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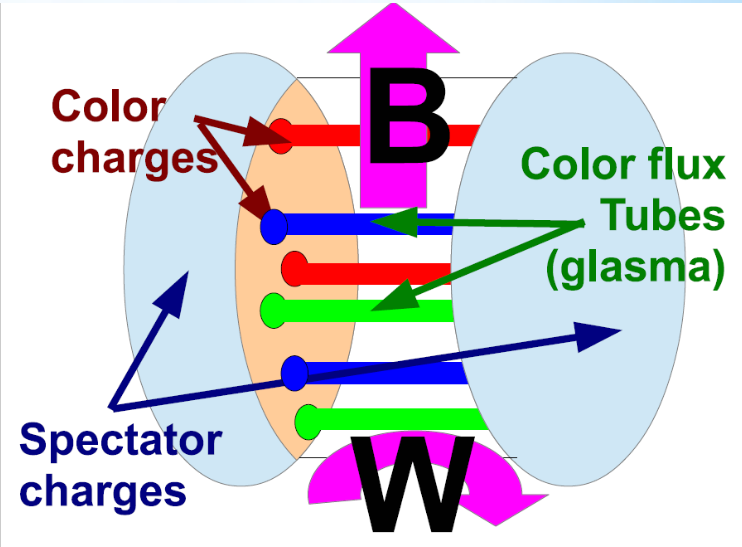
Quantum effects in real-time evolution of gauge theories

Motivation

Glasma state at early stages of HIC
Overpopulated gluon states
Almost "classical" gauge fields

Chaotic Classical Dynamics
[Saviddy,Susskind…]

- Positive Lyapunov exponents
- Gauge fields forget initial conditions
- But not enough for **Thermalization**



Motivation

Dimensionally reduced (9+1) dimensional N=1 Super-Yang-Mills in (d+1) dimensions

$$S_{(d+1)} = \int d^{d+1}x \,\mathrm{Tr}\left(\frac{1}{4}F_{AB}^2 + \frac{i}{2}\bar{\psi}\not{D}\psi + \right.$$
$$\left. + \frac{1}{2}(D_A X_\mu)^2 - \frac{1}{4}[X_\mu, X_\nu]^2 + \frac{1}{2}\bar{\psi}\tilde{\gamma}^\mu[X_\mu, \psi]\right)$$

A,B = 0…d, μ,ν = d+1…9

N x N hermitian matrices

Majorana-Weyl fermions, N x N hermitian matrices

“Holographic” duality [Witten’96]:

- X^{ii}_μ = Dp brane positions
- X^{ij}_μ = open string excitations

Motivation

N=1 Supersymmetric Yang-Mills in D=1+9:

gauge bosons+adjoint Majorana-Weyl fermions

Reduce to a single point = BFSS matrix model

[Banks, Fischler, Shenker, Susskind'1997]

$$L = \frac{1}{2g}\left[\mathrm{tr}\dot{X}^i\dot{X}^i + 2\theta^T\dot{\theta} - \frac{1}{2}\mathrm{tr}[X^i, X^j]^2 - 2\theta^T\gamma_i[\theta, X^i]\right]$$

N x N hermitian matrices

Majorana-Weyl fermions, N x N hermitian

System of *N* D0 branes joined by open strings

Motivation

Stringy interpretation:

Dynamics of self-gravitating D0 branes

Entropy production = black hole formation

Motivation

So far, mostly classical simulations …

Quantum effects?

$$\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

- Thermalization and scrambling?
- Fast thermalization in Yang-Mills plasma?
- Quantum entanglement?
- Evaporation of black holes [Hanada'15]
- … Information paradox?

In this talk:

Numerical methods to address all these questions

- Beyond classical-statistical: quantum fluctuations of gauge fields
- Semi-analytic solutions based on truncated Heisenber equations
- Classical-statistical approximation: effect of fermions → black hole "evaporation"

Going beyond CSFT

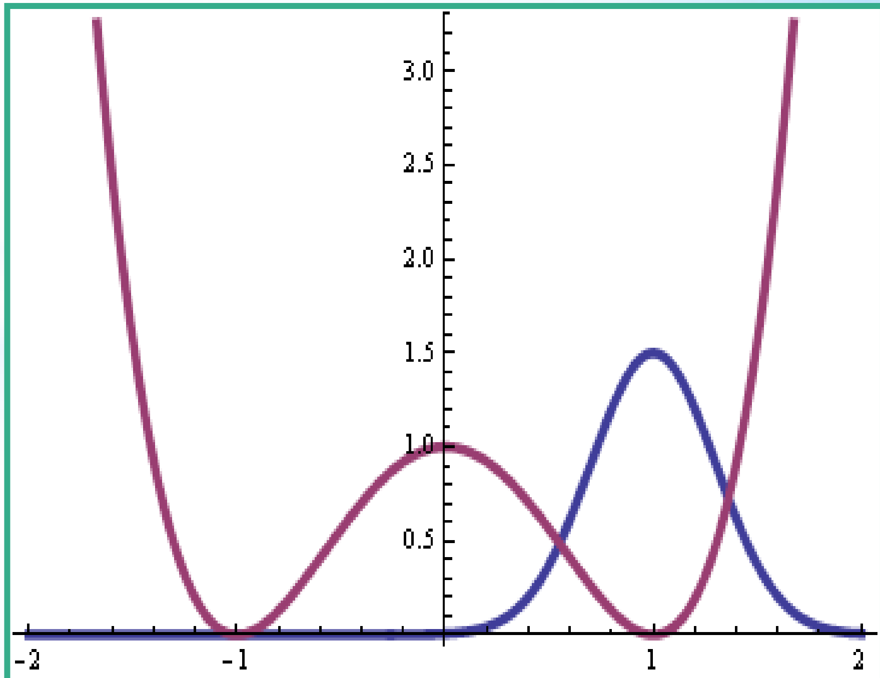
- Return to Heisenberg eqs of motion
- VEV with more non-trivial correlators

<u>First, test this idea on the simplest example</u>

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{a\hat{x}^2}{2} + \frac{b\hat{x}^3}{3} + \frac{c\hat{x}^4}{4}$$

$$\partial_t \hat{x} = \hat{p},$$
$$\partial_t \hat{p} = -a\hat{x} - b\hat{x}^2 - c\hat{x}^3$$

Tunnelling between potential wells? Absent in CSFT!



Next step: Gaussian Wigner function

Assume Gaussian wave function at any t

Simpler: Gaussian Wigner function

$$\langle \hat{x}^2 \rangle = x^2 + \sigma_{xx},$$
$$\langle \hat{p}^2 \rangle = p^2 + \sigma_{pp},$$
$$\langle \frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2} \rangle = xp + \sigma_{xp}$$

For other correlators: use Wick theorem!

$$\langle \hat{x}^4 \rangle = x^4 + 6x^2\sigma_{xx} + 3\sigma_x x^2,$$
$$\langle \hat{x}^2\hat{p} \rangle = x^2 p + 2x\sigma_{xp} + p\sigma_{xx}$$

Derive closed equations for

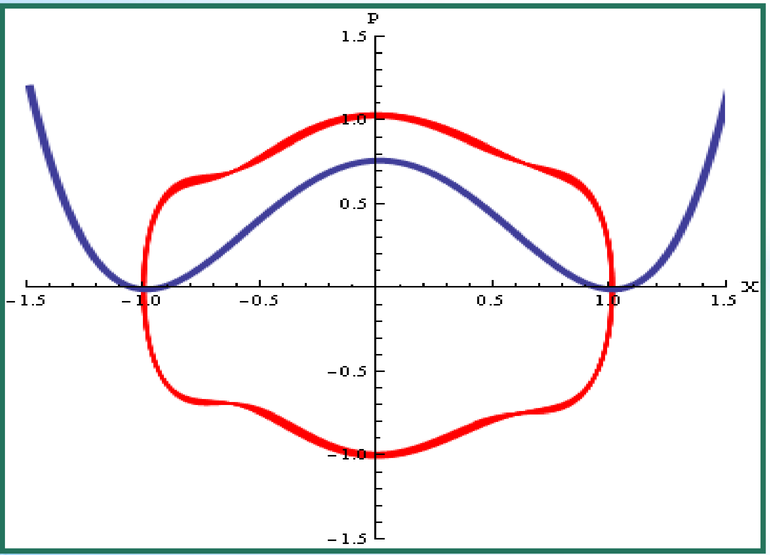
x, p, σ_{xx} , σ_{xp} , σ_{pp}

Origin of tunnelling

$$\partial_t p = -ax - bx^2 - cx^3 - b\sigma_{xx} - 3cx\sigma_{xx},$$
$$\partial_t x = p$$

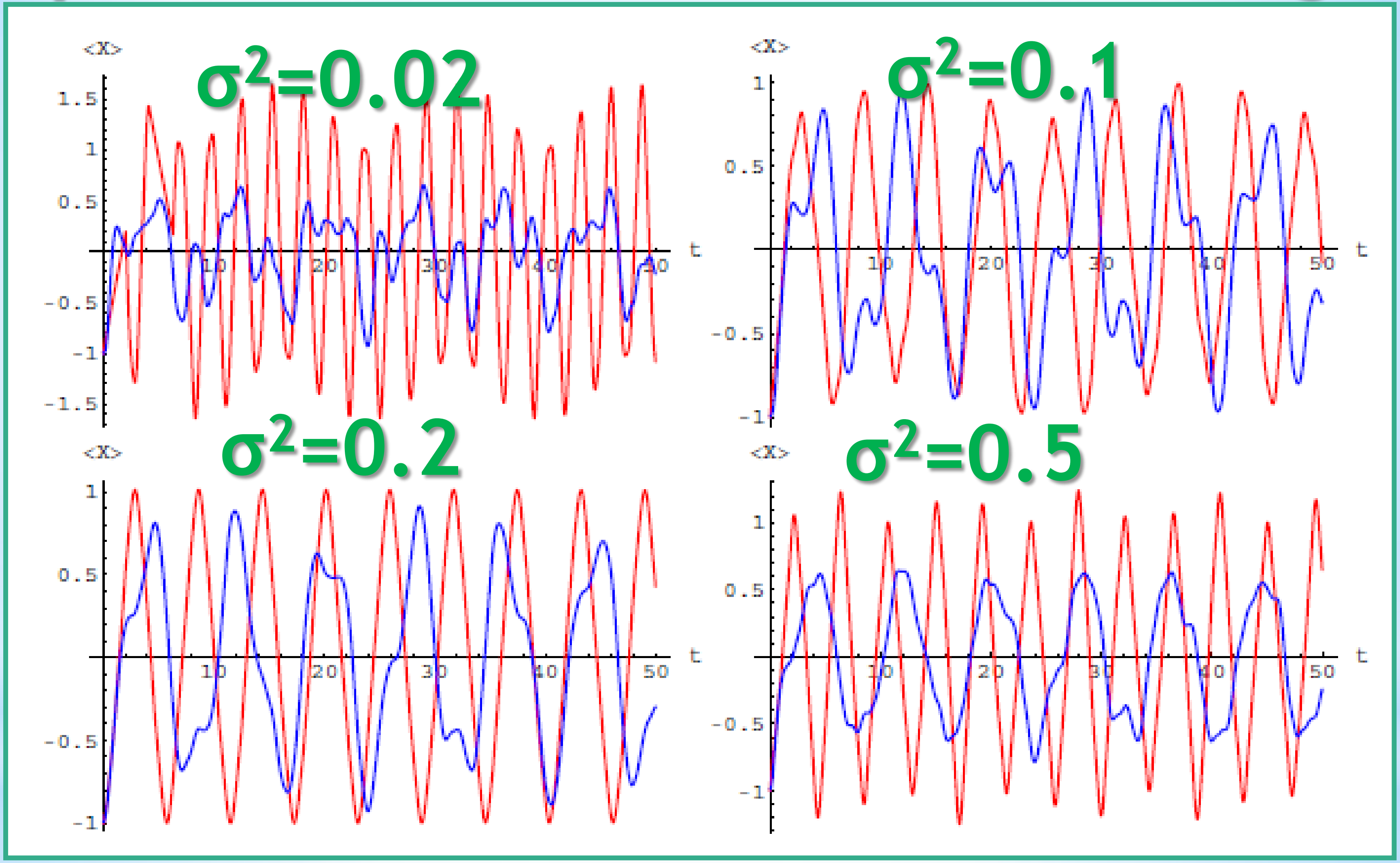
Positive force even at *x=0* (classical minimum)

$$\partial_t \sigma_{xx} = 2\sigma_{xp},$$
$$\partial_t \sigma_{xp} = \sigma_{pp} - a\sigma_{xx} - 2bx\sigma_{xx} - 3cx^2\sigma_{xx} - 3c\sigma_{xx}^2,$$
$$\partial_t \sigma_{pp} = -2\left(a\sigma_{xp} + 2bx\sigma_{xp} + 3cx^2\sigma_{xp} + 3c\sigma_{xx}\sigma_{xp}\right)$$



Quantum force causes classical trajectory to leave classical minimum

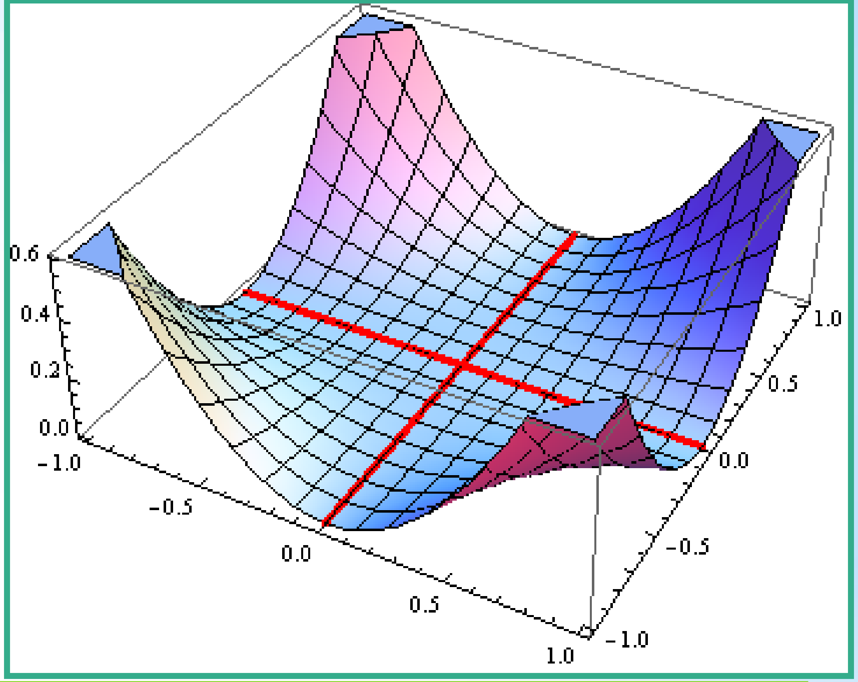
Improved CSFT vs exact Schrödinger



- Early-time evolution OK
- Tunnelling period qualitatively OK

2D potential with flat directions
(closer to BFSS model)

$$\hat{H} = \frac{\hat{p}_x^2}{2} + \frac{\hat{p}_y^2}{2} + \frac{\kappa}{2}\hat{x}^2\hat{y}^2$$

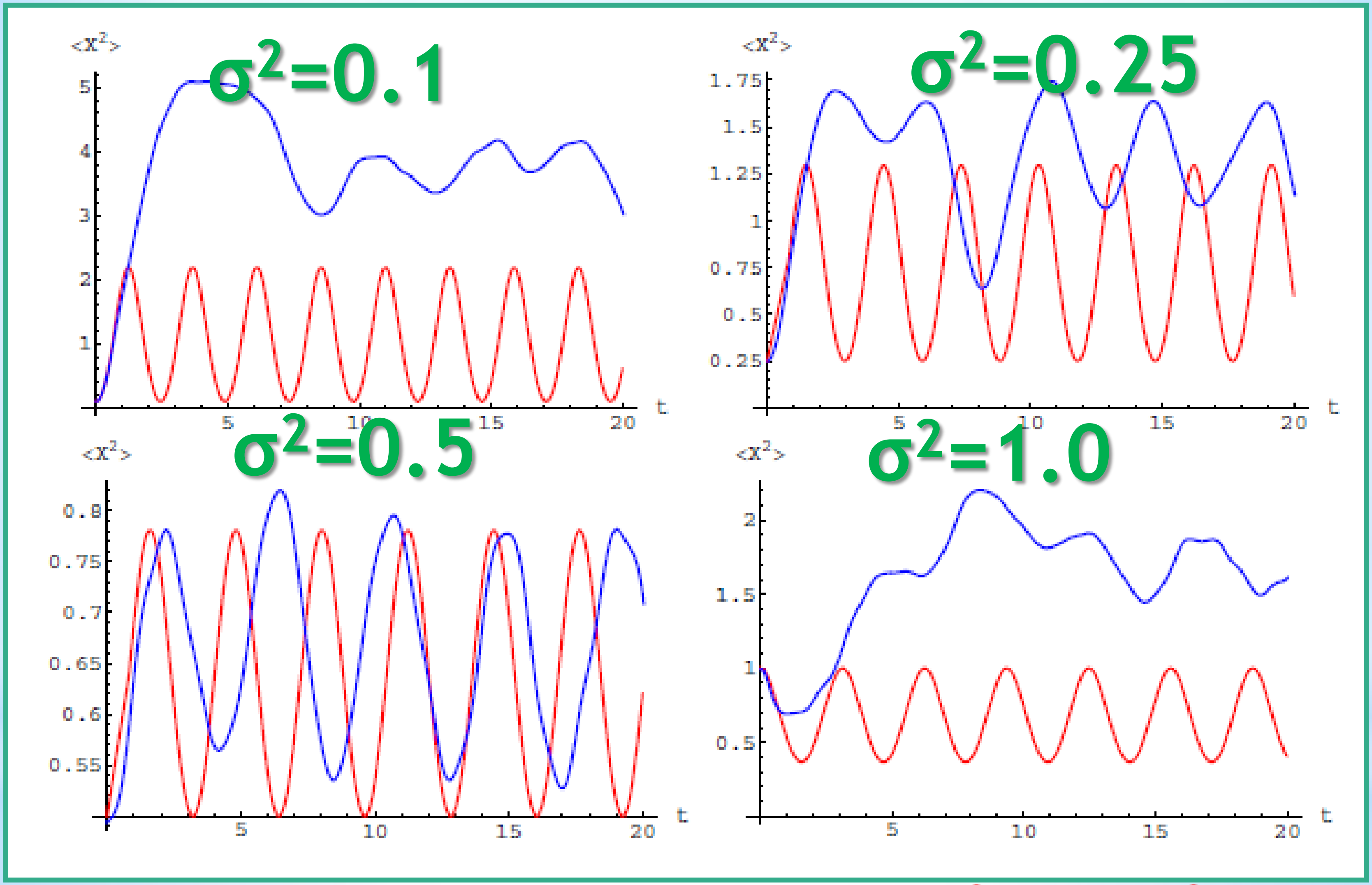


Classic runaway along *x=0 or y=0*

$$\sigma_{xxij} = \delta_{ij}\sigma_{xx}, \quad \sigma_{ppij} = \delta_{ij}\sigma_{pp}, \quad \sigma_{xpij} = \delta_{ij}\sigma_{xp},$$
$$x_i = 0, \quad p_i = 0,$$

Maximally symmetric initial conditions

Improved CSFT vs exact Schrödinger



- OK for wavefuncs with $\langle x^2 \rangle \sim \langle p^2 \rangle$
- Wrong for large $\langle x^2 \rangle$ or $\langle p^2 \rangle$

BFSS matrix model: Hamiltonian formulation

$$\hat{H} = \frac{1}{2}\hat{P}_i^a \hat{P}_i^b + \frac{1}{4} C_{abc} C_{ade} \hat{X}_i^b \hat{X}_j^c \hat{X}_i^d \hat{X}_j^e + \frac{i}{2} C_{abc} \hat{\psi}_\alpha^a \left[\sigma_i\right]_{\alpha\beta} \hat{X}_i^b \hat{\psi}_\beta^c,$$

a,b,c - su(N) Lie algebra indices

Heisenberg equations of motion

$$\partial_t \hat{X}_i^a = \hat{P}_i^a$$

$$\partial_t \hat{P}_i^a = -C_{abc} C_{cde} \hat{X}_j^b \hat{X}_i^d \hat{X}_j^e - \tfrac{i}{2} C_{bac} \sigma^i_{\alpha\beta} \hat{\psi}_\alpha^b \hat{\psi}_\beta^c,$$

$$\partial_t \hat{\psi}_\alpha^a = C_{abc} X_i^b \sigma^i_{\alpha\beta} \hat{\psi}_\beta^c$$

Average assuming that X are classical (“strong field regime”)!!!

$$\langle \hat{X}_j^b \hat{X}_i^d \hat{X}_j^e \rangle = \langle \hat{X}_j^b \rangle \langle \hat{X}_i^d \rangle \langle \hat{X}_j^e \rangle \equiv X_j^b X_i^d X_j^e$$

Improved CSFT for BFSS model

$$\partial_t P^a_i = -C_{abc}C_{cde}X^b_j X^d_i X^e_j - \frac{i}{2}C_{bac}\sigma^i_{\alpha\beta}\langle\psi^b_\alpha\psi^c_\beta\rangle -$$
$$- C_{abc}C_{cde}X^b_j[XX]^{de}_{ij} - C_{abc}C_{cde}[XX]^{be}_{jj}X^d_i - C_{abc}C_{cde}[XX]^{bd}_{ji}X^e_j$$

$$\partial_t [XX]^{ab}_{ij} = [XP]^{ab}_{ij} + [XP]^{ba}_{ji},$$
$$\partial_t [XP]^{af}_{ik} = [PP]^{af}_{ik} - C_{abc}C_{cde}\left(X^d_i X^e_j + [XX]^{de}_{ij}\right)[XX]^{bf}_{jk} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^e_j + [XX]^{be}_{jj}\right)[XX]^{df}_{ik} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^d_i + [XX]^{bd}_{ji}\right)[XX]^{ef}_{jk},$$
$$\partial_t [PP]^{af}_{ik} = -C_{abc}C_{cde}\left(X^d_i X^e_j + [XX]^{de}_{ij}\right)[XP]^{bf}_{jk} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^e_j + [XX]^{be}_{jj}\right)[XP]^{df}_{ik} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^d_i + [XX]^{bd}_{ji}\right)[XP]^{ef}_{jk} + (\{a,i\} \leftrightarrow \{f,k\})$$

- **CPU time ~ *N^5* (double commutators)**
- **RAM memory ~ *N^4***
- ***SUSY still broken …***

Entropy/phase volume conservation

- **Entropy of mixed state in terms of correlators ➡ simplectic eigenvalues of**

$$\Xi = \begin{pmatrix} \langle\langle x \otimes x \rangle\rangle & \langle\langle x \otimes p \rangle\rangle \\ \langle\langle p \otimes x \rangle\rangle & \langle\langle p \otimes p \rangle\rangle \end{pmatrix}$$

- **Simplectic eigenvals = eigenvals of *ε Θ***

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- simplectic form

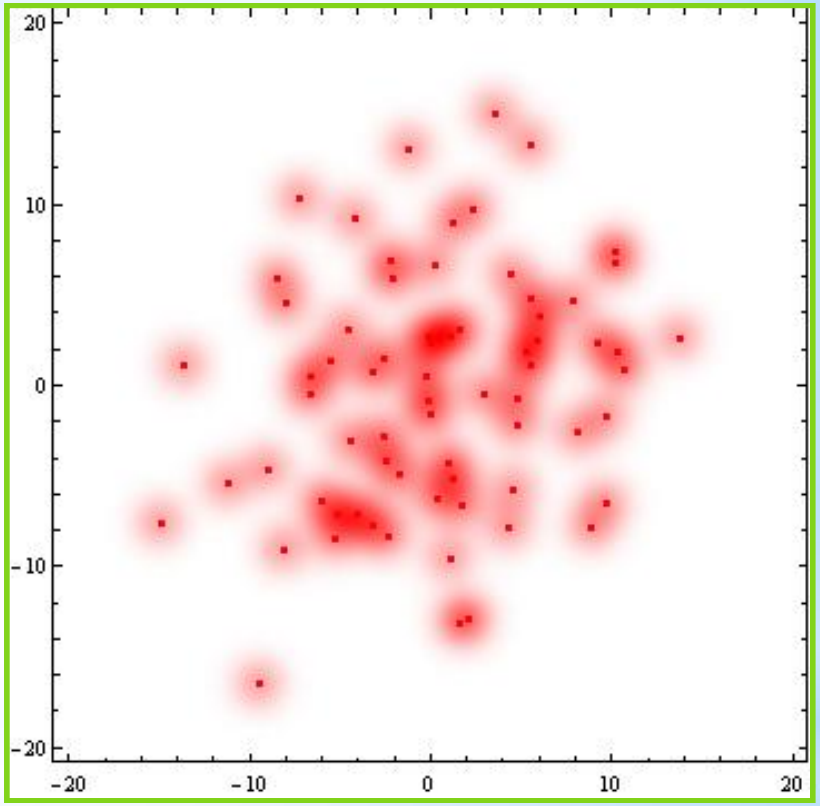
Heisenberg equations take the form

$$\partial_t \left(\varepsilon\Xi\right) = V\left(\varepsilon\Xi\right) - \left(\varepsilon\Xi\right)V$$

Initial conditions

$\langle X^a_i \rangle$ are random Gaussian

$$[XX]^{ab}_{ij} = \sigma^2\, \delta^{ab}\delta_{ij},$$
$$[PP]^{ab}_{ij} = \sigma^2\, \delta^{ab}\delta_{ij},$$
$$[XP]^{ab}_{ij} = 0$$



Minimal quantum dispersion (Uncertainty principle)

(Classical) dispersion of $\langle X^a_i \rangle$ roughly corresponds to temperature

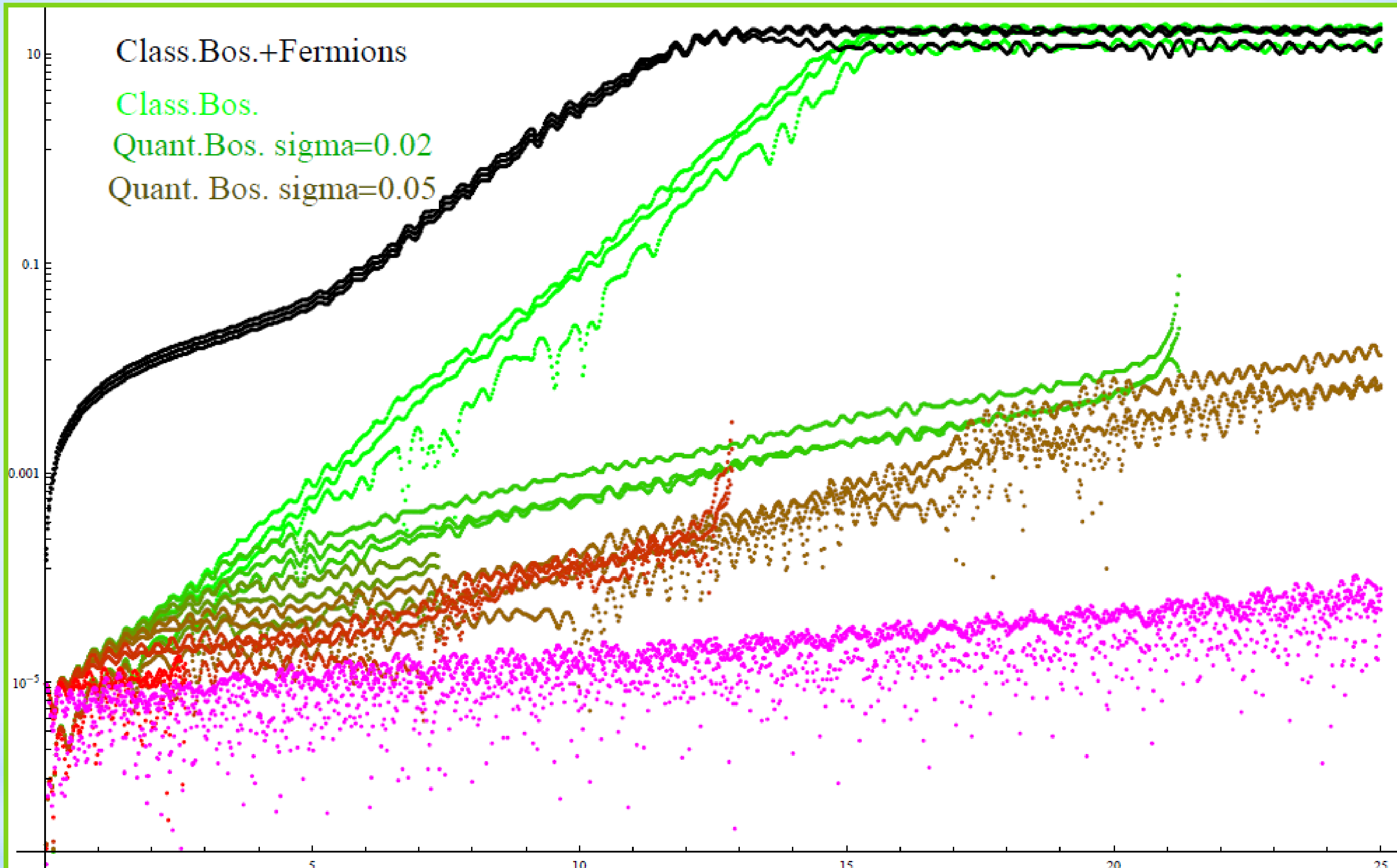
Initial conditions

Choice of initial momentum

P chosen to have

- **zero angular momentum**
- **zero gauge constraint**
- **zero *Tr[P]***
- **minimal *Tr[P²]***

Quantum effects decrease instability



$$\langle in|\, e^{i\hat{p}\delta x}\hat{x}(t)\, e^{-i\hat{p}\delta x}\, |in\rangle - \langle in|\, \hat{x}(t)\, |in\rangle =$$

$$= i\langle in|\, [\hat{p}(0), \hat{x}(t)]\, |in\rangle$$

Lyapunov exponents ~ OTO correlators !!!

Entanglement entropy production

- Separate out some variables, restrict the correlator matrix to them

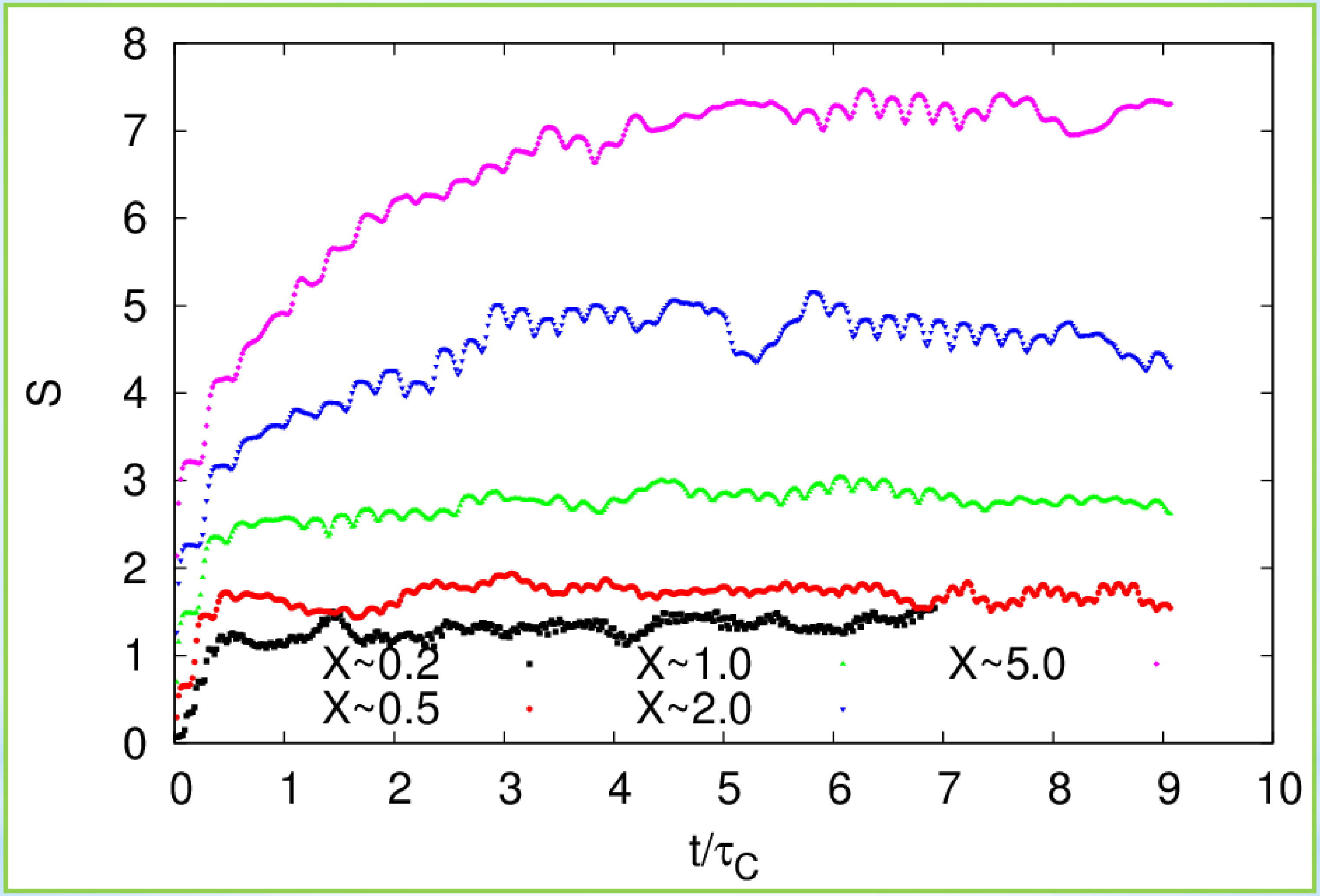
$$\Xi = \begin{pmatrix} \langle \langle x \otimes x \rangle \rangle & \langle \langle x \otimes p \rangle \rangle \\ \langle \langle p \otimes x \rangle \rangle & \langle \langle p \otimes p \rangle \rangle \end{pmatrix}$$

- Restricted correlator, in general, mixed

Entanglement entropy vs simplectic evals of restricted correlators

$$S = \sum_i \left(f_i + \tfrac{1}{2}\right) \ln \left(f_i + \tfrac{1}{2}\right) - \left(f_i - \tfrac{1}{2}\right) \ln \left(f_i - \tfrac{1}{2}\right)$$

Entanglement entropy production



Single matrix entry entangled with others

Initially, *dS/dt* scales as classical Lyapunov t

Intermediate conclusions

- Quantum effects do not speed up thermalization, at least in our Gaussian approximation, as indicated by classical Lyapunov exponents

- Alternative criterion: entanglement entropy, “quantum scrambling”

- Early-time thermalization at most governed by classical Lyapunov exponents

- More general OTO correlators?

Classical-statistical field theory (CSFT)

[Son, Aarts, Smit, Berges, Tanji, Gelis,…]

- Schwinger pair production
- Axial charge generation in glasma
- (Chiral) plasma instabilities

Closed system of equations: $\partial_t X_i^a = P_i^a$

$$\partial_t P_i^a = -C_{abc} C_{cde} X_j^b X_i^d X_j^e - \frac{i}{2} C_{bac} \sigma^i_{\alpha\beta} \langle \hat{\psi}^b_\alpha \hat{\psi}^c_\beta \rangle$$

$$\partial_t \langle \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \rangle = C_{ade} X_i^d \sigma^i_{\alpha\gamma} \langle \hat{\psi}^e_\gamma \hat{\psi}^b_\beta \rangle + C_{bde} X_i^d \sigma^i_{\beta\gamma} \langle \hat{\psi}^a_\alpha \hat{\psi}^e_\gamma \rangle$$

Numerical solution! Fermions are costly!

CPU time + RAM memory scaling ~ *N*^{4}

Parallelization is necessary

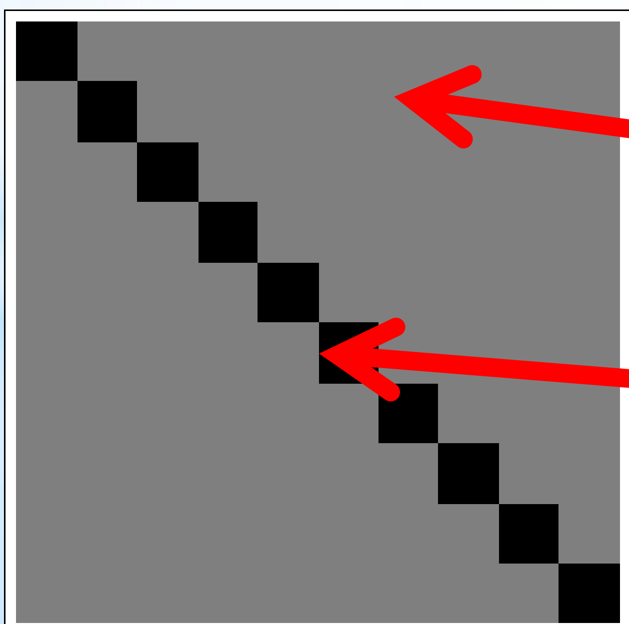
Initial conditions: bosons

Initial state should be excited to allow for nontrivial evolution (thinking about black holes, we still believe in quantum mechanics)

$$\mathcal{Z}^{-1}\mathrm{Tr}\left(e^{-\hat{H}/T}e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t}\right) = \mathcal{Z}^{-1}\mathrm{Tr}\left(e^{-\hat{H}/T}\hat{O}\right)$$

Our initial conditions:

$$X_\mu^{ij} =$$



Gaussian random, dispersion f

Gaussian random, dispersion 1

Initial conditions: fermions

Fermions are in ground state at given X^a_i

$$\langle \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \rangle = \frac{\delta^{ab}\delta_{\alpha\beta}}{2} - i \sum_{\epsilon>0} \left(u^a_\alpha(\epsilon) v^b_\beta(\epsilon) - v^a_\alpha(\epsilon) v^b_\beta(\epsilon) \right)$$

$$h^{ab}_{\alpha\beta} u^b_\beta(\epsilon) = i\epsilon\, v^a_\alpha(\epsilon), \quad h^{ab}_{\alpha\beta} v^b_\beta(\epsilon) = -i\epsilon\, u^a_\alpha(\epsilon)$$

Tricky for Majoranas, no Dirac sea!!! ($\psi = \bar{\psi}$)

Negative energy stored in fermions

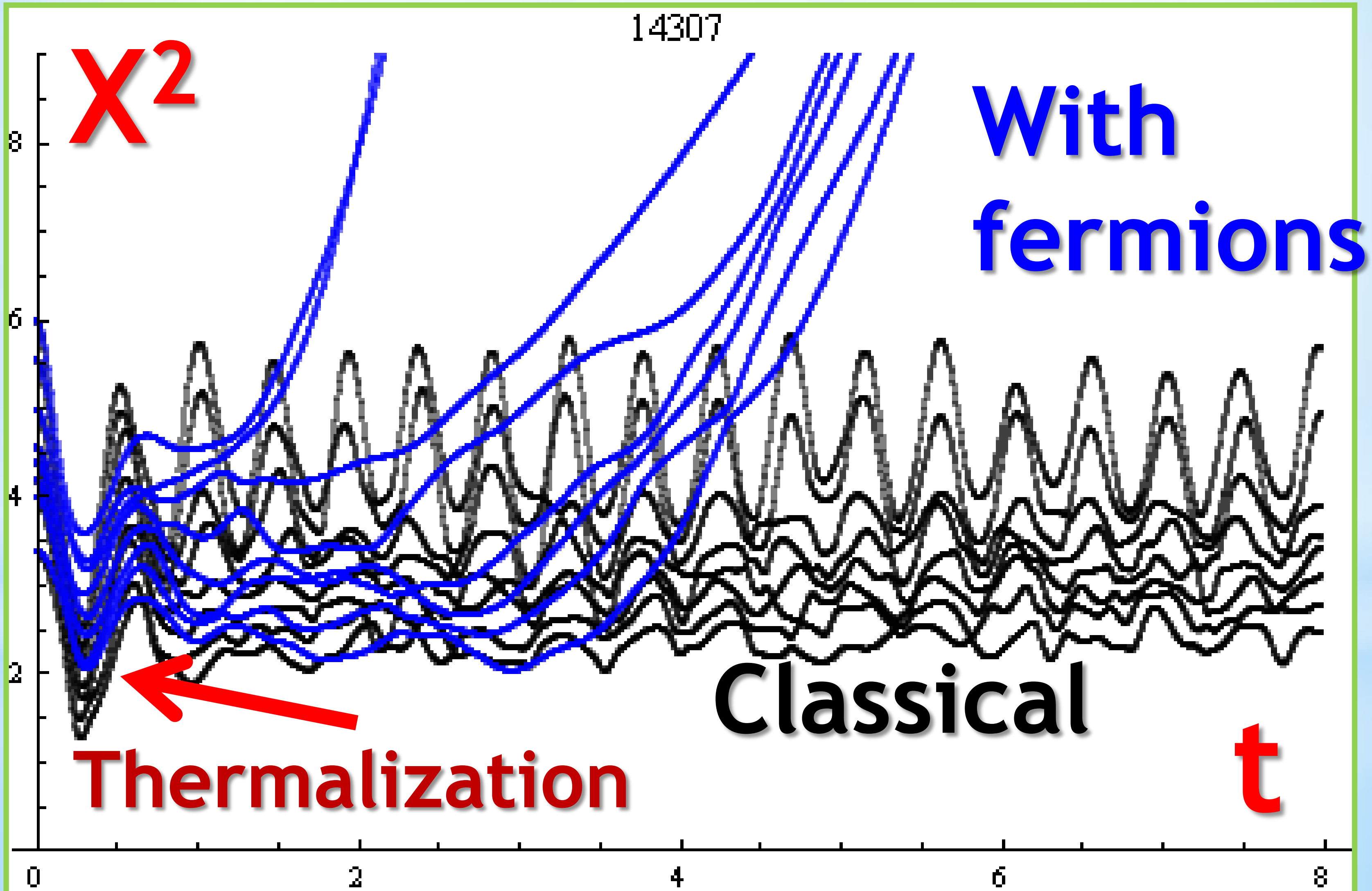
Initial conditions: momenta

For given X and <ψψ>: Choose P such that

- **Zero angular momentum (non-rotating BH)**
- **Gauge constraint** $J^a = C_{abc} X^b_i P^c_i - \sum_{\epsilon>0} C_{abc} u^b_\alpha(\epsilon) v^c_\alpha(\epsilon)$
- ***Tr(P) = 0* (center of mass at rest)**
- **$Tr(P^2)$ is minimal**

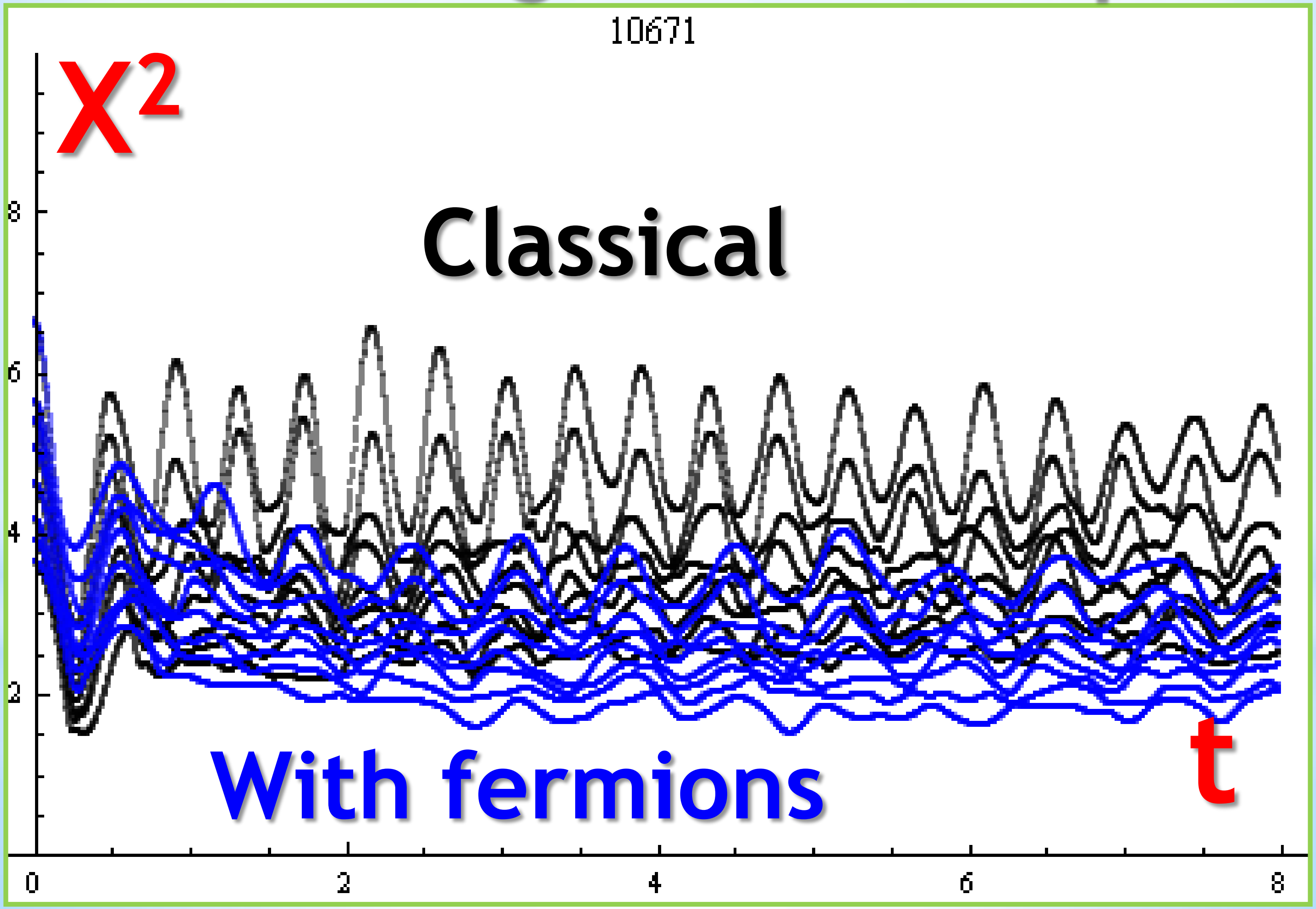
Some results N = 8, f = 0.48

Hawking radiation of D0 branes

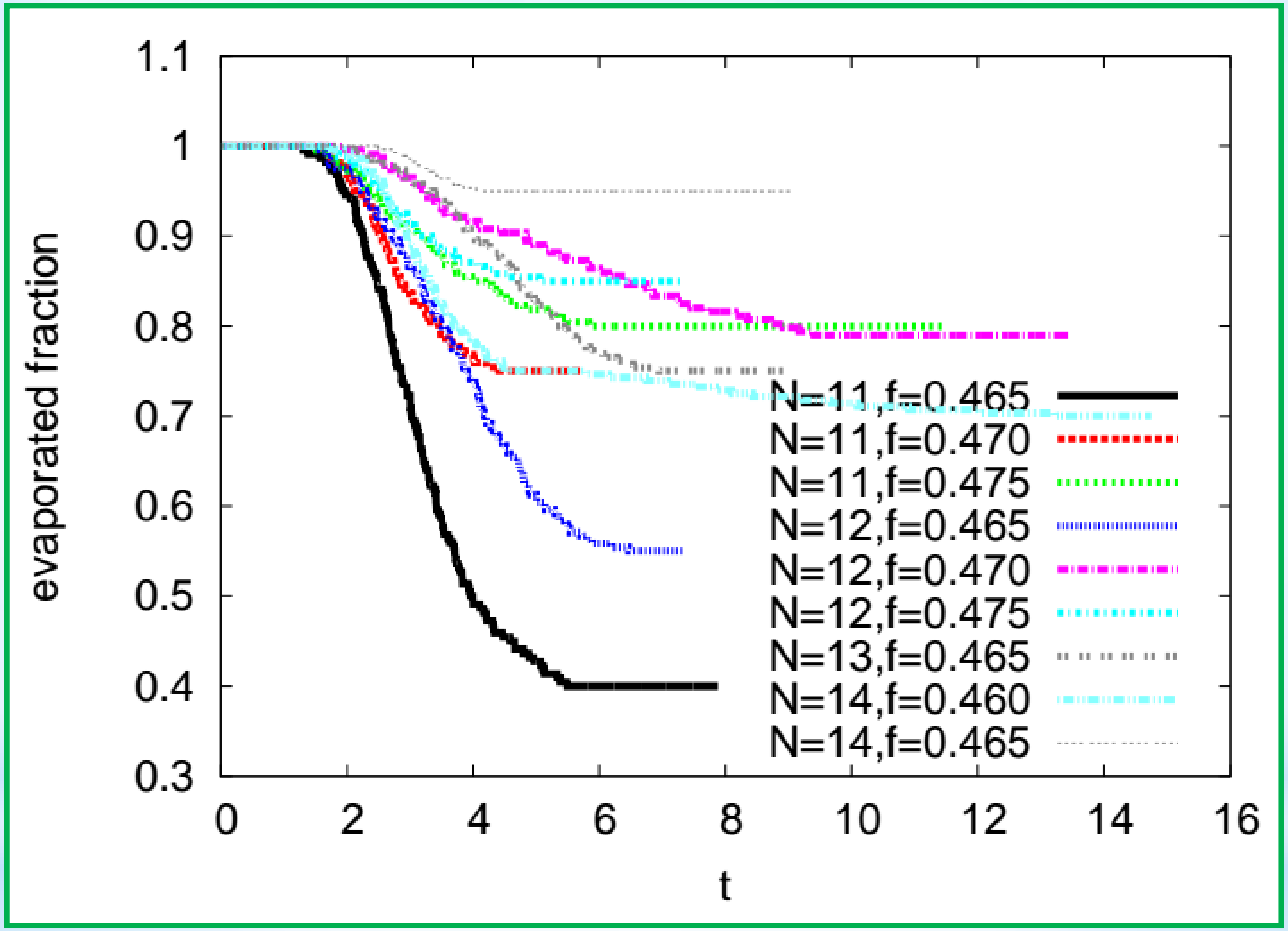


Some results N = 8, f = 0.48

Not all configurations evaporate

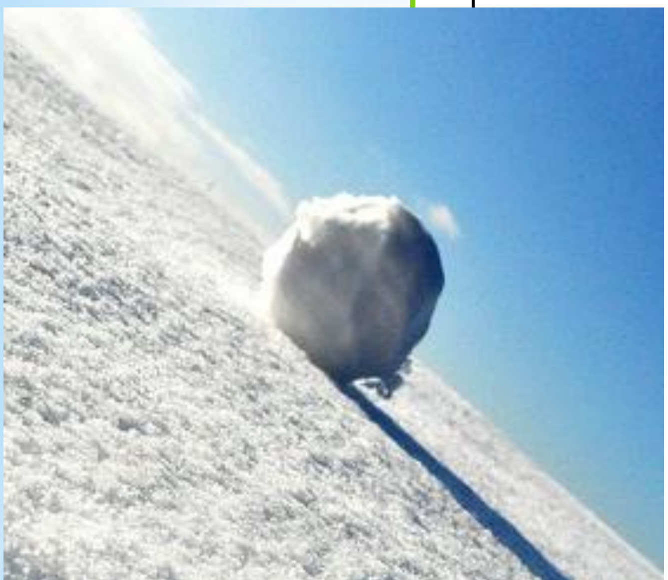


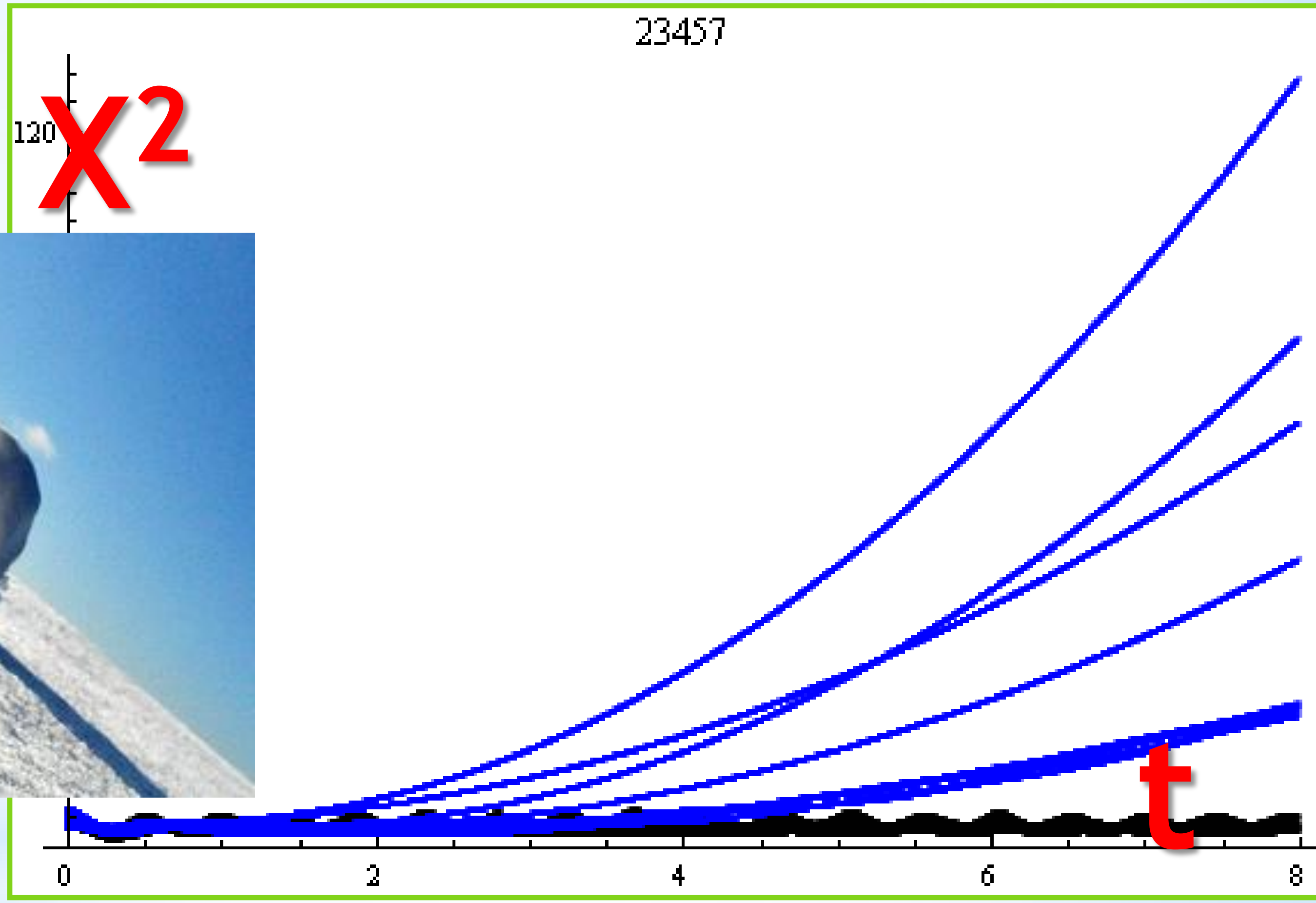
Fraction of evaporating configurations



At f > 0.5 and N->∞ almost no configurations evaporate, phase transition at large N?

Constant acceleration at late times





- Boson kinetic energy grows without bound
- Fermion energy falls down
- Origin of const force – SUSY violation?

CSFT and SUSY

16 supercharges in BFSS model:

$$\hat{Q}_\alpha = \hat{P}_i^a [\sigma_i]_{\alpha\beta} \hat{\psi}_\beta^a - \frac{1}{4} C_{abc} \hat{X}_i^b \hat{X}_j^c [\sigma_{ij}]_{\alpha\beta} \hat{\psi}_\beta^a$$

$$\sigma_{ij} \equiv \sigma_i \sigma_j - \sigma_j \sigma_i$$

$$\left\{ \hat{Q}_\alpha, \hat{Q}_\beta \right\} = 2\delta_{\alpha\beta} \hat{H} - 2 (\sigma_i)_{\alpha\beta} \hat{X}_i^a \hat{J}^a$$

$$\left[\hat{H}, \hat{Q}_\gamma \right] = -i \hat{\psi}_\gamma^a \hat{J}^a$$

Gauge transformations

$$\hat{J}^a = C_{abc} \hat{X}_i^b \hat{P}_i^c - \frac{i}{2} C_{abc} \hat{\psi}_\alpha^b \hat{\psi}_\alpha^c$$

CSFT and SUSY

In full quantum theory

$$\partial_t \hat{Q}_\delta = \frac{i}{2} C_{abc} \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \hat{\psi}^c_\gamma \left(\sigma^i_{\alpha\beta} \sigma^i_{\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \right) = 0$$

Fierz identity (cyclic shift of indices):

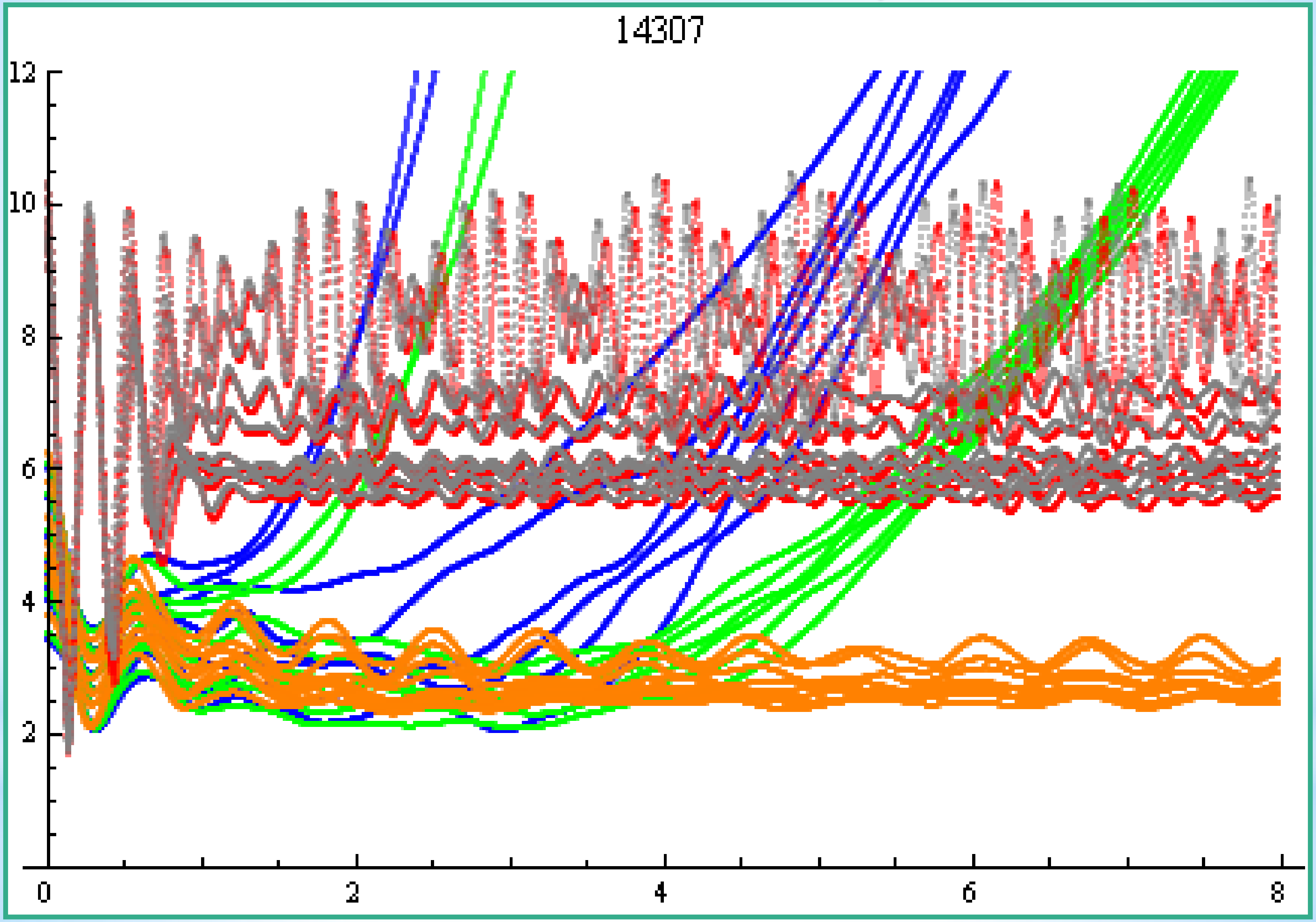
$$[\sigma_i]_{\alpha\beta} [\sigma_i]_{\gamma\delta} + [\sigma_i]_{\alpha\gamma} [\sigma_i]_{\beta\delta} + [\sigma_i]_{\alpha\delta} [\sigma_i]_{\gamma\beta} = \\ = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}$$

In CSFT approximation

$$\partial_t \hat{Q}_\delta = \frac{i}{2} C_{abc} \langle \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \rangle \hat{\psi}^c_\gamma \left(\sigma^i_{\alpha\beta} \sigma^i_{\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \right) \neq 0$$

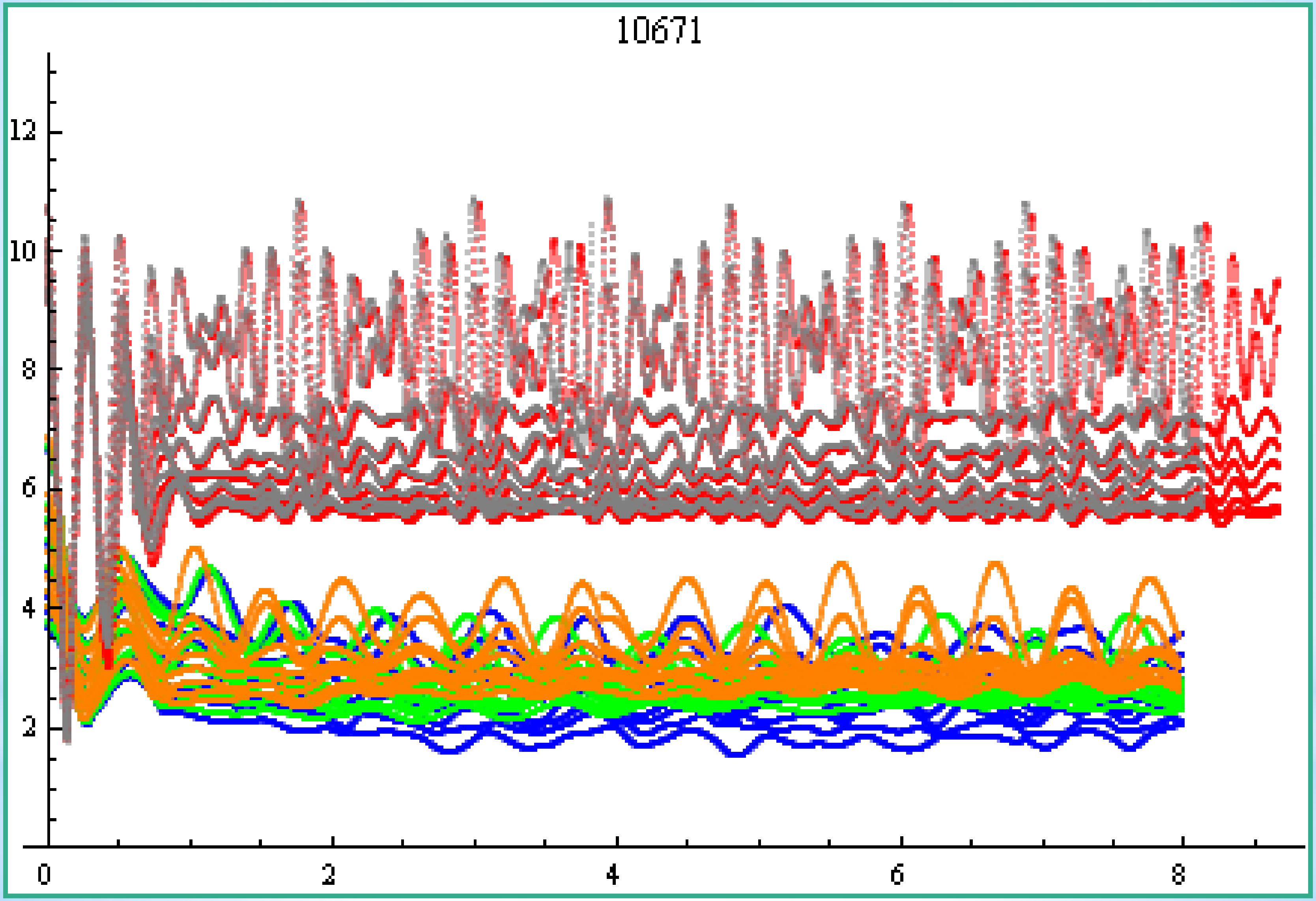
Fermionic 3pt function seems necessary!

Some results N = 8, f = 0.48



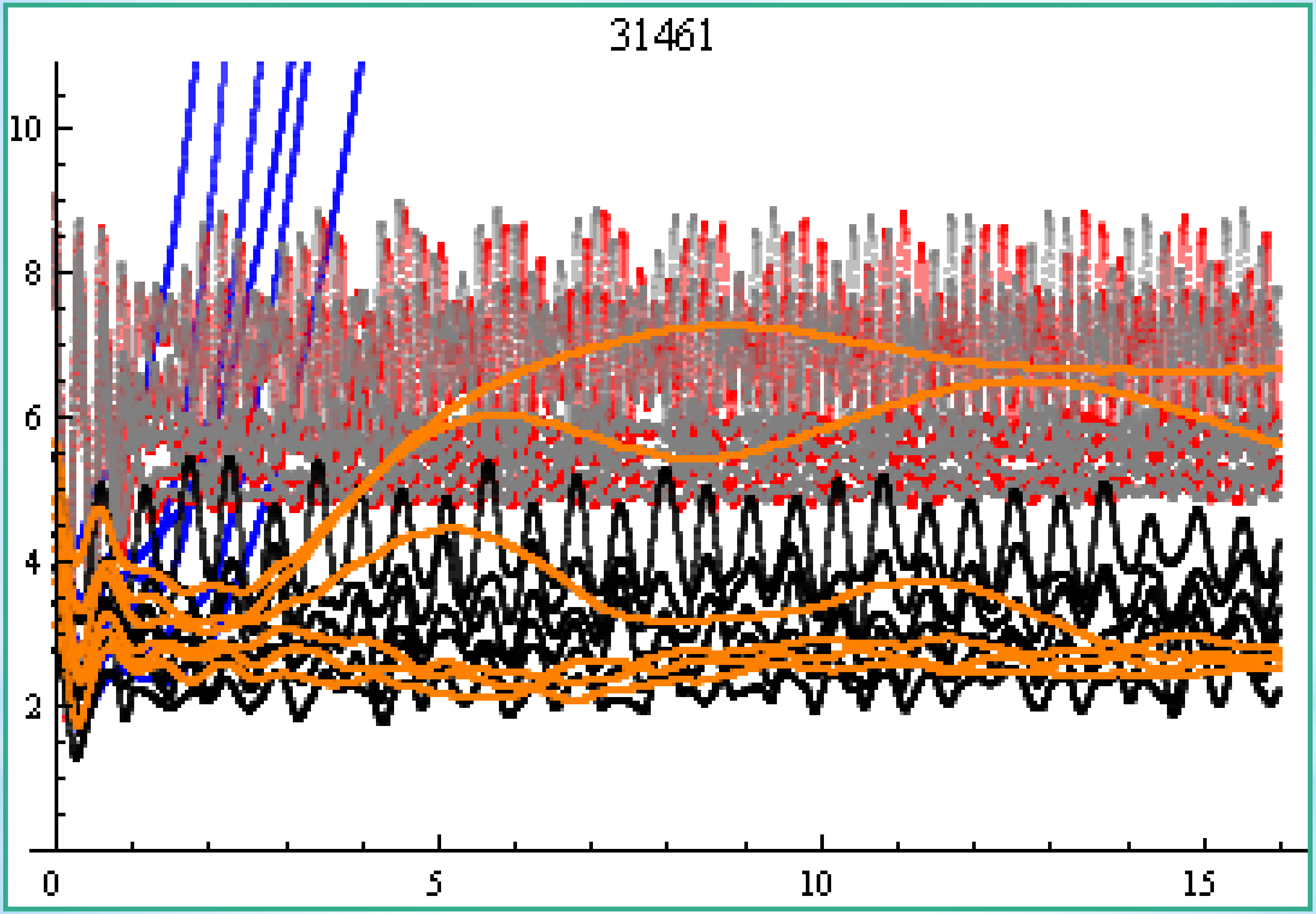
$\sigma 2$ = *0.0*, *0.1*, *0.2*, *1.0* + no fermions

Some results N = 8, f = 0.48



$\sigma 2$ = 0.0, 0.1, 0.2, 1.0 + no fermions

Evaporation threshold: N = 6, f = 0.50



σ2 = 0.0, 0.2, 1.0 **+ no fermions**

Quantum fluctuations vs evaporation

Quantum fluctuations of X variables suppress evaporation!!!

- Small σ^2: evaporation becomes slower
- Intermediate σ^2: no evaporation
- Large σ^2: fermions negligible
- Fine-tuning with realistic initial conditions?

Exact solutions with high symmetry?

What effect quantum fluctuations of X have on classical dynamics?

Exact solution: quantum X (limit of large σ^2)

SU(N) x SO(D) symmetric initial conditions

$$[XX]_{ij}^{ab}(t=0) = \sigma_{xx}(t=0)\,\delta_{ij}\delta^{ab},$$
$$[XP]_{ij}^{ab}(t=0) = \sigma_{xp}(t=0)\,\delta_{ij}\delta^{ab},$$
$$[PP]_{ij}^{ab}(t=0) = \sigma_{pp}(t=0)\,\delta_{ij}\delta^{ab}$$

$$X_i^a(t=0) = 0,$$
$$P_i^a(t=0) = 0,$$

Equations of motion, σ_{xp} can be excluded

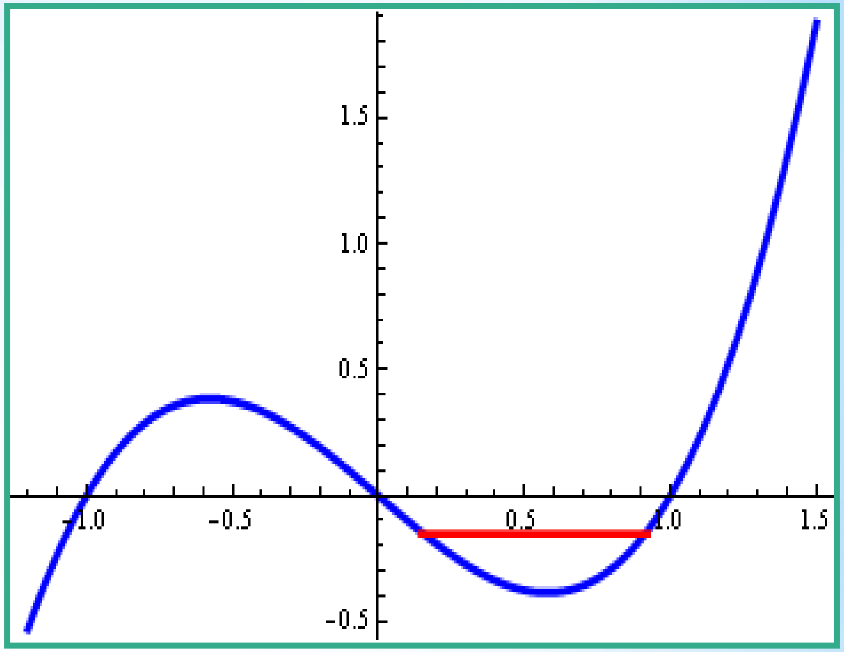
$$\partial_t \sigma_{xx} = 2\sigma_{xp}, \quad \partial_t \sigma_{xp} = \sigma_{pp} - 2N(D-1)\,\sigma_{xx}^2,$$
$$\partial_t \sigma_{pp} = -4N(D-1)\,\sigma_{xx}\sigma_{xp}$$

$$\partial_t\left(\sigma_{pp} + N(D-1)\,\sigma_{xx}^2\right) = 0 \Rightarrow$$
$$\Rightarrow \sigma_{pp} + N(D-1)\,\sigma_{xx}^2 \equiv A = \mathrm{const}$$
$$\partial_t^2 \sigma_{xx} = 2A - 6N(D-1)\,\sigma_{xx}^2$$

Motion in 1D cubic potential!!!

Exact solution: quantum X, maxsym

$$V\left(\sigma_{xx}\right) = 2N\left(D-1\right)\sigma_{xx}^3 - 2A\sigma_{xx}$$



$$E_{1D} = \frac{\left(\partial_t \sigma_{xx}\right)^2}{2} + V\left(\sigma_{xx}\right) =$$
$$= 2\sigma_{xp}^2 + V\left(\sigma_{xx}\right) =$$
$$= -2\left(\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2\right)$$

Larger than ¼,
(uncertainty relation)

- ✓ **Always periodic oscillations**
- ✓ **Always safely within potential well**
- ✓ **“Quantum” tunnelling only to *$X^2<0$*???**

One step further:
quantum X + quantum ψ, maxsym

$$\langle \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \hat{X}^c_\mu \rangle = C_{abc} [\sigma_\mu]_{\alpha\beta} [\psi\psi x],$$
$$\langle \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \hat{P}^c_\mu \rangle = C_{abc} [\sigma_\mu]_{\alpha\beta} [\psi\psi p]$$

$$\partial_t^2 \sigma_{xx} = 2A - 3\kappa \left(\sigma_{xx}^2 + [\psi\psi x] \right),$$
$$\partial_t^2 [\psi\psi x] = -\kappa \sigma_{xx} [\psi\psi x] - 1/4$$

- ✓ **Bounded solutions for small *[ψψx]***
- X **Otherwise $\sigma_{xx}(t)<0$, unphysical**
- X **No "evaporating" solutions $\sigma_{xx}(t) \sim t^2$**
- X **SUSY still not restored**

Summary

- Quantum fermions in BFSS model trigger real-time instability, "black hole evaporation"?
- Not all configurations "evaporate"
- Artificial acceleration at large distances
- Quantum fluctuations of X coordinates suppress instability
- Is "fine tuning" possible with realistic initial conditions?
- Faster "quantum" thermalization?