Multiplicative autocorrelation of the fractional part

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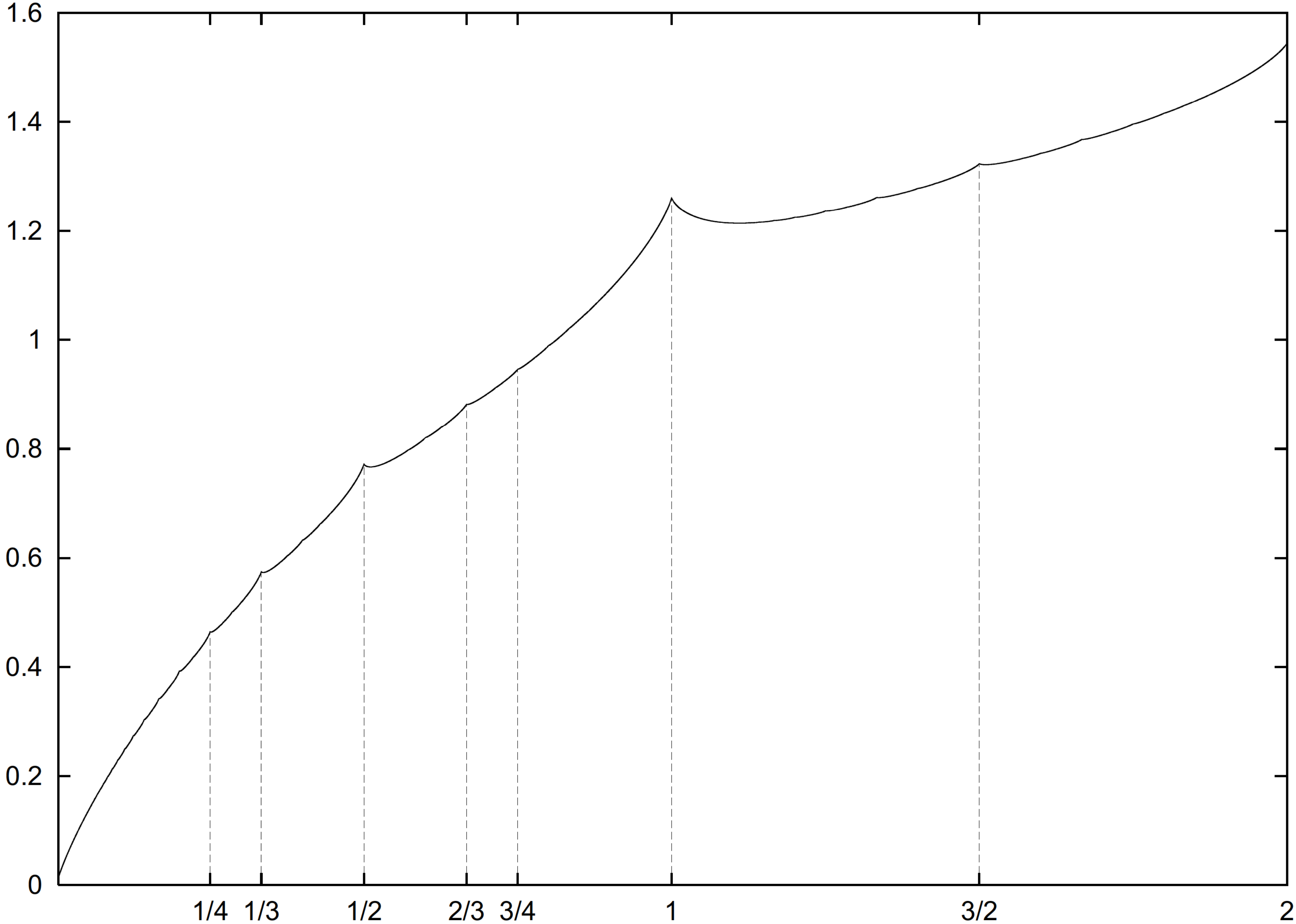
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1/4
1/3
1/2
2/3
3/4
1
3/2
2

$$A(\lambda) = \int_0^\infty \{t\}\{\lambda t\}\frac{dt}{t^2} \quad (\lambda \geqslant 0)$$

$$\{x\} = x - \lfloor x \rfloor \quad \in [0,1[$$

$$\lfloor x \rfloor = \max\,]-\infty, x] \cap \mathbb{Z}$$

(Báez-Duarte, B., Landreau, Saias, 2005)

Motivation

Study of the repartition of prime numbers.

$$\pi(x) = \sum_{p \leqslant x} 1$$

Some facts from a long history

Conjecture of Gauss and Legendre (1792-1808)

$$\pi(x) \sim \mathrm{li}(x) \quad (x \to \infty)$$

Proved by Hadamard and la Vallée-Poussin (1896).

Integral logarithm

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}$$

Asymptotic expansion

$$x\Big(\frac{0!}{\log x} + \frac{1!}{\log^2 x} + \frac{2!}{\log^3 x} + \cdots\Big)$$

Chebyshev ($\sim$ 1848-1852) :

Approximating $\pi(x)$ reduces to approximating

$$\chi(x) = \begin{cases} 0 & 0 < x \leqslant 1 \\ 1 & x \geqslant 1 \end{cases}$$

by linear combinations of $\lfloor x/n \rfloor$, n positive integer.

Example

$$\lfloor x\rfloor - 2\lfloor x/2\rfloor \leqslant \chi(x) \leqslant \lfloor x\rfloor - \lfloor x/2\rfloor - \lfloor x/4\rfloor - \lfloor x/8\rfloor - \cdots$$

$$\Longrightarrow$$

(by Chebyshev's method)

$$\log 2 + o(1) \leqslant \pi(x)/\mathrm{li}(x) \leqslant \log 4 + o(1) \quad (x\to\infty)$$

Remark

$$\lfloor x\rfloor - 2\lfloor x/2\rfloor = 2\{x/2\} - \{x\}$$

$$\lfloor x\rfloor-\lfloor x/2\rfloor-\lfloor x/4\rfloor-\lfloor x/8\rfloor-\cdots = -\{x\}+\{x/2\}+\{x/4\}+\{x/8\}+\cdots$$

Riemann (1859)

$$\pi(x) + \left(\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots\right) =$$

$$\mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} \quad (x > 1,\ x \neq p^{\nu})$$

(proved by von Mangoldt in 1895).

The ρ's are defined by

$$\zeta(\rho) = 0 \quad \text{and} \quad \rho \neq -2, -4, \dots$$

where

$$\zeta(s) = (\text{the analytic continuation of}) \sum_{n \geqslant 1} \frac{1}{n^s}.$$

Riemann's hypothesis (RH)

$$\zeta(\rho) = 0 \text{ and } \rho \neq -2, -4, \ldots \Rightarrow \Re\rho = \frac{1}{2}.$$

In terms of prime numbers, this is equivalent to

$$\pi(x) - \mathrm{li}(x) = O(x^{1/2} \log x) \quad (x \geqslant 2).$$

Linking Chebyshev's approach and Riemann's approaches : Nyman's criterion (1950)

$$(\mathrm{RH}) \;\Leftrightarrow\; \inf \int_0^\infty \Big|\chi(t) - \sum_{\alpha \geqslant 1} c_\alpha \{t/\alpha\}\Big|^2 \frac{dt}{t^2} = 0$$

The lower bound is on all finite linear combinations ($\alpha \geqslant 1$, real).

Báez-Duarte's criterion (2003)
Same as Nyman's, with α integer

$$(\mathrm{RH}) \Leftrightarrow \inf \int_0^\infty \Big|\chi(t) - \sum_{n\geqslant 1} c_n\{t/n\}\Big|^2 \frac{dt}{t^2} = 0$$

The lower bound is on all finite linear combinations (n positive integer).

Expanding the square in Nyman's criterion leads to integrals

$$\int_0^\infty \{t/\alpha\}\{t/\beta\}\frac{dt}{t^2} = \frac{1}{\alpha}A(\alpha/\beta) = \frac{1}{\beta}A(\beta/\alpha)$$

The study of A is motivated by the study of (RH) (via Nyman's criterion).

Known facts about A

$A(\lambda)$ for λ rational : Vasyunin's formula (1995)
$a,\ b$ positive integers, coprime, $\lambda = a/b$

$$A(\lambda) = \frac{1-\lambda}{2}\log\lambda + \frac{\lambda+1}{2}(\log 2\pi - \gamma) - \frac{\pi}{2b}\big(V(a,b) + V(b,a)\big)$$

where

$$V(a,b) = \sum_{k=1}^{b-1}\{ka/b\}\cot k\pi/b$$

(Vasyunin's sum).

Auxiliary functions

$$B_1(x) = \begin{cases} \{x\} - 1/2 & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$$

(first Bernoulli function, saw-teeth function)

$$\varphi_1(x) = \sum_{n \geqslant 1} \frac{B_1(nx)}{n}$$

(variant of a function considered by Riemann in his Habilitationschrift)

$$\alpha(x) = \{1/x\} \quad \text{(Gauss map)}$$

$$\alpha_k(x) = \alpha\big(\alpha_{k-1}(x)\big) \quad \text{(iterates)}$$

$$\gamma_k(x) = \alpha_0(x) \cdots \alpha_{k-1}(x) \log 1/\alpha_k(x)$$

$$\mathcal{W}(x) = \sum_{k \geqslant 0} (-1)^k \gamma_k(x) \quad (0 \leqslant x < 1)$$

$$\mathcal{W}(x) = \mathcal{W}(\{x\})$$

(Wilton 1933)

$$\mathcal{W}(x) = \sum_{k\geqslant 0}(-1)^k \frac{\log q_{k+1}(x)}{q_k(x)} + O(1)$$

$q_k(x)$: k^{th} convergent of the continued fraction of x.

The series $\mathcal{W}(x)$ absolutely converges a.e.

Set of convergence : Wilton numbers.

The Brjuno function

$$\Phi(x) = \sum_{k \geqslant 0} \gamma_k(x) \quad (0 \leqslant x < 1)$$

$$\Phi(x) = \Phi(\{x\})$$

Important in the theory of dynamical systems
Brjuno (1965), Yoccoz (1985)

The series $\varphi_1(x)$ and $\mathcal{W}(x)$ converge for the same values of x.
(la Bretèche, Tenenbaum, 2004)

Moreover, $\varphi_1(x) + \frac{1}{2}\mathcal{W}(x)$

- is bounded
- is continuous at every irrational
- has a left and right limit at every rational.

(B., Martin, work in progress)

Two functional relations link A and φ_1.

$$A(\lambda) + \varphi_1(\lambda) + \lambda\varphi_1(1/\lambda) = \frac{\lambda+1}{2}(\log 2\pi - \gamma) + \frac{\lambda-1}{2}\log 1/\lambda$$

(modular relation)

$$A(\lambda) = \frac{1}{2}\log\lambda + \frac{1-\gamma+\log 2\pi}{2} - \lambda\int_\lambda^\infty \varphi_1(t)\frac{dt}{t^2}$$

(integral relation)

(BD-B-L-S, 2005)

$a,\ b$ positive integers, coprime, $\lambda = a/b$

$$A(a/b+t) \sim \frac{|t|\log|t|}{2b} \quad (t \to 0)$$

(BD-B-L-S, 2005)

Open question
Are the rationals the only local maxima of A?

Study of the differentiability points of A

Theorem (B-M) : The differentiability points of A are precisely the Wilton numbers.

Preliminary study

Theorem (B-M, 2012) : The Lebesgue points of the Brjuno function are precisely the Brjuno numbers.