4D N=2 Yang-Mills /2D CFT Correspondence

A. Belavin

Landau Institute for Theoretical Physics

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Instantons in Yang-Mills theory were discovered by BPST in 1975. They are solutions of the self-duality equation (1975)

$$F_{\mu\nu} = \widetilde{F}_{\mu\nu}$$

Using the results of BZ and AW (1977) the general N-instanton solution was constructed by ADHM(1978).
It is expressed through $N \times N$ matrices B_1, B_2, a $N \times 2$ matrix I and a $2 \times N$ matrix J, which obey

$$[B_1, B_2] + IJ = 0$$

$$[B_1 B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0$$

and can be conveniently organizied to $2N \times (2N+2)$ matrix Δ

$$\Delta = a + bz = \begin{pmatrix} I & B_1 & B_2 \\ J^\dagger & -B_2^\dagger & B_1^\dagger . \end{pmatrix} + \begin{pmatrix} 0 & z_1 & z_2 \\ 0 & -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

Then the solution is expressed via $2N \times 2$ matrix $U(x)$:
$A_\mu = U^\dagger(x)\partial_\mu U(x)$, if $\quad U^\dagger U = 1$ and $\quad \Delta^\dagger U = 0$,

II. Conformal Field Theory

Two-dimensional conformal Liouville field theory arises in non-critical String theory. The Lagrangian of the theory reads

$$\mathcal{L}_{\mathsf{LFT}} = \frac{1}{8\pi}(\partial_a \phi)^2 + \mu e^{2b\phi}$$

Here μ is the cosmological constant and parameter b is related to the central charge c of the Virasoro algebra

$$c = 1 + 6Q^2\ , \ \ Q = b + \frac{1}{b}\ .$$

Virasoro algebra

$$[L_m, L_n] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}\ ,$$

The central problem in CFT is the computation of the correlation functions of the primary fields Φ_Δ . Here Δ is the conformal dimension

$$\Delta(\lambda) = \frac{Q^2}{4} - \lambda^2.$$

4-point correlation function of bosonic primaries Φ_i is expressed in terms so-called Conformal blocks

$$\langle \Phi_1(q)\Phi_2(0)\Phi_3(1)\Phi_4(\infty)\rangle =$$

$$= (q\bar{q})^{\Delta-\Delta_1-\Delta_2} \sum_{\Delta} C_{12}^{\Delta} C_{34}^{\Delta} F(\Delta_i|\Delta|q) F(\Delta_i|\Delta|\bar{q})$$

This function $F(\Delta_i|\Delta|q)$ is defined in Virasoro representation theory terms uniquely.

III. Seiberg-Witten theory

In 1994 Seiberg-Witten proposed an exact expression for the low energy effective action (prepotential $\mathcal{F}$)of $\mathcal{N}=2$ SUSY $d=4$ Yang-Mills gauge theory with spontaneous breaking of non-abelian symmetry $SU(2)->U(1)$

$$L_{N=2}=tr\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+2\mathcal{D}_\mu\Phi^*\mathcal{D}^\mu\Phi+\right.$$
$$\left.+\sum_a(i\bar{\lambda}_a\bar{\sigma}_\mu\mathcal{D}^\mu\lambda^a+g\Phi^*[\lambda_a,\lambda^a]+g\Phi[\bar{\lambda}_a,\bar{\lambda}^a])+2g^2[\Phi^*,\Phi]^2)\right\}$$

Using non-renormalization theorem,holomophicity,electric-magnetic duality and some additional physical assumptions like the conjecture about the connection the analytic properties of the prepotential and vanishing masses of dyons S-W write down the explicit expression for $\mathcal{F}$

$$\mathcal{F}(\Psi)=\frac{i}{2\pi}\Psi^2\log\frac{\Psi^2}{\Lambda^2}-\frac{i}{\pi}\sum_{n=1}^{\infty}\mathcal{F}_N\left(\frac{\Lambda}{\Psi}\right)^{4N}\Psi^2$$

IV. Multi-Instanton calculus of DKM+H

To verify this proposal Dorey-Khoze-Mattis started the direct computation coefficients $\mathcal{F}_N$ quasiclassicaly.They get that N-instanton contribution is

$$\mathcal{F}_N = \int d\mu^{(n)} e^{-S_{ind}^{(N)}}$$

$$S_{ind}^{(N)} = \int d^4 x tr \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2\mathcal{D}_\mu \Phi^* \mathcal{D}^\mu \phi + g\Phi^* [\lambda_a, \lambda^a] \right\}$$

The fields satisfy a reduced set of eq-s of motion

$$F_{\mu\nu} = \widetilde{F}_{\mu\nu};$$

$$\bar{\sigma}_\mu \mathcal{D}^\mu \lambda^a = 0;$$

$$\mathcal{D}_\mu \mathcal{D}^\mu \Phi = -g[\lambda_a, \lambda^a]$$

Due to appearence Weyl zero modes of positive chirality in selfdual Y-M background $S_{ind}^{(N)}$ depends also on grassmann matrices $N \times N$ matrices M_1, M_2, a $N \times 2$ matrix μ and a $2 \times N$ matrix ν which obey

$$[M_1, B_2] + [B_1, M_2] + \mu J + I\nu = 0$$
$$[M_1, B_1^\dagger] + [M_2, B_2^\dagger] + \mu I^\dagger - J^\dagger \nu = 0$$

Taking into account that elements of cotangent bundle (DB_1, DB_2, DI, DJ) satisfy to the same eq-s after the change

$$(M_1, M_2, \mu, \nu) -> (DB_1, DB_2, DI, DJ)$$

DKM transform the integral over super moduli space to integral of the exponential of an mixed differential form.

$$\mathcal{F}_N = \int_{\mathcal{M}_N} e^{-S_{ind}^{(N)}(A, \mathcal{D}A)}$$

V. Nekrasov patition function

It can be proved that $S_{ind}^{(N)}(A, \mathcal{D}A)$ is an exact equivariant form

$$S_{ind}^{(N)}(A, \mathcal{D}A) = d_v \omega$$

and obtain

$$\mathcal{F}_N = \int_{\mathcal{M}_N} e^{-d_v \omega}$$

Using the localization technique the computation of the integral is reduced to finding fixed points of the vector field and its determinants in these fixed points .

The result is Nekrasov explicit formula for coefficients of S-W prepotential in the pure $\mathcal{N} = 2$ SUSY Yang-Mills theory

$$\mathcal{F}_N = \sum_P \frac{1}{\det v(P)}$$

VI. AGT correspondence between $\mathcal{N}=2$ SUSY $4d$ gauge theory and $2d$ Conformal field theory.

Using this explicit expression for the Nekrasov instanton partition function Alday,Gaiotto,Tachikawa in 2009 proposed the remarkable correspondence between this gauge theory and CFT.

In particular they argued that Instanton partition function of
4-dimensional $\mathcal{N}=2$ SUSY gauge theory with four hypemultiplets
and four-point Conformal block of 2-dimensional Liouville field theory coincide.

Moduli space

ADHM date consist of $N \times N$ matrices B_1, B_2, a $N \times 2$ matrix I and a $2 \times N$ matrix J, which are subject of the following set of conditions:

$$[B_1, B_2] + IJ = 0,$$

The solutions related by $GL(N)$ transformations

$$B_i' = gB_ig^{-1}, \;\; I' = gI, \;\; J' = Jg^{-1}; \quad g \in GL(N)$$

are equivalent.

The vectors obtained by the repeated action of B_1 and B_2 on $I_{1,2}$,columns of the matrix I ,span N-dimensional vector space V,a fiber of the N-dimensional fiber bundle ,whose base is the moduli space $\mathcal{M}_N$ itself.

The vector field and its fixed points.

The construction of the instanton partition function involves the determinants of the vector field v on $\mathcal{M}_N$, defined by

$$B_l \to t_l B_l; \quad I \to I t_v; \quad J \to t_1 t_2 t_v^{-1} J,$$

where parameters $t_l \equiv \exp \epsilon_l \tau$, $l = 1, 2$ and $t_v = \exp a\sigma_3 \tau$.

Fixed points, which are relevant for the determinants evaluation, are found from the conditions:

$$t_l B_l = g^{-1} B_l g; \quad I t_v = g^{-1} I; \quad t_1 t_2 t_v^{-1} J = J g.$$

The solutions of this system can be parameterized by pairs of Young diagrams $\vec{Y} = (Y_1, Y_2)$ such that the total number of boxes $|Y_1| + |Y_2| = N$. The cells $(i_1, j_1) \in Y_1$ and $(i_2, j_2) \in Y_2$ correspond to vectors $B_1^{i_1} B_2^{j_1} I_1$ and $B_1^{i_2} B_2^{j_2} I_2$ respectively. It is convenient to use these vectors as a basis in the fiber V attached to some fixed point.

Then the explicit form of the ADHM data for the given fixed point is defined straightaway

$$
\begin{aligned}
&g_{ss'} = \delta_{ss'} t_1^{i_s-1} t_2^{j_s-1},\\
&(B_1)_{ss'} = \delta_{i_s+1,i_{s'}} \delta_{j_s,j_{s'}},\\
&(B_2)_{ss'} = \delta_{i_s,i_{s'}} \delta_{j_{s+1},j_{s'}},\\
&(I_1)_s = \delta_{s,1},\\
&(I_2)_s = \delta_{s,|Y_1|+1},\\
&J = 0,
\end{aligned}
$$

where $s = (i_s, j_s)$.

The first terms of the series for $\chi(q)$ looks as follows

$$\chi(q) = 1 + 2q^{1/2} + 4q + 8q^{3/2} + 16q^2 + 28q^{5/2} + ...$$

The $\chi_B(q)\chi_F(q)$ equals to the character of standard representation of the $\mathcal{NSR}$ algebra with generators L_n, G_r. $\chi_B(q)$ equals to the character of the Fock representation of the Heisenberg algebra. The term $\chi_B(q)\chi_F(q)$ should be related to the fact that $\widehat{sl}(2)$ representation of level 2 can be realized by one bosonic and one fermionic field .

The the generating function the whole space $\mathcal{M}_{\mathrm{sym}}$ has the form

$$\chi(q) = \sum_N |\mathcal{M}_{\mathrm{sym}}(N)| q^{\frac{N}{2}} = \prod_{n \in \mathbb{Z},\, n > 0} \frac{1}{\left(1 - q^{\frac{n}{2}}\right)^2}$$

The result equals to the character of the simple representation of $\widehat{gl}(2)_2 \times \mathcal{NSR}$ namely the tensor product of Fock representation of Heisenberg algebra, vacuum representation of $\widehat{gl}(2)_2$ and NS representation of $\mathcal{NSR}$.

Determinants of the vector field

The form of $\mathcal{N} = 2$ $SU(2)$ instanton partition function was derived to be equal an integral of the equivariantly form, defined in terms of the vector field v acting on the moduli space $\mathcal{M}_N$.

By localization technique , the moduli integral is reduced to the determinants of the vector field v in the vicinity of its fixed points

$$\mathcal{Z}_N(a, \epsilon_1, \epsilon_2) = \sum_n \frac{1}{\det_n v}.$$

We need to find all eigenvectors of the vector field on the tangent space passing through the fixed points

$$\begin{aligned} &t_i \delta B_i = \wedge\, g \delta B_i g^{-1}, \\ &\delta I t = \wedge\, g \delta I, \\ &t_1 t_2 t^{-1} \delta J = \wedge\, \delta J g^{-1}. \end{aligned}$$

This is equivalent to the following set of equations

$$
\begin{aligned}
\lambda\,(\delta B_i)_{ss'} &= (\epsilon_i + \phi_{s'} - \phi_s)\,(\delta B_i)_{ss'}, \\
\lambda\,(\delta I)_{sp} &= (a_p - \phi_s)\,(\delta I)_{sp}, \\
\lambda\,(\delta J)_{ps} &= (\epsilon_1 + \epsilon_2 - a_p + \phi_s)\,(\delta J)_{ps},
\end{aligned}
$$

where $\Lambda = \exp \lambda\tau$, $g_{ss} = \exp \phi_s \tau$ and

$$\phi_s = (i_s - 1)\epsilon_1 + (j_s - 1)\epsilon_2 + a_{p(s)}.$$

We should keep only those eigenvectors which belong to the tangent space. This means excluding variations breaking ADHM constraints. On the Moduli space

$$[\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J = 0.$$

Gauge symmetry can be taken into account in the following way. We fix a gauge in which $\delta B_{1,2}, \delta I, \delta J$ are orthogonal to any gauge transformation of $B_{1,2}, I, J$. This gives additional constraint

$$\left[\delta B_l, B_l^\dagger\right] + \delta I I^\dagger - J^\dagger \delta J = 0.$$

The variations in the LHS of the eq-ns above should be excluded.

The corresponding eigenvalues are defined from the equations

$$t_1t_2([\delta B_1, B_2] + [B_1, \delta B_2] + \delta IJ + I\delta J) =$$
$$= \Lambda\, g\Big([\delta B_1, B_2] + [B_1, \delta B_2] + \delta IJ + I\delta J\Big)g^{-1},$$
$$\left[\delta B_l, B_l^\dagger\right] + \delta II^\dagger - J^\dagger \delta J = \Lambda\, g\Big(\left[\delta B_l, B_l^\dagger\right] + \delta II^\dagger - J^\dagger \delta J\Big)g^{-1}.$$

One finds the following eigenvalues, which should be excluded :

$$\lambda = (\epsilon_1 + \epsilon_2 + \phi_s - \phi_{s'}),$$
$$\lambda = (\phi_s - \phi_{s'}).$$

Thus, the determinant of the vector field is given by

$$\det v =$$
$$\frac{\prod\limits_{s,s'\in\vec{Y}} (\epsilon_1 + \phi_{s'} - \phi_s)(\epsilon_2 + \phi_{s'} - \phi_s) \prod\limits_{l=1,2;s\in\vec{Y}} (a_l - \phi_s)(\epsilon_1 + \epsilon_2 - a_l + \phi_s)}{\prod\limits_{s,s'\in\vec{Y}} (\phi_{s'} - \phi_s)(\epsilon_1 + \epsilon_2 - \phi_{s'} + \phi_s)}$$

Re-expressed in terms of arm-length and leg-length this expression gives

$$\det{}' v = \prod_{\alpha,\beta=1}^{2} \prod_{s \in {}^{\diamondsuit}Y_\alpha(\beta)} E\Big(a_\alpha - a_\beta, Y_\alpha, Y_\beta \Big| s\Big)\big(Q - E\Big(a_\alpha - a_\beta, Y_\alpha, Y_\beta \Big| s\Big)\big),$$

here $E\Big(a, Y_1, Y_2 \Big| s\Big)$ are defined as follows

$$E\Big(a, Y_1, Y_2 \Big| s\Big) = a + b(L_{Y_1}(s) + 1) - b^{-1} A_{Y_2}(s)\ ,$$

where $A_Y(s)$ and $L_Y(s)$ are respectively the arm-length and the leg-length for a cell s in Y. The region ${}^{\diamondsuit}Y_\alpha(\beta)$ is defined as

$${}^{\diamondsuit}Y_\alpha(\beta) = \Big\{(i,j) \in Y_\alpha \Big| P\Big(k'_j(Y_\alpha)\Big) \neq P\Big(k_i(Y_\beta)\Big)\Big\},$$

or, in other words, the boxes having different parity of the leg- and arm-factors. So the contribution of the vector multiplet reads

$$Z^{\text{sym}}_{\text{vec}}(\vec{a}, \vec{Y}) \equiv \frac{1}{\det{}' v}$$

Four-point Super Liouville conformal block

Two-dimensional super conformal Liouville field theory arises in non-critical String theory. The Lagrangian of the theory reads

$$\mathcal{L}_{\text{SLFT}} = \frac{1}{8\pi}(\partial_a\phi)^2 + \frac{1}{2\pi}\left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right) + 2i\mu b^2\bar{\psi}\psi e^{b\phi} + 2\pi b^2\mu^2 e^{2b\phi} .$$

Here μ is the cosmological constant and parameter b is related to the central charge c of the super-Virasoro algebra

$$c = 1 + 2Q^2 , \quad Q = b + \frac{1}{b} .$$

We are interested in the Neveu-Schwarz sector of the super-Virasoro algebra

$$\begin{aligned}
[L_m, L_n] &= (n-m)L_{n+m} + \frac{c}{8}(n^3 - n)\delta_{n+m} , \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2}c(r^2 - \frac{1}{4})\delta_{r+s} , \\
[L_n, G_r] &= (\frac{1}{2}n - r)G_{n+r} .
\end{aligned}$$

where the subscripts m, n –integers and r, s – half-integers. The NS fields belong to highest weight representations of super-Virasoro algebra.

The central problems in CFT is the computation of the correlation functions of the primary fields Φ_Δ and Ψ_Δ has the conformal dimension Δ defined by $L_0|\Delta\rangle = \Delta|\Delta\rangle$, while $\Psi_\Delta \equiv G_{-1/2}\Phi_\Delta$. Together fields Φ_Δ and Ψ_Δ form primary super doublet. The standart parametrization of the conformal dimensions

$$\Delta(\lambda) = \frac{Q^2}{8} - \frac{\lambda^2}{2}.$$

4-point correlation function of bosonic primaries Φ_i is expressed in terms superconformal blocks

$$\langle\Phi_1(q)\Phi_2(0)\Phi_3(1)\Phi_4(\infty)\rangle =$$

$$(q\bar{q})^{\Delta-\Delta_1-\Delta_2} \sum_\Delta \Big(C_{12}^\Delta C_{34}^\Delta F_0(\Delta_i|\Delta|q) F_0(\Delta_i|\Delta|\bar{q}) + \widetilde{C}_{12}^\Delta \widetilde{C}_{34}^\Delta F_1(\Delta_i|\Delta|q) F_1(\Delta_i|\Delta|\bar{q}) \Big) .$$