Dubna, Dec 2012

Superstrings and Supergeometry

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Summary of the Talk

- Pure Spinor Superstring and Supergeometry
- Amplitudes and Integration Forms
- Integral Forms and Cartan Calculus
- Čech Cohomology and de Rham Cohomology
- Applications

Pure Spinor String Theory

We start from the fields $x^m, \theta^\alpha, p_\alpha$ (θ^α are Majorana-Weyl spinors in $d=(9,1)$) and the free field action

$$S = \int d^2z \Big(\partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial}\theta^\alpha + \hat{p}_\alpha \partial \hat{\theta}^\alpha\Big)$$

i) The total conformal charge is $c_T = (10)_x + (-32)_{p,\theta}$

ii) Inserting $p_\alpha = p^*_\alpha \equiv \frac{1}{2}\partial x_m \gamma^m_{\alpha\beta}\theta^\beta + \frac{1}{8}(\gamma^m_{\alpha\beta}\theta^\beta)(\theta\gamma_m\partial\theta)$ in S,

$$S|_{p=p^*} = S_{Green-Schwarz}$$

iii) So, $d_\alpha \equiv p_\alpha - p^*_\alpha \approx 0$ must be identified with the fundamental constraint.

$$Q = \oint dz \lambda^\alpha d_\alpha$$

By requiring the nilpotency of Q

$$\{Q,Q\} = \oint \lambda^\alpha(z) \oint \lambda^\beta(w) d_\alpha(z) d_\beta(w) = \oint \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta \Pi_m = 0$$

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$$

Pure Spinor Constraints

The space of the zero modes for the ghost fields for the Pure Spinor String theory is

- Conifold Space with Base SO(10)/U(5)
- It is a local Calabi-Yau with conifold singularity
- Integration Measure for this space?

Viewed from String Amplitude computations:

Vertex operators: $\int dz d\bar{z} \mathcal{V}^{(0,0)}_{z\bar{z}}$ $\oint dz \mathcal{V}^{(1,0)}_{z}$, $\oint d\bar{z} \mathcal{V}^{(0,1)}_{\bar{z}}$ and $\mathcal{U}^{(1,1)}$

$$\langle\langle \quad \mathcal{U}^{(1,1)}_1 \mathcal{U}^{(1,1)}_2 \mathcal{U}^{(1,1)}_3 \prod_{j=1}^{n} \int dz_j d\bar{z}_j \mathcal{V}^{(0,0)}_j \quad \rangle\rangle$$

To compute these amplitudes we need to perform the OPE contractions until we get to the following expression

$$\langle \mathcal{M} \rangle = \int d^{16}\theta_0 d^{10} x_0 \mathcal{D}\lambda_0 \mu(\theta_0, \lambda_0) \mathcal{M}(x_0, \theta_0, \lambda_0)$$

To integrate over these variables we need to define a suitable measure

for that we need a good understanding of the integration on supermanifolds

Multiloop Amplitudes

The complete N-point g-loop amplitudes can be written in this way

$$\int_{\mathcal{M}_g} dm_i \int [d\mu_\lambda][d\theta][dx] \int \prod_{l=1}^{g} [d\mu_w][dd_l] \prod_{n=1}^{3g-3} \int \mu(z_n) b_B(z_n) \prod_{k=1}^{r} \mathcal{U}^{(1)}(z_k) \times$$
$$\prod_{i=3g-3+1}^{11g} Z(B_{mn} N_i^{mn}) Z(J_i) \prod_{j=1}^{11} Y_{C^j} \prod_{p=1}^{N} \int d\tau_p \mathcal{V}_p^{(0)}$$

where we used

$$Z(X) = \Big[Q, \Theta(X)\Big] = \Big[Q, X\Big]\delta(X)$$

Picture Raising Operators

$$Y_{C^i} = C^i_\alpha \theta^\alpha \delta(C^i_\alpha \lambda^\alpha)$$

Picture Lowering Operators

$$\Big\{Q, b_X\Big\} = Z(X)T$$

Viewed from String Field Theory perspective

$$S = \int d^{10}x_0 d^{16}\theta_0 \mathcal{D}\lambda_0 \mu(\lambda_0, \theta_0)\Big(\Phi^{(1)} Q^{(1)} \Phi^{(1)} + \ldots\Big)$$

Properties:

1) BRST invariance
2) SUSY
3) Saturation of zero modes
   (bosonic and fermionic zero modes)

The result is the following

$$\mathcal{D}\lambda_0 = d\lambda^{\alpha_1} \wedge \ldots \wedge d\lambda^{\alpha_{11}} \epsilon_{\alpha_1 \ldots \alpha_{16}} (\gamma^m \gamma^n \gamma^p \gamma_{mnp})^{[\alpha_{12} \ldots \alpha_{16}](\beta_1 \ldots \beta_3)} \frac{\partial}{\partial \lambda^{\beta_1}} \cdots \frac{\partial}{\partial \lambda^{\beta_3}}$$

$$\mu(\lambda_0, \theta_0) = \prod_{i=1}^{11} (C^i_\alpha \theta^\alpha) \delta(C^i_\alpha \lambda^\alpha)$$

Integration on Manifolds

The usual integration on standard manifolds can be viewed as follows

There is a map between

$$\omega \in \Omega^\bullet(\mathcal{M}) \quad \Longrightarrow \quad \mathcal{C}^\infty\left(\widehat{\mathcal{M}}\right) = \Omega^\bullet(\mathcal{M})$$

by identifying the forms with anticommuting variables

$$dx^i \to \theta^i$$

Then, on the new space we can integrate functions $\widehat{\omega} \in \mathcal{C}^\infty\left(\widehat{\mathcal{M}}\right)$

Since there is a natural measure $\widehat{\mu} = dx^{i_1} \wedge \ldots \wedge dx^{i_n} \wedge d\theta^{i_1} \wedge \ldots \wedge d\theta^{i_n}$

which gives

$$\int_{\widehat{\mathcal{M}}} \widehat{\omega} = \int_{\mathcal{M}} \omega$$

Complexes of Superforms

Given 1-superforms, we have commuting variables $d\theta^i \wedge d\theta^j = d\theta^j \wedge d\theta^i$

So the complex

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} \cdots$$

is infinite. There is no top form.

- How we can define the top form?
- How we can define a sensible integration theory?

It is convenient to introduce a new basic object

$$\delta(d\theta^i)$$

The new object has the following properties

If we denote by dx^I the 1-forms associated to the commuting coordinates of the manifold

$$dx^I \wedge dx^J = -dx^J \wedge dx^I, \qquad dx^I \wedge d\theta^j = d\theta^j \wedge dx^I,$$

$$d\theta^i \wedge d\theta^j = d\theta^j \wedge d\theta^i, \qquad \delta(d\theta) \wedge \delta(d\theta') = -\delta(d\theta') \wedge \delta(d\theta),$$

$$d\theta\,\delta(d\theta) = 0, \qquad d\theta\,\delta'(d\theta) = -\delta(d\theta).$$

the last three equations follow from the usual definition of the Dirac delta function and from the changing-variable formula for Dirac delta's

$$\delta(ax+by)\delta(cx+dy) = \frac{1}{\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\delta(x)\delta(y)$$

Then, we can introduce the new set of forms

$$dx^{[K_1} \cdots dx^{K_l]} d\theta^{(i_{l+1}} \cdots d\theta^{i_r)} \delta(d\theta^{[i_{r+1}}) \cdots \delta(d\theta^{i_{r+s}]})$$

and the complexes

$$\cdots \xrightarrow{d} \Omega^{(r|q)} \xrightarrow{d} \Omega^{(r+1|q)} \cdots \xrightarrow{d} \Omega^{(p+1|q)} \xrightarrow{d} \cdots$$

Form Number

Picture Number

$$\Omega^{(p+1|q)}$$

Notice that the picture number corresponds to the number of Delta functions, while the form number corresponds to the total form degree. It can be negative by considering the derivatives of Delta functions.

$$\int_{\mathbb{C}^{(p+1|q)}} \omega^{(p+1|q)} = \epsilon^{i_1 \cdots i_q} \partial_{\theta^{i_1}} \cdots \partial_{\theta^{i_q}} \int_{\mathbb{C}^{p+1}} f(x, \theta)$$

Čech cohomology of $\mathbb{P}^{1|1}$

- $\mathbb{P}^{\mathbf{1|1}}$ is the simplest non-trivial example of super-projective space. It is defined as usual as an algebraic variety by quotient of the complex superspace $\mathbf{C}^{2|1}$ with respect to a complex number different from zero. It can be covered by two patches

Transition functions from one patch to another

$$U_0 = \{[z_0; z_1] \in \mathbb{P}^1 : z_0 \neq 0\},$$
$$U_1 = \{[z_0; z_1] \in \mathbb{P}^1 : z_1 \neq 0\}.$$

$$\Phi^*(\gamma) = \frac{1}{\widetilde{\gamma}}, \qquad \Phi^*(\psi) = \frac{\widetilde{\psi}}{\widetilde{\gamma}}.$$

The change of patch reflects upon the following transformation

$$\Phi^* \delta^n(d\widetilde{\psi}) = \gamma^{n+1} \delta^n\,(d\psi) - \gamma^n \psi\, d\gamma\; \delta^{n+1}(d\psi).$$

Results

Per interi n non negativi

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong \mathbb{C}^{4n+4},$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong 0.$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \mathbb{C}^{4n}$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong 0,$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}.$$

Notice that $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0})$ and $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1})$ have the same dimension.

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0}) \times \check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \to \check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}$$

Defined in terms of the this pairing

$$\langle d\gamma\ (d\psi)^n, \delta^n(d\psi)\rangle = (-1)^n n!\, d\gamma\ \delta(d\psi),$$

$$\langle (d\psi)^{n+1}, d\gamma\ \delta^{n+1}(d\psi)\rangle = -\ (-1)^n (n+1)!\, d\gamma\ \delta(d\psi),$$

$$\langle d\gamma\ (d\psi)^n, d\gamma\ \delta^{n+1}(d\psi)\rangle = \langle\ (d\psi)^{n+1}, \delta^n(d\psi)\rangle = 0.$$

Super de Rham cohomology

1. d behaves as a differential on functions;
2. $d^2 = 0$;
3. d commutes with δ and its derivatives, and so $d(\delta^{(k)}(d\psi)) = 0$.

For $n \geq 0$, the holomorphic de Rham cohomology groups of $\mathbb{P}^{1|1}$ are as follows:

$$H^{n|0}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H^{-n|1}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H^{1|1}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong 0.$$

In this computation $H^{0|1}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol})$ is generated by the constant sheaf

$$\psi\delta(d\psi)$$

Two Theorems

Given a supermanifold $M^{n|m}$, for $i \geq 0$ we have the following isomorphism

$$H^{i|j}_{\mathrm{DR}}(M^{n|m}) \cong \check{H}^i(M^{n|m}, \mathcal{H}^{0|j}).$$

where the constant sheaf is generated by the integral forms extended globally

$$\psi\delta(d\psi)$$

which is the analogous of the Čech-de Rham isomorphism.

Let $M^{n|m}$ be a supermanifold, such that $\mathcal{H}^{0|j}$ is a constant sheaf
Then the de Rham cohomology of $M^{n|m}$ is

$$H^{*|j}_{\mathrm{DR}}(M^{n|m}) = H^{*}_{\mathrm{DR}}(M) \otimes \mathcal{H}^{0|j}.$$

Now we can use the above geometrical setting in superstring amplitudes -- we define the PCO as follows

$$\mu(\lambda_0, \theta_0) = \prod_{i=1}^{11} (C^i_\alpha \theta^\alpha) \delta(C^i_\alpha \lambda^\alpha)$$

1) -- BRST invariant
2) -- Any change of the gauge paramenters C^i_α: $\delta_{C_\alpha} \mu = \{Q, \theta^\alpha C_\beta \theta^\beta \delta'(C_\gamma \lambda^\gamma)\}$
3) -- It changes the picture, but
what is the picture in this context?

Given a vertex operator $\mathcal{U}^{(n)} = \lambda^{\alpha_1} \dots \lambda^{\alpha_n} A_{\alpha_1 \dots \alpha_n}(x, \theta)$

$$(and \ \ \{Q, \theta^\alpha\} = \lambda^\alpha)$$

Using the usual "dictionary" $Q \leftrightarrow d$ and $\lambda^\alpha \leftrightarrow d\theta^\alpha$ we have

$$\mathcal{U}^{(n)} = d\theta^{\alpha_1} \dots d\theta^{\alpha_n} A_{\alpha_1 \dots \alpha_n}(x, \theta)$$

Path integral on zero modes

$$\int d^n x_0 d^m \theta_0 \mathcal{D}\lambda_0 \prod_k \mathcal{U}^{(k)}$$

Theory Integration
of Superforms on Supermanifold
(Baranov-Schwarz-Bernstein-Leites-Voronov)

- For a bosonic manifold $\mathcal{M}^{(n)}$,$\exists$ a top form ω_n such that any diff. of the manifold $\omega'_n = \mathrm{Det} J \omega_n$, namely it transforms as a measure.

- For a superspace $\mathcal{M}^{(n|m)}$,$\nexists$ a top form in the space of superforms that transforms as Berezinian

The insertion of the cohomology sheafs

$$\theta_0^i \delta(d\theta_0^i)$$

solves all problems.

Cartan Calculus

$$d = d\theta^\alpha D_\alpha + (dx^m + \theta\gamma^m d\theta)\partial_m$$

Even/Odd Vector fields:

$$v = v^\alpha D_\alpha + v^m \partial_m$$ with v^α odd/even, v^m even/odd

Even	$\iota_v \,,$	$\iota_v^2 = 0 \,,$	$\mathcal{L}_v = d\iota_v + \iota_v d$
Odd	$\iota_{\tilde{v}} \,,$	$\iota_{\tilde{v}}^2 \neq 0 \,,$	$\mathcal{L}_{\tilde{v}} = d\iota_{\tilde{v}} - \iota_{\tilde{v}} d$

Finally,

$$\delta(\iota_{\tilde{v}}) = \int_{-\infty}^{\infty} dt\, e^{it\iota_{\tilde{v}}}$$

$$\Gamma_{\tilde{v}} = \Big[d, \Theta(\iota_{\tilde{v}})\Big]$$

The last equation translates into the PCO operators for multiloop string computations

$$Z(X) = \Big[Q, \Theta(X)\Big] = \Big[Q, X\Big]\delta(X)$$

where X is a generic field constructed in terms of zero modes

★The same analysis can be also performed for RNS superstring where the there are 2-d super-Riemann surfaces

★It is useful for twistor string theory to explain the measure for the open sector of the theory

★It is important for Non-Renormalization theorems in 10d and 11d (for supermembranes and topological theories)

Our Results

- Rigorous formulation of integration theory for superforms
- Derivation of Cech de Rham isomorphisms for supermanifolds
- Thom Class and Integration on Sub-Supermanifolds (with M. Marescotti)
- SuperBalanced Varieties and Properties
- Dualities for Super-Cohomologies
- Embedding of a bosonic manifold in a Super-Calabi-Yau manifolds and topological strings
- Pure Spinor Superstring Perturbation theory and multiloop computations (with P.Vanhove).
- Fermionic T-dualities and Super-Cohomologies (Sigma models computations)

Conclusions

- Supergeometry is a powerful tool for several applications, here I used it to motivate the construction of multiloop amplitudes in Pure Spinor Superstrings

- Recent developments by E. Witten for string perturbation theory in RNS formalism

- Applications to Super-Calabi-Yau as the coset manifold PSU(4|4)/SU(3|4) related to AdSxS spaces.