The Thermal-Statistical Model for Particle Production II.

J. Cleymans

25 July 2008 / Dubna, Russia

Outline

Hydrodynamic Flow.

Cooper-Frye formula

From

$$N = \int j^\mu d\sigma_\mu \quad \text{and} \quad j^\mu = \int d^3p \frac{p^\mu}{E} f(r,p,t)$$

obtain

$$E\frac{dN}{d^3p} = \int p^\mu d\sigma_\mu \; f(r,p,t)$$

The number of particles of type *i* is determined by:

$$E\frac{dN_i}{d^3p} = \frac{g_i}{(2\pi)^3}\int d\sigma_\mu p^\mu \exp\left(-\frac{p^\mu u_\mu}{T}+\frac{\mu_i}{T}\right)$$

Integrating this over all momenta

$$N_i = \frac{g_i}{(2\pi)^3}\int d\sigma_\mu \int \frac{d^3p}{E}p^\mu \exp\left(-\frac{p^\mu u_\mu}{T}+\frac{\mu_i}{T}\right)$$

or

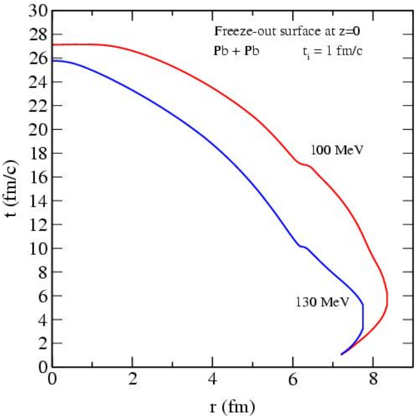
$$N_i = \int d\sigma_\mu u^\mu n_i(T,\mu)$$

If the temperature and chemical potential are unique along the freeze-out curve

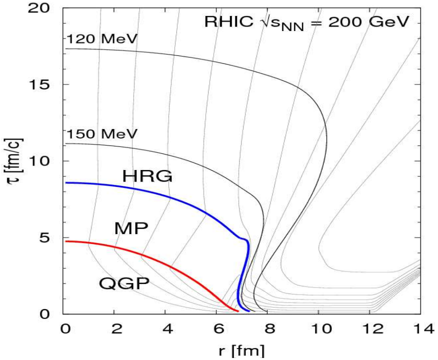
Particle Yield:

$$N_i = n_i(T,\mu)\int d\sigma_\mu u^\mu$$

i.e. integrated (4π) multiplicities are the same as for a single fireball at rest (apart from the volume).



J. Cleymans, K. Redlich, D.K. Srivastava
Phys. Rev. C55 (1997) 1431



K. Eskola, H. Honkanen, H.Niemi, P.V. Ruuskanen, S.S. Räsänen, hep-ph/0506049

Full Hydrodynamic Flow

Bjorken scaling + Transverse expansion after integration over m_T

$$\left(\frac{dN_i}{dy}\right)_{y=0} = \frac{g}{\pi}\int_\sigma r\, dr\, \tau_F(r)$$
$$\left\{\cosh(y_T) - \left(\frac{\partial \tau_F}{\partial r}\right)\sinh(y_T)\right\} m_i^2 T K_2\left(\frac{m_i}{T}\right)$$

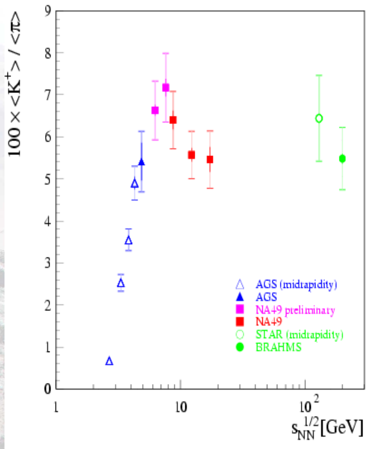
Consequence : $\dfrac{dN_i/dy}{dN_j/dy} = \dfrac{N_i^0}{N_j^0}$

Effects of hydrodynamic flow cancel out in ratio.

The NA49 Collaboration has recently performed a series of measurements of Pb-Pb collisions at 20, 30, 40, 80 and 158 AGeV beam energies . When these results are combined with measurements at lower beam energies from the AGS they reveal an unusually sharp variation with beam energy in the $\Lambda/\langle\pi\rangle$, with $\langle\pi\rangle \equiv 3/2(\pi^+ + \pi^-)$, and K^+/π^+ ratios. Such a strong variation with energy does not occur in pp collisions and therefore indicates a major difference in heavy-ion collisions. This transition has been referred as the “horn”.

The Elephant in the Room

Friese
Dinkelaker
Blume
Speltz



Difficult to avoid, Hard to Model

→ But no unambiguous corroborating evidence

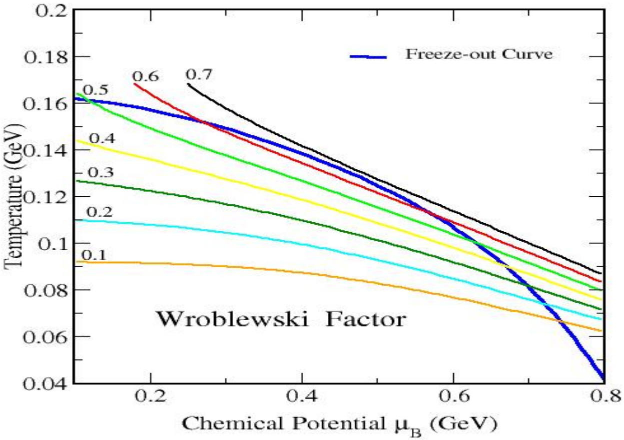
Strangeness in Heavy Ion Collisions
vs
Strangeness in pp - collisions

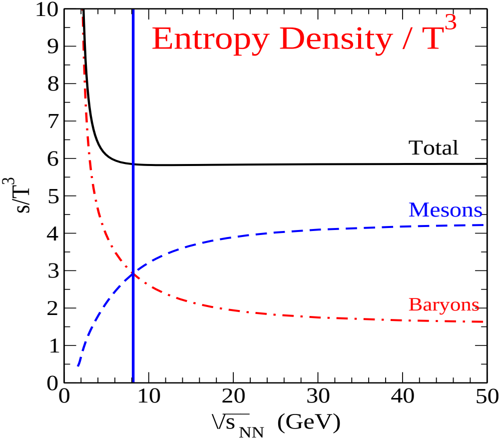
Use the Wroblewski factor

$$\lambda_s = \frac{2\langle s\bar{s}\rangle}{\langle u\bar{u}\rangle + \langle d\bar{d}\rangle}$$

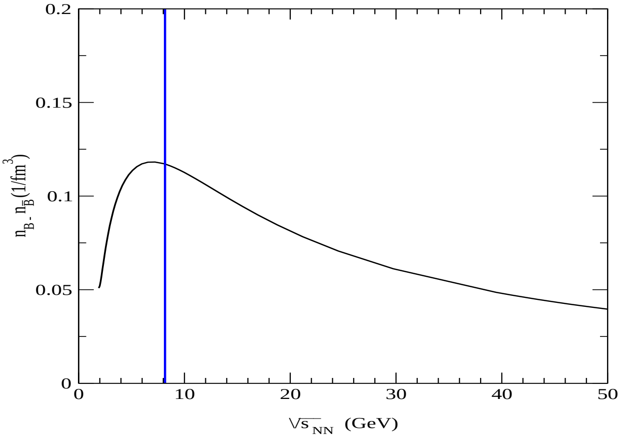
This is determined by the number of **newly** created quark – anti-quark pairs and **before** strong decays, i.e. before ρ's and Δ's decay.

Limiting values :
$\lambda_s = 1$ all quark pairs are equally abundant, SU(3) symmetry.
$\lambda_s = 0$ no strange quark pairs.

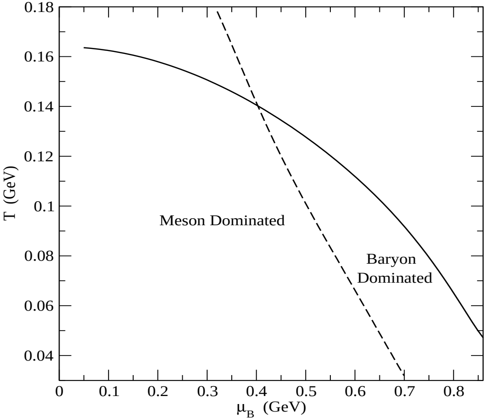

Freeze-out Curve
0.7
0.6
0.5
0.4
0.3
0.2
0.1
Wroblewski Factor
Temperature (GeV)
Chemical Potential μ_B (GeV)
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.2
0.4
0.6
0.8



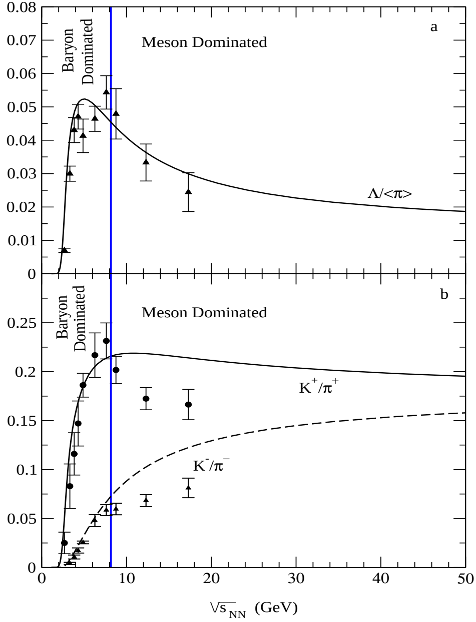
J. C., H. Oeschler, K. Redlich and S. Wheaton, Physics Letters B615 (2005) 50-54.



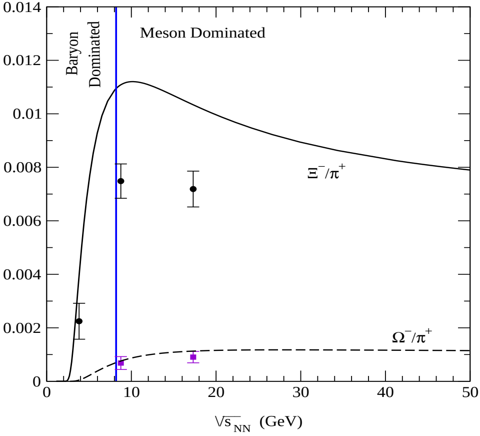
J. C., H. Oeschler, K. Redlich and S. Wheaton, Physics Letters B615 (2005) 50-54.



J. C., H. Oeschler, K. Redlich and S. Wheaton, Physics Letters B615 (2005) 50-54.



J. C., H. Oeschler, K. Redlich and S. Wheaton, Physics Letters B615 (2005) 50-54.



J. C., H. Oeschler, K. Redlich and S. Wheaton, Physics Letters B615 (2005) 50-54.

Maxima in Particle Ratios predicted by the Thermal Model.

Ratio	Maximum at $\sqrt{s_{NN}}$ (GeV)	Maximum Value
$\Lambda/\langle\pi\rangle$	5.1	0.052
Ξ^-/π^+	10.2	0.011
K^+/π^+	10.8	0.22
Ω^-/π^+	27	0.0012

J. C., H. Oeschler, K. Redlich and S. Wheaton, Physics Letters B615 (2005) 50-54.

Excluded Volume Corrections.

$$
\begin{aligned}
Z &= \exp\left\{ V \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}+\frac{\mu}{T}} \right\} \\
&= \sum_{N=0}^{\infty} \frac{V^N}{N!} e^{\mu N/T} \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N
\end{aligned}
$$

with excluded volume corrections

$$
\begin{aligned}
Z \rightarrow & \sum_{N=0}^{\infty} \frac{(V - V_0 N)^N}{N!} e^{\mu N/T} \\
& \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N \theta(V - V_0 N)
\end{aligned}
$$

Excluded Volume Corrections.

It is more convenient to consider these corrections in the pressure ensemble:

$$Z_p \equiv \int_0^\infty dV\ e^{-PV/T} \sum_{N=0}^{\infty} \frac{V^N}{N!} e^{\mu N/T} \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N$$

$$\begin{aligned} Z_p \quad \rightarrow \quad & \sum_{N=0}^{\infty} \int_0^\infty dV\ e^{-PV/T} \\ & \frac{(V - V_0 N)^N}{N!} e^{\mu N/T} \\ & \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N \theta(V - V_0 N) \end{aligned}$$

introduce $x \equiv V - V_0 N$.

Excluded Volume Corrections.

$$
\begin{aligned}
Z_p &= \sum_{N=0}^{\infty}\int_0^{\infty} dx\ e^{-Px/T} \\
&\quad \frac{x^N}{N!} e^{-PV_0N/T} e^{\mu N/T} \\
&\quad \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}}\right]^N
\end{aligned}
$$

a new variable $\bar{\mu} \equiv \mu - PV_0$

Excluded Volume Corrections.

$$Z_p \rightarrow \sum_{N=0}^{\infty} \int_0^{\infty} dx\ e^{-Px/T} \frac{x^N}{N!} e^{\bar{\mu}N/T} \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N$$

which is the original partition function with the replacement

$$\boxed{\bar{\mu} = \mu - P\ V_0}$$

Excluded Volume Corrections.

The particle number density now becomes:

$$\begin{aligned} n &= \frac{T}{V}\frac{\partial}{\partial\mu}\ln\ Z \\ &= \frac{T}{V}\frac{\partial\bar{\mu}}{\partial\mu}\frac{\partial}{\partial\bar{\mu}}\ln\ Z \\ &= \frac{\partial\bar{\mu}}{\partial\mu}n_0 \\ &= [1 - V_0\ n]\,n_0 \end{aligned}$$

$$\boxed{n = \frac{n_0}{1 + V_0 n_0}}$$

Effects Cancel Out in Ratios.

J.C., K. Redlich, H. Satz, E. Suhonen, ZfP C33, 151, (1986)

D.H. Rischke, M.I. Gorenstein, H. Stöcker, W. Greiner, ZfP C51, 485 (1991).

Exact Strangeness Conservation.

For a small system at low temperatures ($T \approx 50$ MeV), e.g. at GSI canonical corrections are necessary.
Instead of

$$N_K \approx \exp -M_K/T$$

one gets

$$N_K \approx \exp -2M_K/T$$

Extra suppression is due to strangeness conservation and lack of a large heat bath. This correction disappears quickly at higher energies and is already small at AGS energies.

Exact Strangeness Conservation.

$$Z = \text{Tr } e^{-\frac{H}{T} + \frac{\mu N}{T}}$$

Insert a Kronecker delta in the trace:

$$\sum_i n_i(S=1) + 2\sum_j n_j(S=2) + 3\sum_k n_k(3) =$$

$$\sum_i \bar{n}_i(S=-1) + 2\sum_j \bar{n}_j(S=-2) + 3\sum_k \bar{n}_k(S=-3)$$

and rewrite it as

$$\delta\left(\sum_i n_i(S=1) + ..., \sum_i \bar{n}_i(S=-1) + ...\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi$$

$$\exp\left(i\phi\sum_i n_i(S=1) + ..\ -i\phi\sum_i \bar{n}_i(S=-1)\right)$$

Exact Strangeness Conservation.

$$Z = \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp \Big\{ Z_1 e^{i\phi} + Z_{-1} e^{-i\phi} + Z_2 e^{2i\phi} + Z_{-2} e^{-2i\phi} + Z_3 e^{3i\phi} + Z_{-3} e^{-3i\phi} \Big\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp \Big\{ \sqrt{Z_1 Z_{-1}} \left[\sqrt{\frac{Z_1}{Z_{-1}}} e^{i\phi} + \sqrt{\frac{Z_{-1}}{Z_1}} e^{-i\phi} \right] + \sqrt{Z_2 Z_{-2}} \left[\sqrt{\frac{Z_2}{Z_{-2}}} e^{2i\phi} + \sqrt{\frac{Z_{-2}}{Z_2}} e^{-2i\phi} \right] + \sqrt{Z_3 Z_{-3}} \left[\sqrt{\frac{Z_3}{Z_{-3}}} e^{3i\phi} + \sqrt{\frac{Z_{-3}}{Z_3}} e^{-3i\phi} \right] \Big\}$$

Z_1: sum of all particles with strangeness 1, e.g. K^+

Exact Strangeness Conservation.

Use

$$\exp\left\{\frac{x}{2}\left(t+\frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} I_m(x)t^m$$

to obtain

$$Z = \frac{1}{2\pi}\int_0^{2\pi} e^{3im\phi+2in\phi+ip\phi} \sum_{p=-\infty}^{\infty} I_p(x_1) \sum_{n=-\infty}^{\infty} I_n(x_2) \sum_{m=-\infty}^{\infty} I_m(x_3)\, y_1^p y_2^n y_3^m$$

where

$$y_i = \sqrt{\frac{Z_i}{Z_{-i}}} \qquad x_i = 2\sqrt{Z_i Z_{-i}}$$

Exact Strangeness Conservation.

$$Z = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{3m+2n}(x_1)\, y_1^{-3m-2n}\, I_n(x_2)\, y_2^n\, I_m(x_3)\, y_3^m$$

Exact Strangeness Conservation.

$$
\begin{aligned}
Z &= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left\{Z_1 e^{i\phi} + Z_{-1} e^{-i\phi}\right\} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left\{\sqrt{Z_1 Z_{-1}}\left[\sqrt{\frac{Z_1}{Z_{-1}}} e^{i\phi} + \sqrt{\frac{Z_{-1}}{Z_1}} e^{-i\phi}\right]\right\}
\end{aligned}
$$

Z_1: sum of all particles with strangeness 1, e.g. K^+
Z_{-1}: sum of all particles with strangeness -1, e.g. Λ

Exact Strangeness Conservation.

Use

$$\exp\left\{\frac{x}{2}\left(t+\frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} I_m(x)t^m$$

to obtain

$$Z = \frac{1}{2\pi}\int_0^{2\pi} e^{ip\phi} \sum_{p=-\infty}^{\infty} I_p(x_1)y_1^p$$

where

$$y_1 = \sqrt{\frac{Z_1}{Z_{-1}}} \qquad x_1 = 2\sqrt{Z_1 Z_{-1}}$$

$$Z = I_0(x_1)$$

Exact Strangeness Conservation.

In more detail, e.g. the multiplicity of K^+

$$N_{K^+} = \frac{T}{Z}\frac{\partial I_0(x_1)}{\partial \mu_{K^+}}\bigg|_{\mu_{K^+}=0}$$

Use

$$\frac{d}{dz} I_0(z) = I_1(z)$$

Exact Strangeness Conservation.

$$
\begin{aligned}
N_{K^+} &= \frac{T}{Z}\frac{\partial}{\partial \mu_{K^+}} I_0(x_1) \\
&= \frac{T}{I_0(x_1)} I_1(x_1) \frac{\partial x_1}{\partial \mu_{K^+}} \\
&= \frac{T}{I_0(x_1)} I_1(x_1) \frac{\partial 2\sqrt{Z_1 Z_{-1}}}{\partial \mu_{K^+}} \\
&= \frac{I_1(x_1)}{I_0(x_1)} \sqrt{\frac{Z_{-1}}{Z_1}} N^0_{K^+}
\end{aligned}
$$

where $N^0_{K^+}$ refers to the "unmodified" kaon multiplicity.

Exact Strangeness Conservation.

In the small volume limit this becomes

$$\lim_{z\to 0} I_0(z) = 1$$

and

$$\lim_{z\to 0} I_1(z) = \frac{z}{2}$$

$$\begin{aligned} \lim_{V\to 0} &= N^0_{K^+} Z_{-1} \\ \lim &= N^0_{K^+} Z_{-1} \\ &= N^0_{K^+} \left[N^0_{K^-} + N^0_{\Lambda} + \cdots \right] \end{aligned}$$

i.e., the particle multiplicity is

- proportional to V^2, and not V^1.
- proportional to $\exp(-2m_K/T)$ or to $\exp(-(m_K + m_\Lambda)/T)$ and not simply $\exp(-m_K/T)$, i.e. there is additional suppression of strange particles.

Exact Strangeness Conservation.

In the small volume limit this becomes

$$\lim_{z\to 0} I_0(z) = 1$$

and

$$\lim_{z\to 0} I_1(z) = \frac{z}{2}$$

$$\begin{aligned}
\lim_{V\to 0} &= N^0_{K^+} Z_{-1} \\
\lim &= N^0_{K^+} Z_{-1} \\
&= N^0_{K^+}\left[N^0_{K^-} + N^0_{\Lambda} + \cdots\right]
\end{aligned}$$

i.e., the particle multiplicity is

- proportional to V^2, and not V^1.
- proportional to $\exp(-2m_K/T)$ or to $\exp(-(m_K+m_\Lambda)/T)$ and not simply $\exp(-m_K/T)$, i.e. there is additional suppression of strange particles.

Exact Strangeness Conservation

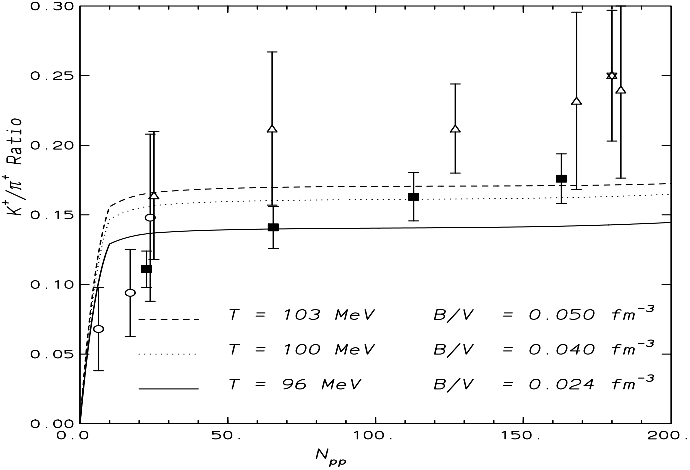


Figure 1

Transverse Momentum in the Thermal Model

$$E_i \frac{dN_i}{d^3p} = \frac{g_i}{(2\pi)^3} V E_i \exp\left(-\frac{E_i}{T} + \frac{\mu_i}{T}\right)$$

$$\frac{dN_i}{dy\ m_T dm_T} = \frac{g_i}{(2\pi)^2} V m_T \cosh y\ e^{-\frac{m_T}{T}\cosh y + \frac{\mu_i}{T}}$$

$$\frac{dN_i}{m_T dm_T} = \frac{g_i}{2\pi^2} V m_T\ K_1\left(\frac{m_T}{T}\right) e^{\frac{\mu_i}{T}}$$

For large values of m_T :

$$\boxed{\frac{dN_i}{dm_T} = \frac{g_i}{2\pi^2} V m_T^{3/2}\ \exp\left(-\frac{m_T}{T}\right)\sqrt{\frac{T\pi}{2}}\, e^{\frac{\mu_i}{T}}}$$

SCALING in m_T

Rapidity Distribution in the Thermal Model

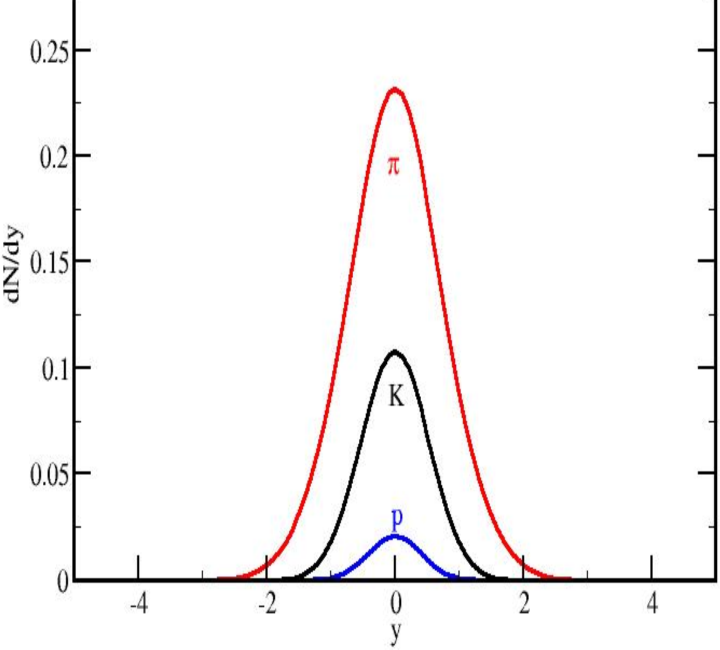
$$\frac{dN_i}{dy\ m_T dm_T} = \frac{g_i}{(2\pi)^2} V m_T \cosh y\ e^{-\frac{m_T}{T}\cosh y + \frac{\mu_i}{T}}$$

$$\frac{dN_i}{dy} = \frac{g_i V}{2\pi^2}\left[\frac{2T^3}{\cosh^2 y} + \frac{2mT^2}{\cosh y} + m^2 T\right] e^{\frac{\mu_i}{T}}$$

$$e^{-\frac{m}{T}\cosh y}$$

Narrow Distribution in Rapidity Approximately Gaussian

R



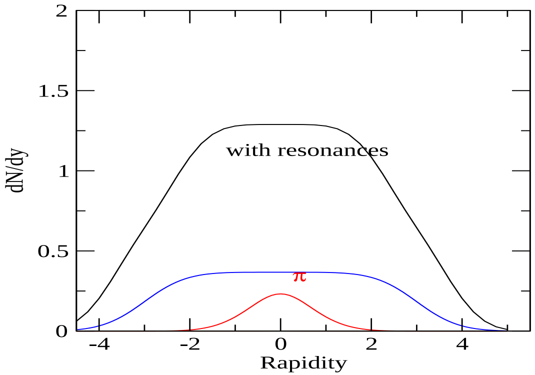
Superposition of Fireballs.

$$\frac{dN_i}{dy} = \int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB}) \frac{dN_i^0}{dy}(y - Y_{FB})$$

where

$$\frac{dN_i^0}{dy} = \frac{g_i V}{2\pi^2}\left[\frac{2T^3}{\cosh^2(y - Y_{FB})} + \frac{2mT^2}{\cosh(y - Y_{FB})} + m^2 T\right] e^{\frac{\mu_i}{T}}$$

$$e^{-\frac{m}{T}\cosh(y - Y_{FB})}$$


2
1.5
1
0.5
0
dN/dy
with resonances
π
-4
-2
0
2
4
Rapidity

Superposition of Fireballs.

$$n_i = \int_{-\infty}^{\infty} dy \int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB}) \frac{dn_i^0}{dy}(y - Y_{FB})$$

$$n_i = \int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB}) \int_{-\infty}^{\infty} dy \frac{dn_i^0}{dy}(y - Y_{FB})$$

$$n_i = g_i\, V\, \frac{1}{2\pi^2} T m_i^2 K_2\left(\frac{m_i}{T}\right) e^{\frac{\mu_i}{T}} \int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB})$$

Equivalent to changing the volume V.

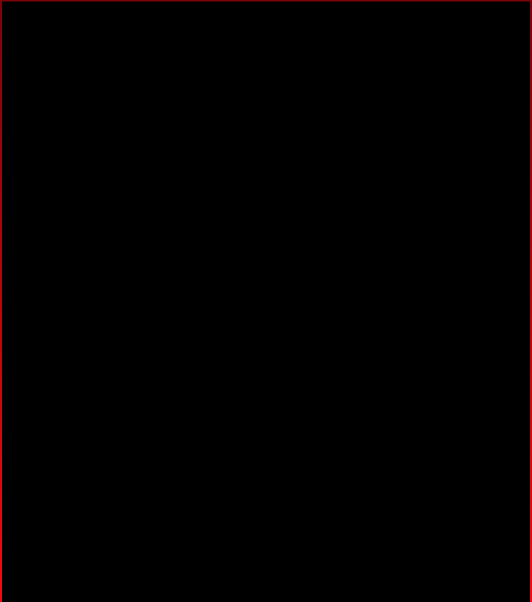
Superposition of Fireballs.

$$\frac{n_i}{n_j} = \frac{\int_{-\infty}^{\infty} dy \int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB}) \frac{dn_i^0}{dy}(y - Y_{FB})}{\int_{-\infty}^{\infty} dy \int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB}) \frac{dn_j^0}{dy}(y - Y_{FB})}$$

$$\frac{n_i}{n_j} = \frac{n_i^0}{n_j^0} \frac{\int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB})}{\int_{-Y}^{Y} dY_{FB}\, \rho(Y_{FB})}$$

$$\frac{n_i}{n_j} = \frac{n_i^0}{n_j^0} = \frac{m_i^2}{m_j^2} \frac{K_2\left(m_i/T\right)}{K_2\left(m_j/T\right)} e^{(\mu_i - \mu_j)/T}$$

Effects Cancel Out in Ratios.



The cartoon shown by Weisskopf at the conclusion of the 1962 ICHEP in Geneva is still timely