Continuous Wavelet Transform for Gauge Theories

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Abstract

We describe the application of the continuous wavelet transform to calculation of the Green functions in quantum field theory: scalar ϕ^4 theory, quantum electrodynamics, quantum chromodynamics. The method of continuous wavelet transform in quantum field theory presented by Altaisky [Phys. Rev. D **81**, 125003 (2010)] for the scalar ϕ^4 theory, consists in substitution of the local fields $\phi(x)$ by those dependent on both the position x and the resolution a. The substitution of the action $S[\phi(x)]$ by the action $S[\phi_a(x)]$ makes the local theory into nonlocal one, and implies the causality conditions related to the scale a, the *region causality* [J.D.Christensen and L.Crane, J.Math.Phys. (N.Y.) **46**, 122502 (2005)]. These conditions make the Green functions $G(x_1, a_1, \ldots, x_n, a_n) = \langle \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n) \rangle$ finite for any given set of regions by means of an effective cutoff scale $A = \min(a_1, \ldots, a_n)$.

References:

This talk is based on

M.V.Altaisky. *Phys. Rev. D* 81(2010) 125003
M.V.Altaisky and N.E.Kaputkina. *JETP Lett.* 94(2011)341
M.V.Altaisky and N.E.Kaputkina. *Phys. Rev. D* 88(2013)025015

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- Causality
- Gauge theories

Let us consider a field theory with 4th power interaction

$$W[J] = \mathcal{N} \int e^{-\int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi\right]} \mathcal{D}\phi$$

The connected Green functions are given by variational derivatives of the generating functional:

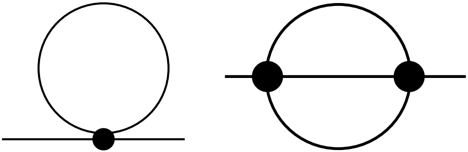
$$\Delta^{(n)} \equiv \langle \phi(x_1) \ldots \phi(x_n) \rangle_c = \left. \frac{\delta^n \ln W[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0}$$

In statistical sense these functions have the meaning of the *n*-point correlation functions [ZJ99].

The divergences of Feynman graphs in the perturbation expansion of the Green functions with respect to the small coupling constant λ emerge at coinciding arguments $x_i = x_k$.
For instance, the bare two-point correlation function

$$\Delta_0^{(2)}(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{\imath p(x-y)}}{p^2+m^2}$$

is divergent at $x = y$ for $d \geq 2$



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Measurement

Have the divergences ever been observed?

- To localize a particle in an interval Δx the measuring device requests a momentum transfer of order $\Delta p \sim \hbar/\Delta x$. $\phi(x)$ at a point x has no experimental meaning. What is meaningful, is the vacuum expectation of product of fields in certain region around x

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- If the particle, described by $\phi(x)$, have been initially prepared on the interval $(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2})$, the probability of registering it on this interval is ≤ 1: for the registration depends on the strength of interaction and the ratio of typical scales related to the particle and to the equipment.

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- Statement of existence: if a measuring equipment with a given resolution a fails to register an object, prepared on spatial interval of width Δx with certainty, then tuning the equipment to *all* possible resolutions a' would lead to the registration.

Regularization

- Implies the dependence on certain scale parameter [tHV72, Wil73, Ram89]

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2 - \Lambda^2}, \quad \int_{\Lambda e^{-\delta l}}^{\Lambda}, \quad g\mu^{2\epsilon} \int d^{4-2\epsilon} p \ldots$$

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- Kadanoff blocking assumes the larger blocks interact with each other in the same way as their sub-blocks [Kad66, Ito85]
- The theory based on the Fourier transform describes the strength of the interaction of all fluctuations *up to* the scale $1/\Lambda$, but says nothing about the interaction strength *at* a given scale

$$g\prod_i \int_{|k|<\Lambda} e^{-\imath k_i x}\tilde{\phi}(k_i)\frac{d^d k}{(2\pi)^d}$$

Translation group and affine group

- Translation group: $G : x' = x + b$

$$\phi(x) = \langle x|\phi\rangle = \int \langle x|k\rangle \frac{d^d k}{(2\pi)^d} \langle k|\phi\rangle$$

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- Arbitrary (locally compact) group [Car76, DM76] acting on Hilbert space $\mathcal{H}$:

$$\hat{1} = \frac{1}{C_g} \int_{q\in G} U(g)|g\rangle d\mu_L(q) \langle g|U^*(q)$$

$g \in \mathcal{H}$ is an admissible vector, such that

$$C_g = \frac{1}{\|g\|_2^2} \int_G |\langle g|U(q)|g\rangle|^2 d\mu(q) < \infty.$$

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- Affine group $G : x' = ax + b, |g; a, b\rangle = U(a, b)|g\rangle$ (coordinate representation with L^1-norm)

$$d\mu_L(a, b) = \frac{da d^d b}{a}, \quad U(a, b)g(x) = \frac{1}{a^d} g\left(\frac{x-b}{a}\right)$$

Resolution-dependent fields

We define the *resolution-dependent fields*

$$\phi_a(x) \equiv \langle g; a, x|\phi\rangle,$$

also referred to as scale components of ϕ, where $\langle g; a, x|$ is the bra-vector corresponding to localization of the measuring device around the point x with the spatial resolution a; g labels the apparatus function of the equipment, an *aperture*.
If the measuring equipment has the best resolution A, *i.e.* all states $\langle g; a \geq A, x|\phi\rangle$ are registered, but those with $a < A$ are not, the regularization of the fields in momentum space, with the cutoff momentum $\Lambda = 2\pi/A$ corresponds to the UV-regularized functions

$$\phi^{(A)}(x) = \frac{1}{C_g}\int_{a\geq A} \langle x|g; a, b\rangle d\mu(a, b)\langle g; a, b|\phi\rangle.$$

The regularized n-point Green functions are

$$\mathcal{G}^{(A)}(x_1, \ldots, x_n) \equiv \langle \phi^{(A)}(x_1), \ldots, \phi^{(A)}(x_n)\rangle_c.$$

Continuous Wavelet Transform

To keep the scale-dependent fields the same physical dimension as the ordinary fields we use the CWT in L^1-norm [FPAA90, Chu92, HM98]:

$$\begin{aligned} \phi(x) &= \frac{1}{C_g}\int \frac{1}{a^d} g\left(\frac{x-b}{a}\right)\phi_a(b)\frac{da d^d b}{a}, \\ \phi_a(b) &= \int \frac{1}{a^d}\overline{g\left(\frac{x-b}{a}\right)}\phi(x)d^d x, \quad \tilde\phi_a(k) = \overline{\tilde g(ak)}\tilde\phi(k) \end{aligned}$$

For isotropic wavelets g the normalization constant C_ψ is readily evaluated using Fourier transform:

$$C_g = \int_0^\infty |\tilde g(ak)|^2 \frac{da}{a} = \int |\tilde g(k)|^2 \frac{d^d k}{S_d |k|} < \infty,$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of unit sphere in $\mathbb{R}^d$.

Continuous Wavelet Transform in D dimensions

$$G : x' = aR(\theta)x + b, x, b \in \mathbb{R}^d, a \in \mathbb{R}_+, \theta \in SO(d),$$

where $R(\theta)$ is the rotation matrix. We define unitary representation of the affine transform:

$$U(a,b,\theta)g(x) = \frac{1}{a^d} g\left(R^{-1}(\theta)\frac{x-b}{a}\right).$$

The wavelet coefficients of the function $\phi(x) \in L^2(\mathbb{R}^d)$ with respect to the basic wavelet $g(x)$ are

$$\phi_{a,\theta}(b) = \int_{\mathbb{R}^d} \frac{1}{a^d} \overline{g\left(R^{-1}(\theta)\frac{x-b}{a}\right)} \phi(x) d^d x.$$

The function $\phi(x)$ can be reconstructed from its wavelet coefficients:

$$\phi(x) = \frac{1}{C_g} \int \frac{1}{a^d} g\left(R^{-1}(\theta)\frac{x-b}{a}\right) \phi_{a\theta}(b) \frac{da d^d b}{a} d\mu(\theta)$$

Feynman Diagram Technique

CWT in Fourier representation

$$\phi(x) = \frac{1}{C_g}\int_0^\infty \frac{da}{a}\int \frac{d^d k}{(2\pi)^d} e^{-\imath kx}\tilde{g}(ak)\tilde{\phi}_a(k)$$

The Feynman rules [Alt03],[Alt10]:

Feynman Diagram Technique

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- each interaction vertex is substituted by its wavelet transform; for the N-th power interaction vertex this gives multiplication by factor $\prod_{i=1}^{N} \overline{\tilde{g}(a_i k_i)}$.

Scalar field example of SDP

Substitution of the CWT into field theory $W[J]$ gives a theory for the fields $\phi_a(x)$ [Alt07]:

$$\begin{aligned} W_W[J_a] &= \mathcal{N}\int \exp\Bigl[-\frac{1}{2}\int \phi_{a_1}(x_1)D(a_1,a_2,x_1-x_2)\phi_{a_2}(x_2)\frac{da_1 d^d x_1}{a_1} \times \\ &\times \frac{da_2 d^d x_2}{a_2} - \frac{\lambda}{4!}\int V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}\phi_{a_1}(x_1)\cdots\phi_{a_4}(x_4)\frac{da_1 d^d x_1}{a_1} \times \\ &\times \frac{da_2 d^d x_2}{a_2}\frac{da_3 d^d x_3}{a_3}\frac{da_4 d^d x_4}{a_4} + \int J_a(x)\phi_a(x)\frac{da d^d x}{a}\Bigr]\mathcal{D}\phi_a, \end{aligned}$$

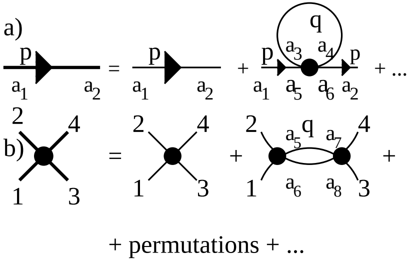
with $D(a_1,a_2,x_1-x_2)$ and $V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}$ denoting the wavelet images of the inverse propagator and that of the interaction potential. The Green functions for scale component fields are given by functional derivatives

$$\langle \phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle_c = \left.\frac{\delta^n \ln W_W[J_a]}{\delta J_{a_1}(x_1)\ldots\delta J_{a_n}(x_n)}\right|_{J=0}.$$

Diagrams: scale-dependent ϕ^4 theory

Let us consider the contribution of the tadpole diagram to the two-point Green function $G^{(2)}(a_1, a_2, p)$ shown in a) below. The bare Green function is

$$G_0^{(2)}(a_1, a_2, p) = \frac{\tilde{g}(a_1 p)\tilde{g}(-a_2 p)}{p^2 + m^2}.$$



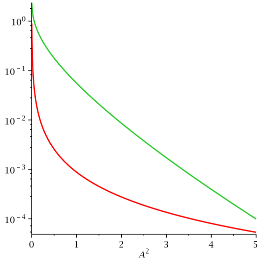
Feynman diagrams for the Green functions $G^{(2)}$ and $G^{(4)}$ for the resolution-dependent fields

g_1 wavelet $g(x) = -\frac{xe^{-x^2/2}}{(2\pi)^{d/2}}, \quad \tilde{g}(k) = \imath k e^{-k^2/2}$

$$T_1^4(\alpha^2) = \frac{-4\alpha^4 e^{2\alpha^2}\mathrm{Ei}_1(2\alpha^2) + 2\alpha^2}{64\pi^2\alpha^4} m^2,$$

$$\lim_{s^2 \gg 4m^2} X_4(\alpha^2) = \frac{\lambda^2}{256\pi^6}\frac{e^{-2\alpha^2}}{2\alpha^2}\left[e^{\alpha^2} - 1 - \alpha^2 e^{2\alpha^2}\mathrm{Ei}_1(\alpha^2) + 2\alpha^2 e^{2\alpha^2}\mathrm{Ei}_1(2\alpha^2)\right],$$

Dimensionless scale factor $\alpha \equiv Am$, A is the minimal scale of all external lines



Scale-decay factors for the two-point and four-point Green functions. The bottom curve is the graph of the tadpole and one-loop vertex as a function of A^2; the top curve is the graph of the vertex divided by $\frac{\lambda^2}{256\pi^6}$ as a function of A^2. $m = s^2 = 1$ is set for both curves. Redrawn from Altaisky PRD 81(2010)125003

Causality and commutation relations

In standard quantum field theory the operator ordering is performed according to the non-decreasing of the time argument in the product of the operator-valued functions acting on vacuum state

$$\underbrace{A(t_n)A(t_{n-1})\dots A(t_2)A(t_1)}_{t_n\ge t_{n-1}\ge\dots\ge t_2\ge t_1}|0\rangle.$$

The quantization is performed by separating the Fourier transform of quantum fields into the positive- and the negative-frequency parts

$$\phi = \phi^+(x) + \phi^-(x),$$

defined as follows

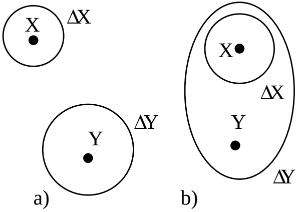
$$\phi(x) = \int \frac{d^d k}{(2\pi)^d}\left[e^{\imath kx}u^+(k) + e^{-\imath kx}u^-(k)\right],$$

where the operators $u^\pm(k) = u(\pm k)\theta(k_0)$ are subjected to canonical commutation relations

$$[u^+(k), u^-(k')] = \Delta(k, k').$$

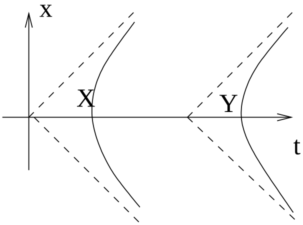
Causality for scale-dependent fields

In case of the scale-dependent fields, because of the presence of the scale argument in new fields $\phi_{a,\eta}(x)$, where a and η label the size and the shape of the region centered at x, the problem arises how to order the operators supported by different regions. This problem was solved in (Altaisky PRD 81(2010)125003) on the base of the *region causality assumption* [CC05].

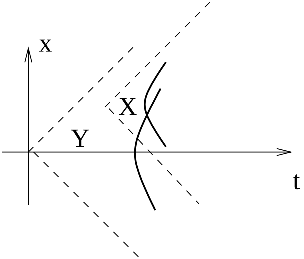


Causal ordering of scale-dependent fields. Space-like regions are drawn in Euclidean space: a) The event regions do not intersect; b) Event X is inside the event Y

Region Causality in Minkowski Space



Disjoint events in (t, x) plane in Minkowski space



Nontrivial intersection of two events $X \subset Y$ in (t, x) plane in Minkowski space

Causality for scale-dependent fields

Causality principle

The coarse acts on vacuum first

d_0^0		d_1^0	
d_{00}^1	d_{01}^1	d_{10}^1	d_{11}^1

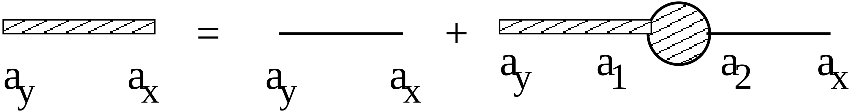
Table: Binary tree of operator-valued functions. Vertical direction corresponds to the scale variable. The causal sequence of the operator-valued functions shown in the table above is: $d_0^0, d_{00}^1, d_{01}^1, d_1^0, d_{10}^1, d_{11}^1$. As it is shown the underlined regions of different scales do not intersect

Green's functions are not singular at coinciding arguments – they are projections from coarser scale to finer scale:

$$G_0^{(2)}(a_1, a_2, b_1 - b_2 = 0) = \int \frac{d^4p}{(2\pi)^4} \frac{\tilde{g}(a_1 p)\tilde{g}(-a_2 p)}{p^2 + m^2} e^{-\imath p \cdot 0},$$

since $|\tilde{g}(p)|$ vanish at $p \to \infty$.

The Dyson-Schwinger equation relating the full propagator with the bare propagator is symbolically drawn in the diagram



The corresponding integral equation can be written as

$$\mathcal{G}(x-y,a_x,a_y) = G(x-y,a_x,a_y) + \int \frac{da_1}{a_1} \int \frac{da_2}{a_2} \int dx_1 dx_2 \times$$
$$\times \quad G(x-x_2,a_x,a_2)\mathcal{P}(x_2-x_1,a_2,a_1)\mathcal{G}(x_1-y,a_1,a_y),$$

where $\mathcal{P}(x_2-x_1,a_2,a_1)$ denotes the vacuum polarization operator if G is the massless boson, or the self-energy diagram otherwise.

$$\tilde{\mathcal{G}}_{a_x,a_y}(p) = \tilde{G}_{a_x,a_y}(p) + \int \frac{da_1}{a_1} \int \frac{da_2}{a_2} \tilde{G}_{a_x,a_2}(p)\tilde{\mathcal{P}}_{a_2,a_1}(p)\tilde{\mathcal{G}}_{a_1,a_y}(p).$$

Wavelet transform in Minkowski space

Light-cone coordinates (x_+, x_-, x, y)

$x_\pm = \frac{t \pm z}{\sqrt{2}}, \quad \mathbf{x}_\perp = (x, y)$

The rotation matrix has a block-diagonal form

$$M(\eta,\phi) = \begin{pmatrix} e^\eta & 0 & 0 & 0 \\ 0 & e^{-\eta} & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix},$$

so that $M^{-1}(\eta,\phi) = M(-\eta,-\phi)$.
We can define the wavelet transform in light-cone coordinates in the same way as in Euclidean space using the representation of the affine group

$$x' = aM(\eta,\phi)x + b$$

Definition of basic wavelets in Minkowski space

In contrast to wavelet transform in Euclidean space, where the basic wavelet g can be defined globally on $\mathbb{R}^d$, the basic wavelet in Minkowski space is to be defined separately in four domains impassible by Lorentz rotations:

$$A_1 : k_+ > 0, k_- < 0; A_2 : k_+ < 0, k_- > 0;$$
$$A_3 : k_+ > 0, k_- > 0; A_4 : k_+ < 0, k_- < 0,$$

where k is wave vector, $k_\pm = \frac{\omega \pm k_z}{\sqrt{2}}$. Whence we have four separate wavelets in these four domains.

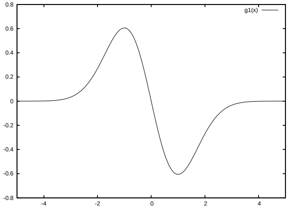
Thus the wavelet coefficiens are

$$W^i_{ab\eta\phi} = \int_{A_i} e^{\imath k_- b_+ + \imath k_+ b_- - \imath \mathbf{k}_\perp \mathbf{b}_\perp} \tilde{f}(k_-, k_+, \mathbf{k}_\perp)$$
$$\bar{\tilde{g}}(ae^\eta k_-, ae^{-\eta} k_+, aR^{-1}(\phi)\mathbf{k}_\perp) \frac{dk_+ dk_- d^2\mathbf{k}_\perp}{(2\pi)^4}.$$

Let us introduce a localized wave packet in Fourier space $\tilde{g}(t,k) = e^{-\imath tk - k^2/2}$. It is a gaussian wave packet at initial time $t=0$. At finite t it can be approximated by

$$\tilde{g}(t,k) = \tilde{g}_0(k) + \frac{t}{1!}\tilde{g}_1(k) + \frac{t^2}{2!}\tilde{g}_2(k) + O(t^3),$$

where $\tilde{g}_n(k) = \frac{d^n}{dt^n}\tilde{g}(t,k)\Big|_{t=0}$ are responsible for the shape of the packet at the times at which 1, 2 or n excitations are significant; with g_1 being the first excitation.



The only restriction is the finiteness of the wavelet cutoff function

$$f(x) = \frac{1}{C_g}\int_x^\infty |\tilde{g}(a)|^2 \frac{da}{a}, f(0) = 1$$

Quantum electrodynamics: one loop

In one-loop approximation the radiation corrections in QED come from three primitive Feynman graphs: fermion self-energy

$$\Sigma(p) = -e^2 \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \frac{-\imath}{\not p - \not q + m} \gamma_\nu \frac{\delta_{\mu\nu}}{q^2},$$

gives the corrections to the bare electron mass m_0 related to irradiation of virtual photons;

vacuum polarization operator

$$\Pi_{\mu\nu}(p) = -e^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{Sp}[\gamma_\mu \frac{1}{\not p + \not q + m} \gamma_\nu \frac{1}{\not q + m}]$$

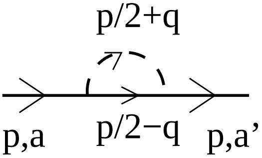
contributes to the Lamb shift of the atom energy levels;

and the vertex function

$$\Gamma_\rho(p,q) = -\imath e^3 \int \frac{d^4f}{(2\pi)^4} \gamma_\tau \frac{1}{\not p + \not f + m} \gamma_\rho \frac{1}{\not f + \not q + m} \gamma_\sigma \frac{\delta_{\tau\sigma}}{f^2}$$

determines the anomalous magnetic moment of the electron

Electron self-energy in scale-dependent theory



$A = \min(a, a')$.

$$\frac{\Sigma^{(A)}(p)}{\tilde g(ap)\tilde g(-a'p)} = -\imath e^2 \int \frac{d^4q}{(2\pi)^4} \frac{F_A(p,q)\gamma_\mu\left[\frac{\not p}{2} - \not q - m\right]\gamma_\mu}{\left[\left(\frac{p}{2}-q\right)^2 + m^2\right]\left[\frac{p}{2}+q\right]^2},$$

For the isotropic wavelet $F_A(p,q) = f^2(A(\frac{p}{2}-q))f^2(A(\frac{p}{2}+q))$

$$\frac{\Sigma^{(A)}(p)}{\tilde g(ap)\tilde g(-a'p)} = -\imath e^2 \int \frac{d^4y}{(2\pi)^4} F_A(p,|p|y)\times$$

$$\times\frac{\not p + 4m - 2|p|\not y}{\left[y^2+\frac{1}{4} - y\cos\theta - \frac{m^2}{p^2}\right]\left[y^2+\frac{1}{4}+y\cos\theta\right]}.$$

where $\theta = \angle(\mathbf{p},\mathbf{q})$ is the Euclidean angle; $\mathbf{y} = \mathbf{q}/|\mathbf{p}|$

Electron self-energy, g_1 wavelet

In high energy limit, $p^2 \gg 4m^2$, the contribution of last term in the numerator vanishes for the symmetry, and the diagram can be easily integrated in angle variable:

$$\frac{\Sigma^{(A)}(p)}{\tilde{g}(ap)\tilde{g}(-a'p)} = -\frac{\imath e^2}{4\pi^2} R_1(p)(\not{p} + 4m) \qquad \text{where:}$$

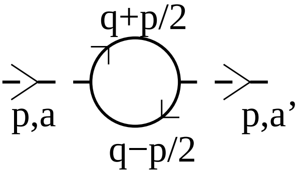
$$R_1(p) = \int_0^\infty dyyF_A(p,|p|y)\left[1 - \sqrt{1 - \frac{1}{\beta^2(y)}}\right],$$

$$\beta(y) = y + \frac{1}{4y}.$$

The integral $R_1(p)$ is finite for any wavelet cutoff function. For the g_1 wavelet we get

$$R_1(p) = \frac{1}{8A^2p^2}\left(2A^2p^2\mathrm{Ei}_1(A^2p^2) - 4A^2p^2\mathrm{Ei}_1(2A^2p^2)\right.$$
$$\left. - e^{-A^2p^2} + 2e^{-2A^2p^2}\right)$$

Vacuum polarization diagram



$$
\begin{aligned}
\frac{\Pi^{(A)}_{\mu\nu}(p)}{\tilde g(ap)\tilde g(-a'p)} &= -e^2 \int \frac{d^4q}{(2\pi)^4} F_A(p,q)\times \\
&\times \frac{\mathrm{Sp}(\gamma_\mu(\not q + \not p/2 - m)\gamma_\nu(\not q - \not p/2 - m))}{[(q+p/2)^2+m^2]\,[(q-p/2)^2+m^2]} \\
&= -4e^2 \int \frac{d^4q}{(2\pi)^4} F_A(p,q)\times \\
&\times \frac{2q_\mu q_\nu - \frac{1}{2}p_\mu p_\nu + \delta_{\mu\nu}(\frac{p^2}{4} - q^2 - m^2)}{\left[(q+\frac{p}{2})^2+m^2\right]\left[(q-\frac{p}{2})^2+m^2\right]}.
\end{aligned}
$$

$$\frac{\Pi^{(A)}_{\mu\nu}(p)}{\tilde g(ap)\tilde g(-a'p)} = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\pi^{(A)}_T + X^{(A)}\frac{p_\mu p_\nu}{p^2}$$

where

$$\begin{aligned}
\pi^{(A)}_T = &\quad -\frac{e^2}{3\pi^2}m^2p^2\int_0^\infty dyyF_A(mp,mpy)\Bigg[y^2+\\
+ &\quad \left(1-\sqrt{\frac{\frac{1}{16}+y^4+\frac{1}{p^4}-\frac{y^2}{2}+\frac{1}{2p^2}+\frac{2y^2}{p^2}}{\left(\frac14+y^2+\frac{1}{p^2}\right)^2}}\right)\\
\times &\quad \left(\frac58-\frac{4}{p^2}-\frac{2}{p^4}-2y^2\left(1+\frac{2}{p^2}\right)-2y^4\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
X^{(A)} = &\quad \frac{e^2m^2p^2}{\pi^2}\int_0^\infty dyyF_A(mp,mpy)\\
\times &\quad \Bigg[y^2-\left(1-\sqrt{\frac{\frac{1}{16}+y^4+\frac{1}{p^4}-\frac{y^2}{2}+\frac{1}{2p^2}+\frac{2y^2}{p^2}}{\left(\frac14+y^2+\frac{1}{p^2}\right)^2}}\right)
\end{aligned}$$

Result for g_1 wavelet at large $p^2 \gg 4$

For g_1 wavelet the regularizing function

$$F_A(p,q) = \exp\left(-A^2p^2 - 4A^2q^2\right).$$

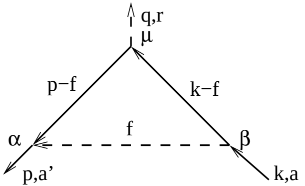
Hence for large $p^2 \gg 4$ the integral can be evaluated by substitution $y^2 = t$

$$\begin{aligned}\pi_T^{(A)} = & -\frac{e^2}{6\pi^2}m^2p^2\left\{\frac{e^{-a^2p^2}}{8a^6p^6}\left(4a^4p^4 - a^2p^2 - 1\right) + \frac{e^{-2a^2p^2}}{8a^6p^6}\right.\\ \times & \left.\left(1 - 4a^4p^4 + 2a^2p^2\right) - \frac{1}{2}\mathrm{Ei}_1\left(a^2p^2\right) + \mathrm{Ei}_1\left(2a^2p^2\right)\right\}.\end{aligned}$$

Similarly, for the longtitudinal term

$$X^A = \frac{e^2m^2p^2}{16\pi^2}\frac{e^{-a^2p^2}\left(a^2p^2 - 1 + e^{-a^2p^2}\right)}{a^6p^6}$$

Vertex function



$$-\imath e \frac{\Gamma^{(A)}_{\mu,r}}{\tilde g(-pa')\tilde g(-qr)\tilde g(ka)} = (-\imath e)^3 \int \frac{d^4 l}{(2\pi)^4} \gamma_\alpha G(p-f)\gamma_\mu \times$$
$$\times G(k-f)\gamma_\beta D_{\alpha\beta} F_A(p-f) F_A(k-f) F_A(f).$$

The explicit substitution with photon propagator taken in Feynman gauge gives

$$\imath e \frac{\Gamma^{(A)}_{\mu,r}}{\tilde g(-pa')\tilde g(-qr)\tilde g(ka)} = (-\imath e)^3 \int \frac{d^4 f}{(2\pi)^4} \gamma_\alpha \frac{\not p - \not f - m}{(p-f)^2 + m^2}$$
$$\times \gamma_\mu \frac{\not k - \not f - m}{(k-f)^2 + m^2} \gamma_\alpha \frac{1}{f^2} F_A(p-f) F_A(k-f) F_A(f)$$

Ward-Takahashi Identities

The standard procedure of the variation of action with a gauge fixing term leads to the equations (Albeverio,Altaisky, 2009):

$$
\begin{aligned}
q_\mu \Gamma_{\mu a_4 a_3 a_1}(p,q,p+q) = & \int \frac{da_2}{a_2} G^{-1}_{a_1 a_2}(p+q) \tilde{M}_{a_2 a_3 a_4}(p+q,q,p) \\
- & \int \frac{da_2}{a_2} \tilde{M}_{a_1 a_3 a_2}(p+q,q,p) G^{-1}_{a_2 a_4}(p),
\end{aligned}
$$

where

$$
\tilde{M}_{a_1 a_2 a_3}(k_1,k_2,k_3) = (2\pi)^d \delta^d(k_1 - k_2 - k_3) \overline{\tilde{g}}(a_1 k_1) \tilde{g}(a_2 k_2) \tilde{g}(a_3 k_3).
$$

$$\Pi^{(\mathcal{A})}_{AB,\mu\nu}(p) = -\frac{g^2}{2} f^{ABC} f^{BDC} \int \frac{d^4l}{(2\pi)^4} \frac{N_{\mu\nu}(l,p)F_{\mathcal{A}}(l+p,l)}{l^2(l+p)^2},$$

This integral can be easily evaluated in infrared limit [AK13] where ordinary QCD is divergent:

$$\Pi^{(\mathcal{A},g_1)}_{AB,\mu\nu}(p\to 0) = -g^2 f^{ACD} f^{BDC} \int \frac{d^4q}{(2\pi)^4} \frac{e^{-4\mathcal{A}^2q^2}}{q^4}[5q_\mu q_\nu + q^2\delta_{\mu\nu}].$$

Making use of isotropy we get

$$\Pi^{(\mathcal{A},g_1)}_{AB,\mu\nu}(p\to 0) = -\frac{9g^2 f^{ACD} f^{BDC}\delta_{\mu\nu}}{32}\int_0^\infty q dq e^{-4\mathcal{A}^2q^2} = -\frac{9g^2 f^{ACD} f^{BDC}\delta_{\mu\nu}}{256\mathcal{A}^2}.$$

Gauge invariance

Abelian theory

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x)$$

In view of linearity of CWT:
$A_{\mu(x);a} \to A_{\mu(x);a} + \partial_\mu \omega_a(x)$

Non-Abelian Theory

$A_\mu(x) \to U(x) A_\mu(x) U^+(x) - \imath U(x) \partial_\mu U^+(x)$

Nonlinear terms give commutator between different scales and we cannot set the gauge invariance over given scale

$$f^{ACB} T^B \int \frac{1}{a^d} g\left(\frac{x-b}{a}\right) \omega_a^A(b) \frac{dadb}{a} \frac{1}{a'^d} g\left(\frac{x-b'}{a'}\right) A^C_{\mu;a'}(b') \frac{da'db'}{a'}$$

Instantons

Pure gauge $A_\mu(x) = \imath U \partial_\mu U^+$
Instanton solution

$$A_\mu(x) = \frac{2\rho^2}{x^2+\rho^2} x_\nu \Lambda_{\mu\nu}$$

Cannot be used as basic wavelet for the nonvanishing behavior of its Fourier image at $k \to 0$. However the integration over affine group is the same as for wavelets.
Quantization with constraints: Bulut and Polyzou. 2013; J.-P. Gazeau et al. 2013;

This gives the Casimir energy

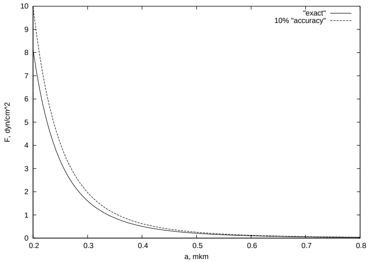
$$\mathcal{E}(a,\delta) = -\frac{\hbar c \pi^2}{720 a^3}\left[1 + \frac{2}{7}\left(\frac{2\pi\delta}{a}\right)^2 + \right.$$
$$\left. + \frac{3}{28}\left(\frac{2\pi\delta}{a}\right)^4 + \ldots\right],$$

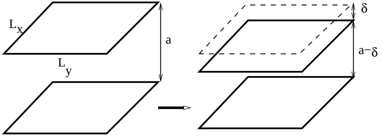
and the Casimir force

$$\mathcal{F}(a,\delta) = -\frac{\hbar c \pi^2}{240 a^4}\left[1 + \frac{10}{21}\left(\frac{2\pi\delta}{a}\right)^2 + \right.$$
$$\left. + \frac{1}{4}\left(\frac{2\pi\delta}{a}\right)^4 + \ldots\right],$$

Altaisky,Kaputkina *JETP Lett.* 94(2011)341

Deviation of Casimir force between two plates of unit area in vacuum. The solid line denotes the "exact" Casimir force ($\delta = 0$), the dashed line denotes the scale-dependent Casimir force with $\delta/a = 0.1$





THANK YOU FOR YOUR ATTENTION !!!

Perspectives

- QCD
- Instantons

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