Some diophantine conditions in Teichmüller dynamics

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Paradigm (I): Rotations on the circle.

$R_\alpha : [0,1) \to [0,1)$, $x \mapsto x + \alpha \mod \mathbf{Z}$.
-Diophantine condition: existence of infinitely many solutions of

$$|R_\alpha^n(x) - y| < \varphi_n.$$

-Define $\mathcal{R}(T, y, \varphi) := \bigcap_{n \in \mathbf{N}} \bigcup_{i > n} R_\alpha^{-i}\big(B(y, \varphi_i)\big)$.

-In *parameter space*: diophantine condition on the *rotation number* α. For $x = y = 0$ we get

$$|q\alpha - p| < \varphi_q.$$

Thm (Khinchin)
$\sum \varphi_n < +\infty \Rightarrow$ *finitely many solutions a.e.* α.
$\sum \varphi_n = +\infty \Rightarrow$ *infinitely many solutions a.e.* α.

-In *phase space*: diophantine condition on pairs (x, y) in $[0,1)^2$.

Thm (Kurzweil)

-If $\sum \varphi_n = +\infty \Rightarrow \text{Leb}\big(\mathcal{R}(\alpha, y, \varphi)\big) = 1$ *a.e.* α *and any* y.
-α *bonded type* $\Leftrightarrow R_\alpha$ *has the MSTP.*

-MSTP: $\text{Leb}\big(\mathcal{R}(\alpha, y, \varphi)\big) = 1$ for any y and any monotone decreasing φ_n with $\sum \varphi_n = +\infty$.

-α bounded type: $\sup a_n < \infty$, in terms of the continued fraction

$$\alpha = a_0 + [a_1, a_2, \ldots].$$

-Equivalently there is a constant $L(\alpha) > 0$ such that

$$L(\alpha) = \limsup_{q,p \to \infty} \frac{1}{q|q\alpha - p|}.$$

-*Lagrange spectrum*: the set $\mathcal{L}$ of values $L(\alpha)$, where α is of bounded type.
-Remark: $L(\alpha) = +\infty$ for a.e. α.

Paradigm (II) flat tori

-Lattice $\Lambda = \zeta_A \mathbf{Z} \oplus \zeta_B \mathbf{Z} \leftrightarrow$ flat torus $X = \mathbf{C}/\Lambda$.

-*Phase space dynamics:* vertical flow f^t_Λ on Λ.

-Standard basis (ζ_A, ζ_B) of Λ. Horizontal interval $I := (0, \Re(\zeta_A + \zeta_B))$ embedded in Λ. The first return of f^t_Λ to I is a rotation $R_\alpha : I \to I$, whose rotation number is $\alpha := \frac{\Re(\zeta_B)}{\Re(\zeta_A + \zeta_B)}$.

-*Dynamics in parameter space:*

$$\mathrm{SL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{Z}) \leftrightarrow T^{(1)}\mathbf{H}/\mathrm{PSL}(2, \mathbf{Z})$$

$$\zeta_A \mathbf{Z} \oplus \zeta_B \mathbf{Z} \leftrightarrow (\zeta_A/\zeta_B, \arg(\zeta_A)),$$

($\zeta_B =$ shortest period, $\zeta_A =$ second shortest.)

-Action of $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \leftrightarrow$ geodesic flow g_t.

-Fundamental domain Ω in $\mathrm{SL}(2, \mathbf{R})$. The first return map $g_t : \Omega \to \Omega$ corresponds to the continued fraction on α.

Diophantine conditions $\leftrightarrow$ *cusp excursions:*

-Let $h(\Lambda)$ be the hyperbolic height of Λ and $\mathrm{Sys}(\Lambda) := \min\{|v|; v \in \Lambda^*\}$ be the *systole*. Remark: $h(\Lambda) = -\ln\left(\mathrm{Sys}(\Lambda)^2\right)$.

-Khinchin theorem $\Rightarrow$ Logarithmic law:

$$\limsup h(g_t\Lambda)/\ln t = 1/2.$$

-$L(\alpha) < +\infty \Leftrightarrow g_t\Lambda$ is bounded. More precisely

$$L(\alpha) = \limsup_{t\to\infty} \frac{2}{\mathrm{Sys}^2(g_t\Lambda)} = \limsup_{t\to\infty} 2e^{h(g_t\Lambda)}.$$

-The *Lagrange spectrum* encodes an information on the distribution on bounded geodesics.

i) $\min \mathcal{L} = \sqrt{5} = L(\frac{\sqrt{5}-1}{2})^{-1}$ (Dirichlet)
ii) $\mathcal{L} \cap (0,3)$ is discrete. (Markov).
iii) $\mathcal{L}$ is closed and equal to the closure of $\{L(\beta)^{-1}; \beta \text{ quadratic irrational }\}$. (Cusick)
iv) Exists $c > 0$ such that $[c, +\infty) \subset \mathcal{L}$. (Hall)
v) $t \mapsto \dim_H \mathcal{L} \cap (0,t)$ is continuous (and Cantor staircase-like). (Moreira)

Translation Surfaces. Equivalent definitions:

i) $X = P/\partial P$, where P is a polygon with sides in ∂P pairwise parallel, with same length and same orientation.
ii) $X = (S, w)$, where S compact Riemann surface, w holomorphic one form.
iii) Flat surface X with conical singularities whose angles are multiple of 2π.

-*Phase space dynamics:* vertical flow f_X^t on X.

-IETs.
-Let $\zeta_A, \zeta_B, \zeta_C, \zeta_D$ be the sides of P.
-Horizontal interval $I := (0, \Re(\zeta_A + \cdots + \zeta_D))$ embedded in X.
-The first return map $T : I \to I$ of f_X^t to I is an IET.

Parameter space: fix positive integers g, r and $k_1, \dots, k_r$ with $k_1 + \cdots + k_r = 2g - 2$.

-Stratum: the set $\mathcal{H}$ of $X = (S, w)$ where w has zeros of order $k_1, \dots, k_r$, modulo $(S, w) \sim (S', w')$ iff $\exists f : S \to S'$ bi-holomorphic with $w' = f_* w$.

-Local coordinates: $\zeta_A, \dots, \zeta_D$, that is a family of sides in ∂P which forms a basis for $H_1(X, \Sigma, \mathbf{Z})$, where $\Sigma =$ zeroes of w.

-Complex orbifold with $\dim_{\mathbf{C}} = 2g + r - 1$. Not compact.

-*Saddle connection* for X: geodesic segment $\gamma : [0, 1] \to X$ with $\gamma^{-1}(\text{ zeros }) = \{0, 1\}$. $\mathrm{Hol}(X)$: the set of *periods* $v = \int_\gamma w$ of saddles connections γ of X.

$$\mathrm{Sys}(X) := \min\{|v|; v \in \mathrm{Hol}(X)\}.$$

-*Malher criterion:* $\mathcal{K} \subset \mathcal{H}$ is compact iff exists $\epsilon > 0$ such that $\mathrm{Sys}(X) \geq \epsilon$ for any $X \in \mathcal{K}$.

Action of $\mathrm{SL}(2,\mathbf{R})$ in parameter space.

-For any X take a polygon P with $X = P/\partial P$.

-Define $A \cdot X := AP/\partial AP$. It does not depend on P but just on X and A.

-The *Teichmüller flow* $\mathcal{F}_t$ on $\mathcal{H}(k_1,\dots,k_r)$ is the action of $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

Thm. (Masur,Veech): *There exists a unique $\mathcal{F}_t$-invariant smooth measure μ on the unit-area locus $\mathcal{H}^{(1)}$. Moreover μ is finite and ergodic on connected components.*

Continued fraction for IETs

-*Parameter space of IETs*: let $\mathcal{A} = \{A, B, C, D\}$ be a finite alphabet with $2g + r - 1$ elements.

$$T = (\pi, \lambda) \in \mathrm{Sym}(\mathcal{A}) \times \mathbf{R}_+^{\mathcal{A}}$$

Veech: $\mathcal{F}_t$ has a fundamental domain Ω in

$$\mathrm{Sym}(\mathcal{A}) \times (\mathbf{R}_+)^{\mathcal{A}} \times \mathbf{R}^{\mathcal{A}}.$$

Rauzy-Veech-Zorich: The first return map $\mathcal{F}_t : \Omega \to \Omega$ projects to a piecewise linear-projective map

$$Q : \mathrm{Sym}(\mathcal{A}) \times \Delta \to \mathrm{Sym}(\mathcal{A}) \times \Delta$$

which generalizes the *Gauss map* (Δ is the standard simplex in $\mathbf{R}_+^{\mathcal{A}}$.). It has an unique finite, smooth (analytic density) and $Q-$invariant measure.

Diophantine conditions. Let φ_n be a positive decreasing sequence/function.

-IETs: for points x, y in the interval T where T acts we set

$$|T^n(x) - y| < \varphi_n.$$

-Translation Surfaces: for points p, p' in X we set

$$|f_X^t(p) - p'| < \varphi_t.$$

-*Phase space:* fix a translation surface X and let f_θ be the flow in direction θ. Set

$$\mathcal{R}(f_\theta, p, \varphi) := \bigcap_{t>0} \bigcup_{s>t} f_\theta^{-s}\big(B(p, \varphi_s)\big).$$

Thm (Chaika) *Fix any X and consider φ_t with $\int \varphi_t dt = +\infty$.*
-*We have $\mathrm{Leb}\big(\mathcal{R}(f_\theta, p, \varphi)\big) = 1$ for any p and almost any θ.*
-*Moreover $p \in \mathcal{R}(f_\theta, p, \varphi)$ for almost any θ and almost any p.*

-*Parameter space:* a *connection* for T is a triple (u_β, u_α, n), where u_β and u_α are singularities of T such that

$$T^n(u_\beta) = u_\alpha.$$

Keane: No connections $\Rightarrow$ T is minimal.
Rauzy: No connections $\Leftrightarrow$ T is infinitely renormalizable.
-It is natural to set the conditions

$$|T^n(u_\beta) - u_\alpha| < \varphi_n.$$

Thm (M.)
-*If $\sum \varphi_n < +\infty$ we have finitely many solutions for a.e. T.*
-*If $\sum \varphi_n = +\infty$ with $n\varphi_n$ decreasing monotone, then for a.e. T we have infinitely many (optimal) solution (u_β, u_α, n) for any fixed pair (β, α).*

Principle: *A datum (T, x, y) with x, y in $[0, 1)$ is an element of an augmented parameter space, thus diophantine conditions in phase space, in a weaker form, follow directly from corresponding one in parameter space.*

-A triple (u_β, u_α, n) is optimal if we can suspend it to a saddle connection γ on X.

Coro I *For almost any X, $\mathrm{Hol}(X)$ contains infinitely many solutions of*

$$|\mathrm{Area}(v)| < |v| \times \varphi(|v|)$$

Coro II (Log. Law) *For almost any X we have*

$$\limsup_{t\to\infty} \frac{-\log\big(\mathrm{Sys}(\mathcal{F}_t X)\big)}{\log(t)} = \frac{1}{2}.$$

-*Lagrange spectra.* Let X be a suspension of T. Following Boshernitzan, we set

$$\mathcal{E}_n(T) := \min\{|T^n(u_\beta) - u_\alpha|; \beta, \alpha \in \mathcal{A}\}.$$

Vorobets: we have

$$L(X) := \limsup_{n\to\infty} \frac{1}{n\mathcal{E}_n(T)} = \limsup_{t\to\infty} \frac{2}{\mathrm{Sys}^2(\mathcal{F}_t X)}.$$

Hubert-M.-Ulcigrai study of the Lagrange spectra

$$\mathcal{L}(\mathcal{I}) := \{L(X); X \in \mathcal{I}\},$$

where $\mathcal{I}$ is a closed and $\mathrm{SL}(2,\mathbf{R})-$invariant locus some stratum $\mathcal{H}$.
-For the classical Lagrange spectrum:

$$L(\alpha) = \limsup_{n\to\infty} [\ldots, a_{n-2}, a_{n-1}] + a_n + [a_{n+1}, a_{n+2}, \ldots].$$

-We prove similar formula for the re-normalization on extremal loci $\mathcal{I}$:
1) $\mathcal{I} = \mathcal{C}$ connected component of $\mathcal{H}$.
2) $\mathcal{I} = \mathrm{SL}(2,\mathbf{R}) \cdot X$ for a *square-tiled surface* X.
Thm (Hubert-M.-Ulcigrai) *For the loci as in 1) and 2), we have properties iii) and iv) as in the classical case.*

-*Phase space characterization of bounded type condition.*

-*Definition:* X is of *bounded type* if $L(X) < +\infty$.

Remark

1. *If* X *is of bounded type then it has the MSTP, that is*

$$\mathrm{Leb}\big(\mathcal{R}(f_X^t, p, \varphi)\big) = 1$$

   for any p *and any decreasing* φ_t *with* $\int \varphi_t dt = +\infty$.

2. *There are translation surfaces* X *which satisfy the MSTP but are not of bounded type.*
3. *Let* $\mathcal{I}$ *be either a closed Teichmuller disc or the stratum* $\mathcal{H}_2(2)$. *Then* X *is of bounded type if and only if it has the MSTP.*

Thank you!