The effective-range theory application to study the nuclear vertex constants for bound and resonant states of the lightest nuclei up to $^8\text{Be}$.

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FBS-Dubna - 2012
The effective-range theory taking into account the Coulomb interaction has been developed and applied to computing the nuclear vertex constants (NVC) for a decay (fuse) $A \leftrightarrow B+C$ and for relating the asymptotic normalization coefficients (ANC) of the radial wave functions for a nucleus $A$. We study the $^3$He $s$-wave bound state and the resonant $s$-wave states of $^2$He, $^3$He, $^8$Be, the $p$-wave states of $^5$He and $^5$Li, and the $d$-wave state of $^8$Be. Experimental phase shifts and pole positions are used as input data.


The effective-range theory

\[ K_l(k^2) = -\frac{1}{\sigma_l} + r_l k^2 / 2 + \ldots \] \quad (1)

In the case of the nucleon-deuteron (Nd) system, the effective-range function \( K_0(k^2) \) has the form

\[ K_0(k^2) = \frac{(-1/A_0 + C_2 k^2 + C_4 k^4)}{(1 + k^2/k_0^2)}. \] \quad (2)

In the absence of Coulomb interaction, the effective-range function \( K_l(k^2) \) is related to the phase shift \( \delta_l(E) \) by the well-known equation

\[ K_l(k^2) = k^{2l+1} \cot \delta_l(E). \] \quad (3)

In the presence of a Coulomb repulsion, the right-hand side of this equation is modified as (see, for example, [10])

\[ K_l(k^2) = k^{2l+1} (c_l \gamma)^{-1} \times \left[ \frac{2 \pi \gamma}{\exp(2 \pi \gamma) - 1} \right] \times \left( \cot \delta_l^C(E) - i + 2 \gamma H(\gamma) \right), \] \quad (4)

where

\[ H(\gamma) = \Psi(i \gamma) + (2i \gamma)^{-1} - \ln(i \gamma), \] \quad (5)

\[ (c_l \gamma)^{-1} = \prod_{n=1}^{l} \left( 1 + \gamma^2 / n^2 \right), \quad (c_0 \gamma)^{-1} = 1, \] \quad (6)

\[ \delta_l^C(E) = \delta_l(E) - \sigma_l(E). \] \quad (7)

Here, we have used the following notation: \( \delta_l(E) \) is the phase shift for the sum of the Coulomb and nuclear potentials; \( \sigma_l(E) \) is the Coulomb phase shift, which is given by the relation

\[ \exp(2i \sigma_l) = \Gamma(l + 1 + i \gamma) / \Gamma(l + 1 - i \gamma); \] \quad (8)

\( \Psi(i \gamma) \) is the logarithmic derivative of the Euler gamma function; \( \gamma = \lambda / k \) is the Sommerfeld Coulomb parameter; \( \lambda = \mu \alpha Z_1 Z_2 \) where \( \alpha = e^2 / \hbar c \) is the fine-structure constant; and \( Z_1 \) is the charge number of nucleus \( I \). We will also use the Bohr radius \( a_B = 1 / \lambda \).

This notation was adopted in [11]. Using an explicit expression for \( \Psi(i \gamma) \) in the form of a semi-infinite sum, we can recast \( H(\gamma) \) into the form (see, for example, [12])

\[ H(\gamma) = \frac{i \pi}{\exp(2i \pi \gamma) - 1} + \gamma^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + \gamma^2)} - \ln \gamma - \zeta, \] \quad (9)

where \( \zeta \approx 0.5772 \) is the Euler constant.


2. METHOD FOR CALCULATING THE RENORMALIZED VERTEX CONSTANT

\[ f_1^C = \exp(2i\sigma_i)f_i, \quad (14) \]

\[ f_i = (\exp(2i\delta_i^{C}) - 1)/2ik \]
\[ = 1/(k \cot \delta_i^{C} - ik). \quad (15) \]

Let us determine \((k \cot \delta_i^{C} - ik)\) from (4) and substitute the result in (14) and (15). We obtain the formulas

\[ f_i^C = \tilde{f}_{1N}k^{2l}\phi_i(k), \quad (16) \]

where

\[ \tilde{f}_{1N} = [K_l(k^2) - 2k^{2l+1}(ct_{l\gamma})^{-1}\gamma H(\gamma)]^{-1}, \quad (17) \]

\[ \phi_i(k) = \exp(2i\sigma_i)[2\pi\gamma/[\exp(2\pi\gamma) - 1]](ct_{l\gamma})^{-1}. \quad (18) \]

We simplify the expression for \(\phi_i(k)\). After making some simple transformations and using the relation {see formula (3.2) in [24], p. 54}

\[ \Gamma(l+1+i\gamma)\Gamma(l+1-i\gamma) = \frac{\pi P_{l+1}(i\gamma)}{\gamma \sinh(\pi\gamma)}, \quad (19) \]

where \(P_n\) (not a Legendre polynomial) is defined by the formulas

\[ P_1(i\gamma) = \gamma^2, \quad P_{l+1}(i\gamma) = \gamma^2 \prod_{n=1}^l (n^2 + \gamma^2), \quad (20) \]

we arrive at the expression

\[ \phi_i(k) = \left(\frac{\Gamma(l+1+i\gamma)}{l!}\right)^2 e^{-\pi\gamma}. \quad (21) \]

\[ \tilde{f}_{1N} = \tilde{f}_{1N}k^{2l} = \frac{k^{2l}}{[K_l(k^2) - 2\lambda k^{2l} H(\gamma)(ct_{l\gamma})^{-1}]} \quad (22) \]

It is obvious from this formula [see also (4)] that the amplitude in question has a pole at \(k = p\) if the denominator in (22) vanishes. For the standard effective-range expansion, the pole position can be determined from the equation [with allowance for (1)]

\[ \frac{1}{a_1} + np^2/2 + \ldots \]
\[ = 2\lambda p^{2l} H(\lambda/p)[(ct_{l\gamma})^{-1}]_{k=p}. \quad (23) \]

Using the definitions of the vertex constant that are known from [26], we obtain the following expression for the renormalized vertex constant:

\[
\tilde{G}_i^2 = -\left(\pi/\mu^2\right) \lim_{k \to -i\kappa} (k^2 + \kappa^2) \tilde{f}_{IN}(k).
\] (26)

One can recast expression (26) into the form

\[
\tilde{G}_i^2 = \frac{(-2\pi/\mu^2)p^{2l+1}}{d/dk [K_1(k^2) - 2\lambda k^2 H(\gamma)(e_\gamma)^{-1}]_{k=p}},
\] (27)

where, for the standard expansion in (1), we have

\[
\frac{d}{dk} [K_1(k^2)]_{k=p} = r_1 p + \ldots.
\]

\[
\tilde{G}_0^2 = \frac{2\pi \kappa/\mu^2}{\varphi(x) - 2\kappa \kappa_0^2 - C_0 + C_2 \kappa_0^2 - C_4 \kappa^2 (2\kappa_0^2 - \kappa^2)/(\kappa_0^2 - \kappa^2)}.
\] (30)

The function \(\varphi(x)\) can be written as (\(x = \lambda/\kappa = 1/a_0 \kappa\))

\[
\varphi(x) = -1 - 2x + 2x^2 \Psi'(x).
\] (31)

It should be noted that, by virtue of relation (15), the positions of the poles \(k_{\text{res}} = i\kappa\) in the scattering amplitude \(f(k)\) for a bound and a resonance (or a virtual) state are given by

\[
\cot \delta_i^C(E) = i.
\] (32)

\[
K_0(k^2) = -1/a_0 + r_0 k^2/2 - P r_0^3 k^4 + Q r_0^5 k^6 - \ldots,
\] (28)

whence it follows

\[
\tilde{G}_0^2 = \frac{2\pi \kappa/\mu^2}{\varphi(x) - [r_0 \kappa + 4P(r_0 \kappa)^3 + 6Q(r_0 \kappa)^5 + \ldots]},
\]

and the effective-range function in the form (2), which has a pole and which leads to the relation \((C_0 = -1/a_0)^4\)

The function \(\varphi(x)\) can be written as (\(x = \lambda/\kappa = 1/a_0 \kappa\))

3.1. Proton–Proton System: Ground State of $^2$He

The results obtained in [2] by solving the transcendental equation (23) with the set of parameters borrowed from [17] (without errors), $k_{2\text{He}} = 0.0644 - i0.0871$ fm$^{-1}$ and $E_{2\text{He}} = -142 - i465$ keV, differ from the results presented in [11], $k_{2\text{He}} = 0.0647 - i0.0870$ fm$^{-1}$ and $E_{2\text{He}} = -140 - i467$ keV, only in the last decimal place. In [2], the square of the corresponding renormalized vertex constant was calculated for the first time. It was found to be $\tilde{G}^2 = -(0.060 + i0.051)$ fm. The results of the calculations change only slightly if we set $P = Q = 0$.


3.2.1. Properties of the $^3$He bound state. The renormalized vertex constant calculated in [2] for the bound state of the $^3$He nucleus in the effective-range approximation specified by Eq. (2) appears to be strongly (approximately by a factor of two) underestimated with respect to the results of both experimental analyses and theoretical calculations. This is because the pole corresponding to the $^3$He bound state lies far from the region of convergence of the expansion in powers of $k^2$ in the numerator in (2). Unfortunately, other data that can be found in the literature for the $^3$H and $^3$He vertex constants were obtained more than 20 years ago. Moreover, one can find either the values of the renormalized vertex constant $\tilde{G}^2$ or the values of the asymptotic normalization constant $C^2$ in various studies. In [2], both the renormalized vertex constant and the asymptotic normalization constant for the bound state of the $^3$He nucleus are presented in Table 1. For the vertex constant and the asymptotic normalization coefficient, use was made there of the relation

$$\tilde{G}^2 = \frac{(27/4)\pi\kappa_{\tau}}{\lambda_{N}} C^2, \quad (39)$$

which is adopted in the literature (see, for example, [28, 29]) and where $\lambda_{N} = \hbar/mc$ and $\kappa_{\tau}$ is the wave number for the bound state of the $^3$He nucleus. This relation corresponds to the following asymptotic behavior for the normalized (to unity) radial wave function for $Nd$ relative motion (in the case of the $S$ wave):

$$\psi(r) \rightarrow C_{\tau} N_{ZR} \left[W_{\tau,1/2}(2\kappa_{\tau}r)\right]/r, \quad (40)$$

$$r \rightarrow \infty, \quad N_{ZR} = (2\kappa_{\tau})^{1/2}.$$
3.2.2. Subthreshold resonance in the pd system If we use the set of parameters for the effective-range function from [30], our calculations of the subthreshold-resonance energy lead to the value of $E_{\text{res}}^{pd} = -(0.315 + i0.102)$ MeV, which differs only by a few thousandths from the corresponding value of $E_{\text{res}}^{pd} = -(0.319 + i0.099)$ MeV for version no. 8 from Table 1 in [2], where the $^3$He binding energy was used as an additional adjustable parameter. We will now compare our result, $E_{\text{res}}^{pd} \approx -(0.32 + i0.10)$ MeV, with other results available from the literature. The value of $E_{\text{res}}^{pd} = -(0.432 + i0.32) $ MeV was obtained in [31] by the $N/D$ method. The absolute value of the resonance energy is slightly overestimated in the $N/D$ method, whereas the absolute value of the imaginary part is less by a factor of about three. In [22], the value of $E_{\text{res}}^{pd} = -(0.432 + i0.56)$ MeV was obtained by solving the three-body equation with the Eikemeier–Hackenbroich $NN$ potential. The real part of this value agrees with the result presented in [31], but the absolute value of the imaginary part is larger by a factor of about five than that in [19] and is larger by an order of magnitude than that obtained by the $N/D$ method [31]; this requires an explanation. It is noteworthy that the position found in [22] for the virtual triton pole ($B_v = 1.62$ MeV) also exceeds considerably (by a factor greater than three) other estimates available for this quantity in the literature.


On the convergence of an effective-range function expansion for the Nd scattering

\[ K(k^2) = \frac{-1/a + C_2 k^2 + C_4 k^4}{1 + k^2/\kappa_0^2}, \quad (1) \]

The dynamics of the spin-doublet nucleon–deuteron system at low energies is determined by the Feynman diagram of one-nucleon exchange in the elastic-scattering amplitude. In the partial-wave amplitude \( f(k) \), this diagram generates a singularity closest to the threshold. The corresponding logarithmic branch point occurs at an energy of \( E_{\text{cm}} = -E_d = -\epsilon_d/3 \approx -0.74 \text{ MeV} \), which is equivalent to the region \( |E_{\text{lab}}| < 1.1 \text{ MeV} \) in the laboratory frame (\( \epsilon_d \) is the deuteron binding energy). The position of this singularity determines the radius of convergence of the numerator on the right-hand side of (1) in powers of \( k^2 \) in the complex plane of momentum. Strictly speaking, it is the boundary that specifies the region of lower energies from the point of view of convergence of the expansion in (1). In the literature, the term “low energies” was treated rather widely. For example, the energy range \( 1 \leq E_{\text{lab}} \leq 46.3 \text{ MeV} \) was considered in [J. Arvieux, Nucl. Phys. A 221, 253 (1974)] in performing a partial-wave analysis of proton–deuteron scattering.
Fig. 5. Results of a partial-wave analysis [41] of nucleon scattering on $^4$He nuclei in the $P$ wave: the closed circles and boxes represent data on $n\alpha$ scattering for $j = 3/2$ and $1/2$, respectively, while the inverted and right closed triangles stand for the results on $p\alpha$ scattering for $j = 3/2$ and $1/2$, respectively.

Table 2. Complex values of the resonance energies \( E_r - i\Gamma/2 \) (in MeV units) for \( J^P = 3/2^- \) and \( 1/2^- \) in the c.m. frame with respect to the threshold in the \( \alpha + N \) (\(^5\)He, \(^5\)Li) channels and residues of the \( T \) matrix \( R_t = |R_t| \exp(i\varphi_t) \) (\( \varphi_t \) is given in degrees)

| Method                      | \( E_r \) | \( \Gamma \) | \( |R_t| \) | \( \varphi_t \) |
|-----------------------------|----------|----------|----------|----------|
| \(^5\)He nucleus, \( J^P = 3/2^- \) |           |          |          |          |
| Eff. range [16]             | 0.778    | 0.639    |          |          |
| \( N/D \) [25, 37]          | 0.607    | 0.542    | 0.160    | 132      |
| \( d, \gamma \) [38]        | 0.80     | 0.65     |          |          |
| \( R \) function [39]       | 0.761    | 0.643    |          |          |
| \( R \) matrix [36]         | 0.80     | 0.65     |          |          |
| Eff. range (our study)      | 0.778    | 0.641    | 0.171    | -133     |
| \(^5\)He nucleus, \( J^P = 1/2^- \) |           |          |          |          |
| Eff. range [16]             | 1.999    | 4.534    |          |          |
| \( N/D \) [25, 37]          | 1.875    | 5.296    | 0.237    | 177      |
| \( R \) function [39]       | 1.970    | 5.218    |          |          |
| \( R \) matrix [36]         | 2.07     | 5.57     |          |          |
| Eff. range (our study)      | 2.000    | 4.533    | 0.199    | 178      |
| \(^5\)Li nucleus, \( J^P = 3/2^- \) |           |          |          |          |
| Eff. range [16]             | 1.637    | 1.292    |          |          |
| \( N/D \) [25, 37]          | 1.655    | 1.278    | 0.304    | -136     |
| \( d, \gamma \) [38]        | 1.72     | 1.28     |          |          |
| \( R \) matrix [36]         | 1.69     | 1.23     |          |          |
| Eff. range (our study)      | 1.630    | 1.437    | 0.278    | -148     |
| \(^5\)Li nucleus, \( J^P = 1/2^- \) |           |          |          |          |
| Eff. range [16]             | 2.858    | 6.082    |          |          |
| \( N/D \) [25, 37]          | 2.691    | 6.448    | 0.313    | -178     |
| \( R \) matrix [36]         | 3.18     | 6.60     |          |          |
| Eff. range (our study)      | 2.34     | 6.01     | 0.287    | 4.2      |


The phase-shift-analysis data for the $^3$He$^4$He scattering are analyzed using the effective-range theory, with the Coulomb interaction taken into account. We find both the renormalized nuclear vertex constants for the vertex $^7$Be $\rightarrow$ $^3$He + $^4$He in the ground ($3/2^-$) and in the first excited ($1/2^-$) bound states of $^7$Be, and the corresponding asymptotic normalization constants of the radial wave functions in these states. The results obtained can be used in an astrophysical S-factor calculation for radiative capture reaction $^4$He($^3$He, $\gamma$)$^7$Be.

![Graph 1](image1.png)

**Fig. 1.** The dependence of the effective-range function $K_{3/2}(E_3)$ on the energy of the incident nucleus $^3$He for the $^7$Be ground state $j^\pi = 3/2^-$. The points are the results of the phase shift analysis (x—from [3], •—from [4]). The solid curve is the result of fitting $\delta$ [3, 4]. The circle at the negative energy corresponds to the $^7$Be ground state pole. The symbol “+” on the energy axis here and on Fig. 2 shows the position of the energy maximum of the convergence region for the effective-range expansion (1) (this point is accidentally close to the narrow $7/2^-$ resonance position).

![Graph 2](image2.png)

**Fig. 2.** As in Fig. 1 except for the first excited bound state $j^\pi = 1/2^-$ of the $^7$Be.

Fig. 3. The dependence of the effective-range function $K_{3/2}(k^2)$ on the $^3$He-$^4$He relative momentum squared for the $^7$Be ground state. The points are presented for the phase shift analysis results from [3] with the experimental uncertainties taken into account (○ for $\delta_+$, • for $\delta_-$). The dashed and solid curves are the results of fitting $\delta_+$ and $\delta_-$ respectively. The circle at the negative energy corresponds to the $^7$Be ground state pole.

Fig. 4. As in Fig. 3 except for the first excited bound state $j^\pi = 1/2^-$ of the $^7$Be. The circle at the negative energy corresponds to the first excited $^7$Be bound state pole.
The results of our calculations are given for the absolute values of the asymptotic normalization constant \((C_j)\), renormalized nuclear vertex constant \(G_{ren}^2\) for the ground \((j^\pi = 3/2^-)\) and first excited \((j^\pi = 1/2^-)\) bound states of \(^{7}\)Be, and also the parameters of the effective-range function (1), found by fitting the experimental phase shifts \(\delta_j\) and the binding energies \(\varepsilon_j\) (see text).

| \(j^\pi, \delta\) | \(a_j, \text{Fm}^3\) | \(-r_j, \text{Fm}^{-1}\) | \(-P_j, \text{Fm}\) | \(|G_{ren}^2(j)|, \text{Fm}\) | \(|C_j|, \text{Fm}^{-1/2}\) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| 3/2\(^-\), \(\delta_+\) [3] | 90.7 | 0.115 | 2.26 | 0.072 | 1.85 |
| 3/2\(^-\), \(\delta_-\) [3] | 88.1 | 0.095 | 2.57 | 0.065 | 1.75 |
| 3/2\(^-\), \(\delta\) [3, 4] | 109 | 0.097 | 2.04 | 0.099 | 2.17 |
| 1/2\(^-\), \(\delta_+\) [3] | 167 | 0.080 | 1.97 | 0.087 | 2.22 |
| 1/2\(^-\), \(\delta_-\) [3] | 193 | 0.049 | 2.26 | 0.089 | 2.25 |
| 1/2\(^-\), \(\delta\) [3, 4] | 230 | 0.040 | 2.08 | 0.120 | 2.61 |

We consider it necessary to obtain more precise measurements within the low-energy area. In particular, solving the problem of the narrow 7/2\(^-\) resonance influence on the phase shift in other states including the states studied in the present work needs to be done. On the whole, the results obtained for the vertex constants are smaller in absolute value in comparison with those of model calculations in the literature. The fact that the ANC for the excited state 1/2\(^-\) are larger than those for the ground state 3/2\(^-\) is unexpected in some ways.
We find the vertex constant for the synthesis $\alpha + \alpha \rightarrow ^8\text{Be}$ and the corresponding asymptotic normalization coefficient of the Gamov wave function for $^8\text{Be}$ in the ground state. We use modern data on the position and width of the narrow resonance of this state as well as the energy dependence of the $\alpha$–$\alpha$ scattering phase shift in the $s$–wave known from the literature. The effective-range theory was applied with the Coulomb interaction taken into account. The parameters of the standard effective-range function expansion up to the member with $k^4$ ($k$ being the relative momentum) were found.

![Graph](image)

**Fig. 2.** Dependence of the phase shifts $\delta_0$ and $\delta_2$ on the $\alpha\alpha$ scattering energy $E_\alpha$ (in MeV). Experimental data with uncertainties: points represent $\delta_0$, open circles—$\delta_2$. Curves describe energy behavior $\delta_0$ on $E_\alpha$ for the effective–range approximation (3): dash-dots are for variant A, solid line—for variant B. An arrow shows the maximum energy (convergence radius $E_\alpha \approx 5$ MeV) where the expansion (3) converges.
THE CHARACTERISTICS OF THE $^8$Be GROUND STATE

Resonance energy position for $^8$Be ground state, parameters $a$, $r$, $P$ of effective—range function (3), found for variants A and B at given fixed input values $E_0$ and $\Gamma \pm \Delta \Gamma$ from (1), computing results with parameters of effective—range function which are complex values of renormalized NVC $G_{\text{ren}}^2$ (in fm), absolute values of ANC ($|C|$) (in fm$^{-1/2}$) and of approximated values ANC ($|C_f|$) (in fm$^{-1/2}$) (see text)

| Variant, method | $E_0$, keV | $\Gamma$, eV | $-a$ | $r$ | $-P$ | $\text{Re} G_{\text{ren}}^2$, $\text{Fm}$ | $\text{Im} G_{\text{ren}}^2 \cdot 10^4$, $\text{Fm}^{-1/2}$ | $|C|$, $\text{Fm}^{-1/2}$ | $|C_f|$, $\text{Fm}^{-1/2}$ |
|-----------------|------------|-------------|------|-----|------|-----------------|-----------------|-----------------|-----------------|
| A               | 91.84      | 5.57        | 1920 | 1.096 | 0.398 | 0.560           | 1.52            | 1.46            | 2.40            |
| A$_{-}$         | 91.84      | 5.32        | 1902 | 1.092 | 0.400 | 0.543           | 1.43            | 1.43            | 2.34            |
| A$_{+}$         | 91.84      | 5.82        | 1966 | 1.099 | 0.391 | 0.574           | 1.59            | 1.47            | 2.45            |
| B               | 91.84      | 5.57        | 1936 | 1.100 | 0.311 | 0.561           | 1.52            | 1.46            | 2.40            |
| $\delta = \pi/2$ | 97.79      | 4.65        | 1760 | 1.096 | 0.314 | 0.538           | 2.27            | 1.52            | 1.87            |

Note: As the third fitting quantity we use the phase shift $\delta_0 = 68^\circ$ at $E_{\text{lab}} = 7.88$ MeV (variant A) $\delta_0 = 96.6^\circ$ at $E_{\text{lab}} = 5.26$ MeV (variant B).
The first excited state of $^8$Be ($2^+$)

The effective-range parameters ($a_2, r_2, P_2$), renormalized NVC and the abs. value of ANC

| №  | $E_2$, MeV | $\Gamma_2$, MeV | $a_2$, fm$^5$ | $r_2$, fm$^{-3}$ | $P_2$, fm$^8$ | Re$G_{2ren}^2$, fm | Im$G_{2ren}^2$, fm | $|C_2|$, fm$^{-1/2}$ |
|----|-----------|-----------------|------------|----------------|-------------|-----------------|-----------------|------------------|
| 1  | 3.02      | 1.47            | 172        | 0.233          | 2.08        | 0.0188          | 0.0121          | 1.136            |
| 2  | 3.02      | 1.51            | 188        | 0.220          | 0.83        | 0.0190          | 0.0127          | 1.146            |
| 3  | 3.04      | 1.47            | 107        | 0.283          | 3.10        | 0.0192          | 0.0113          | 1.134            |
| 4  | 3.04      | 1.51            | 116        | 0.267          | 2.64        | 0.0194          | 0.0119          | 1.143            |
| 5  | 2.87      | 1.31            | –32.7      | –0.275         | 19.8        | 0.0113          | 0.0154          | 1.062            |
| 6  | 2.77      | 1.24            | –43.1      | –0.196         | 45.9        | 0.0125          | 0.0151          | 1.081            |


In variants 5-6 values we calculated $E_2$ and $\Gamma_2$ using the found effective-range function parameters. While fitting $a_2, r_2$ and $P_2$ we use the minimal values of $\delta_2$ in var. 5 and maximum values of $\delta_2$ in var. 6 (see Afzal S.A., Ahmad A.A.Z., Ali S., *Rev. Mod. Phys.* 1969. V. 41. P. 247; Warburton E.K., *Phys. Rev. C*, 1986. V. 33, P. 303).
The points no (1-4) on the complex energy plane are taken from the review [Tilley, D.R., Kelley, J.H., Godwin, J.L., et al., Nucl.Phys. A, 2004, vol. 745, p. 155]. One can see that the fitting the phase shift in the full energy region leads to poles shift to the smaller Re$E_2$ and Im$E_2$. 