

II Workshop on Precision Physics and Fundamental Physical Constants

Fine and hyperfine structure of the energy spectrum of light muonic atoms

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Outline

1. Hyperfine splitting of 1S- and 2S-states in muonic atoms
2. Fine structure of P-wave energy levels in muonic atoms.
3. Hyperfine splitting of P-wave states in muonic atoms.
4. Hyperfine splitting of the ground state in muonic helium.
5. Proton polarizability contribution to HFS in muonic atoms.

- ▶ R.N. Faustov, A.P. Martynenko, EPJ **C24**, 281, (2002).
- ▶ E.V. Cherednikova, R.N. Faustov, A.P. Martynenko, Nucl. Phys. **A703**, 365 (2002).
- ▶ R.N. Faustov, A.P. Martynenko, JETP **98**, 39 (2004).
- ▶ A.P. Martynenko, Phys. Rev. **A71**, 022506 (2005).
- ▶ A.P. Martynenko, ZhETF **128**, 1169 (2005).
- ▶ A.P. Martynenko, Phys. Rev. **A76**, 012505 (2007).
- ▶ A.P. Martynenko, Phys. Atom. Nucl. **71**, 126 (2008).
- ▶ A.P. Martynenko, ZhETF **133**, 794 (2008).
- ▶ A.A. Krutov, A.P. Martynenko, Phys. Rev. **A78**, 032513 (2008).
- ▶ E.N. Elekina, A.P. Martynenko, arXiv:0909.2759

Light muonic atoms: muonic hydrogen (μp), muonic deuterium (μd), ions of muonic helium ($\mu_2^3\text{He}^+$), ($\mu_2^4\text{He}^+$), muonic helium.

Experimental investigations:

1. D.Bakalov et al. Proc. III Int. Symp. on Weak and Electr. Inter. in Nuclei Dubna, 1992. Measurement of the hyperfine splitting of muonic hydrogen; A. Adamczak, D. Bakalov, K. Bakalova et al. Hyperfine Interact. **136**, 1 (2001); A.Dupays et al. Phys. Rev. A **68**, 052503 (2003).
2. F. Kottmann et al. QED and Physics of the vacuum, AIP Conf. Proc. v.564 p.13., Muonic hydrogen Lamb shift (2P-2S) measurements at PSI with a precision of 30 ppm.
3. G. Carboni et al. Nucl. Phys. A278, 381 (1977), Measurements of the energy intervals ($2P_{3/2} - 2S_{1/2}$) (1527.5 meV) and ($2P_{1/2} - 2S_{1/2}$) (1381.29 meV) in the ion of muonic helium ($\mu_2^4\text{He}^+$) at the muon beam of the CERN.
4. P. Hauser et al. PRA 46, 2363 (1992).
5. Hyperfine splitting of the ground state of muonic helium was measured many years ago with sufficiently high accuracy: $\Delta\nu_{\text{exp}}^{\text{hfs}} = 4465.004(29)$ MHz. C.J.Gardner et al., Phys. Rev.

Light muonic atoms. Theoretical investigations:

1. A.Di Giacomo, Nucl. Phys. B **11**, 411 (1969).
2. E. Borie, Z. Phys. A **275**, 347 (1975).
3. E. Borie, G. A. Rinker, Phys. Rev. A **18**, 324 (1978); Rev. Mod. Phys. **54**, 67 (1982).
4. E. Borie, Z. Phys. A **278**, 127 (1976); A **297**, 17 (1980).
5. S.D. Lakdawala, P.J. Mohr, Phys. Rev. A **24**, 2224 (1981); Phys. Rev. A **22**, 1572 (1980); K.N. Huang, V.W. Hughes, Phys. Rev. A **20**, 706 (1979); R. J. Drachman, Phys. Rev. A **22**, 1755 (1980); M. Ya. Amusia et al. J. Phys. B **16**, L71 (1983); M.K. Chen, Phys. Rev. A **45**, 1479 (1992); A.M. Frolov, Phys. Rev. A **61**, 022509 (2000).
6. K. Jungmann, V.W. Hughes, G. zu Pultitz (Eds.), Proc. Int. Symp. "The future of muon physics" 1991, Heidelberg.
7. K. Pachucki, Phys. Rev. A **53**, 2092 (1996); A. Veitia, K. Pachucki, Phys. Rev. A **69**, 042501 (2004).
8. S.G. Karshenboim, V.B. Smirnov (Eds.) Precision physics of simple atomic systems, LNP, v.627, 2003.

Hyperfine splitting of 1S- and 2S-states in muonic helium ion ($\mu_2^3\text{He}$)⁺

All contributions to the energy spectrum of light muonic atoms can be divided into two groups:

1. Corrections obtained in analytical form when studying the spectra of muonium and hydrogen atom.

- ▶ M.I. Eides, H. Grotch, V.A. Shelyuto, Phys. Rep. **342**, 62 (2001).
- ▶ S.G. Karshenboim, Phys. Rep. **422**, 1 (2005).

2. Contributions of the electron vacuum polarization, nuclear structure and recoil effects which are specific for each muonic atom.

Modern numerical values of fundamental physical constants are taken from

P.J.Mohr, B.N.Taylor, D.B. Newell, Rev. Mod. Phys. **80**, 633 (2008).

- ▶ the electron mass $m_e = 0.510998910(13) \cdot 10^{-3} \text{ GeV}$
- ▶ the muon mass $m_\mu = 0.1056583668(38) \text{ GeV}$
- ▶ the fine structure constant $\alpha^{-1} = 137.035999679(94)$
- ▶ the proton mass $m_p = 0.938272013(23) \text{ GeV}$
- ▶ the helion mass $2.808391383(70) \text{ GeV}$
- ▶ the helion magnetic moment $\mu_h = -2.127497723(25)$
- ▶ the muon anomalous magnetic moment $a_\mu = 1.16592069(60) \cdot 10^{-3}$.

Basic contribution to the interaction operator in two-particle bound S-state is determined by the Breit Hamiltonian:

$$\begin{aligned}
 H_B &= H_0 + \Delta V_B^{fs} + \Delta V_B^{hfs}, \quad H_0 = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r}, \\
 \Delta V_B^{fs} &= -\frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta(\mathbf{r}) - \\
 &\quad - \frac{Z\alpha}{2m_1 m_2 r} \left(\mathbf{p}^2 + \frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2} \right), \\
 \Delta V_B^{hfs} &= \frac{8\pi\alpha\mu_h}{3m_1 m_p} \frac{\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2}{4} \delta(\mathbf{r}), \\
 \psi_{100}(r) &= \frac{W^{3/2}}{\sqrt{\pi}} e^{-Wr}, \quad W = \mu Z\alpha, \\
 \psi_{200}(r) &= \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-Wr/2} \left(1 - \frac{Wr}{2} \right),
 \end{aligned}$$

The energy difference of the triplet and singlet states:

$$\Delta E_F^{hfs}(nS) = \frac{8\mu^3 Z^3 \alpha^4 \mu_h}{3m_1 m_p n^3} = \begin{cases} 1S: -1370.725 \text{ meV} \\ 2S: -171.341 \text{ meV} \end{cases}.$$

$$\Delta E_{a_\mu}^{hfs}(nS) = a_\mu \Delta E_F^{hfs}(nS) = \begin{cases} 1S: -1.598 \text{ meV} \\ 2S: -0.200 \text{ meV} \end{cases}.$$

The contribution of the relativistic effects of order α^6 :

$$\begin{aligned} \Delta E_{rel}^{hfs}(nS) &= \left[1 + \frac{11n^2 + 9n - 11}{6n^2} (Z\alpha)^2 + \dots \right] \Delta E_F^{hfs}(nS) = \\ &= \begin{cases} 1S: -0.438 \text{ meV} \\ 2S: -0.078 \text{ meV} \end{cases} \end{aligned}$$

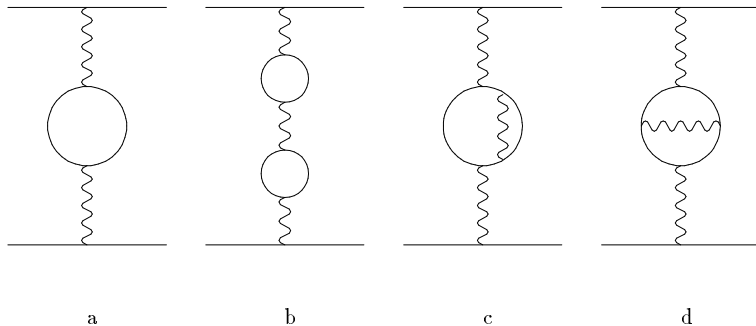


Figure: Effects of one-loop and two-loop vacuum polarization in the one-photon interaction.

One-loop electron vacuum polarization correction:

$$\Delta V_{1\gamma,VP}^{hfs}(r) = \frac{8\alpha\mu_h}{3m_1m_2} \frac{\sigma_1\sigma_2}{4} \frac{\alpha}{3\pi} \int_1^\infty \rho(s) ds \left[\pi\delta(\mathbf{r}) - \frac{m_e^2\xi^2}{r} \right] e^{-2m_e\xi r},$$

$$\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \frac{1}{k^2 + 4m_e^2\xi^2}, \rho(\xi) = \sqrt{\xi^2 - 1}(2\xi^2 + 1)/\xi^4.$$

$$\Delta E_{1\gamma,VP}^{hfs}(1S) = \frac{8\mu^3 Z^3 \alpha^5 \mu_h}{9m_1 m_p \pi} \int_1^\infty \rho(\xi) d\xi \times$$

$$\left[1 - \frac{4m_e^2\xi^2}{W^2} \int_0^\infty x dx e^{-x(1+\frac{m_e\xi}{W})} \right] = -4.203 \text{ meV},$$

$$\Delta E_{1\gamma,VP}^{hfs}(2S) = \frac{\mu^3 Z^3 \alpha^5 \mu_h}{9m_1 m_p \pi} \int_1^\infty \rho(\xi) d\xi \times$$

$$\left[1 - \frac{4m_e^2\xi^2}{W^2} \int_0^\infty x \left(1 - \frac{x}{2}\right)^2 dx e^{-x(1+\frac{2m_e\xi}{W})} \right] = -0.540 \text{ meV}.$$

Two-loop electron vacuum polarization correction:

- M.I.Eides, H.Grotch, V.A.Shelyuto, Phys. Rev. D **65**, 013003 (2001)

$$\begin{aligned} \Delta V_{1\gamma, VP-VP}^{hfs}(r) &= \frac{8\pi\alpha\mu_h}{3m_1m_p} \frac{\sigma_1\sigma_2}{4} \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times \\ &\times \left[\delta(\mathbf{r}) - \frac{m_e^2}{\pi r(\eta^2 - \xi^2)} \left(\eta^4 e^{-2m_e\eta r} - \xi^4 e^{-2m_e\xi r} \right) \right]. \\ \Delta E_{1\gamma, VP-VP}^{hfs}(1S) &= \frac{8\alpha^6\mu^3Z^3\mu_h}{27m_1m_p} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times \\ &\times \left[1 - \frac{4m_e^2}{W^2(\eta^2 - \xi^2)} \int_0^\infty e^{-2x} x dx \left(\eta^4 e^{-2m_e\eta x/W} - \xi^4 e^{-2m_e\xi x/W} \right) \right] = \\ &= -0.017 \text{ meV}, \end{aligned}$$

$$\Delta E_{1\gamma, VP-VP}^{hfs}(r)(2S) = \frac{\alpha^6 \mu^3 Z^3 \mu_h}{27 m_1 m_p} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \times$$

$$\times \left[1 - \frac{4m_e^2}{W^2(\eta^2 - \xi^2)} \int_0^\infty e^{-x} x dx \left(\eta^4 e^{-2m_e \eta x/W} - \right. \right.$$

$$\left. \left. - \xi^4 e^{-2m_e \xi x/W} \right) \left(1 - \frac{x}{2} \right)^2 \right] = -0.002 \text{ meV}.$$

$$\Delta V_{1\gamma, 2-loop VP}^{hfs}(r) = \frac{8\alpha^3 \mu_h}{3\pi m_1 m_p} \int_0^1 \frac{f(v) dv}{1-v^2} \left[\delta(\mathbf{r}) - \frac{m_e^2}{\pi r(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}} \right],$$

$$f(v) = v \left\{ (3-v^2)(1+v^2) \left[Li_2 \left(-\frac{1-v}{1+v} \right) + 2Li_2 \left(\frac{1-v}{1+v} \right) + \right. \right.$$

$$\left. \left. + \frac{3}{2} \ln \frac{1+v}{1-v} \ln \frac{1+v}{2} - \ln \frac{1+v}{1-v} \ln v \right] + \left[\frac{11}{16} (3-v^2)(1+v^2) + \frac{v^4}{4} \right] \right.$$

$$\left. \times \ln \frac{1+v}{1-v} + \left[\frac{3}{2} v(3-v^2) \ln \frac{1-v^2}{4} - 2v(3-v^2) \ln v \right] + \frac{3}{8} v(5-3v^2) \right\}$$

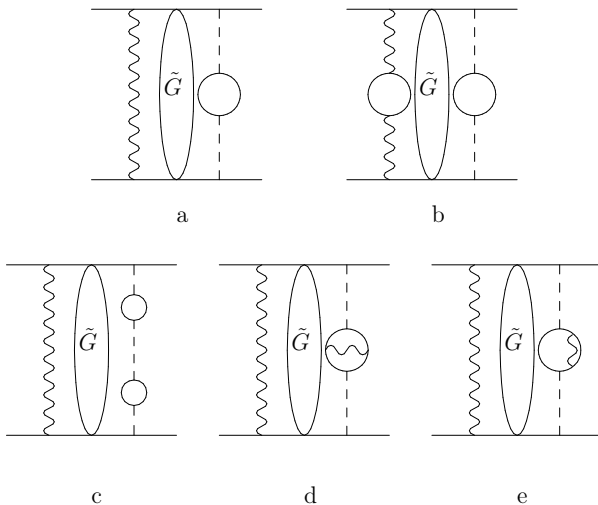


Figure: Effects of one-loop and two-loop vacuum polarization in the second order perturbation theory. The dashed line represents the Coulomb photon. $\tilde{G}$ is the reduced Coulomb Green function.

Coulomb Green's function:

- ▶ S.A.Zapryagaev, N.L.Manakov, V.G.Pal'chikov, Theory of multicharge ions with one and two electrons, Moscow, Energoatomizdat, 1985.
- ▶ H.F. Hameka, J. Chem. Phys. **47**, 2728 (1967)

$$\tilde{G}_n(\mathbf{r}, \mathbf{r}') = \sum_{l,m} \tilde{g}_{nl}(r, r') Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}').$$

$$\tilde{g}_{10}(r, r') = -4\mu^2 Z\alpha \left(\sum_{m=2}^{\infty} \frac{L_{m-1}^1(x) L_{m-1}^1(x')}{m(m-1)} + \frac{5}{2} - \frac{x}{2} - \frac{x'}{2} \right) e^{-\frac{x+x'}{2}}$$

$$\tilde{g}_{20}(r, r') = -2\mu^2 Z\alpha \left[\sum_{m=1, m \neq 2}^{\infty} \frac{L_{m-1}^1(x) L_{m-1}^1(x')}{m(m-2)} + \right.$$

$$\left. \left(\frac{5}{2} + x \frac{\partial}{\partial x} + x' \frac{\partial}{\partial x'} \right) L_1^1(x) L_1^1(x') \right] e^{-\frac{x+x'}{2}}, \quad x = 2\mu Z\alpha r/n.$$

Second order perturbation theory:

- V.G.Ivanov, S.G.Karshenboim, JETP **82**, 656 (1996).

$$\Delta E_{SOPT\ VP\ 1}^{hfs} = 2 \langle \psi | \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_B^{hfs} | \psi \rangle,$$

$$\Delta V_{VP}^C(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \left(-\frac{Z\alpha}{r} \right) e^{-2m_e \xi r}.$$

$$\tilde{G}_{1S}(\mathbf{r}, 0) = \frac{Z\alpha\mu^2}{4\pi} \frac{e^{-x}}{x} g_{1S}(x),$$

$$g_{1S}(x) = \left[4x(\ln 2x + C) + 4x^2 - 10x - 2 \right],$$

$$\tilde{G}_{2S}(\mathbf{r}, 0) = -\frac{Z\alpha\mu^2}{4\pi} \frac{e^{-x/2}}{2x} g_{2S}(x),$$

$$g_{2S}(x) = \left[4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4 \right],$$

One-loop VP corrections in second order perturbation theory:

$$\Delta E_{VP}^{hfs}{}_1(1S) = -\Delta E_F^{hfs}(1S) \frac{2\alpha}{3\pi} (1 + a_\mu) \int_1^\infty \rho(\xi) d\xi \times$$

$$\int_0^\infty e^{-2x(1+\frac{m_e\xi}{W})} g_{1S}(x) dx = -9.260 \text{ meV},$$

$$\Delta E_{VP}^{hfs}{}_1(2S) = \Delta E_F^{hfs}(2S) \frac{\alpha}{3\pi} (1 + a_\mu) \int_1^\infty \rho(\xi) d\xi \times$$

$$\int_0^\infty e^{-x(1+\frac{2m_e\xi}{W})} g_{2S}(x) dx = -0.869 \text{ meV}.$$

Two-loop VP corrections in second order perturbation theory:

- ▶ K.Pachucki, Phys. Rev. **A53**, 2092 (1996)
- ▶ H.F. Hameka, J. Chem. Phys. **47**, 2728 (1967)

$$\tilde{G}_{1S}(r, r') = -\frac{Z\alpha\mu^2}{\pi} e^{-(x_1+x_2)} g_{1S}(x_1, x_2),$$

$$g_{1S}(x_1, x_2) = \frac{1}{2x_{<}} - \ln 2x_{>} - \ln 2x_{<} + Ei(2x_{<}) + \frac{7}{2} -$$

$$-2C - (x_1 + x_2) + \frac{1 - e^{2x_{<}}}{2x_{<}},$$

$$\tilde{G}_{2S}(r, r') = -\frac{Z\alpha\mu^2}{16\pi x_1 x_2} e^{-(x_1+x_2)} g_{2S}(x_1, x_2),$$

$$g_{2S}(x_1, x_2) = 8x_{<} - 4x_{<}^2 + 8x_{>} + 12x_{<}x_{>} - 26x_{<}^2x_{>} + 2x_{<}^3x_{>} - 4x_{>}^2 -$$

$$26x_{<}x_{>}^2 + 23x_{<}^2x_{>}^2 - x_{<}^3x_{>}^2 + 2x_{<}x_{>}^3 - x_{<}^2x_{>}^3 + 4e^{x_{<}}(1-x_{<})(x_{>} - 2)x_{>} +$$

$$4(x_{<} - 2)x_{<}(x_{>} - 2)x_{>}[-2C + Ei(x_{<}) - \ln(x_{<}) - \ln(x_{>})].$$

Two-loop VP corrections in second order perturbation theory:

$$\Delta E_{SOPT VP 21}^{hfs}(1S) = -\frac{16\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu)}{27\pi^2 m_1 m_p} \int_1^\infty \rho(\xi) d\xi \times$$

$$\times \int_1^\infty \rho(\eta) d\eta \int_0^\infty dx e^{-2x(1 + \frac{m_e \xi}{W})} g_{1S}(x),$$

$$\Delta E_{SOPT VP 22}^{hfs}(1S) = -\frac{256\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu) m_e^2}{27\pi^2 m_1 m_p W^2} \int_1^\infty \rho(\xi) d\xi \times$$

$$\times \int_1^\infty \rho(\eta) \eta^2 d\eta \int_0^\infty x_1 dx_1 e^{-2x_1(1 + \frac{m_e \xi}{W})} \times$$

$$\int_0^\infty x_2 dx_2 e^{-2x_2(1 + \frac{m_e \xi}{W})} g_{1S}(x_1, x_2).$$

$$\Delta V_{1\gamma, VP}^{hfs}(r) = \frac{8\alpha \mu_h}{3m_1 m_2} \frac{\sigma_1 \sigma_2}{4} \frac{\alpha}{3\pi} \int_1^\infty \rho(s) ds \left[\pi \delta(\mathbf{r}) - \frac{m_e^2 \xi^2}{r} \right] e^{-2m_e \xi r}$$

Two-loop VP corrections in second order perturbation theory:

$$\begin{aligned}\Delta E_{SOPT\ VP\ 21}^{hfs}(2S) &= \frac{\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu)}{27 \pi^2 m_1 m_\rho} \int_1^\infty \rho(\xi) d\xi \times \\ &\times \int_1^\infty \rho(\eta) d\eta \int_0^\infty \left(1 - \frac{x}{2}\right) dx e^{-x(1 + \frac{2m_e \xi}{W})} g_{2S}(x), \\ \Delta E_{SOPT\ VP\ 22}^{hfs}(2S) &= -\frac{2\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu) m_e^2}{27 \pi^2 m_1 m_\rho W^2} \int_1^\infty \rho(\xi) d\xi \times \\ &\times \int_1^\infty \rho(\eta) \eta^2 d\eta \int_0^\infty \left(1 - \frac{x_1}{2}\right) dx_1 e^{-x_1(1 + \frac{2m_e \xi}{W})} \times \\ &\int_0^\infty \left(1 - \frac{x_2}{2}\right) dx_2 e^{-x_2(1 + \frac{2m_e \xi}{W})} g_{2S}(x_1, x_2).\end{aligned}$$

Two-loop VP corrections in second order perturbation theory:

$$\Delta V_{VP-VP}^C(r) = \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left(-\frac{Z\alpha}{r}\right) \times$$

$$\times \frac{1}{\xi^2 - \eta^2} \left(\xi^2 e^{-2m_e \xi r} - \eta^2 e^{-2m_e \eta r} \right),$$

$$\Delta V_{2-loop VP}^C(r) = -\frac{2Z\alpha^3}{3\pi^2 r} \int_0^1 \frac{f(v) dv}{(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}.$$

Nuclear structure corrections:

- ▶ A.C. Zemach, Phys. Rev. **104**, 1771 (1956)
- ▶ J.L. Friar, Lec. Notes in Phys. **627**, 285 (2003)

What atomic physics can do for nuclear physics?

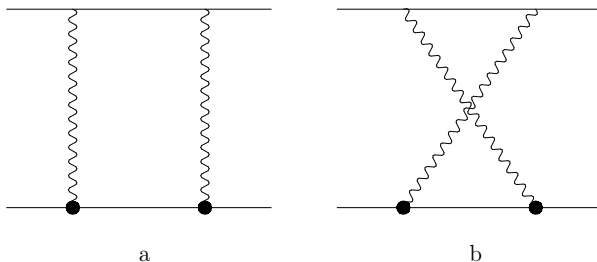


Figure: Effects of the nuclear structure of order α^5 .

Two possibilities to improve the theory:

1. Measurement of the electric G_E and magnetic G_M proton form factors

The Zemach correction

$$\Delta E_Z = E^F \frac{2\mu\alpha}{\pi^2} \int \frac{d\mathbf{p}}{p^4} \left[\frac{G_E(-\mathbf{p}^2)G_M(-\mathbf{p}^2)}{\mu_p} - 1 \right] = E^F(-2\mu\alpha)R_p$$

2. Using the muonic hydrogen to obtain new value of the Zemach radius with the accuracy 10^{-3}

Table: The values of the Zemach radii.

Zemach radius	HFS in ep Dupays...	HFS in ep Volotka...	e-p scat. Friar, Sick	e-p scat. Brodsky...
R_p, fm	1.047(19)	1.040(16)	1.086(12)	1.013(16)

Nuclear structure corrections in 2γ amplitudes:

$$\Delta E_{str}^{hfs} = -\frac{(Z\alpha)^5 \mu^3}{3\pi m_1 m_2 n^3} \delta_{l0} \int_0^\infty \frac{dk}{k} V(k),$$

$$V(k) = \frac{2F_2^2 k^2}{m_1 m_2} + \frac{\mu}{(m_1 - m_2)k(k + \sqrt{4m_1^2 + k^2})} \times$$

$$\begin{aligned} & \left[-128F_1^2 m_1^2 - 128F_1 F_2 m_1^2 + 16F_1^2 k^2 + 64F_1 F_2 k^2 + 16F_2^2 k^2 + \right. \\ & \left. + \frac{32F_2^2 m_1^2 k^2}{m_2^2} + \frac{4F_2^2 k^4}{m_1^2} - \frac{4F_2^2 k^4}{m_2^2} \right] + \frac{\mu}{(m_1 - m_2)k(k + \sqrt{4m_2^2 + k^2})} \times \\ & \left[128F_1^2 m_2^2 + 128F_1 F_2 m_2^2 - 16F_1^2 k^2 - 64F_1 F_2 k^2 - 48F_2^2 k^2 \right]. \end{aligned}$$

$$\Delta E_{iter, str}^{hfs} = - \langle V_{1\gamma} \times G^f \times V_{1\gamma} \rangle_{str}^{hfs} = -\frac{64}{3} \frac{\mu^4 Z^4 \alpha^5 \mu_h}{m_1 m_p \pi n^3} \int_0^\infty \frac{dk}{k^2}.$$

Nuclear structure corrections:

$$\Delta E_{str}^{hfs} = \begin{cases} 1S : 48.376 \text{ meV} \\ 2S : 6.047 \text{ meV} \end{cases}$$

The nuclear charge radius $r_N(^3_2\text{He}) = 1.844 \pm 0.045 \text{ fm}$. The theoretical errors 1.5 meV (1S) and 0.020 meV (2S).

The contribution of the polarizability of the ^3_2He nucleus?

The interval $\Delta_{12} = 8\Delta E_{str}^{hfs}(2S) - \Delta E_{str}^{hfs}(1S)$.

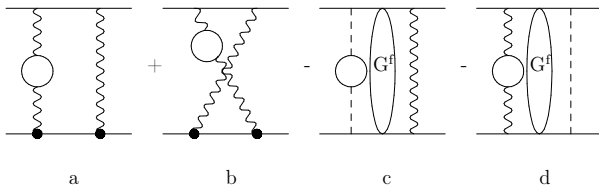


Figure: Nuclear structure and vacuum polarization effects of order α^6 . The dashed line represents the Coulomb photon. G^f is the free two-particle propagator.

Nuclear structure and VP corrections:

$$\Delta E_{str,VP}^{hfs} = -\frac{2\alpha(Z\alpha)^5\mu^3}{m_1 m_2 \pi^2 n^3} \int_0^\infty V_{VP}(k) dk \int_0^1 \frac{v^2 \left(1 - \frac{v^2}{3}\right) dv}{k^2(1 - v^2) + 4m_e^2},$$

$$\Delta E_{iter,str VP}^{hfs} = -2 \langle V^C \times G^f \times \Delta V_{VP}^{hfs} \rangle^{hfs} =$$

$$= -2 \langle V_{VP}^C \times G^f \times \Delta V_B^{hfs} \rangle^{hfs} =$$

$$= -\Delta E_F^{hfs} \frac{4\mu\alpha(Z\alpha)}{m_e \pi^2} \int_0^\infty dk \int_0^1 \frac{v^2 \left(1 - \frac{v^2}{3}\right) dv}{k^2(1 - v^2) + 1}.$$

$$\Delta E_{str,VP}^{hfs} + 2\Delta E_{iter,str VP}^{hfs} = \begin{cases} 1S: 0.760 \text{ meV} \\ 2S: 0.095 \text{ meV} \end{cases}$$

Nuclear structure corrections of order α^6 :

$$\Delta V_{str} = \frac{2\pi(Z\alpha)}{3} r_N^2 \delta(\mathbf{r}), \quad r_N^2 = 6 \frac{dG_E(q^2)}{dq^2} \Big|_{q^2=0}.$$

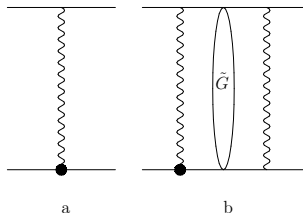


Figure: Nuclear structure effects of order α^6 in the one-photon interaction and the second order PT. $\tilde{G}$ is the reduced Coulomb Green function.

$$\begin{aligned} \Delta E_{str\ SOPT}^{hfs}(nS) &= 2 \langle \psi_n | \Delta V_B^{hfs} \cdot \tilde{G} \cdot \Delta V_{str} | \psi_n \rangle = \\ &= \frac{4\pi(Z\alpha)}{3} \Delta E_F^{hfs}(nS) r_N^2 \tilde{G}(0,0). \end{aligned}$$

Dimensional regularization in the calculation of $\tilde{G}(0, 0)$

- ▶ A.H. Hoang, Phys. Rev. D **57**, 1615 (1998).
- ▶ A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys. Rev. A **59**, 4316 (1999).

$$\Delta E_{str\ SOPT}^{hfs}(1S) = \frac{4}{3}(Z\alpha)^2 m_1^2 r_N^2 \Delta E_F^{hfs}(1S) \left[\ln(Z\alpha) - \frac{3}{2} \right],$$

$$\Delta E_{str\ SOPT}^{hfs}(2S) = \frac{4}{3}(Z\alpha)^2 m_1^2 r_N^2 \Delta E_F^{hfs}(2S) [\ln(Z\alpha) - \ln 2].$$

$$\Delta V_{1\gamma\ str}^{hfs}(r) = -\frac{4\pi\alpha(1 + a_\mu)}{9m_1 m_p} r_M^2 \frac{\sigma_1 \sigma_2}{4} \nabla^2 \delta(\mathbf{r}),$$

Nuclear structure corrections:

$$\Delta V_{1\gamma str}^{hfs}(r) = -\frac{4\pi\alpha(1+a_\mu)}{9m_1m_p}r_M^2\frac{\sigma_1\sigma_2}{4}\nabla^2\delta(\mathbf{r}),$$

r_M is the nuclear magnetic radius.

$$\int \nabla^2\delta(\mathbf{r})d\mathbf{r}|\psi_n(\mathbf{r})|^2 = 2\left(\psi(0)\nabla^2\psi(0) + \left(\frac{d\psi_n}{dr}\right)^2|_{r=0}\right),$$

- I.B.Khriplovich, A.I.Milstein, E-preprint hep-ph/9607374, 1996.

$$\nabla^2\psi(0) = \psi(0)\mu^2(Z\alpha)^2\frac{3+2(n^2-1)}{n^2}$$

$$\Delta E_{1\gamma, str}^{hfs}(nS) = -\frac{4}{3}(Z\alpha)^2\mu^2r_M^2\Delta E_F^{hfs}(nS)\frac{1-n^2}{4n^2}.$$

Muon self-energy and recoil corrections

- ▶ M.M.Sternheim, Phys. Rev. **130**, 211 (1963)
- ▶ S.G.Karshenboim, Phys. Lett. A **225**, 97 (1997)

$$\Delta E_{str SE}^{hfs} = \frac{5}{2} \frac{\alpha(Z\alpha)}{\pi} m_1 R_Z \Delta E_F^{hfs} = \begin{cases} 1S: -0.145 \text{ meV} \\ 2S: -0.018 \text{ meV} \end{cases},$$

$$\Delta E_{rec}^{hfs}(1S) = (Z\alpha)^2 \frac{\mu^2}{m_1 m_2} \Delta E_F^{hfs}(1S) \left[-\frac{17}{12} + \frac{25}{3\zeta} + \frac{31\zeta}{72} + \right. \\ \left. + \ln 2 \left(\frac{1}{2} - \frac{23}{2\zeta} - \frac{11\zeta}{8} \right) + \ln \left(\frac{1}{Z\alpha} \right) \left(-\frac{3}{2} + \frac{7}{2\zeta} + \frac{7\zeta}{8} \right) \right] = 0.330 \text{ meV},$$

$$\Delta E_{rec}^{hfs}(2S) = (Z\alpha)^2 \frac{\mu^2}{m_1 m_2} \Delta E_F^{hfs}(2S) \left[-\frac{265}{96} + \frac{821}{96\zeta} - \frac{809\zeta}{1152} + \right. \\ \left. + \ln 2 \left(1 - \frac{12}{\zeta} - \frac{\zeta}{2} \right) + \ln \left(\frac{1}{Z\alpha} \right) \left(-\frac{3}{2} + \frac{7}{2\zeta} + \frac{7\zeta}{8} \right) \right] = 0.038 \text{ meV},$$

$$\zeta = 2m_2\mu_h/m_p Z.$$

Table 1: Hyperfine structure of $1S$ and $2S$ states in $(\mu \text{ } ^3_2He)^+$. $\Delta_{12} = (8\Delta E^{hfs}(2S) - \Delta E^{hfs}(1S))$.

Contribution to HFS	$1S$, meV	$2S$, meV	Interval Δ_{12}
The Fermi energy	-1370.725	-171.341	0
Muon AMM correction of orders α^5 and α^6	-1.598	-0.200	0
Relativistic correction of order α^6	-0.438	-0.078	-0.183
One-loop VP contribution in 1γ interaction of order α^5	-4.203	-0.540	-0.119
Two-loop VP contribution in 1γ interaction of order α^6	-0.050	-0.004	0.016
One-loop muon VP contribution in 1γ interaction of order α^6	-0.052	-0.007	0
One-loop VP contribution in the second order PT of order α^5	-9.260	-0.869	2.305
Two-loop VP contribution in the second order PT of order α^6	-0.105	-0.010	0.022
Nuclear structure correction of order α^5	48.376	6.047	0
Contribution of VP and nuclear structure of order α^6	0.760	0.095	0
Nuclear structure correction of order α^6	2.553	0.272	-0.377
Nuclear structure and muon self-energy correction of order α^6	-0.145	-0.018	0
Nuclear recoil correction of order α^6	0.330	0.038	-0.026
Summary contribution	-1334.560	-166.615	1.638

Fine structure of P-energy levels in $(\mu_2^3\text{He})^+$

$$\Delta V^{fs}(r) = \frac{Z\alpha}{4m_1^2 r^3} \left[1 + \frac{2m_1}{m_2} + 2a_\mu \left(1 + \frac{m_1}{m_2} \right) \right] (\mathbf{L}\boldsymbol{\sigma}_1),$$

$$\begin{aligned} \Delta E^{fs} = E(2P_{3/2}) - E(2P_{1/2}) = & \frac{\mu^3(Z\alpha)^4}{32m_1^2} \left[1 + \frac{2m_1}{m_2} + \right. \\ & \left. 2a_\mu \left(1 + \frac{m_1}{m_2} \right) \right] + \frac{5m_1(Z\alpha)^6}{256} - \frac{m_1^2(Z\alpha)^6}{64m_2} + \\ & + \frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^2} \left[\ln \frac{\mu(Z\alpha)^2}{m_1} + \frac{1}{5} \right] + \alpha(Z\alpha)^4 A_{VP} + \alpha^2(Z\alpha)^4 B_{VP}. \end{aligned}$$

One-loop VP correction to FS

$$\Delta V_{VP}^C(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(s) ds \left(-\frac{Z\alpha}{r} \right) e^{-2m_e sr},$$

$$\Delta V_{VP}^{fs}(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(s) ds \frac{Z\alpha}{4m_1^2 r^3} \left[1 + \frac{2m_1}{m_2} + 2a_\mu \left(1 + \frac{m_1}{m_2} \right) \right] \times \\ \times e^{-2m_e sr} (1 + 2m_e sr) (\mathbf{L}\boldsymbol{\sigma}_1),$$

$$\Delta E_1^{fs} = \frac{\mu^3 (Z\alpha)^4}{32m_1^2} \left[1 + \frac{2m_1}{m_2} + 2a_\mu \left(1 + \frac{m_1}{m_2} \right) \right] \times \\ \times \frac{\alpha}{3\pi} \int_1^\infty \rho(s) ds \int_0^\infty x dx e^{-x(1+\frac{2m_e s}{W})} \left(1 + \frac{2m_e s}{W} x \right) = 129.25 \text{ } \mu\text{eV}.$$

$$\Delta E_2^{fs} = -\frac{\alpha (Z\alpha)^4 \mu^3}{3456\pi m_1 m_2} \left[1 + 2a_\mu + (1 + a_\mu) \frac{2m_1}{m_2} \right] \times \\ \times \int_1^\infty \rho(s) ds \int_0^\infty dx e^{-x(1+\frac{2m_e s}{W})} \int_0^\infty \frac{dx'}{x'^2} e^{-x'} g(x, x') = 140.56 \text{ } \mu\text{eV}.$$

Two-loop VP correction to FS

$$\begin{aligned}
 \Delta V_{VP-VP}^{fs}(r) &= \frac{Z\alpha}{r^3} \left[\frac{1+2a_\mu}{4m_1^2} + \frac{1+a_\mu}{2m_1m_2} \right] (\mathbf{L}\boldsymbol{\sigma}_1) \times \\
 &\times \left(\frac{\alpha}{3\pi} \right)^2 \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \frac{1}{(\xi^2 - \eta^2)} \times \\
 &\times \left[\xi^2 (1 + 2m_e \xi r) e^{-2m_e \xi r} - \eta^2 (1 + 2m_e \eta r) e^{-2m_e \eta r} \right]. \\
 \Delta E_3^{fs} &= \frac{\mu^3 \alpha^2 (Z\alpha)^4}{72\pi^2} \left[\frac{1+2a_\mu}{4m_1^2} + \frac{1+a_\mu}{2m_1m_2} \right] \int_1^\infty \rho(\xi) d\xi \times \\
 &\times \int_1^\infty \rho(\eta) d\eta \frac{1}{(\xi^2 - \eta^2)} \times \int_0^\infty x dx \left[\xi^2 \left(1 + \frac{2m_e \xi}{W} x \right) e^{-x(1 + \frac{2m_e \xi}{W})} \right. \\
 &\left. - \eta^2 \left(1 + \frac{2m_e \eta}{W} x \right) e^{-x(1 + \frac{2m_e \eta}{W})} \right] = 0.20 \text{ } \mu\text{eV}.
 \end{aligned}$$

Two-loop VP correction to FS

$$\Delta V_{2-loop, VP}^{fs}(r) = \frac{2\alpha^2(Z\alpha)}{3\pi^2 r^3} \left[\frac{1 + 2a_\mu}{4m_1^2} + \frac{1 + a_\mu}{2m_1 m_2} \right]$$

$$\int_0^1 \frac{f(v)dv}{1-v^2} e^{-\frac{2m_e r}{\sqrt{1-v^2}}} \left(1 + \frac{2m_e r}{\sqrt{1-v^2}} \right) (\mathbf{L}\boldsymbol{\sigma}_1),$$

$$\Delta E_4^{fs} = \frac{\mu^3 \alpha^2 (Z\alpha)^4}{12\pi^2} \left[\frac{1 + 2a_\mu}{4m_1^2} + \frac{1 + a_\mu}{2m_1 m_2} \right] \int_0^\infty x dx \times$$

$$\int_0^1 \frac{f(v)dv}{1-v^2} e^{-x \left(1 + \frac{2m_e}{W\sqrt{1-v^2}} \right)} \left(1 + \frac{2m_e}{W\sqrt{1-v^2}} x \right) = 0.78 \text{ } \mu\text{eV}.$$

$$\Delta E_5^{fs} = \frac{\mu^3 \alpha^2 (Z\alpha)^4}{1296\pi^2} \left[\frac{1 + a_\mu}{2m_1 m_2} + \frac{1 + 2a_\mu}{4m_1^2} \right] \int_1^\infty \rho(\xi) d\xi \rho(\eta) d\eta \int_0^\infty dx \times$$

$$e^{-x \left(1 + \frac{2m_e \xi}{W} \right)} \int_0^\infty \frac{g(x, x') dx'}{x'^2} \left(1 + \frac{2m_e \eta x'}{W} \right) e^{-x' \left(1 + \frac{2m_e \eta}{W} \right)} = 0.28 \text{ } \mu\text{eV}.$$

Contribution to the fine splitting ΔE^{fs}	Numerical value of the contribution in μeV
Contribution of order $(Z\alpha)^4$ $\frac{\mu^3(Z\alpha)^4}{32m_1^2} \left(1 + \frac{2m_1}{m_2}\right)$	144186.48
Muon AMM contribution $\frac{\mu^3(Z\alpha)^4}{16m_1^2} a_\mu \left(1 + \frac{m_1}{m_2}\right)$	324.46
Contribution of order $(Z\alpha)^6$: $\frac{5m_1(Z\alpha)^6}{256}$	19.94
Contribution of order $(Z\alpha)^6 m_1/m_2$: $-\frac{m_1^2(Z\alpha)^6}{64m_2}$	-0.60
Contribution of order $\alpha(Z\alpha)^4$ in the first order PT $\langle \Delta V_{VP}^{fs} \rangle$	129.25
Contribution of one-loop muon VP in the first order PT $\langle \Delta V_{MVP}^{fs} \rangle$	0.01
Contribution of order $\alpha(Z\alpha)^4$ in the second order PT $\langle \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V^{fs} \rangle$	140.56
Contribution of order $\alpha(Z\alpha)^6$ $\frac{\alpha(Z\alpha)^6 \mu^3}{32\pi m_1^2} \left[\ln \frac{\mu(Z\alpha)^2}{m_1} + \frac{1}{5} \right]$	-0.55
VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_{VP}^{fs} \rangle$	0.28
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{VP-VP}^{fs} \rangle$	0.20
VP Contribution from 1γ interaction of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{2-loop,VP}^{fs} \rangle$	0.78
VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{VP-VP}^C \cdot \tilde{G} \cdot \Delta V^{fs} \rangle$	0.03
VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V_{2-loop,VP}^C \cdot \tilde{G} \cdot \Delta V^{fs} \rangle$	2.31
Summary contribution	144803.15

Hyperfine splitting of P-wave levels in $(\mu_2^3\text{He})^+$.

$$\Delta V_B^{hfs}(r) = \frac{\alpha\mu_h}{2m_1m_2r^3} \left[1 + \frac{m_1}{m_2} - \frac{Zm_1m_p}{2m_2^2\mu_h} \right] (\mathbf{L}\boldsymbol{\sigma}_2) - \frac{\alpha\mu_h(1+a_\mu)}{4m_1m_p r^3} [(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) - 3(\boldsymbol{\sigma}_1\mathbf{n})(\boldsymbol{\sigma}_2\mathbf{n})].$$

Diagonal matrix elements $\langle 2P_{1/2} | \Delta V_B^{hfs} | 2P_{1/2} \rangle$ and $\langle 2P_{3/2} | \Delta V_B^{hfs} | 2P_{3/2} \rangle$:

$$\mathbf{s}_1 \rightarrow \mathbf{J} \frac{\overline{(\mathbf{s}_1\mathbf{J})}}{J^2}, \quad \mathbf{L} \rightarrow \mathbf{J} \frac{\overline{(\mathbf{L}\mathbf{J})}}{J^2},$$

The averaging over angles:

$$\langle \delta_{ij} - 3n_i n_j \rangle = -\frac{1}{5} (4\delta_{ij} - 3\overline{L_i L_j} - 3\overline{L_j L_i}).$$

Diagonal matrix elements ($E_F = Z^3 \alpha^4 \mu^3 \mu_h / 3m_1 m_p$)

$$\begin{aligned}\Delta E^{hfs}(2P_{1/2}) &= E(2^3 P_{1/2}) - E(2^1 P_{1/2}) = \\ &= E_F \left[\frac{1}{3} + \frac{a_\mu}{6} + \frac{m_1}{6m_2} - \frac{Zm_1 m_p}{12m_2^2 \mu_h} + \frac{m_1^3}{\mu^3} A_{rel}^{1/2} (Z\alpha)^2 + A_{VP}^{1/2} \alpha + B_{VP}^{1/2} \alpha^2 \right], \\ \Delta E^{hfs}(P_{3/2}) &= E(2^5 P_{3/2}) - E(2^3 P_{3/2}) = \\ &= E_F \left[\frac{2}{15} - \frac{a_\mu}{30} + \frac{m_1}{6m_2} - \frac{Zm_1 m_p}{12m_2^2 \mu_h} + \frac{m_1^3}{\mu^3} A_{rel}^{3/2} (Z\alpha)^2 + A_{VP}^{3/2} \alpha + B_{VP}^{3/2} \alpha^2 \right].\end{aligned}$$

Relativistic corrections $A_{rel}^{1/2}$, $A_{rel}^{3/2}$

$$\Delta V_D^{hfs} = e\mu \frac{[\mathbf{r} \times \boldsymbol{\alpha}]}{r^3}, \quad \mu = g_N \mu_N \mathbf{s}_2, \mu_N = e/2m_p.$$

$$\Delta E_{rel}^{hfs}(2P_{1/2}) = \frac{4\alpha\mu_h}{m_p} R_{1/2}, \quad \Delta E_{rel}^{hfs}(2P_{3/2}) = -\frac{16\alpha\mu_h}{15m_p} R_{3/2},$$

The radial integrals $R_k = \int_0^\infty g_k f_k dr$.

$$A_{rel}^{1/2} = 47/72, \quad A_{rel}^{3/2} = 7/180.$$

One-loop VP contribution to hyperfine part of the Breit potential

$$\begin{aligned} \Delta V_{VP}^{hfs}(r) &= \frac{\alpha\mu_h}{2m_1m_p r^3} \left[1 + \frac{m_1}{m_2} - \frac{Zm_1m_p}{2m_2^2\mu_h} \right] (\mathbf{L}\sigma_2) \int_1^\infty \rho(s) ds \times \\ &e^{-2m_e sr} (1 + 2m_e sr) - \frac{\alpha\mu_h(1 + a_\mu)}{4m_1m_p r^3} \int_1^\infty \rho(s) ds e^{-2m_e sr} \times \\ &\left[4m_e^2 s^2 r^2 (\sigma_1\sigma_2 - (\sigma_1\mathbf{n})(\sigma_2\mathbf{n})) + (1 + 2m_e sr) (\sigma_1\sigma_2 - 3(\sigma_1\mathbf{n})(\sigma_2\mathbf{n})) \right]. \\ \Delta E_1^{hfs}(2P_{1/2}) &= E_F \frac{\alpha}{18\pi} \int_1^\infty \rho(s) ds \int_0^\infty x dx e^{-x(1 + \frac{2m_e s}{W})} \times \\ &\left[\left(1 + \frac{m_1}{m_2} - \frac{Zm_1m_p}{2m_2^2\mu_h} \right) \left(1 + \frac{2m_e s}{W} x \right) + \right. \\ &\left. + (1 + a_\mu) \left(\frac{2m_e^2 s^2 x^2}{W^2} + 1 + \frac{2m_e s x}{W} \right) \right] = -65.21 \text{ } \mu\text{eV}. \end{aligned}$$

$$\begin{aligned} \Delta E_1^{hfs}(2P_{3/2}) = E_F \frac{\alpha}{18\pi} \int_1^\infty \rho(s) ds \int_0^\infty x dx e^{-x(1+\frac{2m_e s}{W})} \times \\ \times \left[\left(1 + \frac{m_1}{m_2} - \frac{Zm_1 m_p}{2m_2^2} \right) \left(1 + \frac{2m_e s}{W} x \right) - \right. \\ \left. - \frac{(1+a_\mu)}{5} \left(\frac{8m_e^2 s^2 x^2}{W^2} + 1 + \frac{2m_e s x}{W} \right) \right] = -11.15 \text{ } \mu\text{eV}, \end{aligned}$$

$$\begin{aligned} \Delta E_2^{hfs}(2P_{1/2}) = E_F \frac{\alpha}{324\pi} \int_1^\infty \rho(s) ds \int_0^\infty dx e^{-x(1+\frac{2m_e s}{W})} \times \\ \int_0^\infty \frac{dx'}{x'^2} e^{-x'} g(x, x') \left[2 + \frac{m_1}{m_2} - \frac{Zm_1 m_p}{2m_2^2 \mu_h} + a_\mu \right] = -56.79 \text{ } \mu\text{eV}. \end{aligned}$$

$$\begin{aligned} \Delta E_2^{hfs}(2P_{3/2}) = E_F \frac{\alpha}{324\pi} \int_1^\infty \rho(s) ds \int_0^\infty dx e^{-x(1+\frac{2m_e s}{W})} \times \\ \int_0^\infty \frac{dx'}{x'^2} e^{-x'} g(x, x') \left[\frac{4}{5} + \frac{m_1}{m_2} - \frac{Zm_1 m_p}{2m_2^2 \mu_h} - \frac{a_\mu}{5} \right] = -23.42 \text{ } \mu\text{eV}. \end{aligned}$$

Two-loop VP contribution to the potential

$$\Delta V_{VP-VP}^{hfs}(r) = \frac{Z\alpha\mu_h}{2m_1m_2r^3} \left(\frac{\alpha}{\pi}\right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \frac{1}{\xi^2 - \eta^2} \times$$

$$\left\{ \left[1 + \frac{m_1}{m_2} - \frac{Zm_1m_p}{2m_2^2\mu_h} \right] (\mathbf{L}\sigma_2) \left[\xi^2(1+2m_e\xi r)e^{-2m_e\xi r} - \eta^2(1+2m_e\eta r)e^{-2m_e\eta r} \right] \right.$$

$$- \frac{1+a_\mu}{2} \left[\left(\sigma_1\sigma_2 - 3(\sigma_1\mathbf{n})(\sigma_2\mathbf{n}) \right) \left(\xi^2(1+2m_e\xi r)e^{-2m_e\xi r} - \eta^2(1+2m_e\eta r)e^{-2m_e\eta r} \right) \right.$$

$$\left. \left. + 4m_e^2r^2 \left(\sigma_1\sigma_2 - 3(\sigma_1\mathbf{n})(\sigma_2\mathbf{n}) \right) \left(\xi^4e^{-2m_e\xi r} - \eta^4e^{-2m_e\eta r} \right) \right] \right\},$$

$$\Delta V_{2-loop,VP}^{hfs}(r) = \frac{Z\alpha\mu_h}{2m_1m_2r^3} \frac{2}{3} \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{f(v)dv}{1-v^2} e^{-\frac{2m_er}{\sqrt{1-v^2}}} \times$$

$$\left\{ \left[1 + \frac{m_1}{m_2} - \frac{Zm_1m_p}{2m_2^2\mu_h} \right] \left(1 + \frac{2m_er}{\sqrt{1-v^2}} \right) (\mathbf{L}\sigma_2) - \frac{1+a_\mu}{2} \times \right.$$

$$\left[\frac{4m_e^2r^2}{1-v^2} \left(\sigma_1\sigma_2 - (\sigma_1\mathbf{n})(\sigma_2\mathbf{n}) \right) + \left(1 + \frac{2m_er}{\sqrt{1-v^2}} \right) \left(\sigma_1\sigma_2 - 3(\sigma_1\mathbf{n})(\sigma_2\mathbf{n}) \right) \right]$$

Nuclear structure effects

$$\Delta V_{str}^{hfs}(\mathbf{k}) = -\frac{\pi Z\alpha(1+a_\mu) \langle r_M^2 \rangle}{6m_1 m_2} \left[(\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2) \mathbf{k}^2 - (\boldsymbol{\sigma}_1 \mathbf{k})(\boldsymbol{\sigma}_2 \mathbf{k}) \right],$$

$$\Delta E_{str}^{hfs} = \frac{\mu^5 (Z\alpha)^6 (1+a_\mu) \langle r_M^2 \rangle}{144 m_1 m_2} (1 - \delta_{l0}) \times \\ \times \frac{[F(F+1) - j(j+1) - \frac{3}{4}][j(j+1) - \frac{5}{4}]}{j(j+1)}.$$

$r_E = 1.9642(11)$ fm, D.C.Morton, Q.Wu and G.W.F.Drake,
Physical Review A **73**, 034502 (2006).

Off-diagonal matrix element

$$\gamma = \langle 2^3 P_{1/2} | \Delta V^{hfs} | 2^3 P_{3/2} \rangle = E_F \left(-\frac{\sqrt{2}}{48} \right) \left[1 - a_\mu + \frac{2m_1}{m_2} - \right. \\ \left. - \frac{Zm_1 m_p}{m_2^2 \mu_h} + \frac{m_1^3}{\mu^3} C_{rel} (Z\alpha)^2 + C_{VP\alpha} \right]$$

Relativistic and VP corrections to off-diagonal matrix element

$$R_{\frac{1}{2}\frac{3}{2}} = \int_0^\infty \left(g_{\frac{1}{2}}(r) f_{\frac{3}{2}}(r) + g_{\frac{3}{2}}(r) f_{\frac{1}{2}}(r) \right) dr, \quad C_{rel} = 9/16.$$

The vacuum polarization correction in γ :

$$T_1 = (\mathbf{L}\sigma_2), \quad T_2 = [\sigma_1\sigma_2 - 3(\sigma_1\mathbf{n})(\sigma_2\mathbf{n})], \quad T_3 = [\sigma_1\sigma_2 - (\sigma_1\mathbf{n})(\sigma_2\mathbf{n})].$$

Upon averaging over angles they can be written in terms of the $6j$ - symbols:

$$\langle T_3 \rangle = -\langle T_2 \rangle = -\langle T_1 \rangle = 6\hat{j}\hat{j}' \left\{ \begin{matrix} l & F & 1 \\ \frac{1}{2} & \frac{1}{2} & j \end{matrix} \right\} \left\{ \begin{matrix} l & F & 1 \\ \frac{1}{2} & \frac{1}{2} & j' \end{matrix} \right\} = \frac{2\sqrt{2}}{3},$$

total momentum $F = 1$ ($\mathbf{F} = \mathbf{s}_2 + \mathbf{J}$), $l = 1$, $\hat{j} = \sqrt{2j+1}$,
 $\hat{j}' = \sqrt{2j'+1}$.

$$\gamma_1 = \langle 2^3P_{1/2} | \Delta V_{VP}^{hfs} | 2^3P_{3/2} \rangle = E_F \left(-\frac{\sqrt{2}}{72} \right) \frac{\alpha}{\pi} \int_1^\infty \rho(s) ds \int_0^\infty x dx \times$$

$$e^{-x \left(1 + \frac{2m_e s}{W} \right)} \left[\left(1 + \frac{m_1}{m_2} - \frac{Zm_1 m_p}{2m_2^2 \mu_h} \right) \left(1 + \frac{2m_e s}{W} x \right) - \right. \\ \left. - \frac{1 + a_\mu}{2} \left(1 + \frac{2m_e s}{W} x - \frac{4m_e^2 s^2}{W^2} x^2 \right) \right] = 9.49 \mu eV,$$

$$\gamma_2 = \langle 2^3P_{1/2} | \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_B^{hfs} | 2^3P_{3/2} \rangle = E_F \left(-\frac{\sqrt{2}}{2592} \right) \frac{\alpha}{\pi} \times$$

$$\left[1 + \frac{2m_1}{m_2} - \frac{Zm_1 m_p}{m_2^2 \mu_h} - a_\mu \right] \int_1^\infty \rho(s) ds \int_0^\infty x dx \times$$

$$e^{-x \left(1 + \frac{2m_e s}{W} \right)} \int_0^\infty \frac{dx'}{x'^2} e^{-x'} g(x, x') = 5.33 \mu eV.$$

$\gamma = 5497.28 \mu eV$, the shift $2^3P_{3/2}$ and $2^3P_{1/2}$ by
 $\delta = 173.0 \mu eV$.

Contribution to hyperfine splitting	Numerical value of the contribution to $\Delta E^{hfs}(2P_{1/2}), \mu eV$	Numerical value of the contribution to $\Delta E^{hfs}(2P_{3/2}), \mu eV$
Contribution of order α^4	-58356.61	-24088.50
Muon AMM contribution	-33.29	6.66
Relativistic correction of order α^6	-26.62	-1.59
Contribution of order α^5 in the first order PT $\langle \Delta V_{VP}^{hfs} \rangle$	-65.21	-11.15
Contribution of order α^5 in the second order PT $\langle \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V^{hfs} \rangle$	-56.79	-23.42
VP Contribution in the second order PT of order α^6 $\langle \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_{VP}^{hfs} \rangle$	-0.12	-0.07
VP Contribution of 1γ interaction of order α^6 $\langle \Delta V_{VP-VP}^{hfs} \rangle$	-0.10	-0.01
VP Contribution of 1γ interaction of order α^6 $\langle \Delta V_{2-loop,VP}^{hfs} \rangle$	-0.37	-0.08
VP Contribution in the second order PT of order α^6 $\langle \Delta V_{VP-VP}^C \cdot \tilde{G} \cdot \Delta V^{hfs} \rangle$	-0.01	-0.004
VP Contribution in the second order PT of order α^6 $\langle \Delta V_{2-loop,VP}^C \cdot \tilde{G} \cdot \Delta V^{hfs} \rangle$	-0.45	-0.19
Nuclear structure correction	-0.33	0.66
Summary contribution	-58539.90	-24117.69

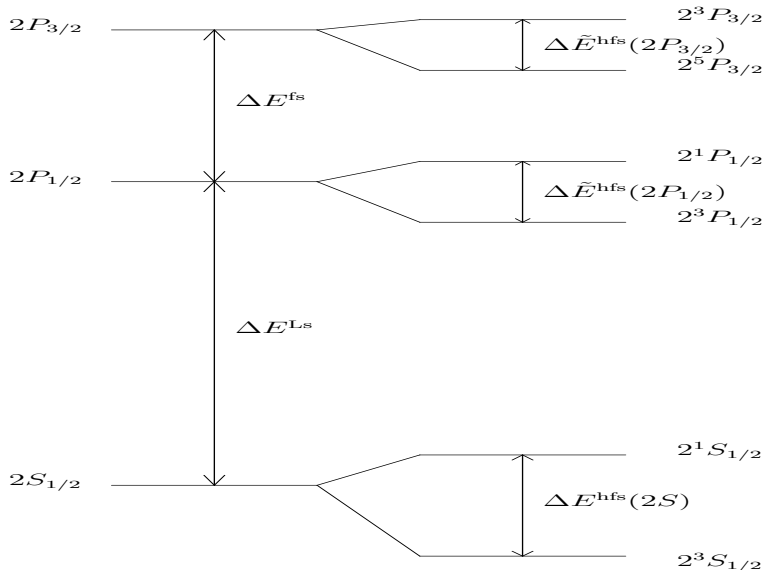


Figure: The structure of S -wave and P -wave energy levels in muonic helium ion $(\mu_2^3\text{He})^+$ for $n = 2$.

$$\Delta \tilde{E}^{hfs}(2P_{1/2}) = \Delta E^{hfs}(2P_{1/2}) - \delta = -58712.90 \mu\text{eV},$$

$$\Delta \tilde{E}^{hfs}(2P_{3/2}) = \Delta E^{hfs}(2P_{3/2}) - \delta = -24290.69 \mu\text{eV}.$$

Table: $(2S - 2P)$ transition energies for muonic helium ion $(\mu_2^3\text{He})^+$.

Transition	Energy (meV)	E.Borie
$^1S_{1/2} - ^3P_{1/2}$	1167.33	1167.3
$^1S_{1/2} - ^3P_{3/2}$	1342.19	1342.3
$^3S_{1/2} - ^1P_{1/2}$	1392.66	1392.4
$^3S_{1/2} - ^3P_{1/2}$	1333.95	1334.1
$^3S_{1/2} - ^3P_{3/2}$	1508.80	1509.1
$^3S_{1/2} - ^5P_{3/2}$	1484.51	1484.6

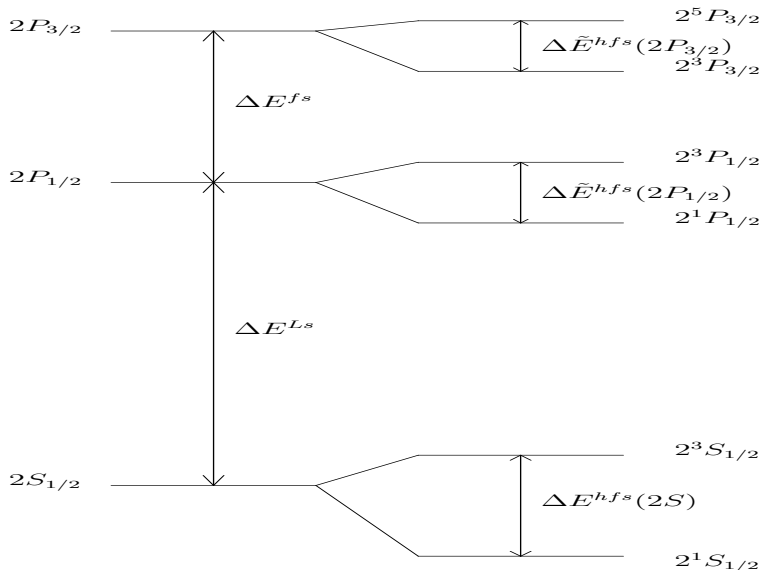


Figure: The structure of S -wave and P -wave energy levels in muonic hydrogen for $n = 2$.

4. Hyperfine splitting of the ground state in muonic helium.

The hyperfine structure of ground-state muonic helium is calculated numerically using the variational approach:

- ▶ V.I. Korobov, Phys. Rev. A **61**, 064503 (2000)
- ▶ A.M. Frolov, Phys. Rev. A **61**, 022509 (2000)
- ▶ K. Pachucki, Phys. Rev. A **63**, 032508 (2001)

$$\psi(r_1, r_2, r) = \sum_i v_i e^{-\alpha_i r_1} e^{-\beta_i r_2} e^{-\gamma_i r}.$$

N	Energy	$\langle \pi \delta(r) \rangle$
100	-402.63726963200300530	0.985756016
600	-402.63726963219007283	0.985707641
1200	-402.63726963219007450	0.985707756

The stochastic variational method

K.Varga, Y.Suzuki, CPC **106**, 157 (1997).

Perturbation theory in the hyperfine splitting calculation.

S.D.Lakdawala, P.J.Mohr, Phys. Rev. A **29**, 1047 (1984)

$$H = H_0 + \Delta H + \Delta H_{rec}, \quad H_0 = -\frac{1}{2M_\mu} \nabla_\mu^2 - \frac{1}{2M_e} \nabla_e^2 - \frac{2\alpha}{x_\mu} - \frac{\alpha}{x_e},$$

$$\Delta H = \frac{\alpha}{x_{\mu e}} - \frac{\alpha}{x_e}, \quad \Delta H_{rec} = -\frac{1}{m_\alpha} \nabla_\mu \cdot \nabla_e$$

$$M_e = m_e m_\alpha / (m_e + m_\alpha), \quad M_\mu = m_\mu m_\alpha / (m_\mu + m_\alpha).$$

$$\Delta H^{hfs} = -\frac{8\pi\alpha}{3m_e m_\mu} \frac{(\sigma_e \sigma_\mu)}{4} \delta(\mathbf{x}_\mu - \mathbf{x}_e),$$

$$\Delta \nu_0^{hfs} = \langle \frac{8\pi\alpha}{3m_e m_\mu} \delta(\mathbf{x}_\mu - \mathbf{x}_e) \rangle = \frac{\nu_F}{\left(1 + \frac{M_e}{2M_\mu}\right)^3}, \quad \nu_F = \frac{8\alpha^4 M_e^3}{3m_e m_\mu}.$$

Numerically the Fermi splitting is equal $\nu_F = 4516.915$ MHz.

VP correction to hyperfine splitting.

$$\Psi_0(\mathbf{x}_e, \mathbf{x}_\mu) = \psi_e(\mathbf{x}_e) \psi_\mu(\mathbf{x}_\mu) = \frac{1}{\pi} (2\alpha^2 M_e M_\mu)^{3/2} e^{-2\alpha M_\mu x_\mu} e^{-\alpha M_e x_e}.$$

$$\Delta V_{VP}^{e\alpha}(x_e) = \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) \left(-\frac{2\alpha}{x_e} \right) e^{-2m_e \xi x_e} d\xi,$$

$$\Delta V_{VP}^{\mu\alpha}(x_\mu) = \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) \left(-\frac{2\alpha}{x_\mu} \right) e^{-2m_e \xi x_\mu} d\xi,$$

$$\Delta V_{VP}^{e\mu}(|\mathbf{x}_e - \mathbf{x}_\mu|) = \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) \frac{\alpha}{x_{e\mu}} e^{-2m_e \xi x_{e\mu}} d\xi,$$

$$\Delta V_{VP}^{hfs}(\mathbf{x}_{e\mu}) = -\frac{8\alpha}{3m_e m_\mu} \frac{(\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2)}{4} \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \times$$

$$\left[\pi \delta(\mathbf{x}_{e\mu}) - \frac{m_e^2 \zeta^2}{x_{e\mu}} e^{-2m_e \xi x_{e\mu}} \right].$$

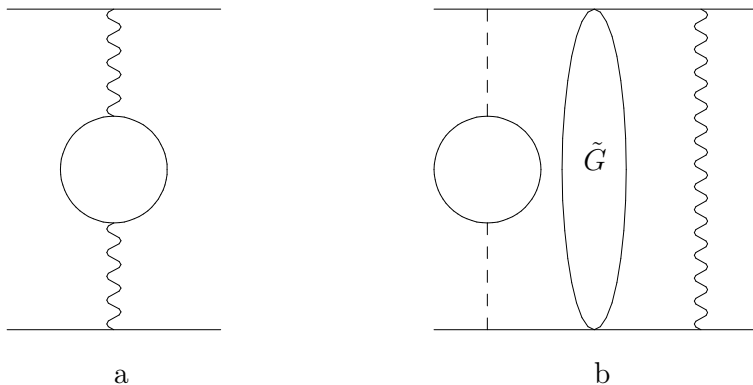


Figure: The vacuum polarization effects. The dashed line represents the Coulomb photon. The wave line represents the hyperfine part of the Breit potential. $\tilde{G}$ is the reduced Coulomb Green's function.

VP contribution in 1γ interaction.

$$\Delta\nu_{VP}^{hfs} = \frac{8\alpha^2}{9m_em_\mu} \frac{(\alpha M_e)^3 (2\alpha M_\mu)^3}{\pi^3} \int_1^\infty \rho(\xi) d\xi \int d\mathbf{x}_e \int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu} \times$$

$$e^{-2\alpha M_e x_e} \left[\pi \delta(\mathbf{x}_\mu - \mathbf{x}_e) - \frac{m_e^2 \xi^2}{|\mathbf{x}_\mu - \mathbf{x}_e|} \right] e^{-2m_e \xi |\mathbf{x}_\mu - \mathbf{x}_e|}.$$

$$I_1 = \int d\mathbf{x}_e \int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu} e^{-2\alpha M_e x_e} \pi \delta(\mathbf{x}_\mu - \mathbf{x}_e) = \frac{\pi^2}{8\alpha^3 M_\mu^3 \left(1 + \frac{M_e}{2M_\mu}\right)^3}$$

$$I_2 = \int d\mathbf{x}_e \int d\mathbf{x}_\mu e^{-4\alpha M_\mu x_\mu} e^{-2\alpha M_e x_e} \frac{1}{|\mathbf{x}_\mu - \mathbf{x}_e|} e^{-2m_e \xi |\mathbf{x}_\mu - \mathbf{x}_e|} =$$

$$= \frac{32\pi^2}{(4\alpha M_\mu)^5} \frac{\left[\frac{M_e^2}{4M_\mu^2} + \left(1 + \frac{m_e \xi}{2M_\mu \alpha}\right)^2 + \frac{M_e}{2M_\mu} \left(3 + \frac{m_e \xi}{M_\mu \alpha}\right) \right]}{\left(1 + \frac{M_e}{2M_\mu}\right)^3 \left(1 + \frac{m_e \xi}{2M_\mu \alpha}\right)^2 \left(\frac{M_e}{2M_\mu} + \frac{m_e \xi}{2M_\mu \alpha}\right)^2}.$$

$$\Delta\nu_{VP}^{hfs} = \nu_F \frac{\alpha M_e}{6\pi M_\mu \left(1 + \frac{M_e}{2M_\mu}\right)^3} \int_1^\infty \rho(\xi) d\xi \times$$

$$\frac{\left[\frac{M_e}{2M_\mu} + 2 \frac{m_e \xi}{2M_\mu \alpha} \frac{M_e}{2M_\mu} + \frac{m_e \xi}{2M_\mu \alpha} \left(2 + \frac{m_e \xi}{2M_\mu \alpha}\right) \right]}{\left(1 + \frac{m_e \xi}{2M_\mu \alpha}\right)^2 \left(\frac{M_e}{2M_\mu} + \frac{m_e \xi}{2M_\mu \alpha}\right)^2} = 0.035 \text{ MHz}.$$

VP contribution in second order PT.

$$\Delta\nu_{VP}^{hfs} \text{ SOPT } e\alpha = \frac{16\pi\alpha}{3m_e m_\mu} \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \frac{\alpha}{3\pi} \rho(\xi) d\xi \psi_{\mu 0}^*(\mathbf{x}_3) \psi_{e 0}^*(\mathbf{x}_3) \times$$

$$\sum_{n,n' \neq 0} \frac{\psi_{\mu n}(\mathbf{x}_3) \psi_{e n'}(\mathbf{x}_3) \psi_{\mu n}^*(\mathbf{x}_2) \psi_{e n'}^*(\mathbf{x}_1)}{E_{\mu 0} + E_{e 0} - E_{\mu n} - E_{e n'}} e^{-2m_e \xi x_1} \psi_{\mu 0}(\mathbf{x}_2) \psi_{e 0}(\mathbf{x}_1).$$

$$\Delta\nu_{VP}^{hfs} SOPT_{e\alpha} = \nu_F \frac{32\alpha M_e^2}{3\pi M_\mu^2} \int_1^\infty \rho(\xi) d\xi \int_0^\infty x_3^3 dx_3 \int_0^\infty x_1 dx_1 \times$$

$$e^{-x_1 \frac{M_e}{M_\mu} \left(1 + \frac{m_e \xi}{\alpha M_\mu}\right)} e^{-2x_3 \left(1 + \frac{M_e}{2M_\mu}\right)} \left[\frac{M_\mu}{M_e x_{>}} - \ln\left(\frac{M_e}{M_\mu} x_{<}\right) - \ln\left(\frac{M_e}{M_\mu} x_{>}\right) + \right.$$

$$\left. Ei\left(\frac{M_e}{M_\mu} x_{<}\right) + \frac{7}{2} - 2C - \frac{M_e}{2M_\mu} (x_1 + x_3) + \frac{1 - e^{\frac{M_e}{M_\mu} x_{<}}}{\frac{M_e}{M_\mu} x_{<}} \right] = 0.151 \text{ MHz},$$

$$G_e(\mathbf{x}_1, \mathbf{x}_3) = \sum_{n \neq 0}^\infty \frac{\psi_{en}(\mathbf{x}_3) \psi_{en}^*(\mathbf{x}_1)}{E_{e0} - E_{en}} = -\frac{\alpha M_e^2}{\pi} e^{-\alpha M_e (x_1 + x_3)} \left[\frac{1}{2\alpha M_e x_{>}} - \right.$$

$$\left. - \ln(2\alpha M_e x_{>}) - \ln(2\alpha M_e x_{<}) + Ei(2\alpha M_e x_{<}) + \frac{7}{2} - 2C - \right.$$

$$\left. \alpha M_e (x_1 + x_3) + \frac{1 - e^{2\alpha M_e x_{<}}}{2\alpha M_e x_{<}} \right].$$

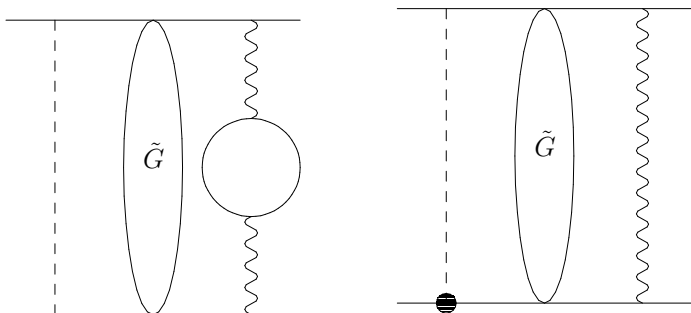


Figure: Vacuum polarization and nuclear structure effects in the second order perturbation theory.

$$\Delta V_{str,\mu}(\mathbf{r}_\mu) = \frac{2}{3}\pi Z\alpha < r_\alpha^2 > \delta(\mathbf{r}_\mu).$$

$$\Delta\nu_{str,\mu}^{hfs} = \frac{64\pi^2\alpha^2 r_\alpha^2}{9m_e m_\mu \sqrt{\pi}} \left(2\alpha M_\mu\right)^{\frac{3}{2}} \int d\mathbf{x}_3 \psi_{\mu 0}^*(\mathbf{x}_3) |\psi_{e0}(\mathbf{x}_3)|^2 G_\mu(\mathbf{x}_3, 0, E_{\mu 0})$$

$$\Delta\nu_{str,\mu}^{hfs} = -\nu_F \frac{8}{3} \alpha^2 M_\mu^2 r_\alpha^2 \left(3 \frac{M_e}{M_\mu} - \frac{11}{2} \frac{M_e^2}{M_\mu^2} + \dots\right) = -0.007 \text{ MHz}.$$

$$\Delta\nu_{str,e}^{hfs} = \frac{64\pi^2\alpha^2}{9m_e m_\mu} r_\alpha^2 \int d\mathbf{x}_1 \int d\mathbf{x}_3 |\psi_{\mu 0}^*(\mathbf{x}_3)|^2 \psi_{e0}(\mathbf{x}_3) \times \\ G_\mu(\mathbf{x}_3, \mathbf{x}_1, E_{e0}) \psi_{e0}(\mathbf{x}_1) \delta(\mathbf{x}_1).$$

$$\Delta\nu_{str,e}^{hfs} = -\nu_F \frac{4}{3} \alpha^2 M_e^2 r_\alpha^2 \left[5 - \ln \frac{M_e}{M_\mu} + \frac{M_e^2}{M_\mu^2} \left(3 \ln \frac{M_e}{M_\mu} - 7\right) + \right. \\ \left. + \frac{M_e^2}{M_\mu^2} \left(\frac{17}{2} - 3 \ln \frac{M_e}{M_\mu}\right) \dots\right] = -0.003 \text{ MHz}.$$

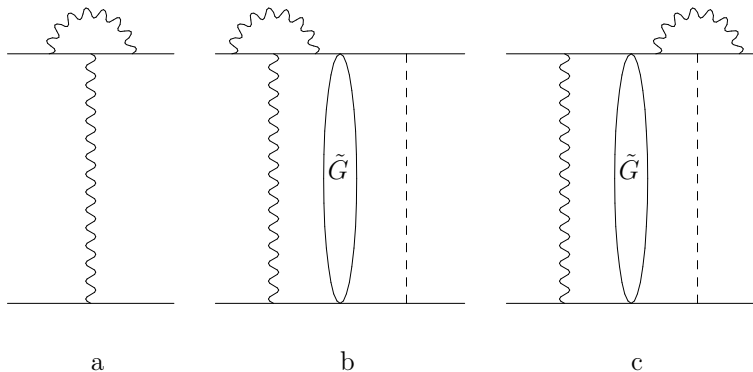


Figure: The electron vertex corrections. The dashed line represents the Coulomb photon. The wave line represents the hyperfine part of the Breit potential. $\tilde{G}$ is the reduced Coulomb Green's function.

Electron vertex corrections.

$$\Delta V_{vertex}^{hfs}(k^2) = -\frac{8\alpha^2}{3m_em_\mu} \left(\frac{\sigma_e \sigma_\mu}{4} \right) \left[G_M^{(e)}(k^2) - 1 \right],$$

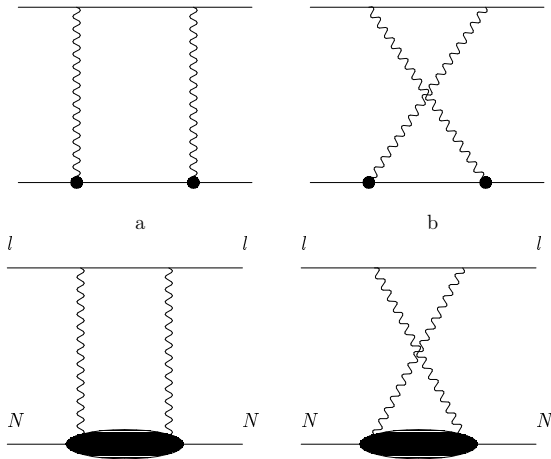
$$\begin{aligned} \Delta \nu_{vertex}^{hfs} &= \nu_F \frac{\alpha}{32\pi^2} \left(\frac{M_e}{M_\mu} \right) \left(\frac{m_e}{\alpha M_\mu} \right)^3 \int_0^\infty k^2 dk \left[G_M^{(e)}(k^2) - 1 \right] \times \\ &\times \left\{ \left[1 + \left(\frac{m_e}{4\alpha M_\mu} \right)^2 k^2 \right] \left[\left(\frac{M_e}{2M_\mu} \right)^2 + \left(\frac{m_e}{4\alpha M_\mu} \right)^2 k^2 \right]^2 \right\}^{-1} = 4.214 \text{ MHz}. \end{aligned}$$

$$G_M^{(e)}(k^2 = 0) \rightarrow 5.244 \text{ MHz}$$

Table 1: Hyperfine singlet-triplet splitting of the ground state in the muonic helium atom.

Contribution to the HFS	$\Delta\nu^{hfs}$, MHz
The Fermi splitting	4516.915
Recoil correction of order $\alpha^4(m_e/m_\mu)$	-33.525
Correction of muon anomalous magnetic moment of order α^5	5.244
Recoil correction of order $\alpha^4(M_e/m_\alpha)\sqrt{(M_e/M_\mu)}$	0.079
Correction due to the perturbation (3) in the second order PT of order $\alpha^4\frac{M_e}{M_\mu}$	-29.650
Relativistic correction of order α^6	0.040
One-loop VP contribution in 1γ ($e\mu$)-interaction of order $\alpha^5\frac{M_e}{M_\mu}$	0.035
One-loop VP contribution in the electron-muon interaction in the second order PT of order $\alpha^5\frac{M_e}{M_\mu}$	-0.145
One-loop VP contribution in the electron-nucleus interaction in the second order PT of order $\alpha^5\frac{M_e}{M_\mu}$	0.151
One-loop VP contribution in the muon-nucleus interaction in the second order PT of order $\alpha^5\frac{M_e}{M_\mu}$	0.048
Nuclear structure correction of order α^6 in the second order PT	-0.010
Recoil correction of order $\alpha^5(m_e/m_\mu)\ln(m_e/m_\mu)$	0.812
Vertex correction of order α^6	-0.606
Electron vertex contribution of order α^5	6.138
Summary contribution	4465.526

Two-photon Feynman amplitudes determining the corrections of the proton structure and polarizability in the hyperfine splitting of the hydrogen atom.



Proton electromagnetic form factors There exist two possibilities to determine electric G_E and magnetic G_M form factors:

1. The Rosenbluth formula method:

$$\frac{d\sigma}{d\cos\theta} = \frac{\alpha^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2) \epsilon(\tau + 1) (1 + \frac{2E}{M} \sin^2(\theta/2))} [\epsilon G_E^2 + \tau G_M^2],$$

θ is the electron scattering angle, M is the proton mass,

$$\tau = \frac{Q^2}{4M^2}, \quad \epsilon = \left[1 + 2(1 + \tau) \tan^2 \frac{\theta}{2} \right]^{-1}$$

Cross section measurements at different θ allow to determine separately $G_e(Q^2)$, $G_M(Q^2)$. The dipole parameterization formula:

$$G_E(Q^2) = \frac{G_M(Q^2)}{\mu_p} = G_D(Q^2) = \left(1 + \frac{Q^2}{0.71 \text{ GeV}^2} \right)^{-2}.$$

2. Polarization measurements:

The ratio G_E/G_M can be determined in polarization transfer experiments, where longitudinally polarized electrons are scattered from unpolarized protons and both transverse and longitudinal polarization are transferred to the struck proton. Such experiments have been performed at Jefferson Lab up to $Q^2 = 5.6 \text{ GeV}^2$ and found the ratio $\mu_p G_E/G_M$ decreasing with increasing Q^2 in stark contrast to the approximation observed in Rosenbluth measurements.

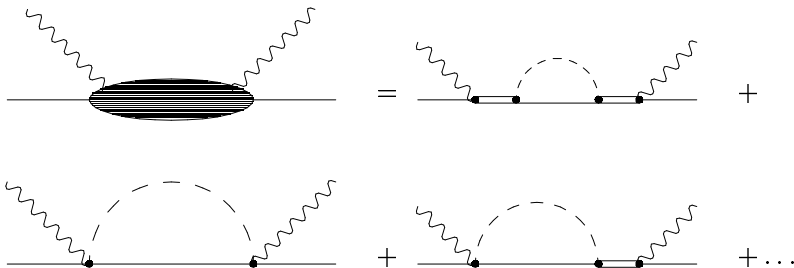
The JLab model is used ([Phys. Rev. Lett. 94, 142301 \(2005\)](#)):

$$G_{JLab}(Q^2) = \left(1 - 0.13 \frac{Q^2}{\text{GeV}^2}\right) G_D(Q^2).$$

The error appeared in the determination of the proton charge and magnetic mean-square radii

$$\langle r^2 \rangle_{E,M} = \int d^3r r^2 \rho_{E,M}(\mathbf{r})$$

is on the level of about 0.75 per cent.



Feynman amplitudes for the proton polarizability correction in the resonance region. Solid, double solid, wave and dashed lines correspond to nucleon, baryon resonance, photon and pion correspondingly.

Proton polarizability contribution to HFS.

What nuclear physics can do for atomic physics?

S.D.Drell, J.D.Sullivan, Phys. Rev. **154**, 1477 (1967)

$$\begin{aligned}\Delta E_{HFS}^P &= E_F \frac{2\alpha m_1}{\pi^3 m_2(1+\kappa)} \int \frac{d^4 Q}{Q^6(1+\frac{4m_1^2}{Q^2} \cos^2 \phi)} \times \\ &\left\{ Q^2(2+\cos^2 \phi) \left[G_1(\nu, Q^2) - G_1(0, 0) \frac{F_2^2(Q^2)}{\kappa^2} \right] - 3Q^4 \cos^2 \phi G_2(\nu, Q^2) \right\} \\ &= \frac{\alpha m_1}{2\pi m_2(1+\kappa)} E_F (\Delta_1 + \Delta_2) = (\delta_1^P + \delta_2^P) E_F = \delta^P E_F \\ \Delta_1 &= \int_0^\infty \frac{dQ^2}{Q^2} \left\{ \frac{9}{4} F_2^2(Q^2) \beta_0(\sigma) - 4m_2^3 \int_{\nu_{th}}^\infty \frac{d\nu}{\nu} \beta_1(\sigma, \theta) G_1(\nu, Q^2) \right\} \\ \Delta_2 &= -12m_2^2 \int_0^\infty \frac{dQ^2}{Q^2} \int_{\nu_{th}}^\infty d\nu \beta_2(\sigma, \theta) G_2(\nu, Q^2)\end{aligned}$$

$G_{1,2}(\nu, Q^2)$ are the nucleon polarized structure functions

Functions $\beta_{0,1,2}$ have the form:

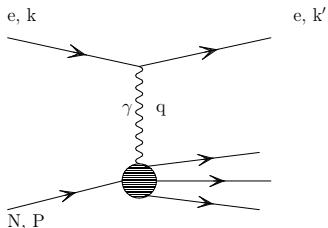
$$\beta_0(\sigma) = \frac{4}{9\sigma^2} \left[-3\sigma + 4\sigma\sqrt{1+\sigma} + 2 - 2\sqrt{1+\sigma} \right],$$

$$\beta_1(\sigma, \theta) = \frac{\theta}{\sigma(-1 + \sigma\theta)} \left[\frac{-2\sqrt{1+\sigma} + 2 - 4\sigma^2 - 2\sigma}{\sqrt{1+\sigma}} + \right. \\ \left. + \frac{2\sigma(-\theta^2 + \theta^{3/2}\sqrt{1+\theta} + \theta + 2)}{\sqrt{\theta(1+\theta)}} \right], \quad \theta = \frac{\nu^2}{Q^2}, \quad \sigma = \frac{4m_1^2}{Q^2},$$

$$\beta_2(\sigma, \theta) = \frac{2}{\sigma(-1 + \sigma\theta)} \left[1 - \sqrt{1+\sigma} + \sigma \left(-\theta + \sqrt{\theta(1+\theta)} \right) \right]$$

- ▶ C.E.Carlson New evaluation of proton structure corrections to hydrogen HFS, arXiv:physics/0610289
- ▶ C.E.Carlson Proton structure corrections to hydrogen HFS, arXiv:physics/0611206
- ▶ C.E. Carlson et al. Phys.Rev. A78, 022517 (2008)

The inclusive reaction of leptons scattered off of nucleons:



$$L_{\mu\nu}(k, s; k', s') = [\bar{u}(k', s')\gamma_\mu u(k, s)]^* [\bar{u}(k', s')\gamma_\nu u(k, s)] =$$

$$= L_{\mu\nu}^{(S)}(k; k') + iL_{\mu\nu}^{(A)}(k, s; k') + L_{\mu\nu}'^{(S)}(k, s; k', s') + iL_{\mu\nu}'^{(A)}(k; k', s'),$$

$$L_{\mu\nu}^{(S)}(k; k') = k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu}(k \cdot k' - m^2),$$

$$L_{\mu\nu}^{(A)}(k, s; k') = m\epsilon_{\mu\nu\alpha\beta}s^\alpha(k - k')^\beta,$$

$$L_{\mu\nu}'^{(S)}(k, s; k', s') = (k \cdot s')(k'_\mu s'_\nu + s_\mu k'_\nu - g_{\mu\nu}k' \cdot s) -$$

$$-(k \cdot k' - m^2)(s_\mu s'_\nu + s'_\mu s_\nu - g_{\mu\nu}s \cdot s') + (k' \cdot s)(s'_\mu k_\nu + k_\mu s'_\nu) -$$

$$-(s \cdot s')(k_\mu k'_\nu + k_\nu k'_\mu), \quad L_{\mu\nu}'^{(A)}(k; k', s') = m\epsilon_{\mu\nu\alpha\beta}s'^\alpha(k - k')^\beta.$$

The polarized structure functions $g_{1,2}(\nu, Q^2)$ enter in the antisymmetric part of hadronic tensor $W_{\mu\nu}$, describing lepton-nucleon DIS

$$W_{\mu\nu} = W_{\mu\nu}^{[S]} + W_{\mu\nu}^{[A]}$$

$$W_{\mu\nu}^{[S]} = (-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}) W_1(\nu, Q^2) +$$

$$+ (P_\mu - \frac{P \cdot q}{q^2} q_\mu) (P_\nu - \frac{P \cdot q}{q^2} q_\nu) \frac{W_2(\nu, Q^2)}{m_2^2}$$

$$W_{\mu\nu}^{[A]} = \epsilon_{\mu\nu\alpha\beta} q^\alpha \left\{ S^\beta \frac{g_1(\nu, Q^2)}{P \cdot q} + [(P \cdot q) S^\beta - (S \cdot q) P^\beta] \frac{g_2(\nu, Q^2)}{(P \cdot q)^2} \right\}$$

$g_1(\nu, Q^2) = m_2^2 \nu G_1(\nu, Q^2)$, $g_2(\nu, Q^2) = m_2 \nu^2 G_2(\nu, Q^2)$, S is the proton spin vector, P is four momentum of the proton, $x = Q^2/2m_2\nu$, the invariant mass of the electroproduced hadronic system W is

$$W^2 = m_2^2 + 2m_2\nu - Q^2 = m_2^2 + Q^2 x(1/x - 1)$$

Hadronic tensor $W_{\mu\nu}$ is proportional to the imaginary part of the off-shell Compton amplitude for the forward scattering of virtual photons on nucleons: $\gamma^* + N \rightarrow \gamma^* + N$. The photon-nucleon interaction depends on the photon polarization as well as on the nucleon one. This gives four independent helicity amplitudes of the form $M_{ab;cd}$ with a, b, c, d values for the helicities of the photon and nucleon initial and final states:

$$M_{1,1/2;1,1/2}, \quad M_{1,-1/2;1,-1/2}, \quad M_{0,1/2;0,1/2}, \quad M_{1,1/2;0,-1/2},$$

These components correspond to the four structure functions W_1 , W_2 , g_1 , g_2 . All other possible combinations of initial and final photon and nucleon helicities are related to the above by time reversal and parity transformation.

We calculate contribution ΔE_{HFS}^P on the basis of the latest experimental data on the structure functions $g_{1,2}(\nu, Q^2)$ and theoretical predictions for the cross sections $\sigma_{1/2,3/2,TL}$

$$g_1(\nu, Q^2) = \frac{m_2 \cdot K}{8\pi^2 \alpha (1 + Q^2/\nu^2)} \left[\sigma_{1/2}(\nu, Q^2) - \sigma_{3/2}(\nu, Q^2) + \frac{2\sqrt{Q^2}}{\nu} \sigma_{TL}(\nu, Q^2) \right],$$

$$g_2(\nu, Q^2) = \frac{m_2 \cdot K}{8\pi^2 \alpha (1 + Q^2/\nu^2)} \left[-\sigma_{1/2}(\nu, Q^2) + \sigma_{3/2}(\nu, Q^2) + \frac{2\nu}{\sqrt{Q^2}} \sigma_{TL}(\nu, Q^2) \right],$$

$\sigma_{1/2}$, $\sigma_{3/2}$ are the virtual photoabsorption transverse cross sections for the total photon-nucleon helicity of 1/2, and 3/2 respectively, σ_{TL} is the interference term between the transverse and longitudinal photon-nucleon amplitudes.

Resonance region $W^2 \leq 4 \text{ GeV}^2$. Unitary isobar model.

$$\sigma_{1/2,3/2} = \left(\frac{k_R}{k} \right)^2 \frac{W^2 \Gamma_\gamma \Gamma_{R \rightarrow N\pi}}{(W^2 - M_R^2)^2 + W^2 \Gamma_{tot}^2} \frac{4m_p}{M_R \Gamma_R} |A_{1/2,3/2}|^2$$

$A_{1/2,3/2}$ are transverse electromagnetic helicity amplitudes,

$$\Gamma_\gamma = \Gamma_R \left(\frac{k}{k_R} \right)^{j_1} \left(\frac{k_R^2 + X^2}{k^2 + X^2} \right)^{j_2}, \quad X = 0.3 \text{ GeV}$$

The parameterization of one-pion decay width is

$$\Gamma_{R \rightarrow N\pi}(q) = \Gamma_R \frac{M_R}{M} \left(\frac{q}{q_R} \right)^3 \left(\frac{q_R^2 + C^2}{q^2 + C^2} \right)^2, \quad C = 0.3 \text{ GeV}$$

for the $P_{33}(1232)$

For $D_{13}(1520)$, $P_{11}(1440)$, $F_{15}(1680)$

$$\Gamma_{R \rightarrow N\pi}(q) = \Gamma_R \left(\frac{q}{q_R} \right)^{2l+1} \left(\frac{q_R^2 + \delta^2}{q^2 + \delta^2} \right)^{l+1}$$

l is the pion angular momentum, $\delta^2 = (M_R - m_p - m_\pi)^2 + \Gamma_R^2/4$.
Here $q(k)$ and q_R (k_R) denote the c.m.s. pion (photon) momenta of resonances with mass M and M_R respectively. In the case of $S_{11}(1535)$ we take into account πN and ηN decay modes:

$$\Gamma_{R \rightarrow \pi, \eta} = \frac{q_{\pi, \eta}}{q} b_{\pi, \eta} \Gamma_R \frac{q_{\pi, \eta}^2 + C_{\pi, \eta}^2}{q^2 + C_{\pi, \eta}^2}$$

The two-pion decay modes

The high lying nucleon resonance R can decay first into N^* ($P_{33}(1232)$ or $P_{11}(1440)$) and a pion or into a nucleon and ρ - or σ -meson. Then the new resonances decay into a nucleon and a pion or two pions:

$$R \rightarrow r + a = \begin{cases} N^* + \pi \rightarrow N + \pi + \pi, \\ \rho(\sigma) + N \rightarrow N + \pi + \pi. \end{cases}$$

The total decay width ($r = N^*, \rho, \sigma$ ($a = \pi, N$)):

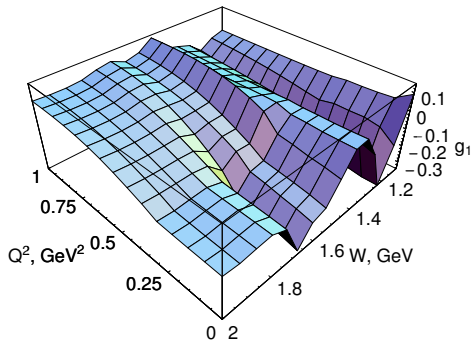
$$\Gamma_{R \rightarrow r+a}(W) = \frac{P_{2\pi}}{W} \int_0^{W-m_a} d\mu \cdot p_f \frac{2}{\pi} \frac{\mu^2 \Gamma_{r,tot}(\mu)}{(\mu^2 - m_r^2)^2 + \mu^2 \Gamma_{r,tot}^2(\mu)} \times \\ \times \frac{(M_R - m_2 - 2m_\pi)^2 + C^2}{(W - m_2 - 2m_\pi)^2 + C^2}$$

The decay width of the meson resonance:

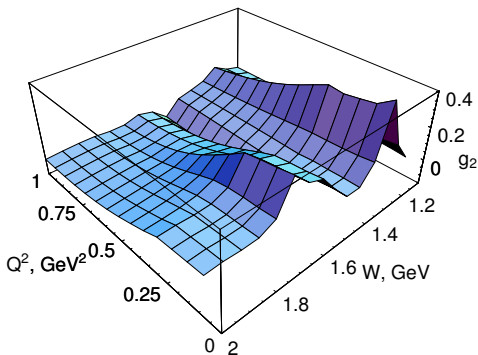
$$\Gamma(\mu) = \Gamma_r \frac{m_r}{\mu} \left(\frac{q}{q_r} \right)^{2J_r+1} \frac{q_r^2 + \delta^2}{q^2 + \delta^2}, \quad \delta = 0.3 \text{ GeV}$$

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Main nonresonant contribution to the cross sections $\sigma_{T,L}$ in the resonance region is determined by the Born terms constructed on the basis of Lagrangians of γNN , $\gamma\pi\pi$, πNN interactions. Another part of nonresonant background comprises the t -channel contributions of ρ , ω mesons obtained by means of effective Lagrangians $\gamma\pi V$, VNN interactions ($V = \rho, \omega$). In the unitary isobar model accounting the Born terms, the vector meson, nucleon resonance contributions and the interference terms we calculated the cross sections $\sigma_{T,L}$ by means of numerical program MAID (<http://www.kph-uni-mainz.de/MAID>) in the resonance region as the functions of two variables W and Q^2 . The obtained nucleon polarized structure functions $g_{1,2}(W, Q^2)$ are presented in Figs.



The proton polarized structure function $g_1(W, Q^2)$ as the function of variables Q^2 ($0 \div 1$) GeV^2 and W ($1.1 \div 2.0$) GeV .



The proton polarized structure function $g_2(W, Q^2)$ as the function of variables Q^2 ($0 \div 1$) GeV^2 and W ($1.1 \div 2.0$) GeV .

Deep-inelastic region

To construct the spin-dependent structure functions of the proton in deep inelastic region we can use the Q^2 evolution equations for the quark and gluon distributions:

$$\frac{dq_i(x, Q^2)}{dt} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left[q_i(y, Q^2) P_{qq} \left(\frac{x}{y} \right) + g(y, Q^2) P_{qg} \left(\frac{x}{y} \right) \right]$$

$$\frac{dg(x, Q^2)}{d \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left[\sum_i q_i(y, Q^2) P_{gq} \left(\frac{x}{y} \right) + g(y, Q^2) P_{gg} \left(\frac{x}{y} \right) \right]$$

The structure function g_1 is related to the polarized quark and gluon distributions:

$$g_1(x, Q^2) = \frac{\langle e^2 \rangle}{2} [C_{NS} \otimes \Delta q_{NS} + C_S \otimes \Delta \Sigma + 2n_f C_g \otimes \Delta g]$$

The nonsinglet and singlet quark distributions are defined as:

$$\Delta q_{NS} = \sum_{i=1}^{n_f} \left(\frac{e_i^2}{\langle e^2 \rangle} - 1 \right) (\delta q_i + \delta \bar{q}_i)$$

$$\Delta \Sigma = \sum_{i=1}^{n_f} (\delta q_i + \delta \bar{q}_i).$$

$\langle e^2 \rangle = n_f^{-1} \sum_{i=1}^n e_i^2$. $\Delta q_i(x, Q^2) = q_+(x, Q^2) - q_-(x, Q^2)$ measures to what degree the parton of flavour q "remembers" its parent proton polarization, $\Delta g(x, Q^2)$ is the longitudinally polarized gluon density.

The evolution equations were solved numerically using the method:

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In the region of $Q^2 \leq 1 \text{ GeV}^2$ we used the experimental data for $g_1(x, Q^2)$. All of the data, including the SMC data at $Q^2 \leq 1 \text{ GeV}^2$, were fitted by the parameterization:

$$g_1(x, Q^2) = a_1 x^{a_2} (1 + a_3 x + a_4 x^2) [1 + a_5 f(Q^2)] F_1(x, Q^2)$$

The numerical integration was performed with the $f(Q^2) = -\ln Q^2$ fit IV corresponding to the perturbative QCD behaviour.

Wandzura-Wilczek relation between spin structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$:

$$g_2^{WW}(x, Q^2) = -g_1(x, Q^2) + \int_x^1 g_1(t, Q^2) \frac{dt}{t}, \quad g_2^{WW} \approx g_2$$

Numerical results

a) The isobar model describing the processes of low-energy scattering of virtual photons on nucleons.

b) The evolution equations for the parton distributions.

► Electronic hydrogen hyperfine splitting

$$\delta_1^P = 2.6 \text{ ppm}, \quad \delta_2^P = -0.4 \text{ ppm}, \quad \delta^P = (2.2 \pm 0.8) \text{ ppm}.$$

► Muonic hydrogen hyperfine splitting

$$\delta_1^P = 5.18 \times 10^{-4}, \quad \delta_2^P = -0.48 \times 10^{-4}, \quad \delta^P = (4.70 \pm 1.04) \times 10^{-4}$$

The main theoretical uncertainty is related with the calculation of the helicity amplitudes $A_{1/2}(Q^2)$, $A_{3/2}(Q^2)$, $S_{1/2}(Q^2)$ in the quark model.

Hyperfine splittings of 1S and 2S states in muonic hydrogen

1. The effects of the vacuum polarization play important role in the case of muonic hydrogen ($\frac{\hbar^2}{\mu e^2} : \frac{\hbar}{m_e c} = 0.737384$)
2. The proton structure effects are taken into account in the loop amplitudes by means of electromagnetic form factors.
3. The proton polarizability contribution is calculated.

$$1S - state : \Delta E_{HFS}(1S) = 182.623 \text{ meV}$$

$$2S - state : \Delta E_{HFS}(2S) = 22.8129 \text{ meV}$$

$$8\Delta E_{HFS}(2S) - \Delta E_{HFS}(1S) = -0.120 \text{ meV}$$