Pseudotoric structures and exotic lagrangian tori

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Let \((X, \omega)\) — be a symplectic manifold of real dimension \(2n\). We understand it as *the phase space of a classical mechanical system*. We are interested in the case of compact phase space. The main problem we have in mind — **Quantization** of such systems.

The main approach — lagrangian quantization:

· for \(\mathbb{R}^{2n}\), \(\omega = \sum dp \wedge dq\) — **V. Maslov**, *semiclassical approximation*;
· for \(T^*S\), \(d\alpha\) — **S. Dobrokhotov, A. Shafarevich**,
· for general compact \((X, \omega)\) — **N.T.** (*algebraic lagrangian geometry*)

Basic geometrical idea — lagrangian submanifolds in \(X\) look and behave like points (Darboux - Weinstein theorem) of an infinite dimensional variety, and any classical Hamiltonian function on \(X\) generates the corresponding dynamics on this variety.
Lagrangian geometry — questions about lagrangian submanifolds of $X$:

1) which homology classes from $H_n(X, \mathbb{Z})$ can be realized by smooth lagrangian submanifolds;

2) what are the topological types of these lagrangian submanifolds;

3) classification up to lagrangian deformations of lagrangian submanifolds of the same topological type and homology class;

4) classification up to Hamiltonian isotopy of lagrangian submanifolds of the same deformation type.

5) unification of all lagrangian submanifolds in an appropriate category
Recall that $S \subset X$ is lagrangian if

$$\omega|_S \equiv 0 \quad \text{and} \quad \dim S = n$$

Thus at least $[S]$ is perpendicular to $[\omega]$. Two lagrangian submanifolds $S_0, S_1 \subset X$ are of the same deformation type if there is a lagrangian film

$$S \subset X \times \mathbb{C}, \quad \omega \oplus dz \wedge d\bar{z}$$

such that $p(S \cap X \times \{i\}) = S_i, \quad i = 0, 1$.

Thus at least $[S_0] = [S_1]$ and $S_0 \simeq S_1$.

Hamiltonian isotopy of lagrangian submanifold $S_0 \subset X$ is given by a time dependent Hamiltonian function $H(x, t) : X \times \mathbb{R} \to \mathbb{R}$ which generates the flow $\phi_H^t$, and $S_t = \phi_H^t(S_0)$ is the corresponding isotopy.
**Toy example:** $\dim = 2$. Let $\Sigma$ be a Riemann surface equipped with a symplectic form. Then since every loop is lagrangian (dimensional reason):

1) every primitive homology class from $H_1(\Sigma, \mathbb{Z})$ is realizable by a smooth lagrangian submanifold;

2) every smooth lagrangian submanifold is isomorphic to $S^1$;

3) two loops from the same homology class are deformation equivalent;

4) two loops are Hamiltonian isotopic if the symplectic area of the oriented film bounded by the loops is zero;

5) the Fukaya category for a curve of any genus exists

**thus for this case the problem is completely solved**
**Example:** $\mathbb{CP}^2$. The projective plane is the simplest compact symplectic manifold in dimension 4:

1) since $H^2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$, any lagrangian submanifold must present trivial homology class;

2) vanishing results for 2- spheres (M. Gromov), riemann surfaces of genus $> 1$ (M. Audin), Klein bottle (S. Nemirovskiy, V. Shevchishin) — they are not realizable as lagrangian submanifolds;

3) — 4) it was believed that well known Clifford tori are unique examples of lagrangian tori in $\mathbb{CP}^2$ since in 1996 Yu. Chekanov proposed a construction of lagrangian torus which is not Hamiltonian isotopic to a Clifford torus — and nobody knows are there other types of lagrangian tori;

5) nevertheless certain constructions of appropriate categories exist (Fukaya - Seidel).

*thus even for this basic case in dimension 4 the problem is not solved yet*
Why we are interested in lagrangian geometry?

If we would like to proceed in the **lagrangian approach to Geometric Quantization** —

*there lagrangian submanifolds represent quantum states* — it is necessary to know all these states = all types of lagrangian submanifolds.

F.e. in **ALAG** the Chekanov result ensures that the moduli space of half weighted Bohr - Sommerfeld lagrangian cycles of level 3, $\mathcal{B}_{S,3}^{hw,r}$, has **at least two disjoint components**

*and may be there is a tunneling between these components?*

As well for **Homological Mirror Symmetry** — one should try to describe all objects in the Fukaya category, so all types of nonisotopic lagrangian tori.
Well known Clifford tori in $\mathbb{CP}^2$ comes from the toric geometry:
there are two real Morse functions $f_1, f_2$ in involution:

$$
f_1 = \frac{|z_1|^2 - |z_2|^2}{\sum_{i=0}^{2} |z_i|^2}, \quad f_2 = \frac{|z_0|^2 - |z_1|^2}{\sum_{i=0}^{2} |z_i|^2}, \quad \{f_1, f_2\}_\omega = 0
$$

in homogeneous coordinates $[z_0 : z_1 : z_2]$;
the degeneration set

$$
\Delta(f_1, f_2) = \{df_1 \wedge df_2 = 0\} \subset \mathbb{CP}^2
$$

is formed by three lines $l_i, l_i = \{z_i = 0\}$;
the action map $F = (f_1, f_2) : \mathbb{CP}^2 \to P_{\mathbb{CP}^2} \subset \mathbb{R}^2$ sends
$\Delta(f_1, f_2)$ to the boundary component $\partial P_{\mathbb{CP}^2}$, and the preimage of any inner point $p \in P_{\mathbb{CP}^2}$ is a smooth lagrangian torus, labeled by values of $f_1, f_2$.

*It is the standard picture for a toric manifold*
Exotic Chekanov tori — the first version for $\mathbb{R}^4$:

- fix a complex structure, so we have $\mathbb{C}^2$ with a coordinate system $(z_1, z_2)$;
- choose a smooth contractible loop $\gamma \subset \mathbb{C}^*$, which lies in a half plane so $\text{Re}\gamma > 0$;
- consider two-dimensional subset given in the coordinates by $(z_1, z_2) = (e^{i\phi}\gamma, e^{-i\phi}\gamma)$ — it is a lagrangian torus;

Remark. If $\gamma$ is not contractible, we get a standard torus.

since $\mathbb{CP}^2 \setminus I$ is symplectomorphic to an open ball in $\mathbb{R}^4$ one implements the construction to the projective plane;

and the last step:

- using Hofer's capacity technique, Chekanov proved that this torus is not equivalent to the standard one.

This torus is called the Chekanov torus; the forthcoming paper by Yu. Chekanov and F. Schlenk contains the details how to construct these nonstandard tori in $\mathbb{CP}^n$ for certain $n$, the products $S^1 \times \ldots \times S^1$, and some other cases.
An alternative description of the Chekanov tori based on the notion of **pseudotoric structure:**

- again we take $\mathbb{C}^2$ and consider pencil $\{Q_w\}$,
  
  \[ Q_w = \{z_1z_2 = w\} \subset \mathbb{C}^2 \text{ of quadrics}; \]

- take real Morse function $F = |z_1|^2 - |z_2|^2$;

- note that the Hamiltonian vector field $X_F$ of this function $F$ preserves each quadric $Q_w$ from the pencil;

- take a smooth contractible loop $\gamma' \subset \mathbb{C}^*_w$ where $\mathbb{C}_w$ parameterizes our pencil $\{Q_w\}$;

- on each quadric $Q_w$, $w \in \gamma'$, mark the level set
  
  \[ S_w = \{F = 0\} \cap Q_w \text{ which is a smooth loop}; \]

- collect these loops along $\gamma'$:

  \[ T(\gamma') = \bigcup_{w \in \gamma'} S_w, \text{ getting a torus} \]

  *it is not hard to see, that we again get the Chekanov torus from the previous slide*, if we put $\gamma = \sqrt[\gamma']$. 
Let us repeat the construction for the projective plane:
· consider pencil of quadrics \( \{ Q_p \} \), \( p \mapsto [\alpha : \beta] \subset \mathbb{CP}^1_{\alpha,\beta} \)
  
  \( Q_p = \{ \alpha z_1 z_2 = \beta z_0^2 \} \subset \mathbb{CP}^2 \); 

· consider real Morse function 
  \[ F = \frac{|z_1|^2 - |z_2|^2}{\sum_{i=0}^{2} |z_i|^2} \];

· note that its Hamiltonian vector field \( X_F \) preserves each element of the pencil;

· choose a smooth contractible loop \( \gamma \subset \mathbb{CP}^1_{\alpha,\beta} \setminus \{ [1 : 0], [0 : 1] \} \);

· on each quadric \( Q_p, p \in \gamma \) take the level set 
  \( S_p = \{ F = 0 \} \cap Q_p \) which is a smooth loop;

· collect the level sets \( S_p \) along the loop \( \gamma \)
  \[ T(\gamma) = \bigcup_{p \in \gamma} S_p \] getting again a lagrangian torus.

The resulting torus is exactly the Chekanov torus, given by the identification of symplectic ball in \( \mathbb{R}^4 \) and \( \mathbb{CP}^2 \setminus \text{line} \).

Another remark: if \( \gamma \subset \mathbb{CP}^1_{\alpha,\beta} \) is non contractible, then the resulting torus is equivalent to a Clifford torus.

Thus equivalence classes \( \Rightarrow \pi_1(\mathbb{CP}^1_{\alpha,\beta} \setminus \{ [0 : 1], [1 : 0] \}) \).
What is the difference between toric and pseudo toric considerations?

\[
\begin{array}{c|c|c}
\mathbb{R} & \text{real Morse function } f & \mathbb{C} \\
\| & f : X \to \mathbb{R} & \text{Lefschetz pencil } \{ Q_p \} \\
\downarrow & \text{toric case} & \downarrow \\
(f_1, f_2) \text{ on } \mathbb{CP}^2 & \text{such that } \{ f_1, f_2 \}_\omega = 0 & (f, \{ Q_p \}) \text{ on } \mathbb{CP}^2 \\
\uparrow & \text{standard commutation rel.} & \uparrow \\
\end{array}
\]

New commutation relation: pencil \( \{ Q_p \} \) commutes with real function \( f \) if the Hamiltonian vector field \( X_f \) is parallel to each element \( Q_p \) of the pencil at each point.
In other words, *pseudotoric structure* (of rank one) is a combination of

- real data \((f_1, \ldots, f_{n-1})\) — first integrals in involution
- complex data \(\{Q_p\}\) — a pencil of symplectic divisors, covering whole \(X\) s.t.

\[
\psi : X \setminus B \rightarrow \mathbb{CP}^1
\]

has generically smooth symplectic fibers

\[
Q_p = \overline{\psi^{-1}(p)} = \psi^{-1}(p) \cup B
\]

and \(H_{f_i}\) is parallel to \(Q_p\) at each point (for all \(i, p\))

Distinguished points \(p_1, \ldots, p_k \in \mathbb{CP}^1\) - singular fibers - form

\(D_{\text{Sing}} \subset \mathbb{CP}^1\)

- \(B \subset X\) is the base set of pencil \(\{Q_p\}\)
- \(Q_p, (f_i|_{Q_p})\) — toric manifold with the same convex polytop.
Now we have

**Theorem (S. Belyov, N.T.)** Let \((f_1, \ldots, f_{n-1}, \psi)\) be a regular pseudotoric structure of rank one on a compact symplectic manifold \(X\). Let \(S \subset \mathbb{CP}^1\) be a smooth lagrangian torus which doesn’t pass through \(p_i\). Then the choice of non critical values \((c_1, \ldots, c_{n-1})\) of \(f_1, \ldots, f_{n-1}\) defines a smooth lagrangian torus \(T(S, c_1, \ldots, c_{n-1}) \subset X\).

Thus we get a correspondence

\[ H_1((\mathbb{CP}^1 \setminus D_{\text{Sing}}), \mathbb{Z}) \rightarrow \text{different types of lagrangian tori} \]

For example, coming back to \(\mathbb{CP}^2\), Clifford and Chekanov tori:

- **Clifford type = primitive elem.**

\[ H_1(\mathbb{CP}^1 \setminus ([1 : 0], [0 : 1]), \mathbb{Z}) \]

- **Chekanov type = trivial elem.**
This hints how to construct non standard lagrangian tori in toric symplectic manifolds in view of the following

**Theorem (S. Belyov, N.T.):**
1. Any smooth compact toric symplectic manifold admits regular pseudotoric structure \((f_1, \ldots, f_{n-1}, \psi, \mathbb{C}P^1)\) of rank one.
2. For this structure the singular divisor \(D_{\text{sing}} \subset \mathbb{C}P^1\) consists of exactly two distinct points, \(p_N, p_S \subset \mathbb{C}P^1\).
3. The primitive and the trivial elements of \(H_1(\mathbb{C}P^1 \setminus (p_N \cup p_S), \mathbb{Z})\) generates lagrangian tori of the standard type and of the Chekanov type respectively.

Suppose additionally that our given toric \((X, \omega_X)\) is monotone, so

\[ K_X = k[\omega_X] \subset H^2(X, \mathbb{Z}) \] — f.e. Fano varieties in AG — then
1. if there is a standard monotone lagrangian torus then there exists a monotone lagrangian torus of the Chekanov type.

**Main conjecture:** these monotone tori are not Hamiltonian isotopic.
Outline of the proof:

· take for a given toric $X$ the set of commuting Morse moment maps $(f_1, \ldots, f_n)$, which give the action map by “action coordinates” $F = (f_1, \ldots, f_n) : X \to P_X$ to convex moment polytop $P_X \subset \mathbb{R}^n$;

· for the components $D_i$ of the boundary divisor $D = F^{-1}(\partial P_X)$ find an integer combination $\sum \lambda_i D_i$ equals to zero;

· rearrange this to the form $\sum_{\lambda_i > 0} \lambda_i D_i = \sum_{\lambda_j < 0} |\lambda_j| D_j$, $D_i \neq D_j$, thus we have two divisors from the same linear system $D_+ = \sum_{\lambda_i > 0} \lambda_i D_i$, $D_- = \sum_{\lambda_j < 0} |\lambda_j| D_j \in |\sum_{\lambda_i > 0} \lambda_i D_i|$;

· take the pencil $< D_+, D_- >$ with the base set $B = D_+ \cap D_-$ — it is our pencil $\psi$, and for generic point $p \in \mathbb{C}P^1$, $p \neq [1 : 0](\mapsto D_+), [0 : 1](\mapsto D_-)$, the divisor $\psi^{-1}(p) \subset X$ is smooth outside the base set $B$;

· the same linear combination $\sum \lambda_i D_i$ after substitution of linear forms $l_i$ which correspond to $D_i$ in $\mathbb{R}^n$ gives a linear relation on $x_i$ — and this relation derive our real data $f'_1, \ldots, f'_{n-1}$ from $f_1, \ldots, f_n$. 
Example: $\mathbb{CP}_3^2$ — del Pezzo surface of degree 6 can be realized in the direct product $\mathbb{CP}_x^2 \times \mathbb{CP}_y^2 \supset U = \{x_0y_0 = x_1y_1 = x_2y_2\}$ with the projection $p_x : U \to \mathbb{CP}_x^2$, $p_x(x_i, y_j) = [x_0 : x_1 : x_2]$. $p_x^0 : U \setminus \text{three lines} \simeq \mathbb{CP}_x^2 \setminus \text{three points}$, but $(p_x^0)^{-1}(T_{Ch}) \subset U$ is not lagrangian — we can’t lift the Chekanov torus, but we can lift the corresponding pseudotoric structure!

· take the pencil $\{Q_{\alpha, \beta}\} = \{\alpha x_0x_1y_2 = \beta x_2^2y_0y_1\} \subset \mathbb{CP}_x^2 \times \mathbb{CP}_y^2$, and the intersections $Q_{\alpha, \beta} \cap U$ gives the Lefschetz pencil $\psi$ on $U$;

· the real Morse function $F = \frac{|x_0|^2 - |x_1|^2}{\sum_{i=0}^2 |x_i|^2} + \frac{|y_1|^2 - |y_0|^2}{\sum_{i=0}^2 |y_i|^2}$ preserves by the Hamiltonian action $U$ and each element $Q_{\alpha, \beta}$ of the pencil, and the restriction $f = F|_U$ gives the real data;

· the choice of a smooth loop $\gamma \subset \mathbb{CP}_1^1 \setminus ([1 : 0], [0 : 1])$ gives a lagrangian torus $T(0, \gamma) = \bigcup_{p \in \gamma} \{f|_{\psi^{-1}(p)} = 0\}$, and if $\gamma$ is contractible, we get a Chekanov torus in $\mathbb{CP}_3^2$. 
Another usage of pseudotoric structure — in construction of **special lagrangian fibrations** on Fano varieties.

D. Auroux, an approach to Mirror Symmetry conjecture: 

\((X, l, \omega, g) — \text{Kahler manifold, } |K_X^{-1}| \supset D\)

\(D \mapsto \Theta_D — \text{holomorphic form with pole along } D.\)

Lagrangian fibration \(\pi: X \setminus D \to B\) is said to be *special* if the proportionality coefficient \(\rho\) from

\[ \Theta_D|_{\pi^{-1}(p)} = \rho \text{Vol}(g|_{\pi^{-1}(p)}) \]

has the same phase: \(\text{Arg}\rho = \text{const}\) for each \(p \in B.\)

**Example:** standard toric fibration.

\(X\) with collection of Morse commuting moment maps \((f_1, \ldots, f_n)\)

with the degeneration locus \(\Delta(f_1, \ldots, f_n) = D \in |K_D^{-1}|\)

The corresponding form \(\Theta_D\) is preserved by the moment maps, so

\[ \Theta_D(X_{f_1} \wedge \ldots X_{f_n}) = \text{const on } X \setminus D \]

but essentially this constant is our \(\rho.\)

**Question:** what about another elements from \(|K_X^{-1}|?\)

**Auroux’s conjecture for \(\mathbb{CP}^2\):** each \(D \in |3H|\) is realizable.
Example: the flag variety. Take $F^3$ — full flag in $\mathbb{C}^3$, realize it as $\mathcal{U} \subset \mathbb{CP}^2_x \times \mathbb{CP}^2_y$, given by the equation $\sum_{i=0}^3 x_i y_i = 0$.

**Pseudotoric structure** on $\mathcal{U}$: two real Morse functions

$$f_1 = \frac{|x_0|^2 - |x_1|^2}{\sum |x_i|^2} + \frac{|y_1|^2 - |y_0|^2}{\sum |y_i|^2}, \quad f_2 = \frac{|x_1|^2 - |x_2|^2}{\sum |x_i|^2} + \frac{|y_2|^2 - |y_1|^2}{\sum |y_i|^2};$$

Lefschetz pencil $\psi : \mathcal{U} \to \mathbb{CP}^1$ given by

$$\psi([x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2]) = [x_0 y_0 : x_1 y_1 : x_2 y_2], \sum_{i=0}^3 x_i y_i = 0.$$

The base set $B \subset \mathcal{U}$ is a hexagon, general element of the pencil is toric del Pezzo surface $\mathbb{CP}^2_3$; three singular elements correspond to points $[1 : -1 : 0], [1 : 0 : -1], [0 : 1 : -1] \in \mathbb{CP}^1$ have the form $\mathbb{CP}^2_2 \cup \mathbb{CP}^2_2$ with intersection along a diagonal of hexagon $B$.

Now take a Morse function $h$ on $\mathbb{CP}^1$ which preserve the Kahler structure by the Hamiltonian action and which has critical points at $p_1 = [1 : -1 : 0]$ and $p_2 = [1 : 0 : -1]$. 
Then
  · we get a lagrangian fibration on $U \setminus D_1 \cup D_2$ where $D_i = \psi^{-1}(p_i)$ has the type $\mathbb{CP}^2 \cup \mathbb{CP}^2$;
  · in the fibration there is a 1-dimensional subfamily of singular lagrangian tori while generic fiber is smooth;
  · the boundary divisor $D_1 \cup D_2$ lies in the anticanonical system $|K_U^{-1}|$;
  · and this fibration is special.

In contrast with the previous examples: $F^3$ is not toric, but it admits pseudotoric structure. It is natural to call such a manifold pseudotoric since it carries a lagrangian fibration which looks similiar to the standard toric lagrangian fibrations. Another examples of pseudotoric manifolds are complex quadrics and certain complete intersections in $\mathbb{CP}^n$; it is reasonable to ask: which symplectic manifolds are pseudotoric?