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Lamb shift in muonic deuterium atom

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Outline

1. Effects of vacuum polarization in the one-photon interaction

2. Relativistic corrections with the vacuum polarization effects

3. Nuclear structure and vacuum polarization effects

4. Recoil corrections, muon self-energy and VP effects

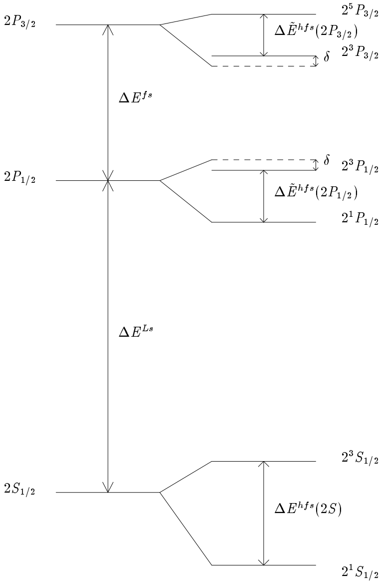
5. Numerical results and discussion

- R. Pohl, A. Antognini, F. Nez et al., Nature **466**, 213 (2010).
- E. Borie, Phys. Rev. A **72**, 052511 (2005); Phys. Rev. A **71**, 032508 (2005).
- K. Pachucki, Phys. Rev. A **53**, 2092 (1996).
- A.P. Martynenko, JETP **101**, 1021 (2005); Phys. Atom. Nucl. **69**, 1309 (2006).
- U.D. Jentschura, Phys. Rev. A **84**, 012505 (2011); arXiv.1107.1737.
- S.G. Karshenboim, V.G. Ivanov, E. Yu. Korzinin and V.A. Shelyuto, Phys. Rev. A **81**, 060501 (2010).
- K. Pachucki, Phys. Rev. Lett. **106**, 193007 (2011).

R. Pohl, A. Antognini, F. Nez et al., Nature **466**, 213 (2010).
The measurement of the (μp) Lamb shift transition energy $(2P^{F=2}_{3/2} - 2S^{F=1}_{1/2})$ at PSI (Paul Scherrer Institute) produced the result 49881.88 (76) GHz (206.2949 (32) meV).

- New value of the proton charge radius $r_p = 0.84184(36)(56)$ fm (experimental uncertainty of 0.76 GHz, the theoretical uncertainty 0.0049 meV).
- New value of proton radius differs by 5.0 standard deviations from the CODATA value $r_p = 0.8768(69) fm$

The structure of S-wave and P-wave energy levels in (μp).



In perturbation theory the basic contribution to the muon-deuteron interaction operator is determined by the Breit Hamiltonian:

$$H_B=\frac{\mathbf{p}^2}{2\mu}-\frac{Z\alpha}{r}-\frac{\mathbf{p}^4}{8m_1^3}-\frac{\mathbf{p}^4}{8m_2^3}+\frac{\pi Z\alpha}{2}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\delta(\mathbf{r})-$$

$$-\frac{Z\alpha}{2m_1m_2r}\left(\mathbf{p}^2+\frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2}\right)+\frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)(\mathbf{L}\boldsymbol{\sigma}_1)=H_0+\Delta V^B$$

The wave functions of $2S$- and $2P$-states are equal to

$$\psi_{200}(r)=\frac{W^{3/2}}{2\sqrt{2\pi}}e^{-\frac{Wr}{2}}\left(1-\frac{Wr}{2}\right),\psi_{21m}(r)=\frac{W^{3/2}}{2\sqrt{6}}e^{-\frac{Wr}{2}}WrY_{1m}(\theta,\phi)$$

The deuteron factor $\delta_I=0$ because we used the common definition of the deuteron charge radius $r_d^2=-6\frac{dF_C}{dQ^2}|_{Q^2=0}$ (K. Pachucki and S.G. Karshenboim, J. Phys. B **28**, L221 (1995); I.B. Khriplovich et al., PLA **221**, 370 (1996)).

The modification of the Coulomb potential due to the vacuum polarization in the coordinate representation

$$\frac{1}{k^2} \to \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi) d\xi \frac{1}{k^2+4m_e^2\xi^2}.$$

$$V_{VP}^C(r) = \frac{\alpha}{3\pi}\int_1^\infty d\xi \rho(\xi)\left(-\frac{Z\alpha}{r}e^{-2m_e\xi r}\right), \rho(\xi) = \frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4}$$

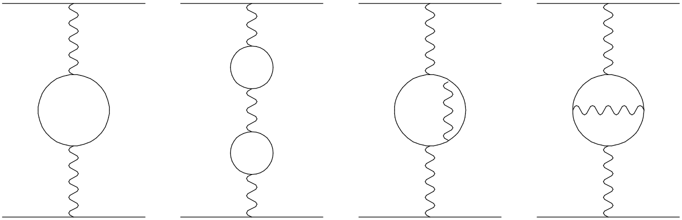


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Figure: Effects of one-loop and two-loop vacuum polarization in the one-photon interaction.

$$\Delta E_{VP}(2S)=-\frac{\mu(Z\alpha)^2\alpha}{6\pi}\int_1^\infty\rho(\xi)d\xi\int_0^\infty xdx\left(1-\frac{x}{2}\right)^2e^{-x\left(1+\frac{2m_e\xi}{W}\right)}=$$

$$=-245.3194\ meV.$$

$$\Delta E_{VP}(2P)=-\frac{\mu(Z\alpha)^2\alpha}{72\pi}\int_1^\infty\rho(\xi)d\xi\int_0^\infty x^3dxe^{-x\left(1+\frac{2m_e\xi}{W}\right)}=$$

$$=-17.6847\ meV.$$

$$\Delta E_{VP}(2P-2S)=227.6347\ meV,$$

$$V^C_{VP-VP}(r)=\frac{\alpha^2}{9\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\left(-\frac{Z\alpha}{r}\right)\times$$

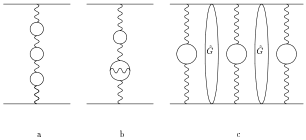
$$\times\frac{1}{(\xi^2-\eta^2)}\left(\xi^2e^{-2m_e\xi r}-\eta^2e^{-2m_e\eta r}\right)$$

$$\Delta V^C_{2-loop\ VP}=-\frac{2}{3}\frac{Z\alpha}{r}\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{(1-v^2)}e^{-\frac{2m_er}{\sqrt{1-v^2}}},$$

$$\Delta E_{2-loop\ VP}(2P-2S)=1.3704\ meV.$$

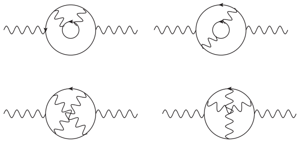
$$\Delta E_{VP-VP}(2P-2S)=0.2956\ meV.$$

Three loop VP contribution



$$\Delta E_{VP-VP-VP}(2P-2S) = 0.0005\ meV,$$

$$\Delta E_{VP-2-loop\ VP}(2P-2S) = 0.0034\ meV,$$

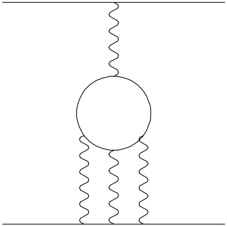


T. Kinoshita and M. Nio, PRL **82**, 3240 (1999); PRL **103**, 079901 (2009) (E)

S.G. Karshenboim et al., PRA **81**, 060501 (2010).

Our estimation of 3-loop VP contribution to the Lamb shift in (μd) is 0.0021 meV. Total 3-loop contribution in 1γ-interaction is 0.0060 meV

The Wichmann-Kroll correction



The particle interaction potential can be written in this case in the integral form:

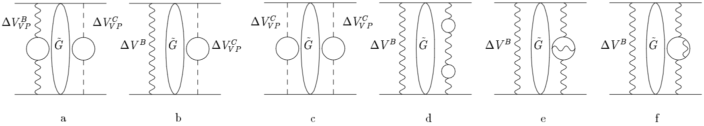
$$\Delta V^{WK}(r) = \frac{\alpha(Z\alpha)^3}{\pi r}\int_0^\infty \frac{d\zeta}{\zeta^4}e^{-2m_e\zeta r}\times$$

$$\times\left[-\frac{\pi^2}{12}\sqrt{\zeta^2-1}\theta(\zeta-1)+\int_0^\zeta dx\sqrt{\zeta^2-x^2}f^{WK}(x)\right].$$

The contribution to the Lamb shift:

$$\Delta E^{WK}(2P-2S) = -0.0011\ meV.$$

The one-loop vacuum polarization corrections in the Breit interaction



$$\Delta V^B_{VP}(r)=\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\sum_{i=1}^4\Delta V^B_{i,VP}(r),$$

$$\Delta V^B_{1,VP}=\frac{Z\alpha}{8}\left(\frac{1}{m_1^2}+\frac{\delta_l}{m_2^2}\right)\left[4\pi\delta(\mathbf{r})-\frac{4m_e^2\xi^2}{r}e^{-2m_e\xi r}\right],$$

$$\Delta V^B_{2,VP}=-\frac{Z\alpha m_e^2\xi^2}{m_1m_2r}e^{-2m_e\xi r}(1-m_e\xi r),$$

$$\Delta V^B_{3,VP}=-\frac{Z\alpha}{2m_1m_2}p_i\frac{e^{-2m_e\xi r}}{r}\left[\delta_{ij}+\frac{r_ir_j}{r^2}(1+2m_e\xi r)\right]p_j,$$

$$\Delta V^B_{4,VP}=\frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)e^{-2m_e\xi r}(1+2m_e\xi r)(\mathbf{L}\boldsymbol{\sigma}_1)$$

In first order perturbation theory the potentials $\Delta V^B_{i,VP}(r)$ give the contributions of order $\alpha(Z\alpha)^4$ to the shift $(2P-2S)$:

$$\Delta E^B_{1,VP}(2P-2S) = -0.0353\ meV,$$

$$\Delta E^B_{2,VP}(2P-2S) = 0.0011\ meV,$$

$$\Delta E^B_{3,VP}(2P-2S) = 0.0012\ meV,$$

$$\Delta E^B_{4,VP}(2P-2S) = -0.0023\ meV.$$

Two-loop modification of the Breit Hamiltonian

$$\Delta V^B_{2-loop\ VP}(r) = \frac{\alpha^2(Z\alpha)}{12\pi^2}\left(\frac{1}{m_1^2}+\frac{\delta_l}{m_2^2}\right)\int_0^1 \frac{f(v)dv}{1-v^2}\times$$

$$\times\left[4\pi\delta(\mathbf{r}) - \frac{4m_e^2}{(1-v^2)r}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\right].$$

Corresponding $(2P-2S)$ shift is the following:

$$\Delta E^B_{2-loop\ VP}(2P-2S) = -0.0002\ meV.$$

VP corrections in second order perturbation theory of orders $\alpha^2(Z\alpha)^2$ and $\alpha(Z\alpha)^4$

$$\Delta E^{VP}_{SOPT} =<\psi|\Delta V^C_{VP}\tilde G\Delta V^C_{VP}|\psi> +2<\psi|\Delta V^B\tilde G\Delta V^C_{VP}|\psi> .$$

$$\Delta E^{VP,VP}_{SOPT}(2S) = -\frac{\mu\alpha^2(Z\alpha)^2}{72\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\times$$

$$\times\int_0^\infty\left(1-\frac{x}{2}\right)e^{-x\left(1+\frac{2m_e\xi}{W}\right)}dx\int_0^\infty\left(1-\frac{x'}{2}\right)e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}dx' g_{2S}(x,x') =$$

$$= -0.1750\ meV,$$

$$\Delta E^{VP,VP}_{SOPT}(2P) = -\frac{\mu\alpha^2(Z\alpha)^2}{7776\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\times$$

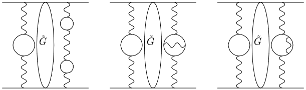
$$\times\int_0^\infty e^{-x\left(1+\frac{2m_e\xi}{W}\right)}dx\int_0^\infty e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}dx' g_{2P}(x,x') = -0.0030\ meV,$$

VP corrections in second order perturbation theory of orders $\alpha^2(Z\alpha)^2$ and $\alpha(Z\alpha)^4$

$$\Delta E^{B,VP}_{SOPT}(2P-2S)=0.0530\ meV.$$

$$\Delta E^{VP,VP;\Delta V^B}_{SOPT}(2P-2S)=0.0004\ meV.$$

$$\Delta E^{VP,\Delta V^B_{VP}}_{SOPT}(2P-2S)=-0.00001\ meV.$$



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$$\Delta E^{VP-VP,VP}_{SOPT}(2S)=-\frac{\mu\alpha^3(Z\alpha)^2}{108\pi^3}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\int_1^\infty\rho(\zeta)d\zeta\int_0^\infty dx(1-\frac{x}{2})\times$$

$$\int_0^\infty dx'(1-\frac{x'}{2})e^{-x'(1+\frac{2m_e\zeta}{W})}\frac{1}{\xi^2-\eta^2}\left[\xi^2e^{-x(1+\frac{2m_e\xi}{W})}-\eta^2e^{-x(1+\frac{2m_e\eta}{W})}\right]g_{2S}(x,x')=-0.0007\,meV,$$

$$\Delta E^{2-loop\ VP,VP}_{SOPT}(2S)=-\frac{\mu\alpha^3(Z\alpha)^2}{18\pi^3}\int_0^1\frac{f(v)dv}{1-v^2}\int_1^\infty\rho(\xi)d\xi\times$$

$$\times\int_0^\infty dx\left(1-\frac{x}{2}\right)e^{-x(1+\frac{2m_e}{\sqrt{1-v^2}W})}\int_0^\infty dx'\left(1-\frac{x'}{2}\right)e^{-x'(1+\frac{2m_e\xi}{W})}g_{2S}(x,x')=-0.0018\,meV,$$

VP correction in third order perturbation theory

$$\delta E^{(1)}_{TOPT} = < \psi_n|\Delta V_{VP}\tilde{G}_n\delta V_{VP}\tilde{G}_n\delta V_{VP}|\psi_n >$$

Another type of third order correction (V.G. Ivanov et al. PRD 80, 027702 (2009)):

$$\delta E^{(2)}_{TOPT} = < \psi_n|\Delta V_{VP}|\psi_n >< \psi_n|\delta V_{VP}\tilde{G}_n\tilde{G}_n\delta V_{VP}|\psi_n >$$

$$\Delta E^{VP,VP,VP}_{TOPT}(2P-2S) = 0.0001\ meV$$

We present the charge radius corrections at two values of r_d:
$r_d = 2.1424(21)$ fm (CODATA 2010) and $r_d = 2.130(9)$ fm (I. Sick):

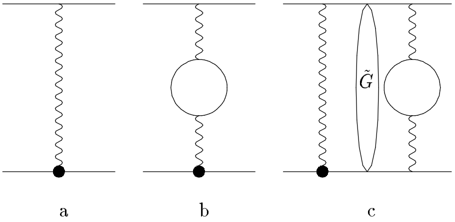


Figure: The leading order nuclear structure and vacuum polarization corrections.

$$\Delta E_{str}(2P-2S) = -\frac{\mu^3(Z\alpha)^4}{12} r_d^2 = -6.07313 \cdot r_d^2 =$$
$$= -27.8749(-27.5532)\ meV.$$

VP and deuteron structure corrections

$$\Delta V^{VP}_{str}(r)=\frac{2Z\alpha^2}{9}r_d^2\int_1^\infty\rho(\xi)d\xi\left[\delta(\mathbf{r})-\frac{m_e^2\xi^2}{\pi r}e^{-2m_e\xi r}\right].$$

$$\Delta E^{VP}_{str}(2P-2S)=-\frac{\mu^3\alpha(Z\alpha)^4}{36\pi}r_d^2\int_1^\infty\rho(\xi)d\xi\left[1-\frac{16m_e^4\xi^4}{(2m_e\xi+W)^2}\right]=$$

$$=-0.01350\cdot r_d^2=-0.0620(-0.0612)\ meV,$$

$$\Delta E^{VP}_{str,SOPT}(2P-2S)=-\frac{\mu^3\alpha(Z\alpha)^4}{36\pi}r_d^2\int_1^\infty\rho(\xi)d\xi\times$$

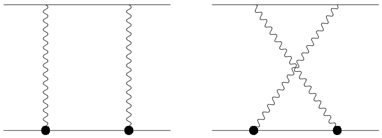
$$\times\frac{-12+23b_1-8b_1^2-4b_1^3+4b_1^4+4b_1(3-4b_1+2b_1^2)\ln b_1}{b_1^5}=$$

$$=-0.020487\cdot r_d^2\ meV=-0.0940(-0.0929)\ meV,\ b_1=1+\frac{2m_e}{W}\xi.$$

$$\Delta E_{str}(2P-2S)+\Delta E^{VP}_{str}(2P-2S)+\Delta E^{VP}_{str,SOPT}(2P-2S)=$$

$$=-6.10712\cdot r_d^2=-28.0309(-27.7074)\ meV.$$

Nuclear structure corrections of order $(Z\alpha)^5$

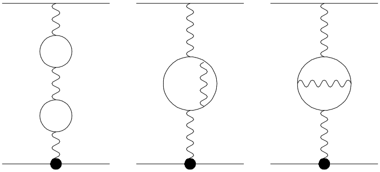


An investigation of elastic contribution to the Lamb shift and the deuteron polarizability contribution:

- J.L. Friar, PRC **16**, 1540 (1977).
- J.L. Friar, J. Martorell and D.W.L. Sprung, PRA **56**, 4579 (1997).
- Y. Lu and R. Rosenfelder, PLB **319**, 7 (1993); W. Leidemann and R. Rosenfelder, PRC **51**, 427 (1995).
- R.N. Faustov and A.P. Martynenko, PRA **67**, 052506 (2003); Phys. Atom. Nucl. **67**, 457 (2004).
- K. Pachucki, PRL**106**, 193007 (2011).

The elastic two-photon contribution is canceled by part of the deuteron polarizability contribution (K. Pachucki 2011): 1.680(16) meV

Two-loop VP and structure corrections



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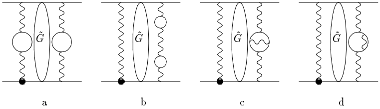
$$\Delta V_{str}^{VP-VP}(r) = \frac{2Z\alpha^3}{27\pi^2} r_d^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times$$

$$\times \left[\pi\delta(\mathbf{r}) - \frac{m_e^2}{r(\xi^2-\eta^2)} \left(\xi^4 e^{-2m_e\xi r} - \eta^4 e^{-2m_e\eta r}\right)\right],$$

$$\Delta V_{str}^{2-loop\ VP}(r) = \frac{4Z\alpha^3}{9\pi^2} r_d^2 \int_0^1 \frac{f(v)dv}{1-v^2} \left[\pi\delta(\mathbf{r}) - \frac{m_e^2}{r(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\right].$$

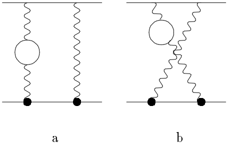
$$\Delta E_{str}^{VP,VP}(2P-2S) = -10.5 \cdot 10^{-5} \cdot r_d^2 = -0.0005(0.0005)\ meV.$$

Two-loop vacuum polarization and nuclear structure corrections of order $\alpha^2(Z\alpha)^4$ in second order PT



$$\Delta E^{VP,VP}_{str,SOPT}(2P-2S) = -9.5\cdot 10^{-5}\cdot r_d^2 = -0.0004(0.0004)\ meV.$$

Nuclear structure correction of order $\alpha(Z\alpha)^5$ from two-photon exchange diagrams with VP insertion



The muon matrix element P_{VP} for nonrelativistic two-photon exchange with an account of the vacuum polarization:

$$P_{VP} = \frac{2\alpha^3}{3\pi}\phi^2(0)\int_1^\infty \rho(\xi)d\xi\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{(4\pi)^2}{q^2(q^2+4m_e^2\xi^2)}\frac{1}{E+\frac{q^2}{2m_1}}\left[e^{i\mathbf{q}(\mathbf{R}-\mathbf{R}')}-1+\frac{q^2}{6}(\mathbf{R}-\mathbf{R}')^2\right]$$

Integrating over q and expanding the resulting expression over small parameter $\sqrt{2m_1E}|\mathbf{R}-\mathbf{R}'|$ we obtain:

$$P_{VP}=\frac{32\alpha^3}{3}m_1\phi^2(0)|\mathbf{R}-\mathbf{R}'|^3\int_1^\infty\rho(\xi)d\xi\Bigg[\frac{a_\xi^3-3a_\xi^2+6a_\xi+6e^{-a_\xi}-6}{12a_\xi^4}-$$

$$-2m_1E|\mathbf{R}-\mathbf{R}'|^2\frac{a_\xi^4-4a_\xi^3+12a_\xi^2-24a_\xi-24e^{-a_\xi}+24}{48a_\xi^6}\Bigg],\quad a_\xi=2m_e\xi|\mathbf{R}-\mathbf{R}'|.$$

Expanding second term at small a_ξ we obtain $(-\pi/240a_\xi)$. Integrating over ξ and summing over excited deuteron states we obtain the contribution to the Lamb shift:

$$\delta E^{VP}_{pol}(2P-2S)=-\frac{m_1^2\alpha^3\phi^2(0)}{1024m_e}\Bigg[\frac{1}{3}\langle\phi_D|R^2H_DR^2|\phi_D\rangle-\frac{4}{5}\langle\phi_D|R_iH_DR^2R_i|\phi_D\rangle+$$

$$+\frac{2}{5}\langle\phi_D|(R_iR_j-\frac{1}{3}\delta_{ij}R^2)H_D(R_iR_j-\frac{1}{3}\delta_{ij}R^2)|\phi_D\rangle\Bigg]=-0.0001\,meV,$$

ϕ_D is the deuteron wave function: (I.B. Khriplovich and A.I. Milstein, JETP **98**, 181 (2004))

$$\phi_D(r)=\sqrt{\frac{\kappa}{2\pi}}\frac{1}{r}e^{-\kappa r},$$

$\kappa=0.0457$ GeV is the inverse deuteron size.

Muon-line radiative correction to the nuclear size effect of order $\alpha(Z\alpha)^5$ (M.I. Eides et al., PRA **55**, 2447 (1997)):

$$\Delta E_{str}^{\alpha(Z\alpha)^5}(2P-2S)=1.985\frac{\alpha(Z\alpha)^5\mu^3}{8}r_d^2=9.62\cdot 10^{-4}\cdot r_d^2=0.0044(0.0044)\,meV.$$

Nuclear structure corrections of order $(Z\alpha)^6$

- J.L. Friar, Ann. Phys. **122**, 151 (1979).
- J.L. Friar and G.L. Payne, PRA **56**, 5173 (1997).

$$\Delta E_{str}^{(Z\alpha)^6}(2P-2S)=\frac{(Z\alpha)^6}{12}\mu^3\left\{r_d^2\left[\langle\ln\mu Z\alpha r\rangle+C-\frac{3}{2}\right]-\frac{1}{2}r_d^2+\frac{1}{3}\langle r^3\rangle\langle\frac{1}{r}\rangle-\right.$$

$$\left.-I_2^{rel}-I_3^{rel}-\mu^2F_{NR}+\frac{1}{40}\mu^2\langle r^4\rangle\right\}=-21.28\cdot 10^{-4}\cdot r_d^2+0.0029=-0.0069(-0.0068)\,meV,$$

The recoil correction of order $(Z\alpha)^4\mu^3/m_2^2$(U.D. Jentschura, PRA **84**, 012505 (2011)):

$$\Delta E_{rec}(2P-2S)=\frac{\mu^3(Z\alpha)^4}{12m_2^2}=0.0672\ meV.$$

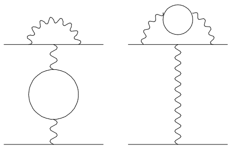


Figure: Radiative corrections with the vacuum polarization effects.

The recoil correction of fifth order in $(Z\alpha)$

$$\Delta E_{rec}^{(Z\alpha)^5}=\frac{\mu^3(Z\alpha)^5}{m_1m_2\pi n^3}\left[\frac{2}{3}\delta_{l0}\ln\frac{1}{Z\alpha}-\frac{8}{3}\ln k_0(n,l)-\frac{1}{9}\delta_{l0}-\frac{7}{3}a_n-\frac{2}{m_2^2-m_1^2}\delta_{l0}(m_2^2\ln\frac{m_1}{\mu}-m_1^2\ln\frac{m_2}{\mu})\right],$$

$$\ln k_0(2S)=2.811769893120563, \ln k_0(2P)=-0.030016708630213,$$

$$a_n=-2\left[\ln\frac{2}{n}+(1+\frac{1}{2}+\ldots+\frac{1}{n})+1-\frac{1}{2n}\right]\delta_{l0}+\frac{(1-\delta_{l0})}{l(l+1)(2l+1)}.$$

$$\Delta E_{rec}^{(Z\alpha)^5}(2P-2S)=-0.0266\ meV.$$

The recoil correction of the sixth order in $(Z\alpha)$:

$$\Delta E_{rec}^{(Z\alpha)^6}(2P-2S)=\frac{(Z\alpha)^6 m_1^2}{8m_2}\left(\frac{23}{6}-4\ln 2\right)=0.0001\ meV.$$

The energy contributions from radiative corrections to the lepton line, the Dirac and Pauli form factors and muon vacuum polarization:

$$\Delta E_{MVP,MSE}(2S)=\frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left[\frac{4}{3}\ln\frac{m_1}{\mu(Z\alpha)^2}-\frac{4}{3}\ln k_0(2S)+\frac{38}{45}+\right.$$

$$\left.+\frac{\alpha}{\pi}\left(-\frac{9}{4}\zeta(3)+\frac{3}{2}\pi^2\ln 2-\frac{10}{27}\pi^2-\frac{2179}{648}\right)+4\pi Z\alpha\left(\frac{427}{384}-\frac{\ln 2}{2}\right)\right]=0.7647\ meV,$$

$$\Delta E_{MVP,MSE}(2P)=\frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left[-\frac{4}{3}\ln k_0(2P)-\frac{m_1}{6\mu}-\right.$$

$$\left.-\frac{\alpha}{3\pi}\frac{m_1}{\mu}\left(\frac{3}{4}\zeta(3)-\frac{\pi^2}{2}\ln 2+\frac{\pi^2}{12}+\frac{197}{144}\right)\right]=-0.0100\ meV.$$

Electron loop polarization insertion in the radiative photon can be expressed in terms of F_1' and F_2

$$\Delta E_{rad+VP}(nS) = \frac{\mu^3}{m_1^2}\frac{(Z\alpha)^4}{n^3}\left[4m_1^2F_1'(0)\delta_{l0} + F_2(0)\frac{C_{jl}}{2l+1}\right],$$

$$C_{jl} = \delta_{l0} + (1-\delta_{l0})\frac{[j(j+1) - l(l+1) - \frac{3}{4}]}{l(l+1)}\frac{m_1}{\mu}.$$

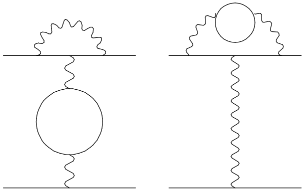
Two-loop contribution to the form factors $F_1'(0)$ and $F_2(0)$:

$$m_1^2F_1'(0) = \left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{9}\ln^2\frac{m_1}{m_e} - \frac{29}{108}\ln\frac{m_1}{m_e} + \frac{1}{9}\zeta(2) + \frac{395}{1296}\right],$$

$$F_2(0) = \left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{3}\ln\frac{m_1}{m_e} - \frac{25}{36} + \frac{\pi^2}{4}\frac{m_e}{m_1} - 4\frac{m_e^2}{m_1^2}\ln\frac{m_1}{m_e} + 3\frac{m_e^2}{m_1^2}\right].$$

Then the correction to the Lamb shift is equal to

$$\Delta E_{rad+VP}(2P-2S) = -0.0018\, meV.$$



a b

The estimatation of the muon self-energy and electron vacuum polarization contribution in Fig.11(a)

$$\Delta E^{VP}_{MSE}=\frac{\alpha}{3\pi m_1^2}\ln\frac{m_1}{\mu(Z\alpha)^2}\left[<\psi_n|\Delta\cdot\Delta V^C_{VP}|\psi_n>+2<\psi_n|\Delta V^C_{VP}\tilde{G}\Delta\left(-\frac{Z\alpha}{r}\right)|\psi_n>\right].$$

The sum of all matrix elements leads to the following shift $(2P-2S)$:

$$\Delta E^{VP}_{MSE}(2P-2S)=-0.0047\;meV.$$

TABLE I: Lamb shift $(2P_{1/2} - 2S_{1/2})$ in muonic deuterium atom.

Contribution to the splitting	$\Delta E(2P-2S)$, meV	Equation, Reference
1	2	3
VP contribution of order $\alpha(Z\alpha)^2$ in one-photon interaction	227.6347	(6), [2]
Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in one-photon interaction	1.6660	(9), (14), [2]
VP and MVP contribution in one-photon interaction	0.0001	(11) , [2]
Three-loop VP contribution in one-photon interaction	0.0060	(17), (18), [29, 31]
The Wichmann-Kroll correction	-0.0011	(21), [2, 31]
Light-by-light contribution	0.0001	[31]
Relativistic and VP corrections of order $\alpha(Z\alpha)^4$ in first order PT	-0.0353	(27)-(30), [5]
Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in first order PT	-0.0002	(32)
Two-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in second order PT	0.1720	(39)-(41) , [31]
Relativistic and one-loop VP corrections of order $\alpha(Z\alpha)^4$ in second order PT	0.0530	(43), [5]
Relativistic and two-loop VP corrections of order $\alpha^2(Z\alpha)^4$ in second order PT	0.0004	(44)-(45)
Three-loop VP contribution in second order PT of order $\alpha^3(Z\alpha)^2$	0.0025	(46)-(47), [31]
Three-loop VP contribution in third order PT of order $\alpha^3(Z\alpha)^2$	0.0001	(48), [29, 31],
Nuclear structure contribution of order $(Z\alpha)^4$	-27.8749	(49), [2, 6]
Nuclear structure and polarizability contribution of order $(Z\alpha)^5$	1.6800	[51]
Nuclear structure and VP contribution in 1γ interaction of order $\alpha(Z\alpha)^4$	-0.0620	(51)
Nuclear structure and VP contribution in second order PT of order $\alpha(Z\alpha)^4$	-0.0940	(52)
Nuclear structure and two-loop VP contribution in 1γ interaction of order $\alpha^2(Z\alpha)^4$	-0.0005	(56)
Nuclear structure and two-loop VP contribution in second order PT of order $\alpha^2(Z\alpha)^4$	-0.0004	(57)

Table I (continued).

1	2	3
Nuclear structure and polarizability contribution of order $\alpha(Z\alpha)^5$ with VP correction	-0.0001	(60)
Nuclear structure contribution of order $\alpha(Z\alpha)^5$ with muon-line radiative correction	0.0044	(62), [61]
Nuclear structure contribution of order $(Z\alpha)^6$	-0.0069	(63), [22, 54]
Recoil correction of order $(Z\alpha)^4$	0.0672	(64), [5]
Recoil correction of order $(Z\alpha)^5$	-0.0266	(69), [2, 6, 55]
Recoil correction of order $(Z\alpha)^6$	0.0001	(70), [6]
Recoil correction to VP of order $\alpha(Z\alpha)^5$ (seagull term)	0.0002	[5]
Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$	-0.0026	(71), Table 9 [6]
Muon self-energy and MVP contribution	-0.7747	(72)-(73), [2, 6]
VP contribution to muon form factors $F_1'(0)$, $F_2(0)$ of order $\alpha^2(Z\alpha)^4$	-0.0018	(78), [4, 6, 63]
VP correction to muon self-energy	-0.0047	(80), [4, 6]
HVP contribution	0.0129	[64, 65]
Total contribution	202.4139 ± 0.0573 ($r_d = 2.1424(21)$ fm) 202.7375 ± 0.2352 ($r_d = 2.130(9)$ fm)	

TABLE II: Fine structure of $2P$-state in muonic deuterium atom.

Contribution to fine splitting ΔE^{fs}	Numerical value in meV	Equation, Reference
Contribution of order $(Z\alpha)^4$ $\frac{\mu^3(Z\alpha)^4}{32m_1^2}\left(1+\frac{2m_1}{m_2}\right)$	8.83848	(82), [2, 6]
Muon AMM contribution $\frac{\mu^3(Z\alpha)^4}{16m_1^2}a_\mu\left(1+\frac{m_1}{m_2}\right)$	0.01957	(82), [2, 6]
Contribution of order $(Z\alpha)^6$	0.00031	(82), [2, 6]
Contribution of order $(Z\alpha)^6 m_1/m_2$	-0.00001	(82), [2, 6]
Contribution of order $\alpha(Z\alpha)^4$ in first order PT $\langle\Delta V^{fs}_{VP}\rangle$	0.00346	(84)
Contribution of order $\alpha(Z\alpha)^4$ in second order PT $\langle\Delta V^C_{VP}\cdot\tilde G\cdot\Delta V^{fs}\rangle$	0.00229	(85)
Contribution of order $\alpha(Z\alpha)^6$ $\frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^2}\left[\ln\frac{\mu(Z\alpha)^2}{m_1}+\frac{1}{5}\right]$	-0.00001	(82), [6]
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^{fs}_{VP-VP}\rangle$	0.000003	(87)
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^{fs}_{2-loop,VP}\rangle$	0.00002	(89)
VP Contribution in second order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{VP}\cdot\tilde G\cdot\Delta V^{fs}_{VP}\rangle$	0.000002	Fig.4(a), $\Delta V^B\to\Delta V^{fs}$
VP Contribution in second order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{VP-VP}\cdot\tilde G\cdot\Delta V^{fs}\rangle$	-0.000001	Fig.4(d), $\Delta V^B\to\Delta V^{fs}$
VP Contribution in second order PT of order $\alpha^2(Z\alpha)^4$ $\langle\Delta V^C_{2-loop,VP}\cdot\tilde G\cdot\Delta V^{fs}\rangle$	0.000026	(90), Fig.4(e-f), $\Delta V^B\to\Delta V^{fs}$
Nuclear structure correction in 1γ interaction	-0.00028	(91), [22]
Summary contribution	8.86386	

Total numerical value 202.4139 meV of the Lamb shift $(2P-2S)$ in muonic deuterium atom agrees with the E. Borie result 202.263 meV. The difference is connected with the calculation of new contributions of higher order in α and m_1/m_2, the proton structure and polarizability correction and slightly different numerical value of the charge radius of the deuteron r_d ($r_d = 2.139(3)$). Two-loop vacuum polarization contribution 0.1720 meV of order $\alpha^2(Z\alpha)^2$ in second order PT is absent. Total value of the Lamb shift $(2P_{1/2} - 2S_{1/2})$ for $r_d = 2.130$ fm amounts 202.9440 meV (E. Borie 2011) This value exceeds our result 202.7375 meV by 0.2065 meV.

The Zemach correction 0.4329 meV and polarizability correction 1.5 meV. It was shown by K. Pachucki that the Zemach correction is canceled by the part of deuteron excited states contribution. As a result the nuclear structure and polarizability contribution is equal to 1.680 meV.

Final theoretical value of the $(2P-2S)$ Lamb shift in the form $\Delta E^{Ls}(2P-2S) = (230.4511 - 6.108485 \cdot r_d^2)$ meV with the value of the deuteron charge radius defined in fm. Then, comparing this expression with the experimental value of the Lamb shift measured with the precision 0.01 meV (50 ppm) we can obtain more accurate value of r_d with an accuracy 0.0005 fm.