec{p}}-\mu}{T}}$$

- The constrained maximization of the entropy of the ideal gas with respect to the single-particle distribution function leads to the results of the Boltzmann-Gibbs statistics

Tsallis-factorized Statistics

- **Ideal Gas (Maxwell-Boltzmann):**

J. Cleymans, D. Worku, Eur. Phys. J. A 48, 160 (2012)

q_c − real parameter

$$S = -\sum_{\vec{p}\sigma}\left[f_{\vec{p}\sigma}^{q_c} \ln_{q_c} f_{\vec{p}\sigma} - f_{\vec{p}\sigma}\right], \qquad f_{\vec{p}\sigma}^{q_c} \equiv \langle n_{\vec{p}\sigma}\rangle$$

$$\langle N\rangle = \sum_{\vec{p}\sigma} f_{\vec{p}\sigma}^{q_c} \qquad \ln_{q_c}(x) = \frac{x^{1-q_c}-1}{1-q_c}, \qquad 0 < q_c < \infty$$

$$E = \sum_{\vec{p}\sigma} f_{\vec{p}\sigma}^{q_c}\varepsilon_{\vec{p}}$$

$$\Omega = E - TS - \mu\langle N\rangle$$

$$= T\sum_{\vec{p}\sigma} f_{\vec{p}\sigma}^{q_c}\left[q_c \ln_{q_c} f_{\vec{p}\sigma} - 1 + \frac{\varepsilon_{\vec{p}}-\mu}{T}\right]$$

$$\frac{\partial\Omega}{\partial f_{\vec{p}\sigma}} = 0, \rightarrow \langle n_{\vec{p}\sigma}\rangle = \left[1 + (q_c - 1)\frac{\varepsilon_{\vec{p}}-\mu}{T}\right]^{\frac{q_c}{1-q_c}}$$

- The constrained maximization of the Tsallis-factorized entropy of the ideal gas (generalized from the Boltzmann-Gibbs entropy of the ideal gas) with respect to the single-particle distribution function should lead to the results of the Tsallis-2 statistics
- Is it indeed the Tsallis-factorized distribution equivalent to the distribution of the Tsallis-2 statistics?
- The Tsallis-factorized statistics should be equivalent to the Tsallis-2 statistics

Ultrarelativistic Ideal Gas: Tsallis-2 statistics $q_c > 1$

cut-off

A.S.P., arXiv:1607.07670

1. Tsallis-2 Statistics:

- **Exact results:**

$$\tilde{\omega} = \frac{gVT^3}{\pi^2}$$

$$Z = 1 + \sum_{N=1}^{N_0} \frac{\tilde{\omega}^N}{N!} \frac{\Gamma\left(\frac{1}{q_c-1} - 3N\right)}{(q_c-1)^{3N}\,\Gamma\left(\frac{1}{q_c-1}\right)} \left[1-(q_c-1)\frac{\mu N}{T}\right]^{\frac{1}{1-q_c}+3N}$$

- The partition function is divergent
- We truncate the series
- In the partition function and the mean occupation numbers only the physical terms are preserved

$$\langle n_{\vec{p}\sigma} \rangle = \frac{1}{Z^{q_c}} \left[1+(q_c-1)\frac{\varepsilon_{\vec{p}}-\mu}{T}\right]^{\frac{q_c}{1-q_c}} + \frac{1}{Z^{q_c}} \sum_{N=1}^{N_0} \frac{\tilde{\omega}^N}{N!} \frac{\Gamma\left(\frac{q_c}{q_c-1} - 3N\right)}{(q_c-1)^{3N}\,\Gamma\left(\frac{q_c}{q_c-1}\right)} \left[1+(q_c-1)\frac{\varepsilon_{\vec{p}}-\mu(N+1)}{T}\right]^{\frac{q_c}{1-q_c}+3N}$$

- The mean occupation numbers in the Tsallis-2 statistics

- **Zeroth term approximation:** (Definition: All terms with $N \geq 1$ in the series given above are deleted by hand)

$$N = 0, \quad Z = 1$$

$$\langle n_{\vec{p}\sigma} \rangle = \left[1+(q_c-1)\frac{\varepsilon_{\vec{p}}-\mu}{T}\right]^{\frac{q_c}{1-q_c}}$$

- The mean occupation numbers in the zeroth term approximation of the Tsallis-2 statistics
- The zeroth term approximation is valid only for $q_c > 3/2$

2. Tsallis-factorized Statistics:

- The constrained maximization of the Tsallis-factorized entropy of the ideal gas (generalized from the Boltzmann-Gibbs entropy of the ideal gas) with respect to the single-particle distribution function does not lead to the true results for the Tsallis-2 statistics

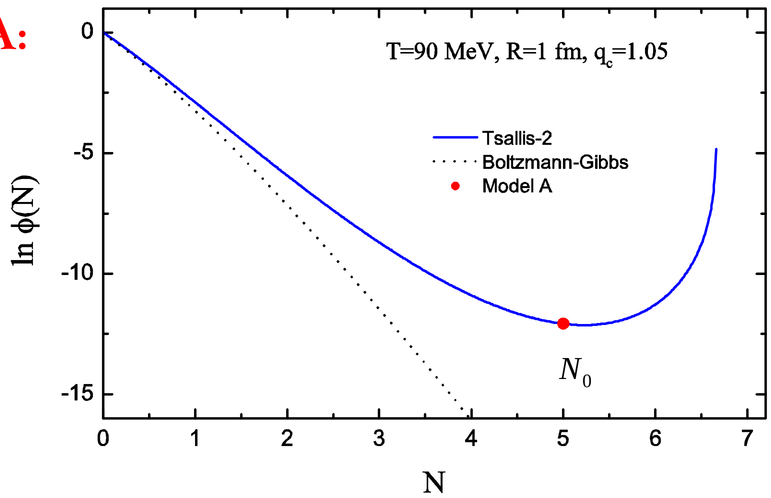
J. Cleymans, D. Worku, Eur. Phys. J. A 48, 160 (2012)

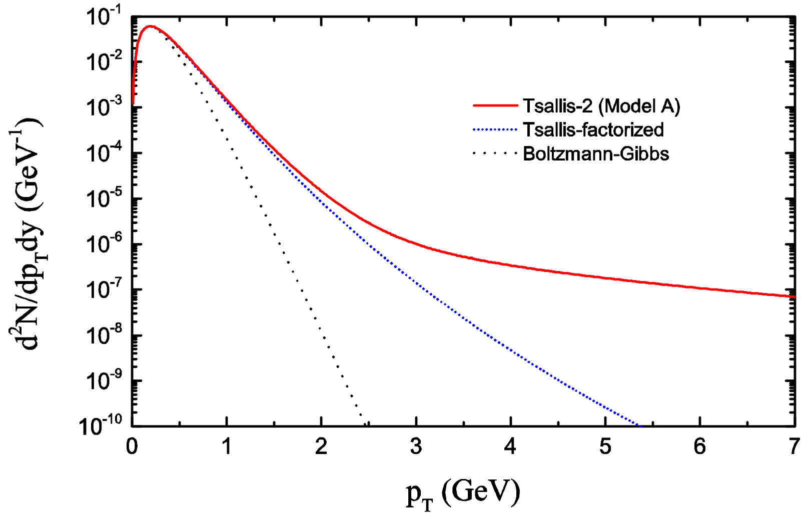
$$\langle n_{\vec{p}\sigma} \rangle = \left[1+(q_c-1)\frac{\varepsilon_{\vec{p}}-\mu}{T}\right]^{\frac{q_c}{1-q_c}}$$

- The Tsallis-factorized distribution is not equivalent to the distribution of the Tsallis-2 statistics
- The Tsallis-factorized statistics is not equivalent to the Tsallis-2 statistics
- The Tsallis-factorized statistics can serve as a particular statistics independent from the Tsallis statistics

The cut-off prescriptions in the Tsallis-2 statistics

1. Model A:

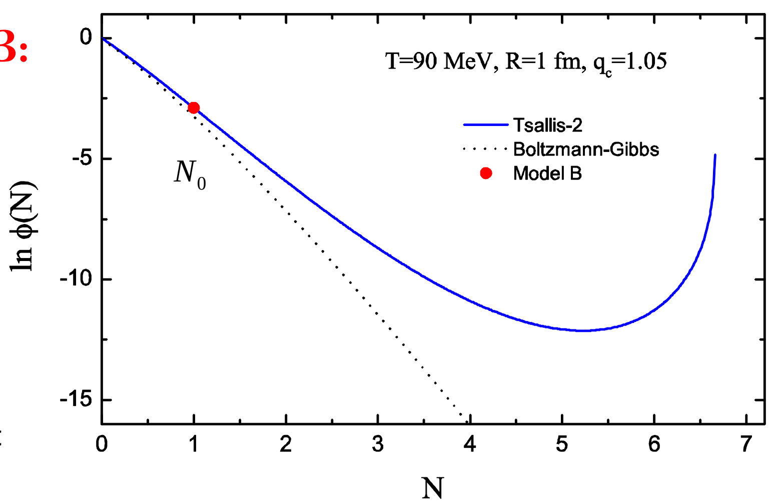


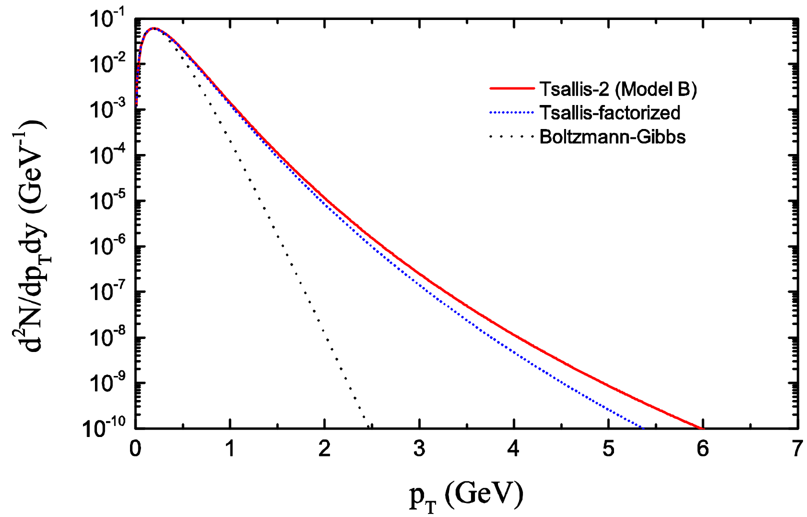


$\min(\ln\phi(N))|_{N=N_0}$

- from minimum

2. Model B:





$$\left.\frac{\partial^2 \ln\phi(N)}{\partial N^2}\right|_{N=N_0} = 0$$

- The inflection point

$$Z=\sum_{N=0}^{N_0}\phi(N), \quad \phi(N)=\frac{\tilde{\omega}^N}{N!}\frac{\Gamma\left(\frac{1}{q_c-1}-3N\right)}{(q_c-1)^{3N}\,\Gamma\left(\frac{1}{q_c-1}\right)}\left[1-(q_c-1)\frac{\mu N}{T}\right]^{\frac{1}{1-q_c}+3N}$$ -Tsallis-2

$$\phi(N)=\frac{\tilde{\omega}^N}{N!}e^{\frac{\mu N}{T}}$$ -Boltzmann-Gibbs

Ultrarelativistic Ideal Gas: Tsallis-1 statistics $q<1$

- **The norm equation:**

❑ Initial:

$$\sum_{N=0}^{N_0} \phi(N) + \sum_{N=N_0+1}^{\infty} \phi(N) = 1$$

unphysical

- The norm equation is divergent

A.S.P., arXiv:1607.07670

$$\phi(N) = \frac{\tilde{\omega}^N}{N!} \frac{\left(\frac{q}{1-q}\right)^{3N} \Gamma\left(\frac{1}{1-q} - 3N\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left[1 + \frac{q-1}{q} \frac{\Lambda + \mu N}{T}\right]^{\frac{1}{q-1}+3N}$$

$$\tilde{\omega} = \frac{gVT^3}{\pi^2}$$

❑ Regularization:

$$\sum_{N=0}^{N_0} \phi(N) + \sum_{N=N_0+1}^{\infty} \phi(N) = 1$$

- We truncate the series
- In the norm equation only the physical terms are preserved

Ultrarelativistic Ideal Gas: Tsallis-1 statistics

$$q<1$$

A.S.P., arXiv:1607.07670

1. Tsallis-1 Statistics:

- **Exact results:**

$$\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]^{\frac{1}{q-1}}+\sum_{N=1}^{N_0}\frac{\tilde{\omega}^N}{N!}\frac{\left(\frac{q}{1-q}\right)^{3N}\Gamma\left(\frac{1}{1-q}-3N\right)}{\Gamma\left(\frac{1}{1-q}\right)}\left[1+\frac{q-1}{q}\frac{\Lambda+\mu N}{T}\right]^{\frac{1}{q-1}+3N}=1$$

- The norm equation

$$\tilde{\omega}=\frac{gVT^3}{\pi^2}$$

-The mean occupation numbers in the Tsallis-1 statistics

$$\langle n_{\vec{p}\sigma}\rangle=\left[1+\frac{q-1}{q}\frac{\Lambda-\varepsilon_{\vec{p}}+\mu}{T}\right]^{\frac{1}{q-1}}+\sum_{N=1}^{N_0}\frac{\tilde{\omega}^N}{N!}\frac{\left(\frac{q}{1-q}\right)^{3N}\Gamma\left(\frac{1}{1-q}-3N\right)}{\Gamma\left(\frac{1}{1-q}\right)}\left[1+\frac{q-1}{q}\frac{\Lambda-\varepsilon_{\vec{p}}+\mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

- **Zeroth term approximation:** (Definition: All terms with $N\geq 1$ in the series given above are deleted by hand)

$$N=0,\quad \Lambda=0$$

$$\langle n_{\vec{p}\sigma}\rangle=\left[1-\frac{q-1}{q}\frac{\varepsilon_{\vec{p}}-\mu}{T}\right]^{\frac{1}{q-1}}$$

- The mean occupation numbers in the zeroth term approximation of the Tsallis-1 statistics
- The zeroth term approximation is valid only for $N_0=0$ at large deviations of q from the unity

$$q\to 1/q_c$$

2. Tsallis-factorized Statistics:

-The Tsallis-factorized distribution is not equivalent to the distribution of the Tsallis-1 statistics

-The Tsallis-factorized statistics is not equivalent to the Tsallis statistics (Tsallis-1 and Tsallis-2 statistics)

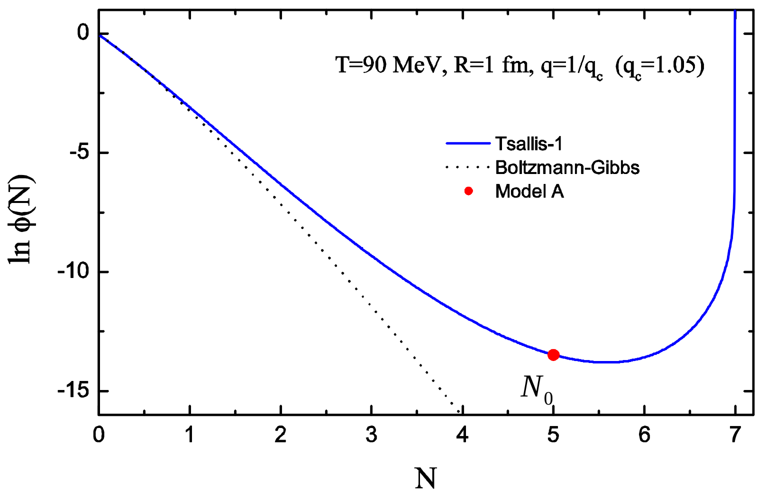
J. Cleymans, D. Worku, Eur. Phys. J. A 48, 160 (2012)

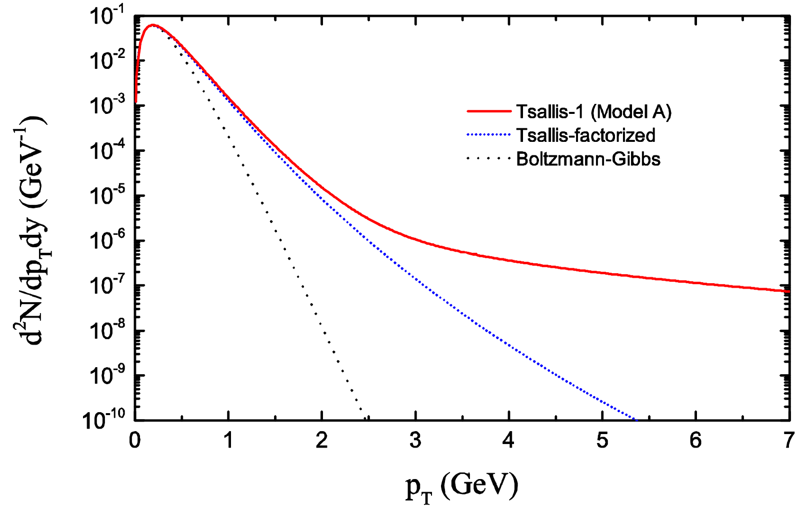
$$\langle n_{\vec{p}\sigma}\rangle=\left[1+(q_c-1)\frac{\varepsilon_{\vec{p}}-\mu}{T}\right]^{\frac{q_c}{1-q_c}}$$

- The mean occupation numbers of the Tsallis-factorized statistics

The cut-off prescriptions in the Tsallis-1 statistics

1. Model A:

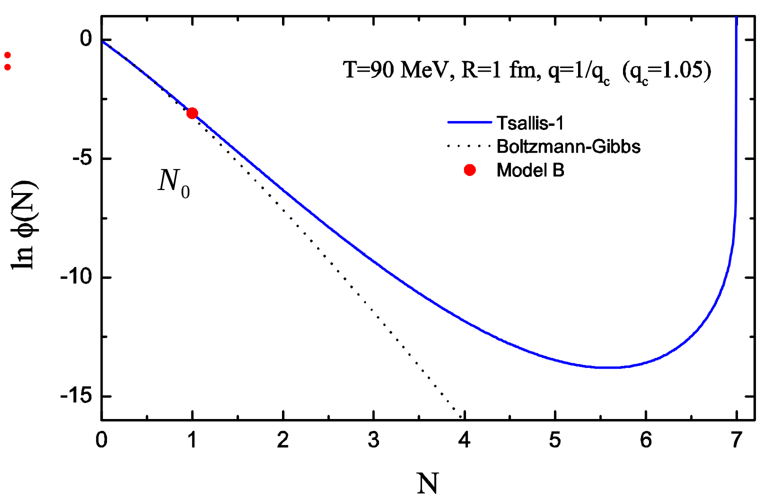


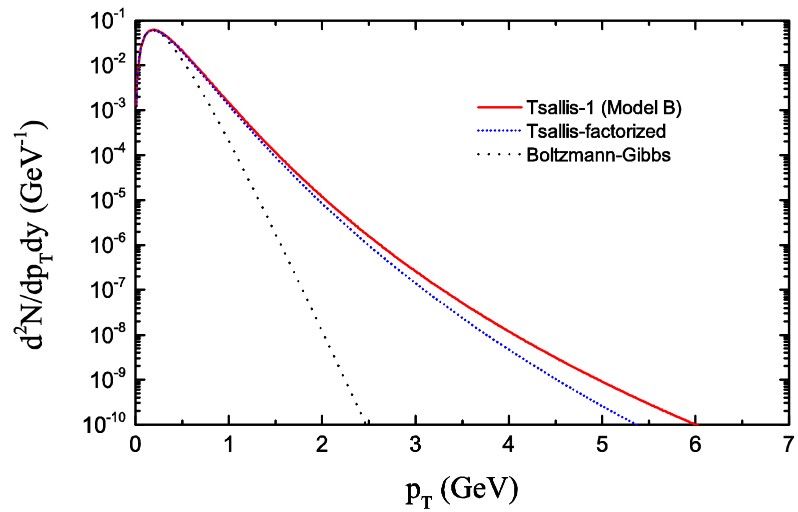


$$\min(\ln\phi(N))\,|_{N=N_0}$$

- from minimum

2. Model B:





$$\left.\frac{\partial^2 \ln\phi(N)}{\partial N^2}\right|_{N=N_0}=0$$

- The inflection point

$$\sum_{N=0}^{N_0}\phi(N)=1,\qquad \phi(N)=\frac{\tilde{\omega}^N}{N!}\frac{\left(\frac{q}{1-q}\right)^{3N}\Gamma\left(\frac{1}{1-q}-3N\right)}{\Gamma\left(\frac{1}{1-q}\right)}\left[1+\frac{q-1}{q}\frac{\Lambda+\mu N}{T}\right]^{\frac{1}{q-1}+3N}$$ -Tsallis-1

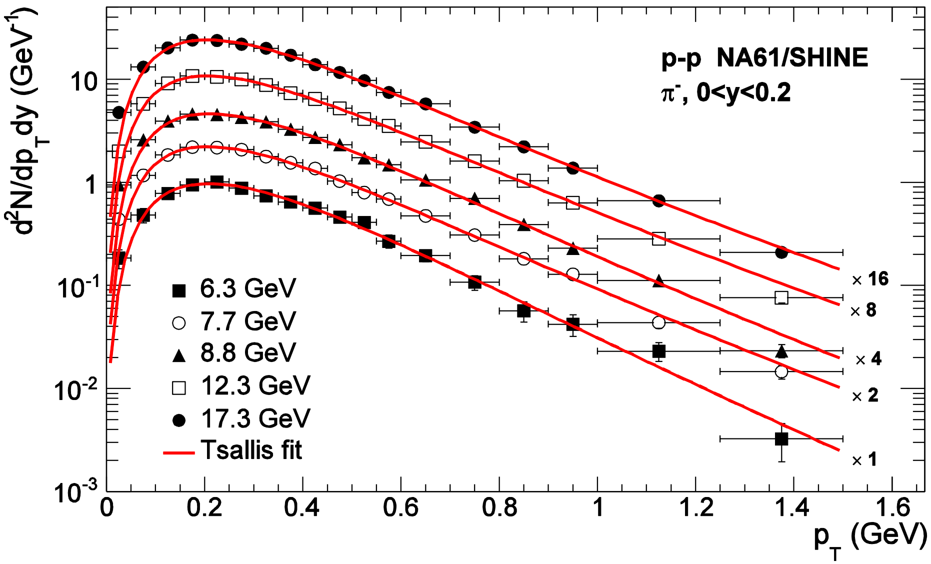
$$\phi(N)=\frac{\tilde{\omega}^N}{N!}e^{\frac{\Omega+\mu N}{T}}$$ -Boltzmann-Gibbs

Comparison of Tsallis-factorized statistics with Tsallis-1 statistics (Model B): Charged pions

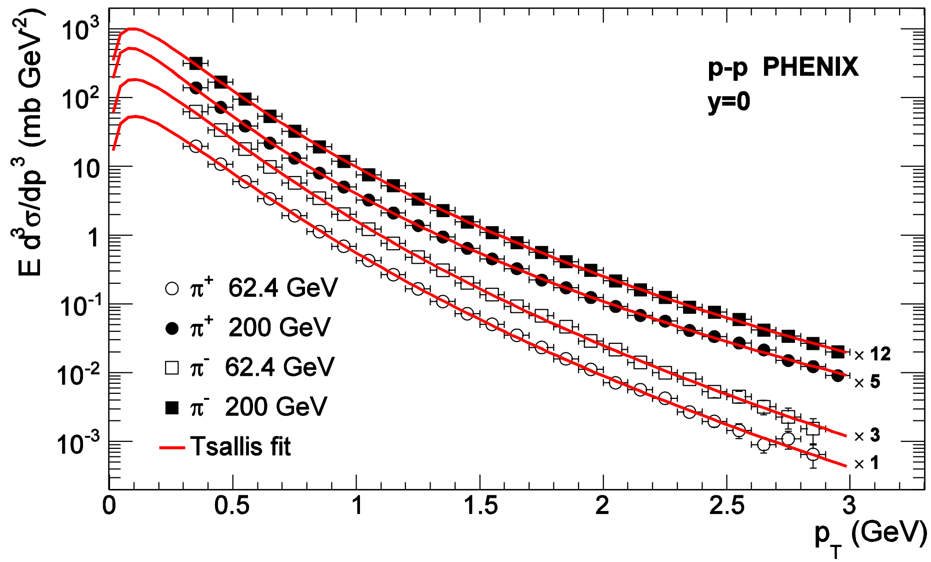
$p + p$

A.S.P., arXiv:1608.01888

Ex. NA61/SHINE, Eur. Phys. J. C **74**, 2794 (2014)



Ex. PHENIX, Phys. Rev. C **83**, 064903 (2011)



Ultrarelativistic distributions of the Tsallis-1 statistics:

$$\left.\frac{d^2N}{dp_T dy}\right|_{y_0}^{y_1} = \frac{gV}{(2\pi)^2} p_T^2 \int_{y_0}^{y_1} dy \cosh y \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} \frac{\left(\frac{q}{1-q}\right)^{3N} \Gamma\left(\frac{1}{1-q} - 3N\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left[1 + \frac{q-1}{q} \frac{\Lambda - p_T \cosh y + \mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

$$\frac{1}{2\pi p_T} \frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^3} p_T \cosh y \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} \frac{\left(\frac{q}{1-q}\right)^{3N} \Gamma\left(\frac{1}{1-q} - 3N\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left[1 + \frac{q-1}{q} \frac{\Lambda - p_T \cosh y + \mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

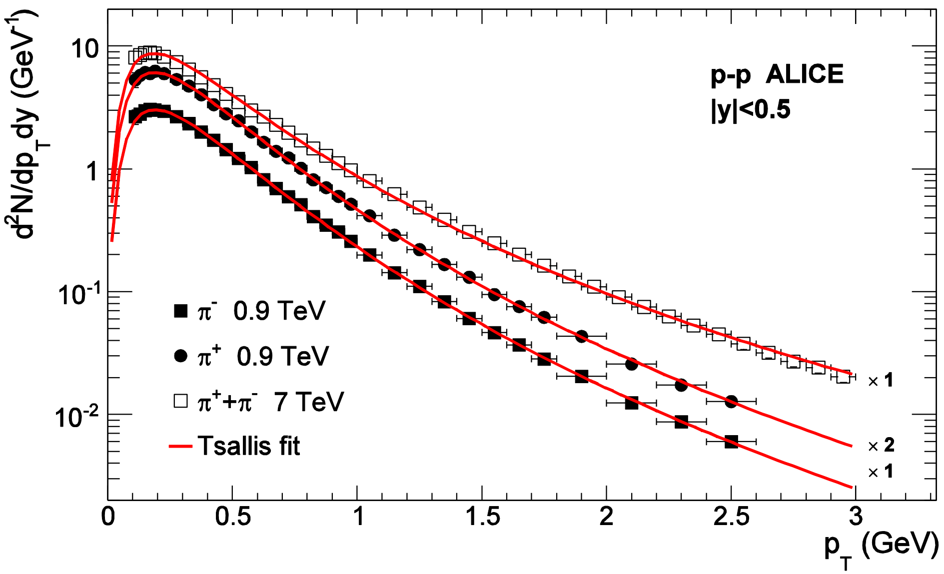
Comparison of Tsallis-factorized statistics with Tsallis-1 statistics (Model B): Charged pions

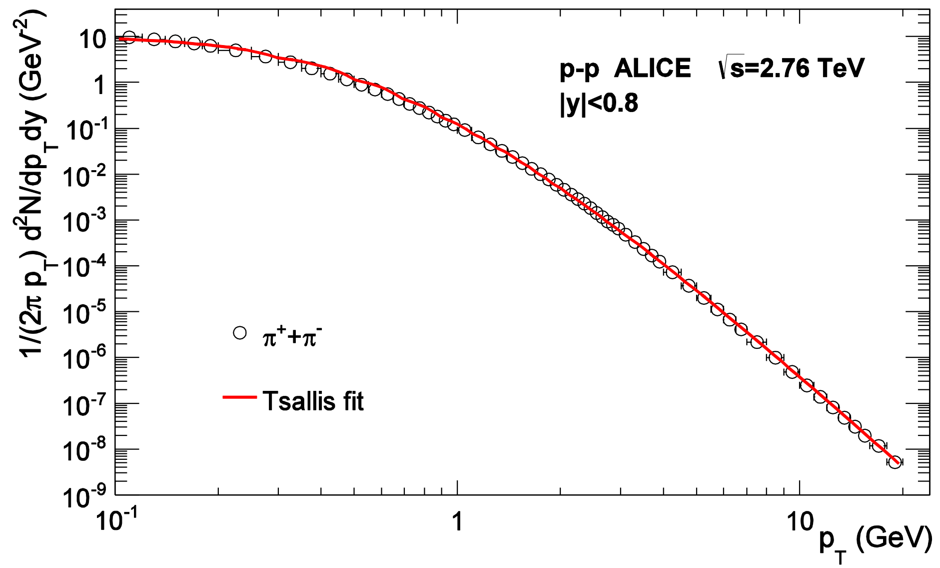
A.S.P., arXiv:1608.01888

$p + p$

Ex. ALICE, Eur. Phys. J. C **71**, 1655 (2011);
Eur. Phys. J. C **75**, 226 (2015)

Ex. ALICE, Phys. Lett. B **736**, 196 (2014)





Ultrarelativistic distributions of the Tsallis-1 statistics:

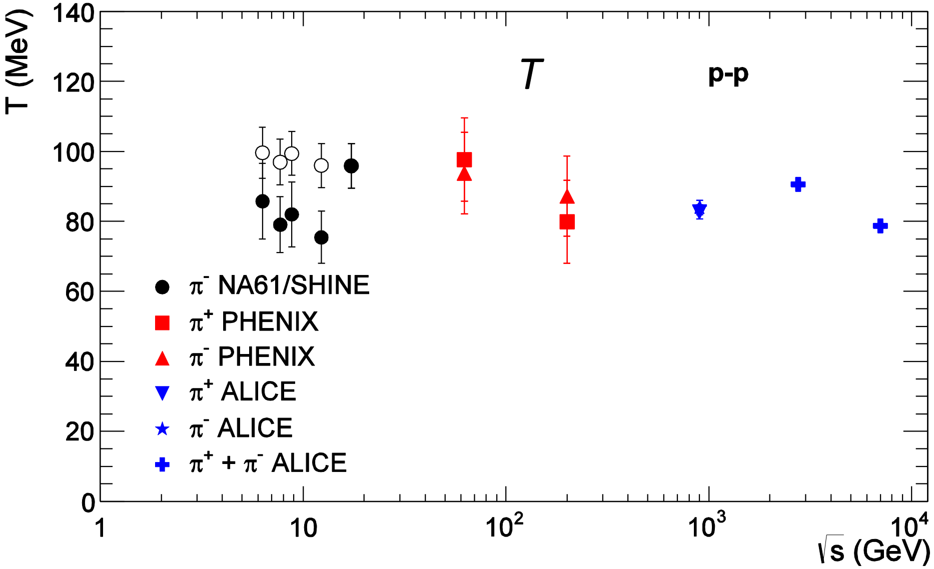
$$\left.\frac{d^2N}{dp_T dy}\right|_{y_0}^{y_1} = \frac{gV}{(2\pi)^2} p_T^2 \int_{y_0}^{y_1} dy \cosh y \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} \frac{\left(\frac{q}{1-q}\right)^{3N} \Gamma\left(\frac{1}{1-q} - 3N\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left[1 + \frac{q-1}{q} \frac{\Lambda - p_T \cosh y + \mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

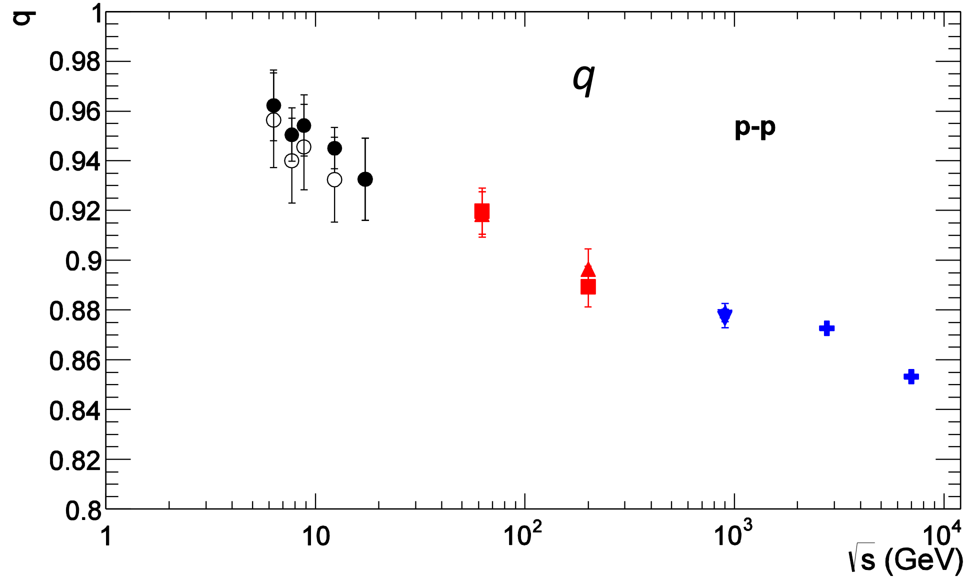
$$\frac{1}{2\pi p_T} \left.\frac{d^2N}{dp_T dy}\right|_{y_0}^{y_1} = \frac{gV}{(2\pi)^3} p_T \int_{y_0}^{y_1} dy \cosh y \sum_{N=0}^{N_0} \frac{\omega^N}{N!} \frac{\left(\frac{q}{1-q}\right)^{3N} \Gamma\left(\frac{1}{1-q} - 3N\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left[1 + \frac{q-1}{q} \frac{\Lambda - p_T \cosh y + \mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

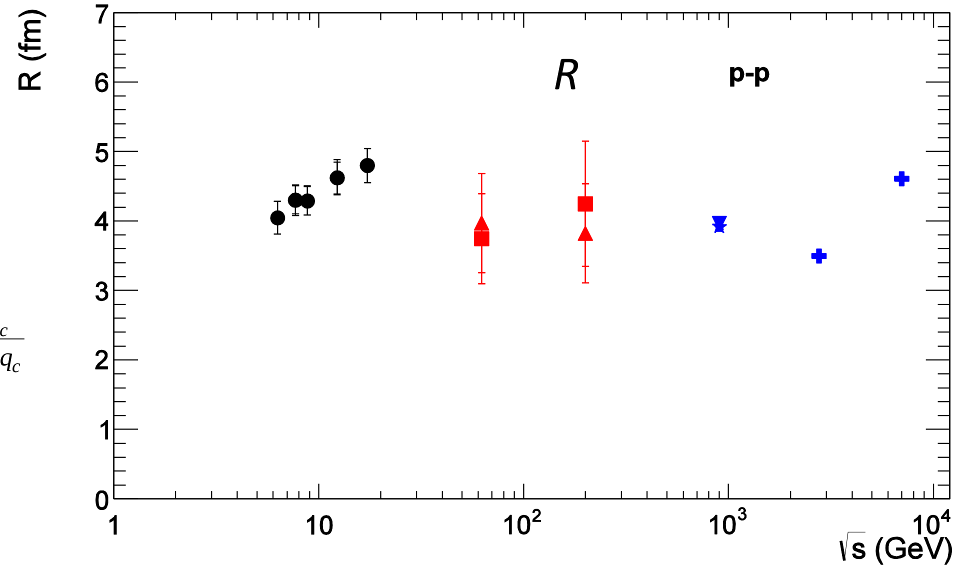
Energy dependence of the parameters of the Tsallis-1 statistics (Model B) and the Tsallis-factorized statistics

$p+p$

A.S.P., arXiv:1608.01888







Solid points – Tsallis-1 statistics
Open symbols – Tsallis-factorized statistics

- Tsallis-factorized distribution
  (the zeroth term approximation, $q_c = \frac{1}{q}$)

$$\frac{d^2N}{dp_T dy} = \frac{gVp_T^2 \cosh y}{(2\pi)^2}\left[1+(q_c-1)\frac{p_T\cosh y-\mu}{T}\right]^{\frac{q_c}{1-q_c}}$$

J. Cleymans, D. Worku, Eur. Phys. J. A 48, 160 (2012)

Parameters of the Tsallis-1 statistics (Model B)

$p + p$

A.S.P., arXiv:1608.01888

Parameters of the Tsallis-1 statistics fit for the pions produced in *pp* collisions at different energies

Collaboration	Type	$\sqrt{s}$, GeV	T, MeV	R, fm	q	χ^2/ndf
NA61/SHINE	π^-	6.3	85.78±10.79	4.047±0.235	0.9623±0.0142	2.821/15
NA61/SHINE	π^-	7.7	79.05±8.01	4.304±0.204	0.9505±0.0107	1.472/15
NA61/SHINE	π^-	8.8	82.01±9.28	4.294±0.212	0.9542±0.0123	1.821/15
NA61/SHINE	π^-	12.3	75.47±7.41	4.627±0.253	0.9451±0.0083	1.152/15
NA61/SHINE	π^-	17.3	95.83±6.38	4.798±0.246	0.9326±0.0166	0.865/15
PHENIX	π^+	62.4	97.62±11.92	3.744±0.648	0.9197±0.0093	1.654/23
PHENIX	π^-	62.4	93.76±11.69	3.971±0.716	0.9184±0.0091	0.878/23
PHENIX	π^+	200.0	79.89±11.80	4.247±0.899	0.8894±0.0082	0.987/24
PHENIX	π^-	200.0	87.20±11.48	3.823±0.714	0.8965±0.0081	0.691/24
ALICE	π^+	900.0	82.72±2.01	3.965±0.069	0.8766±0.0037	3.609/30
ALICE	π^-	900.0	83.92±2.02	3.918±0.068	0.8790±0.0036	1.610/30
ALICE	$\pi^+ + \pi^-$	2760.0	90.61±1.45	3.496±0.057	0.8726±0.0012	12.18/60
ALICE	$\pi^+ + \pi^-$	7000.0	78.75±1.86	4.606±0.093	0.8533±0.0024	9.775/38

Parameters of the Tsallis-factorized statistics

$p + p$

A.S.P., arXiv:1608.01888

Parameters of the fit by the distribution of the Tsallis-factorized statistics (the zero term approximation of Tsallis-1 statistics) for the pions produced in pp collisions at different energies

Collaboration	Type	$\sqrt{s}$, GeV	T, MeV	R, fm	q	$q_c = 1/q$	χ^2/ndf
NA61/SHINE	π^-	6.3	99.59±7.32	4.045±0.234	0.9563±0.0190	1.0457±0.0208	2.825/15
NA61/SHINE	π^-	7.7	96.93±6.49	4.300±0.222	0.9400±0.0171	1.0638±0.0194	1.481/15
NA61/SHINE	π^-	8.8	99.37±6.29	4.290±0.204	0.9455±0.0172	1.0576±0.0193	1.838/15
NA61/SHINE	π^-	12.3	95.92±6.29	4.619±0.228	0.9324±0.0170	1.0725±0.0196	1.175/15
NA61/SHINE	π^-	17.3	95.83±6.38	4.798±0.246	0.9326±0.0166	1.0722±0.0191	0.865/15
PHENIX	π^+	62.4	97.62±11.92	3.744±0.648	0.9197±0.0093	1.0874±0.0110	1.654/23
PHENIX	π^-	62.4	93.76±11.69	3.971±0.715	0.9184±0.0091	1.0888±0.0108	0.878/23
PHENIX	π^+	200.0	79.89±11.81	4.247±0.899	0.8894±0.0082	1.1244±0.0104	0.987/24
PHENIX	π^-	200.0	87.20±11.49	3.823±0.714	0.8965±0.0081	1.1155±0.0101	0.691/24
ALICE	π^+	900.0	82.72±2.01	3.965±0.069	0.8766±0.0037	1.1408±0.0048	3.609/30
ALICE	π^-	900.0	83.92±2.02	3.918±0.068	0.8790±0.0036	1.1376±0.0047	1.610/30
ALICE	$\pi^+ + \pi^-$	2760.0	90.61±1.45	3.496±0.057	0.8726±0.0012	1.1460±0.0016	12.18/60
ALICE	$\pi^+ + \pi^-$	7000.0	78.75±1.86	4.606±0.093	0.8533±0.0024	1.1719±0.0032	9.775/38

- **The results of the Tsallis-factorized statistics (the zeroth term approximation of the Tsallis-1 statistics) deviate from the results of the Tsallis-1 statistics only at low NA61/SHINE energies when the value of the parameter q is close to unity.**
- **At higher energies, when the value of the parameter q deviates essentially from the unity, the Tsallis-factorized statistics satisfactorily recovers the results of the Tsallis-1 statistics because at this values of q in the series of the Tsallis-1 statistics only one term $N = 0$ is physical.**

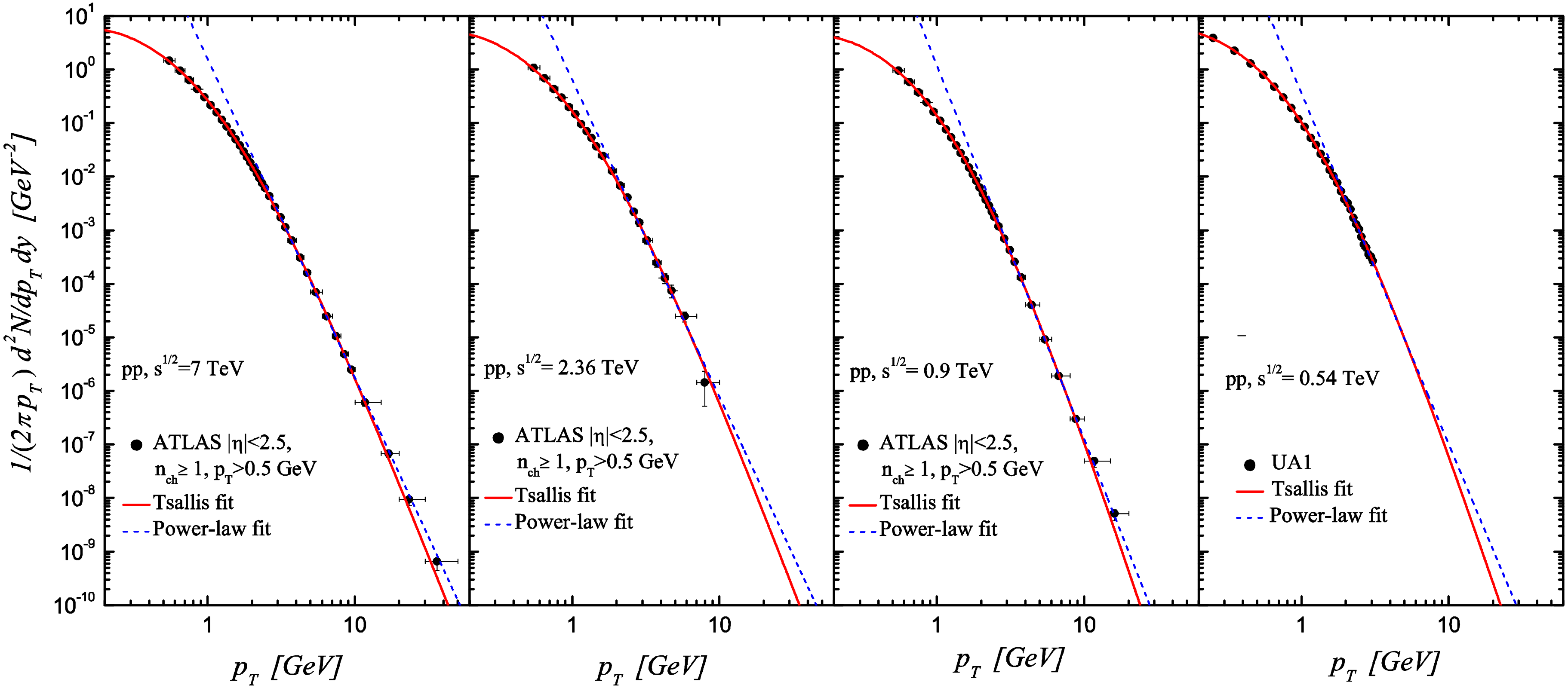
Charged-hadron yields in the Tsallis-factorized statistics

In collaboration with
J. Cleymans, G.I. Lykasov, A.S. Sorin, O.V. Teryaev and D. Worku

Charged-hadron yield at different energies: ATLAS and UA1

J. Cleymans, G.I. Lykasov, A.P., A.S. Sorin, O.V. Teryaev, D. Worku, Phys. Lett.B 723 (2013) 351

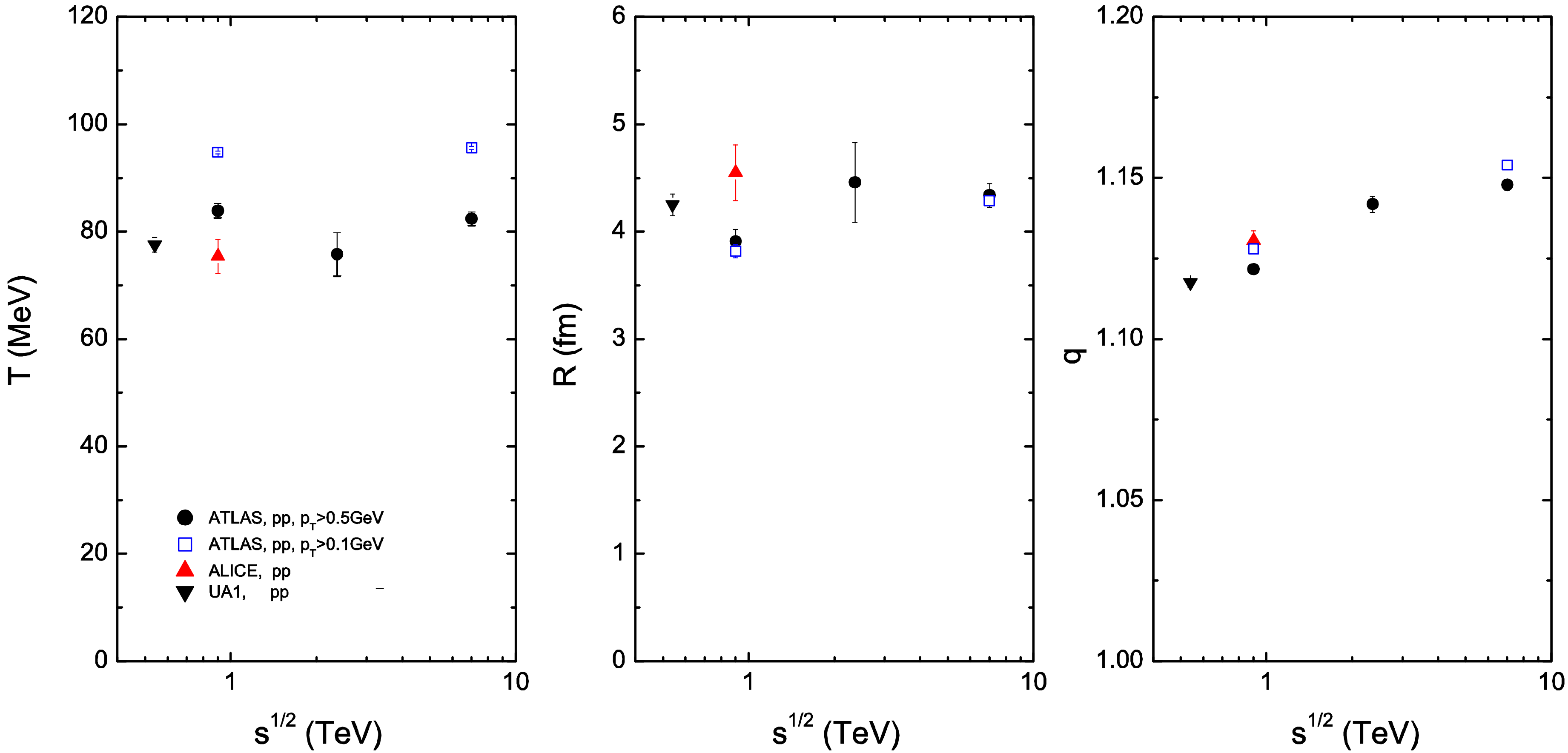
Tsallis-factorized statistics fit



Charged-hadron yield parameters

J. Cleymans, G.I. Lykasov, A.P., A.S. Sorin, O.V. Teryaev, D. Worku, Phys. Lett.B 723 (2013) 351

Tsallis-factorized statistics fit



Conclusions

1. **The analytical expressions for the ultrarelativistic transverse momentum distribution of the Tsallis-1 and Tsallis-2 statistics were obtained**

2. **We found that the transverse momentum distribution of the Tsallis-factorized statistics in the ultrarelativistic case is not equivalent to the transverse momentum distribution of both the Tsallis-1 and Tsallis-2 statistics**

3. **The transverse momentum distribution of the Tsallis-factorized statistics is equivalent only to the distribution in the zeroth term approximation of the Tsallis-2 statistics and the Tsallis-1 statistics with transformation of the parameter q to $1/q_c$**

4. **We have demonstrated on the base of the ultrarelativistic ideal gas that the Tsallis –factorized statistics is not equivalent to the Tsallis statistics (Tsallis-1 and Tsallis-2 statistics)**

Thank you for your attention