Quantum Fields at Finite T and μ

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“Dense Matter in Heavy Ion Collisions and Astrophysics”

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Contents

Reference

FINITE-TEMPERATURE FIELD THEORY
Joseph I. Kapusta

Cambridge University Press 1989

Contents

Quantum Statistical Mechanics

The Second Law of Thermodynamics

Each process ends in an equilibrium state, while the entropy increases for the irreversible processes and stays the same for the reversible.

Quantum Statistical Mechanics
Ensembles

An ensemble is an infinite number of independent systems in equilibrium.

Quantum Statistical Mechanics
Ensembles

An ensemble is an infinite number of independent systems in equilibrium.

Ensembles

- Microcanonical
- Canonical
- Grand canonical

Quantum Statistical Mechanics
Ensembles

Microparameters: Energy E, number of particles (charge) N

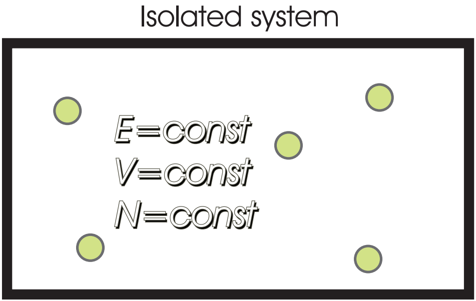
Quantum Statistical Mechanics
Ensembles

Microparameters: Energy E, number of particles (charge) N

Macroparameters: Temperature T, chemical potentials μ_i, pressure P

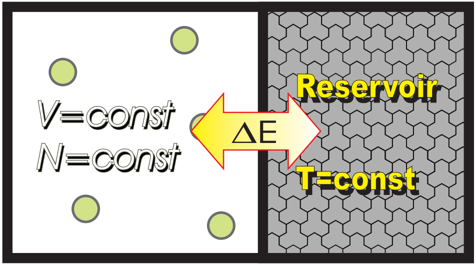
Quantum Statistical Mechanics

Microcanonical Ensemble



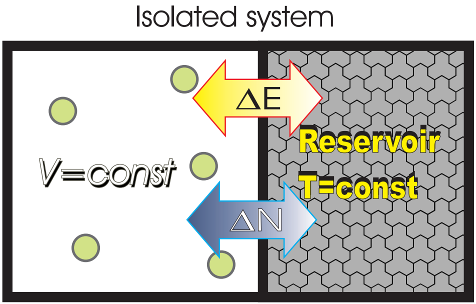
Quantum Statistical Mechanics
Canonical Ensemble

Isolated system



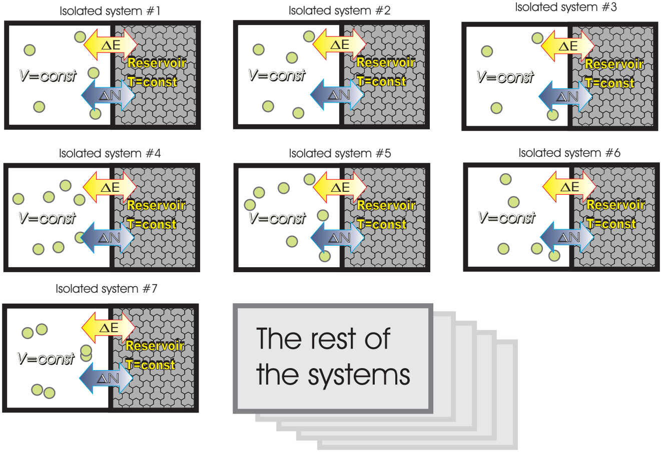
Quantum Statistical Mechanics

Grand Canonical Ensemble

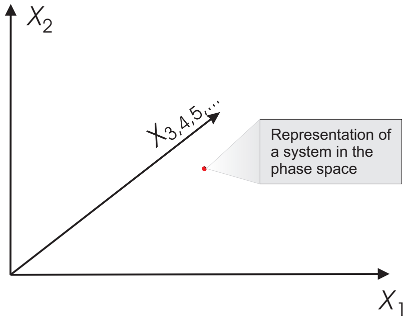


Quantum Statistical Mechanics

Grand Canonical Ensemble



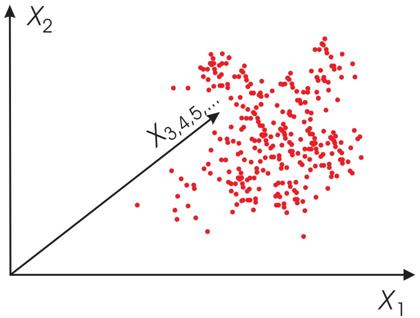
Quantum Statistical Mechanics
Phase Space



A point represents a system with an infinitely large number of particles

Quantum Statistical Mechanics

Phase Space



Each point represents an independent system from the ensemble

Contents

Quantum Statistical Mechanics
Density Matrix

Quantum state

$$|n\rangle, \qquad n \equiv \{\vec{p}, \text{ spin, color, flavor, Baryon number}, \ldots\}$$

Quantum Statistical Mechanics
Density Matrix

Quantum state

$$|n\rangle, \qquad n \equiv \{\vec{p}, \text{ spin, color, flavor, Baryon number}, \ldots\}$$

$$\hat{\rho}_n \quad \text{— density matrix}, \quad \langle n|\hat{\rho}|n\rangle = \hat{\rho}_n$$

Quantum Statistical Mechanics
Density Matrix

Quantum state

$$|n\rangle, \qquad n \equiv \{\vec{p}, \text{ spin, color, flavor, Baryon number}, \ldots\}$$

$$\hat{\rho}_n \quad \text{— density matrix,} \quad \langle n|\hat{\rho}|n\rangle = \hat{\rho}_n$$

$$\mathrm{Tr}\hat{\rho} \equiv \sum_n \hat{\rho}_n = 1 \tag{1}$$

Quantum Statistical Mechanics

Entropy

Definition

$$S = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho} \equiv -\sum_n \hat{\rho}_n \ln \hat{\rho}_n \qquad (2)$$

Quantum Statistical Mechanics
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Determine $\hat{\rho}_n$ from the maximum entropy principle

$$\max_{\hat{\rho}} S \Longrightarrow \hat{\rho}$$

Quantum Statistical Mechanics
Entropy

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$$S = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho} \equiv -\sum_n \hat{\rho}_n \ln\hat{\rho}_n \qquad (2)$$

Determine $\hat{\rho}_n$ from the maximum entropy principle

$$\max_{\hat{\rho}} S \Longrightarrow \hat{\rho}$$

at

$$\mathrm{Tr}\,\hat{\rho} = 1, \qquad (3)$$

$$\mathrm{Tr}[\hat{E}\hat{\rho}] = E, \qquad (4)$$

$$\mathrm{Tr}[\hat{N}_i\hat{\rho}] = N_i, \quad (i = 1, 2, \ldots) \qquad (5)$$

Quantum Statistical Mechanics
Density Matrix

Conditional extremum of S corresponds to the absolute extremum of W:

$$W = S - a\,E - b_i\,N_i - c\,\mathrm{Tr}\hat{\rho} \tag{6}$$

Quantum Statistical Mechanics
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$$W = S - a\,E - b_i\,N_i - c\,\mathrm{Tr}\hat{\rho} \tag{6}$$

$$\frac{\delta W}{\delta \hat{\rho}_n} = 0 \tag{7}$$

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$$\frac{\delta W}{\delta \hat{\rho}_n} = 0 \tag{7}$$

$$-\ln \hat{\rho}_n - 1 - a\,\hat{E}_n - b_i\,\hat{N}_{i;n} - c = 0 \tag{8}$$

Quantum Statistical Mechanics
Density Matrix

Conditional extremum of S corresponds to the absolute extremum of W:

$$W = S - a\,E - b_i\,N_i - c\,\mathrm{Tr}\hat{\rho} \qquad (6)$$

$$\frac{\delta W}{\delta \hat{\rho}_n} = 0 \qquad (7)$$

$$-\ln \hat{\rho}_n - 1 - a\,\hat{E}_n - b_i\,\hat{N}_{i;n} - c = 0 \qquad (8)$$

Introduce new variables: β, μ_i, Ω,
and express the Lagrange multipliers through them

$$\begin{aligned} a &= \beta \\ b_i &= \beta\mu_i \\ c &= -1 - \beta\Omega \end{aligned}$$

Quantum Statistical Mechanics

Density Matrix

Definition

$$\ln \hat{\rho}_n = \beta(\Omega - \hat{E}_n - \mu_i \hat{N}_{i;n}) \qquad (9)$$

Quantum Statistical Mechanics
Density Matrix

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Normalization

$$\mathrm{Tr}\hat{\rho} = 1 \tag{10}$$

Quantum Statistical Mechanics
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$$\mathrm{Tr}\hat{\rho} = 1 \Longrightarrow e^{-\beta\Omega} = \mathrm{Tr}\exp[-\beta(\hat{E} - \mu_i \hat{N}_i)] \tag{10}$$

Quantum Statistical Mechanics

Density Matrix

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$$\ln \hat{\rho}_n = \beta(\Omega - \hat{E}_n - \mu_i \hat{N}_{i;n}) \tag{9}$$

Normalization

$$\mathrm{Tr}\hat{\rho} = 1 \Longrightarrow e^{-\beta\Omega} = \mathrm{Tr}\exp[-\beta(\hat{E} - \mu_i \hat{N}_i)] \tag{10}$$

$$\begin{aligned} T &= \frac{1}{\beta} && \text{— temperature} \\ \mu & && \text{— chemical potential} \end{aligned} \tag{11}$$

Contents

Quantum Statistical Mechanics
Entropy and Free Energy

Definition (see Eq.(2))

$$S = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho}$$

Quantum Statistical Mechanics
Entropy and Free Energy

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$$S = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho}$$

Substitute (9) into (2)

$$S = \beta(-\Omega + E - \mu_i N_i) \tag{12}$$

Quantum Statistical Mechanics

Entropy and Free Energy

Definition (see Eq.(2))

$$S = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho}$$

Substitute (9) into (2)

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From $\beta = T^{-1}$ one obtains

$$\Omega = E - \mu_i N_i - ST \tag{13}$$

Quantum Statistical Mechanics
Entropy and Free Energy

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Ω — free energy

Quantum Statistical Mechanics
Entropy and Free Energy

$$e^{-\beta\Omega} = \mathrm{Tr}\exp[-\beta(\hat{E} - \mu_i \hat{N}_1)] \tag{14}$$

Quantum Statistical Mechanics

Entropy and Free Energy

$$e^{-\beta\Omega} = \mathrm{Tr}\exp[-\beta(\hat{E} - \mu_i\hat{N}_1)] \qquad (14)$$

Derivative of (14) over β gives

$$\Omega = E - \mu_i N_i - \beta\frac{\partial\Omega}{\partial\beta} \qquad (15)$$

Quantum Statistical Mechanics

Entropy and Free Energy

$$e^{-\beta\Omega} = \mathrm{Tr}\exp[-\beta(\hat{E} - \mu_i \hat{N}_1)] \qquad (14)$$

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Comparing (15)

with (13): $\quad \Omega = E - \mu_i N_i - ST$

Quantum Statistical Mechanics

Entropy and Free Energy

$$e^{-\beta\Omega} = \mathrm{Tr}\exp[-\beta(\hat{E} - \mu_i \hat{N}_1)] \qquad (14)$$

Derivative of (14) over β gives

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Comparing (15)

with (13): $\quad \Omega = E - \mu_i N_i - ST$

one obtains

$$S = -\frac{\partial\Omega}{\partial T} \qquad (16)$$

Quantum Statistical Mechanics

Entropy and Free Energy

$$S = -\frac{\partial \Omega}{\partial T}$$

Quantum Statistical Mechanics

Entropy and Free Energy

$$S = -\frac{\partial \Omega}{\partial T}$$

$$N_i = -\frac{\partial \Omega}{\partial \mu_i} \qquad (17)$$

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Entropy and Free Energy

$$S = -\frac{\partial \Omega}{\partial T}$$

$$N_i = -\frac{\partial \Omega}{\partial \mu_i} \qquad (17)$$

$$E = \Omega + \mu_i N_i + ST \qquad (18)$$

Quantum Statistical Mechanics

Entropy and Free Energy

$$S = -\frac{\partial \Omega}{\partial T}$$

$$N_i = -\frac{\partial \Omega}{\partial \mu_i} \tag{17}$$

$$E = \Omega + \mu_i N_i + ST \tag{18}$$

Once Ω is known, S, N_i and E are determined

Quantum Statistical Mechanics

Entropy and Free Energy

$$S = -\frac{\partial \Omega}{\partial T}$$

$$N_i = -\frac{\partial \Omega}{\partial \mu_i} \tag{17}$$

$$E = \Omega + \mu_i N_i + ST \tag{18}$$

Once Ω is known, S, N_i and E are determined

$$\Omega = -PV \tag{19}$$

Contents

Field Theory at Finite T and μ

Scalar Boson Field

Canonical coordinate

$$|\phi\rangle : \qquad \hat{\phi}_a(\vec{x}, t=0)|\phi\rangle = \phi_a(\vec{x})|\phi\rangle \tag{20}$$

$$\int d\phi\, |\phi\rangle\langle\phi| = \mathbf{1} \tag{21}$$

$$\langle\phi_a|\phi_b\rangle = \delta[\phi_a - \phi_b] \tag{22}$$

Field Theory at Finite T and μ

Scalar Boson Field

Canonical coordinate

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$$\int d\phi\ |\phi\rangle\langle\phi| = \mathbf{1} \tag{21}$$

$$\langle\phi_a|\phi_b\rangle = \delta[\phi_a - \phi_b] \tag{22}$$

Canonical momentum

$$|\pi\rangle : \qquad \hat{\pi}_a(\vec{x}, t = 0)|\pi\rangle = \pi_a(\vec{x})|\pi\rangle \tag{23}$$

$$\int \frac{d\pi}{2\pi}\ |\pi\rangle\langle\pi| = \mathbf{1} \tag{24}$$

$$\langle\pi_a|\pi_b\rangle = \delta[\pi_a - \pi_b] \tag{25}$$

Field Theory at Finite T and μ

Scalar Boson Field

Quantum mechanics

$$\langle x|p\rangle = e^{ipx}$$

Field Theory at Finite T and μ

Scalar Boson Field

Quantum mechanics

$$\langle x|p\rangle = e^{ipx}$$

$$\langle \phi|\boldsymbol{\pi}\rangle = \exp\left[i\int d^3x\ \pi(\vec{x})\phi(\vec{x})\right] \tag{26}$$

Field Theory at Finite T and μ
Scalar Boson Field

Quantum mechanics

$$\langle x|p\rangle = e^{ipx}$$

$$\langle \phi|\pi\rangle = \exp\left[i\int d^3x\ \pi(\vec{x})\phi(\vec{x})\right] \tag{26}$$

$t = 0$: $|\phi_a\rangle$

Field Theory at Finite T and μ

Scalar Boson Field

Quantum mechanics

$$\langle x|p\rangle = e^{ipx}$$

$$\langle \phi|\pi\rangle = \exp\left[i\int d^3x\, \pi(\vec{x})\phi(\vec{x})\right] \tag{26}$$

$t = 0$: $|\phi_a\rangle$

Hamiltonian

$$H = \int d^3x\, \mathcal{H}(\phi, \pi) \tag{27}$$

Field Theory at Finite T and μ

Scalar Boson Field

Quantum mechanics

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$t = 0$: $|\phi_a\rangle$

Hamiltonian

$$H = \int d^3x\,\mathcal{H}(\phi,\pi) \quad (27)$$

$$|\phi_a, t\rangle = e^{-iHt}|\phi_a, 0\rangle \quad (28)$$

Field Theory at Finite T and μ
Path Integral

$$\Delta t = \frac{t}{N}$$

Field Theory at Finite T and μ

Path Integral

$$\Delta t = \frac{t}{N}$$

$$e^{-iHt} = \underbrace{e^{-iH\Delta t} e^{-iH\Delta t} e^{-iH\Delta t} \cdots e^{-iH\Delta t}}_{N} \tag{29}$$

Field Theory at Finite T and μ
Path Integral

$$\Delta t = \frac{t}{N}$$

$$e^{-iHt} = \underbrace{e^{-iH\Delta t} e^{-iH\Delta t} e^{-iH\Delta t} \cdots e^{-iH\Delta t}}_{N} \tag{29}$$

$$e^{-iHt} = e^{-iH\Delta t} \cdot \mathbf{1} \cdot e^{-iH\Delta t} \cdot \mathbf{1} \cdot e^{-iH\Delta t} \cdot \mathbf{1} \cdots e^{-iH\Delta t} \tag{30}$$

$$\int d\phi \, |\phi\rangle\langle\phi| = \mathbf{1} \qquad \int \frac{d\pi}{2\pi} \, |\pi\rangle\langle\pi| = \mathbf{1}$$

Field Theory at Finite T and μ

Matrix Element

$$\langle \phi_b | e^{-iHt} | \phi_a \rangle = \tag{31}$$

Field Theory at Finite T and μ

Matrix Element

$$
\begin{aligned}
\langle \phi_b | e^{-iHt} | \phi_a \rangle = \\
&= \lim_{N\to\infty} \int \prod_{i=1}^{N} \frac{d\pi_i \, d\phi_i}{2\pi} \langle \phi_b | \pi_N \rangle \langle \pi_N | e^{-iH\Delta t} | \phi_N \rangle \times \\
&\times \langle \phi_N | \pi_{N-1} \rangle \langle \pi_{N-1} | e^{-iH\Delta t} | \phi_{N-1} \rangle \times \cdots \\
&\cdots \\
&\times \langle \phi_2 | \pi_1 \rangle \langle \pi_1 | e^{-iH\Delta t} | \phi_1 \rangle \langle \phi_1 | \phi_a \rangle \qquad (31)
\end{aligned}
$$

Field Theory at Finite T and μ

Matrix Element

$$\langle\phi_b|e^{-iHt}|\phi_a\rangle =$$

$$= \lim_{N\to\infty}\int\prod_{i=1}^{N}\frac{d\pi_i\,d\phi_i}{2\pi}\langle\phi_b|\pi_N\rangle\langle\pi_N|e^{-iH\Delta t}|\phi_N\rangle\times$$

$$\times\,\langle\phi_N|\pi_{N-1}\rangle\langle\pi_{N-1}|e^{-iH\Delta t}|\phi_{N-1}\rangle\times\cdots$$

$$\cdots$$

$$\times\,\langle\phi_2|\pi_1\rangle\langle\pi_1|e^{-iH\Delta t}|\phi_1\rangle\langle\phi_1|\phi_a\rangle \quad (31)$$

$$\langle\phi_1|\phi_a\rangle = \delta[\phi_1-\phi_a] \quad (32)$$

$$\langle\phi_{i+1}|\pi_i\rangle = \exp\left[i\int d^3\,\pi_i(\vec{x})\phi_{i+1}(\vec{x})\right] \quad (33)$$

Field Theory at Finite T and μ

Matrix Element

$$\Delta t \to 0$$

Field Theory at Finite T and μ
Matrix Element

$\Delta t \to 0$

$$\begin{aligned}\langle \pi_i | e^{-iH\Delta t} | \phi_i \rangle &\approx \langle \phi_i | (1 - iH\Delta t) | \phi_i \rangle = \\ &= \langle \pi_i | \phi_i \rangle (1 - iH_i \Delta t) = \\ &= (1 - iH_i \Delta t) \exp\left[i \int d^3x \pi_i(\vec{x}) \phi_i(\vec{x}) \right]\end{aligned} \tag{34}$$

$$H_i = \int d^3x \, \mathcal{H}(\pi_i(\vec{x}), \phi_i(\vec{x})) \tag{35}$$

Field Theory at Finite T and μ

Matrix Element

$$\langle \pi_b | e^{-iHt} | \phi_a \rangle = \\ = \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{d\pi_i\, d\phi_i}{2\pi} \delta(\phi_1 - \phi_a) \times \\ \times \exp\left\{ -i\Delta t \sum_{j=1}^{N} \int d^3x \left[\mathcal{H}(\pi_j, \phi_j) - \right.\right. \\ \left.\left. - \pi_j \frac{\phi_{j+1} - \phi_j}{\Delta t} \right] \right\} \tag{36}$$

$$\delta(\phi_1 - \phi_a) \Longrightarrow \phi_1 = \phi_a \tag{37}$$

Field Theory at Finite T and μ

Matrix Element

Path integral definition

$$\lim_{N\to\infty} \int \prod_{i=1}^{N} \frac{d\pi_i \, d\phi_i}{2\pi} \ldots = \int \mathcal{D}\pi \mathcal{D}\phi \ldots \tag{38}$$

Field Theory at Finite T and μ

Matrix Element

Path integral definition

$$\lim_{N\to\infty}\int\prod_{i=1}^{N}\frac{d\pi_i\,d\phi_i}{2\pi}\ldots=\int\mathcal{D}\pi\mathcal{D}\phi\ldots \tag{38}$$

$$\langle\phi_b|e^{-iHt}|\phi_a\rangle = \\ = \int_{\phi(x,0)=\phi_a(x)}^{\phi(x,t)=\phi_b(x)}\mathcal{D}\pi\mathcal{D}\phi\exp\left\{i\int_0^t dt'\int d^3x\left(\pi(\vec{x},t')\frac{\partial\phi(\vec{x},t')}{\partial t'}-\right.\right. \\ \left.\left.-\mathcal{H}(\pi(\vec{x},t'),\phi(\vec{x},t'))\right)\right\} \tag{39}$$

Field Theory at Finite T and μ

Partition Function

Definition

$$Z \equiv e^{-\beta\Omega} = \mathrm{Tr}[e^{-\beta(\hat{H}-\mu_i\hat{N}_i)}] = \\ = \int d\phi_a \langle\phi_a| e^{-\beta(\hat{H}-\mu_i\hat{N}_i)} |\phi_a\rangle \quad (40)$$

Field Theory at Finite T and μ

Partition Function

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Imaginary time: $t = -i\tau$

Field Theory at Finite T and μ

Partition Function

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Imaginary time: $t = -i\tau$

Extended Hamiltonian: $\hat{H} \rightarrow \hat{H} - \mu_i\hat{N}_i$

Field Theory at Finite T and μ

Partition Function

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$$Z \equiv e^{-\beta\Omega} = \mathrm{Tr}[e^{-\beta(\hat{H}-\mu_i \hat{N}_i)}] = \\ = \int d\phi_a \langle\phi_a| e^{-\beta(\hat{H}-\mu_i\hat{N}_i)} |\phi_a\rangle \quad (40)$$

Imaginary time: $t = -i\tau$

Extended Hamiltonian: $\hat{H} \rightarrow \hat{H} - \mu_i \hat{N}_i$

$$Z = \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi \exp\left\{ \int_0^\beta d\tau \int d^3x \left(i\pi \frac{\partial\phi}{\partial\tau} - \mathcal{H} + \mu_i \mathcal{N}_i \right) \right\} \quad (41)$$

$$\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{42}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{42}$$

Momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{\partial \phi}{\partial t} \tag{43}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \qquad (42)$$

Momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{\partial \phi}{\partial t} \qquad (43)$$

$$\mathcal{H} = \pi \frac{\partial \phi}{\partial t} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \qquad (44)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$Z = \int \mathcal{D}\pi \underbrace{\int \mathcal{D}\phi}_{\text{periodic}} \exp\left\{ \int_0^\beta d\tau \int d^3x \left(\pi \frac{\partial \phi}{\partial t} - \mathcal{H} \right) \right\} \quad (45)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

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$$t = -i\tau \Longrightarrow \frac{\partial}{\partial t} = i \frac{\partial}{\partial \tau}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

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$\mathcal{H} = \frac{1}{2}\pi^2 + \dots$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

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$$t = -i\tau \Longrightarrow \frac{\partial}{\partial t} = i \frac{\partial}{\partial \tau}$$

$\mathcal{H} = \frac{1}{2}\pi^2 + \dots$

$$Z = \int \mathcal{D}\pi \exp\left\{ \int_0^\beta d\tau \int d^3x \left(i\pi \frac{\partial \phi}{\partial \tau} - \frac{1}{2}\pi^2 \right) \right\} \int \mathcal{D}\phi (\dots) \quad (46)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$
\begin{aligned}
&\int \mathcal{D}\pi \exp\left\{\int_0^\beta d\tau \int d^3x \left(i\pi\frac{\partial\phi}{\partial\tau} - \frac{1}{2}\pi^2\right)\right\} = \\
&= \int \mathcal{D}\pi \exp\left\{\int_0^\beta d\tau \int d^3x \left[-\frac{1}{2}\left(\pi - i\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2\right]\right\} = \\
&= C \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x \left(\frac{\partial\phi}{\partial\tau}\right)^2\right\}
\end{aligned}
\tag{47}
$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\int \mathcal{D}\pi \exp\left\{\int_0^\beta d\tau \int d^3x \left(i\pi\frac{\partial\phi}{\partial\tau} - \frac{1}{2}\pi^2\right)\right\} =$$

$$= \int \mathcal{D}\pi \exp\left\{\int_0^\beta d\tau \int d^3x \left[-\frac{1}{2}\left(\pi - i\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2\right]\right\} =$$

$$= C\exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x \left(\frac{\partial\phi}{\partial\tau}\right)^2\right\} \tag{47}$$

Omit C and insert into Z:

$$Z = \int_{\text{periodic}} \mathcal{D}\phi \exp\left\{\int_0^\beta d\tau \int d^3x \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial\tau}\right)\right\} \tag{48}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$W = \int_0^\beta d\tau \int d^3x \mathcal{L} = = -\frac{1}{2}\int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2\right] \quad (49)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$W = \int_0^\beta d\tau \int d^3x \mathcal{L} = \\ = -\frac{1}{2}\int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2\right] \quad (49)$$

Integration by parts gives

$$W = \int_0^\beta d\tau \int d^3x \mathcal{L} = \\ = -\frac{1}{2}\int_0^\beta d\tau \int d^3x \phi(\vec{x},\tau)\left[-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right]\phi(\vec{x},\tau) \quad (50)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Momentum space

$$\phi(\vec{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n \tau)} \phi_n(\vec{p}) \tag{51}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Momentum space

$$\phi(\vec{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n \tau)} \phi_n(\vec{p}) \tag{51}$$

Periodic boundary conditions: $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Momentum space

$$\phi(\vec{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x} + \omega_n \tau)} \phi_n(\vec{p}) \qquad (51)$$

Periodic boundary conditions: $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta) \Longrightarrow$

$$\omega_n \beta = 2\pi n \equiv \omega_n = 2\pi n T, \quad n = 0, \pm 1, \pm 2, \ldots \qquad (52)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Momentum space

$$\phi(\vec{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)} \phi_n(\vec{p}) \qquad (51)$$

Periodic boundary conditions: $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$

$$\omega_n \beta = 2\pi n \equiv \omega_n = 2\pi n T, \quad n = 0, \pm 1, \pm 2, \ldots \qquad (52)$$

Matsubara frequencies for bosons

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$W = -\frac{\beta^2}{2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \left(\omega_n^2 + \omega^2\right) \phi_n(\vec{p}) \phi_n^*(\vec{p}) \qquad (53)$$

$$\omega^2 = \vec{p}^2 + m^2$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$W = -\frac{\beta^2}{2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \left(\omega_n^2 + \omega^2\right) \phi_n(\vec{p})\phi_n^*(\vec{p}) \tag{53}$$

$$\omega^2 = \vec{p}^2 + m^2$$

$$\begin{aligned} Z = \int \mathcal{D}\phi e^W = \\ = C \int \mathcal{D} \exp\left(-\frac{1}{2}\phi D \phi\right) \end{aligned} \tag{54}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$W = -\frac{\beta^2}{2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \left(\omega_n^2 + \omega^2\right) \phi_n(\vec{p}) \phi_n^*(\vec{p}) \tag{53}$$

$$\omega^2 = \vec{p}^2 + m^2$$

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$$D = \beta^2 \left(-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + m^2\right) \tag{55}$$

$$\tag{56}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$W = -\frac{\beta^2}{2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \left(\omega_n^2 + \omega^2\right) \phi_n(\vec{p}) \phi_n^*(\vec{p}) \tag{53}$$

$$\omega^2 = \vec{p}^2 + m^2$$

$$\begin{aligned} Z &= \int \mathcal{D}\phi e^W = \\ &= C \int \mathcal{D} \exp\left(-\frac{1}{2}\phi D \phi\right) \end{aligned} \tag{54}$$

$$D = \beta^2 \left(-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + m^2\right) \tag{55}$$

$$D = \beta^2(\omega_n^2 + \omega^2) \tag{56}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Gauss form

$$Z = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\phi D \phi} \equiv (\det D)^{-\frac{1}{2}} \tag{57}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Gauss form

$$Z = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D \phi} \equiv (\det D)^{-\frac{1}{2}} \tag{57}$$

$$\ln Z = -\frac{1}{2} \ln \det D \tag{58}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Gauss form

$$Z = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D \phi} \equiv (\det D)^{-\frac{1}{2}} \tag{57}$$

$$\ln Z = -\frac{1}{2} \ln \det D = -\frac{1}{2} \mathrm{Tr} \ln D \tag{58}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

Gauss form

$$Z = \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D \phi} \equiv (\det D)^{-\frac{1}{2}} \tag{57}$$

$$\ln Z = -\frac{1}{2} \ln \det D = -\frac{1}{2} \mathrm{Tr} \ln D \tag{58}$$

$$\ln Z = -\frac{1}{2} \sum_{n} \sum_{\vec{p}} \ln \left[\beta^2 (\omega_n^2 + \omega^2) \right] \tag{59}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2} \sum_n \sum_{\vec{p}} \ln \left[\beta^2 (\omega_n^2 + \omega^2) \right] \tag{60}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2} \sum_{n} \sum_{\vec{p}} \ln \left[\beta^2 (\omega_n^2 + \omega^2) \right] \tag{60}$$

$$\ln \left[\beta^2 (\omega_n^2 + \omega^2) \right] = \int_1^{\beta^2 \omega^2} \frac{dx}{x + (2\pi n)^2} + \ln \left[1 + (2\pi n)^2 \right] \tag{61}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right] \tag{60}$$

$$\ln\left[\beta^2(\omega_n^2 + \omega^2)\right] = \int_1^{\beta^2\omega^2} \frac{dx}{x + (2\pi n)^2} + \ln\left[1 + (2\pi n)^2\right] \tag{61}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{x + (2\pi n)^2} = ? \tag{62}$$

Field Theory at Finite T and μ

Mathematical Tricks

Theorem

If the function $g(z)$ and has finite number of simple poles ξ_k

$$g(z) = (z - \xi_1)^{-1}(z - \xi_2)^{-1} \cdots (z - \xi_N)^{-1} r(z),$$

where $r(z)$ has no poles, $\xi_k \neq 2\pi ni$ and

$$|g(z)| < \frac{C}{|z|}, \quad |z| \to \infty,$$

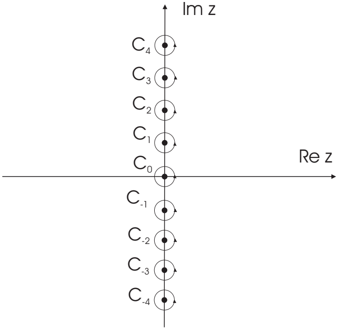
then

$$\sum_{n=-\infty}^{\infty} g(2\pi ni) = -\frac{1}{2} \sum_{k=1}^{N} \coth\left(\frac{\xi_k}{2}\right) \operatorname*{res}_{z=\xi_k} g(z)$$

Field Theory at Finite T and μ

Mathematical Tricks

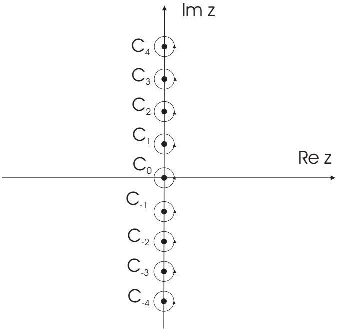
$$\sum_{n=-\infty}^{\infty} g(2\pi n i) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \oint_{C_n} \frac{dz}{2\pi i} g(z) \coth\left(\frac{z}{2}\right) \tag{63}$$



Field Theory at Finite T and μ

Mathematical Tricks

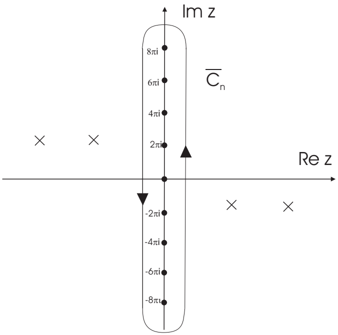
$$\coth\left(\frac{\epsilon - 2\pi n i}{2}\right) \sim \frac{2}{\epsilon}, \qquad \frac{1}{2}\oint_{C_n} \frac{dz}{2\pi i} \coth\left(\frac{z}{2}\right) = 1 \quad (64)$$



Field Theory at Finite T and μ

Mathematical Tricks

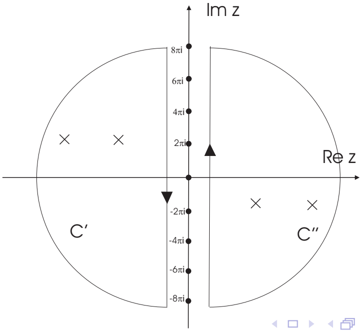
$$\sum_{n=-\infty}^{\infty} g(2\pi n i) = \lim_{n\to\infty} \frac{1}{2} \oint_{\bar{C}_n} \frac{dz}{2\pi i} g(z) \coth\left(\frac{z}{2}\right) \qquad (65)$$



Field Theory at Finite T and μ

Mathematical Tricks

$$\sum_{n=-\infty}^{\infty} g(2\pi n i) = \frac{1}{2}\left(\oint_{C'} \frac{dz}{2\pi i} g(z) \coth\left(\frac{z}{2}\right) + \right.$$
$$\left. + \oint_{C''} \frac{dz}{2\pi i} g(z) \coth\left(\frac{z}{2}\right)\right) \quad (66)$$



Field Theory at Finite T and μ

Mathematical Tricks

$$\frac{1}{2}\oint_{C'}\frac{dz}{2\pi i}g(z)\coth\left(\frac{z}{2}\right)+ + \frac{1}{2}\oint_{C''}\frac{dz}{2\pi i}g(z)\coth\left(\frac{z}{2}\right)= = -\frac{1}{2}\sum_{n=1}^{N}\coth\left(\frac{\xi_n}{2}\right)\operatorname*{res}_{z=\xi_n}g(z) \tag{67}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2} \tag{68}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2}, \quad z \to 2\pi n i \Longrightarrow \frac{1}{x + (2\pi n)^2} \tag{68}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2}, \quad z \to 2\pi n i \Longrightarrow \frac{1}{x + (2\pi n)^2} \tag{68}$$

Poles of $g(z)$

$$\xi_1 = \sqrt{x}, \qquad \xi_2 = -\sqrt{x} \tag{69}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2}, \quad z \to 2\pi n i \Longrightarrow \frac{1}{x + (2\pi n)^2} \tag{68}$$

Poles of $g(z)$

$$\xi_1 = \sqrt{x}, \qquad \xi_2 = -\sqrt{x} \tag{69}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2\pi n)^2} = \\ = -\frac{1}{2}\left(\coth\left(\frac{\sqrt{x}}{2}\right) \operatorname*{res}_{z=\sqrt{x}} \frac{1}{x - z^2} + \right. \\ \left. + \coth\left(\frac{-\sqrt{x}}{2}\right) \operatorname*{res}_{z=-\sqrt{x}} \frac{1}{x - z^2} \right) \tag{70}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x - z^2} = \mp \frac{1}{2\sqrt{x}} \tag{71}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\underset{z=\pm\sqrt{x}}{\text{res}} \frac{1}{x - z^2} = \mp\frac{1}{2\sqrt{x}} \qquad (71)$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2\pi n)^2} = \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) \qquad (72)$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x-z^2} = \mp\frac{1}{2\sqrt{x}} \tag{71}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x+(2\pi n)^2} = \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) \tag{72}$$

$$\int_1^{\beta^2\omega^2} \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) dx \tag{73}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\underset{z=\pm\sqrt{x}}{\mathrm{res}} \frac{1}{x-z^2} = \mp\frac{1}{2\sqrt{x}} \tag{71}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x+(2\pi n)^2} = \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) \tag{72}$$

$$\int_1^{\beta^2\omega^2} \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) dx = $$
$$= 2\ln\sinh\left(\frac{\beta\omega}{2}\right) - 2\ln\sinh 1 \tag{73}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x-z^2} = \mp\frac{1}{2\sqrt{x}} \tag{71}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x+(2\pi n)^2} = \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) \tag{72}$$

$$\int_1^{\beta^2\omega^2} \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) dx = $$
$$= 2\ln\sinh\left(\frac{\beta\omega}{2}\right) - 2\ln\sinh 1 \tag{73}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln \sinh\left(\frac{\beta\omega}{2}\right) \tag{74}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln \sinh\left(\frac{\beta\omega}{2}\right) = \ln\left(e^{\frac{\beta\omega}{2}} - e^{\frac{\beta\omega}{2}}\right) - \ln 2 \tag{74}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln \sinh\left(\frac{\beta\omega}{2}\right) = \ln\left(e^{\frac{\beta\omega}{2}} - e^{\frac{\beta\omega}{2}}\right) - \ln 2 = \\ = \ln\left[e^{-\frac{\beta\omega}{2}}\left(e^{\beta\omega} - 1\right)\right] - \ln 2 \qquad (74)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$
\begin{aligned}
\ln \sinh\left(\frac{\beta\omega}{2}\right) &= \ln\left(e^{\frac{\beta\omega}{2}} - e^{\frac{\beta\omega}{2}}\right) - \ln 2 = \\
&= \ln\left[e^{-\frac{\beta\omega}{2}}\left(e^{\beta\omega} - 1\right)\right] - \ln 2 \\
&= -\frac{\beta\omega}{2} + \ln\left(e^{\beta\omega} - 1\right) - \ln 2
\end{aligned}
\qquad (74)
$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\begin{aligned}
\ln\sinh\left(\frac{\beta\omega}{2}\right) &= \ln\left(e^{\frac{\beta\omega}{2}} - e^{\frac{\beta\omega}{2}}\right) - \ln 2 = \\
&= \ln\left[e^{-\frac{\beta\omega}{2}}\left(e^{\beta\omega} - 1\right)\right] - \ln 2 \\
&= -\frac{\beta\omega}{2} + \ln\left(e^{\beta\omega} - 1\right) - \ln 2
\end{aligned} \tag{74}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2} \sum_{n} \sum_{\vec{p}} \ln \left[\beta^2 (\omega_n^2 + \omega^2) \right]$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right]$$

$$\ln\left[\beta^2(\omega_n^2 + \omega^2)\right] = \int_1^{\beta^2\omega^2} \frac{dx}{x + (2\pi n)^2} + \ln\left[1 + (2\pi n)^2\right]$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right]$$

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$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2\pi n)^2} = \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right)$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right]$$

$$\ln\left[\beta^2(\omega_n^2 + \omega^2)\right] = \int_1^{\beta^2\omega^2} \frac{dx}{x + (2\pi n)^2} + \ln\left[1 + (2\pi n)^2\right]$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2\pi n)^2} = \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right)$$

$$\int_1^{\beta^2\omega^2} \frac{1}{2\sqrt{x}} \coth\left(\frac{\sqrt{x}}{2}\right) dx = -\beta\omega + 2\ln\left(e^{\beta\omega} - 1\right) + const$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\ln Z = -\frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2+\omega^2)\right]$$

$$\ln\left[\beta^2(\omega_n^2+\omega^2)\right] = \int_1^{\beta^2\omega^2} \frac{dx}{x+(2\pi n)^2} + \ln\left[1+(2\pi n)^2\right]$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x+(2\pi n)^2} = \frac{1}{2\sqrt{x}}\coth\left(\frac{\sqrt{x}}{2}\right)$$

$$\int_1^{\beta^2\omega^2} \frac{1}{2\sqrt{x}}\coth\left(\frac{\sqrt{x}}{2}\right)dx = -\beta\omega + 2\ln\left(e^{\beta\omega}-1\right) + const$$

$$\ln Z = \sum_{\vec{p}}\left[\frac{\beta\omega}{2} - \ln\left(e^{\beta\omega}-1\right)\right] \tag{75}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\sum_{\vec{p}} \tag{76}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\sum_{\vec{p}} \rightarrow V \int \frac{d^3p}{(2\pi)^2} \tag{76}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^2} \tag{76}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[\frac{\beta\omega}{2} - \ln\left(e^{\beta\omega} - 1\right) \right] \tag{77}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^2} \tag{76}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[\frac{\beta\omega}{2} - \ln\left(e^{\beta\omega} - 1\right) \right] \tag{77}$$

$$\ln Z = -\beta\Omega$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\sum_{\vec{p}} \to V \int \frac{d^3 p}{(2\pi)^2} \tag{76}$$

$$\ln Z = V \int \frac{d^3 p}{(2\pi)^3} \left[\frac{\beta\omega}{2} - \ln\left(e^{\beta\omega} - 1\right) \right] \tag{77}$$

$$\ln Z = -\beta\Omega$$

$$\Omega = V \int \frac{d^3 p}{(2\pi)^3} \left[-\frac{\omega}{2} + T \ln\left(e^{\beta\omega} - 1\right) \right] \tag{78}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^2} \tag{76}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[\frac{\beta\omega}{2} - \ln\left(e^{\beta\omega} - 1\right) \right] \tag{77}$$

$$\ln Z = -\beta\Omega$$

$$\Omega = V \int \frac{d^3p}{(2\pi)^3} \left[-\frac{\omega}{2} + T \ln\left(e^{\beta\omega} - 1\right) \right] \tag{78}$$

$$P = \int \frac{d^3p}{(2\pi)^3} \left[\frac{\omega}{2} - T \ln\left(e^{\beta\omega} - 1\right) \right] \tag{79}$$

Field Theory at Finite T and μ

Free Scalar Field ($\mu = 0$)

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^2} \tag{76}$$

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$$P = \int \frac{d^3p}{(2\pi)^3} \left[\frac{\omega}{2} - T \ln\left(e^{\beta\omega} - 1\right) \right] \tag{79}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^* \partial_\mu \Phi - m^2 |\Phi|^2 \tag{80}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^* \partial_\mu \Phi - m^2 |\Phi|^2 \tag{80}$$

$$\Phi \to \Phi' = \Phi e^{i\alpha}, \quad \mathrm{Im}\alpha = 0 \tag{81}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^* \partial_\mu \Phi - m^2 |\Phi|^2 \tag{80}$$

$$\Phi \rightarrow \Phi' = \Phi e^{i\alpha}, \quad \mathrm{Im}\alpha = 0 \tag{81}$$

$$j_\mu = i \left(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^* \right), \quad \partial_\mu j^\mu = 0$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Hamilton approach

$$\Phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \tag{82}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Hamilton approach

$$\Phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \tag{82}$$

$$\pi_1 = \frac{\partial \phi_1}{\partial t}, \quad \pi_2 = \frac{\partial \phi_2}{\partial t} \tag{83}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Hamilton approach

$$\Phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \tag{82}$$

$$\pi_1 = \frac{\partial \phi_1}{\partial t}, \quad \pi_2 = \frac{\partial \phi_2}{\partial t} \tag{83}$$

$$\mathcal{H} = \mathcal{L} - \pi_1 \frac{\partial \phi_1}{\partial t} - \pi_2 \frac{\partial \phi_2}{\partial t} \tag{84}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Hamilton approach

$$\Phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \tag{82}$$

$$\pi_1 = \frac{\partial \phi_1}{\partial t}, \quad \pi_2 = \frac{\partial \phi_2}{\partial t} \tag{83}$$

$$\mathcal{H} = \mathcal{L} - \pi_1 \frac{\partial \phi_1}{\partial t} - \pi_2 \frac{\partial \phi_2}{\partial t} \tag{84}$$

$$Q = \int d^3x\, j_0 = \int d^3x (\phi_2 \pi_1 - \phi_1 \pi_2) \tag{85}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Partition function

$$Z = \int \mathcal{D}\pi_1 \mathcal{D}\pi_2 \mathcal{D}\phi_1 \mathcal{D}\phi_2 \times \\ \times \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi_1 \frac{\partial \phi_1}{\partial \tau} + i\pi_2 \frac{\partial \phi_2}{\partial \tau} - \mathcal{H} + \\ + \mu(\phi_2 \pi_1 - \phi_1 \pi_2) \right)\right] \tag{86}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

Partition function

$$Z = \int \mathcal{D}\pi_1 \mathcal{D}\pi_2 \mathcal{D}\phi_1 \mathcal{D}\phi_2 \times \times \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi_1 \frac{\partial\phi_1}{\partial\tau} + i\pi_2 \frac{\partial\phi_2}{\partial\tau} - \mathcal{H} + + \mu(\phi_2\pi_1 - \phi_1\pi_2)\right)\right] \tag{86}$$

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\left\{\int_0^\beta d\tau \int d^3x \left[-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2 - -\frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1\right)^2 - (\nabla\phi_1)^2 - (\nabla\phi_2)^2 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2)\right]\right\} \tag{87}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\phi_1 = \sqrt{2}\zeta\cos\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\phi_{1;n}(\vec{p}) \quad (88)$$

$$\phi_2 = \sqrt{2}\zeta\sin\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\phi_{2;n}(\vec{p}) \quad (89)$$

Field Theory at Finite T and μ

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$$\phi_2 = \sqrt{2}\zeta\sin\theta + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\phi_{2;n}(\vec{p}) \quad (89)$$

$$\phi_{1;n}(\vec{p}=0) = \phi_{2;n}(\vec{p}=0) = 0 \quad (90)$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$Z = C \prod_n \prod_{\vec{p}} \int d\phi_{1;n} d\phi_{2;n} e^W \tag{91}$$

$$W = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2} \sum_n \sum_{\vec{p}} \Phi_{-n}(-\vec{p})^T D \Phi_n(\vec{p}) \tag{92}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$Z = C \prod_n \prod_{\vec{p}} \int d\phi_{1;n} d\phi_{2;n} e^W \tag{91}$$

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$$\Phi_n(\vec{p}) = \begin{pmatrix} \phi_{1;n}(\vec{p}) \\ \phi_{2;n}(\vec{p}) \end{pmatrix} \tag{93}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

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$$\Phi_n(\vec{p}) = \begin{pmatrix} \phi_{1;n}(\vec{p}) \\ \phi_{2;n}(\vec{p}) \end{pmatrix} \tag{93}$$

$$D = \begin{pmatrix} \omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2 \end{pmatrix} \tag{94}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2}\ln\det D \qquad (95)$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2}\ln\det D \tag{95}$$

$$\ln\det D = \ln\left\{\prod_n \prod_{\vec{p}} \beta^4 \left[\left(\omega_n^2 + \omega^2 - \mu^2\right)^2 + 4\mu^2\omega_n^2\right]\right\} = \tag{96}$$

Field Theory at Finite T and μ

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$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2}\ln\det D \tag{95}$$

$$\ln\det D = \ln\left\{\prod_n \prod_{\vec{p}} \beta^4 \left[\left(\omega_n^2 + \omega^2 - \mu^2\right)^2 + 4\mu^2\omega_n^2\right]\right\} =$$
$$= \ln\left\{\prod_n \prod_{\vec{p}} \beta^4 \left[\omega_n^2 + (\omega - \mu)^2\right]\left[\omega_n^2 + (\omega + \mu)^2\right]\right\} = \tag{96}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2}\ln\det D \quad (95)$$

$$\begin{aligned}\ln\det D &= \ln\left\{\prod_n\prod_{\vec{p}}\beta^4\left[\left(\omega_n^2+\omega^2-\mu^2\right)^2+4\mu^2\omega_n^2\right]\right\} = \\ &= \ln\left\{\prod_n\prod_{\vec{p}}\beta^4\left[\omega_n^2+(\omega-\mu)^2\right]\left[\omega_n^2+(\omega+\mu)^2\right]\right\} = \\ &= \ln\left\{\prod_n\prod_{\vec{p}}\beta^2\left[\omega_n^2+(\omega-\mu)^2\right]\right\} + \\ &+ \ln\left\{\prod_n\prod_{\vec{p}}\beta^2\left[\omega_n^2+(\omega+\mu)^2\right]\right\}\end{aligned} \quad (96)$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \\ - \frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left\{\beta^2\left[\omega_n^2 + (\omega - \mu)^2\right]\right\} - \\ - \frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left\{\beta^2\left[\omega_n^2 + (\omega + \mu)^2\right]\right\} \tag{97}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \\ - \frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left\{\beta^2\left[\omega_n^2 + (\omega-\mu)^2\right]\right\} - \\ - \frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left\{\beta^2\left[\omega_n^2 + (\omega+\mu)^2\right]\right\} \tag{97}$$

$\Sigma \to \int$

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 + \\ + V\int \frac{d^3p}{(2\pi)^3}\left[\beta\omega - \ln\left(e^{\beta(\omega-\mu)} - 1\right) - \\ - \ln\left(e^{\beta(\omega+\mu)} - 1\right)\right] \tag{98}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\begin{aligned}\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - \\ - \frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left\{\beta^2\left[\omega_n^2 + (\omega-\mu)^2\right]\right\} - \\ - \frac{1}{2}\sum_n \sum_{\vec{p}} \ln\left\{\beta^2\left[\omega_n^2 + (\omega+\mu)^2\right]\right\}\end{aligned} \tag{97}$$

$$\Sigma \to \int$$

$$\begin{aligned}\ln Z = \beta V(\mu^2 - m^2)\zeta^2 + \\ + V\int \frac{d^3p}{(2\pi)^3}\left[\beta\omega - \ln\left(e^{\beta(\omega-\mu)} - 1\right) - \right. \\ \left. - \ln\left(e^{\beta(\omega+\mu)} - 1\right)\right]\end{aligned} \tag{98}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\begin{aligned}\Omega = & -V(\mu^2 - m^2)\zeta^2 - \\ & - V \int \frac{d^3p}{(2\pi)^3} \left[\omega - T \ln \left(e^{\beta(\omega-\mu)} - 1 \right) - \right. \\ & \left. - T \ln \left(e^{\beta(\omega+\mu)} - 1 \right) \right]\end{aligned} \tag{99}$$

Field Theory at Finite T and μ

Charged Scalar Field ($\mu \neq 0$)

$$\Omega = -V(\mu^2 - m^2)\zeta^2 - \\ - V\int \frac{d^3p}{(2\pi)^3}\left[\omega - T\ln\left(e^{\beta(\omega-\mu)} - 1\right) - \right. \\ \left. - T\ln\left(e^{\beta(\omega+\mu)} - 1\right)\right] \tag{99}$$

Maximum of entropy = minimum of Ω

$$\frac{\partial\omega}{\partial\theta} = 0 \quad \longrightarrow \quad \zeta = 0, \quad |\mu| \neq m \tag{100}$$

Contents

Field Theory at Finite T and μ

Fermions

$$\psi(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{s} \sqrt{\frac{m}{E}} \times \\ \times \left[b(\vec{p},s) u(\vec{p},s) e^{-i\vec{p}\vec{x}} + d^*(\vec{p},s) v(\vec{p},s) e^{i\vec{p}\vec{x}} \right] \quad (101)$$

$$(\not{p} - m) u(\vec{p},s) = 0, \qquad (\not{p} + m) v(\vec{p},s) = 0 \quad (102)$$

$$\bar{u}(\vec{p},s) u(\vec{p},s) = 2m \quad (103)$$

Field Theory at Finite T and μ
Fermions

Lagrangian

$$\mathcal{L} = \bar{\psi}(i \not{\partial} - m)\psi, \qquad \not{\partial} = \gamma^\mu \partial_\mu \tag{104}$$

Field Theory at Finite T and μ

Fermions

Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi, \qquad \not{\partial} = \gamma^\mu \partial_\mu \tag{104}$$

$$\mathcal{L} = \psi^\dagger (i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\vec{p} - m)\psi \tag{105}$$

Field Theory at Finite T and μ

Fermions

Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi, \qquad \not{\partial} = \gamma^\mu \partial_\mu \tag{104}$$

$$\mathcal{L} = \psi^\dagger (i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\vec{p} - m)\psi \tag{105}$$

$$\partial_\mu j^\mu = 0, \qquad j_\mu = \bar{\psi}\gamma_\mu\psi \tag{106}$$

$$Q = \int d^3x\, j_0 = \int d^3x\, \psi^\dagger\psi \tag{107}$$

Field Theory at Finite T and μ

Fermions

Hamilton approach

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial t)} = i\psi^{\dagger} \tag{108}$$

Field Theory at Finite T and μ

Fermions

Hamilton approach

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial t)} = i\psi^\dagger \tag{108}$$

$$\mathcal{H} = \Pi\frac{\partial\psi}{\partial t} - \mathcal{L} = \bar{\psi}(-i\vec{\gamma}\vec{\nabla} + m)\psi \tag{109}$$

Field Theory at Finite T and μ

Fermions

Hamilton approach

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial t)} = i\psi^\dagger \tag{108}$$

$$\mathcal{H} = \Pi\frac{\partial\psi}{\partial t} - \mathcal{L} = \bar{\psi}(-i\vec{\gamma}\vec{\nabla} + m)\psi \tag{109}$$

Partition function

$$Z = \mathrm{Tr} e^{-\beta(\hat{H}-\mu\hat{Q})} \tag{110}$$

Field Theory at Finite T and μ

Fermions

Partition function

$$Z = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \times \\ \times \exp\left\{\int_0^\beta d\tau \int d^3x \, \bar{\psi} \left[-\gamma^0 \frac{\partial}{\partial \tau} + i\vec{\gamma}\vec{\nabla} - m + \mu\gamma^0\right] \psi\right\} \tag{111}$$

Field Theory at Finite T and μ

Fermions

Partition function

$$Z = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \times$$
$$\times \exp\left\{\int_0^\beta d\tau \int d^3x\, \bar{\psi}\left[-\gamma^0 \frac{\partial}{\partial\tau} + i\vec{\gamma}\vec{\nabla} - m + \mu\gamma^0\right]\psi\right\} =$$
$$= \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{\bar{\psi} D \psi}$$
$$D = -\gamma^0 \frac{\partial}{\partial\tau} + i\vec{\gamma}\vec{\nabla} - m + \mu\gamma^0$$
$$D = -i\gamma^0\omega_n - i\vec{\gamma}\vec{p} - m + \mu\gamma^0 \tag{111}$$

$$\{\psi, \psi\} = \{\bar{\psi}, \bar{\psi}\} = \{\bar{\psi}, \psi\} = 0 \tag{112}$$

$$\int d\psi = 0, \quad \int d\psi\, \psi = 1, \quad \int d\psi^\dagger = 0, \quad \int d\psi^\dagger\, \psi = 1 \tag{113}$$

Field Theory at Finite T and μ

Fermions

$$\psi_\alpha(\vec{x},\tau) = \frac{1}{\sqrt{V}} \sum_n \sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)} \tilde{\psi}_{\alpha;n}(\vec{p}) \tag{114}$$

Field Theory at Finite T and μ
Fermions

$$\psi_\alpha(\vec{x},\tau) = \frac{1}{\sqrt{V}} \sum_n \sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)} \tilde{\psi}_{\alpha;n}(\vec{p}) \tag{114}$$

Green's function

$$G_F(\vec{x},\vec{y};\tau,0) = Z^{-1}\mathrm{Tr}\left[\hat{\rho} T_\tau \left[\psi(\vec{x},\tau)\bar{\psi}(\vec{y},0)\right]\right] \tag{115}$$

$$\begin{aligned} T_\tau\left[\psi(\vec{x},\tau_1)\psi(\vec{y},\tau_2)\right] = \\ = \psi(\vec{x},\tau_1)\psi(\vec{y},\tau_2)\theta(\tau_1-\tau_2) - \psi(\vec{y},\tau_2)\psi(\vec{x},\tau_1)\theta(\tau_2-\tau_1) \end{aligned} \tag{116}$$

Field Theory at Finite T and μ

Fermions

$$K = \hat{H} - \mu \hat{Q} \tag{117}$$

$\beta \geqslant \tau$ and $\hat{\rho} = e^{-\beta K}$

$$G_F(\vec{x}, \vec{y}; \tau, 0) = Z^{-1} \mathrm{Tr} \left[e^{-\beta K} T_\tau \left[\psi(\vec{x}, \tau) \bar{\psi}(\vec{y}, 0) \right] \right]$$

$$\tag{118}$$

Field Theory at Finite T and μ

Fermions

$$K = \hat{H} - \mu \hat{Q} \tag{117}$$

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$$\begin{aligned} G_F(\vec{x}, \vec{y}; \tau, 0) &= Z^{-1} \mathrm{Tr} \left[e^{-\beta K} T_\tau \left[\psi(\vec{x}, \tau) \bar{\psi}(\vec{y}, 0) \right] \right] \\ &= Z^{-1} \mathrm{Tr} [\psi(\vec{x}, \tau) \bar{\psi}(\vec{y}, 0) e^{-\beta K}] \end{aligned} \tag{118}$$

Field Theory at Finite T and μ

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Field Theory at Finite T and μ

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$$K = \hat{H} - \mu\hat{Q} \tag{117}$$

$\beta \geqslant \tau$ and $\hat{\rho} = e^{-\beta K}$

$$\begin{aligned} G_F(\vec{x}, \vec{y}; \tau, 0) &= Z^{-1}\mathrm{Tr}\left[e^{-\beta K} T_\tau \left[\psi(\vec{x}, \tau)\bar{\psi}(\vec{y}, 0)\right]\right] \\ &= Z^{-1}\mathrm{Tr}[\psi(\vec{x}, \tau)\bar{\psi}(\vec{y}, 0)e^{-\beta K}] \\ &= Z^{-1}\mathrm{Tr}[\psi(\vec{x}, \tau)e^{-\beta K}e^{\beta K}\bar{\psi}(\vec{y}, 0)e^{-\beta K}] \\ &= Z^{-1}\mathrm{Tr}[\psi(\vec{x}, \tau)e^{-\beta K}\bar{\psi}(\vec{y}, \beta)] \end{aligned} \tag{118}$$

Field Theory at Finite T and μ

Fermions

$$K = \hat{H} - \mu \hat{Q} \tag{117}$$

$\beta \geqslant \tau$ and $\hat{\rho} = e^{-\beta K}$

$$\begin{aligned} G_F(\vec{x}, \vec{y}; \tau, 0) &= Z^{-1}\mathrm{Tr}\left[e^{-\beta K} T_\tau \left[\psi(\vec{x}, \tau)\bar{\psi}(\vec{y}, 0)\right]\right] \\ &= Z^{-1}\mathrm{Tr}[\psi(\vec{x}, \tau)\bar{\psi}(\vec{y}, 0)e^{-\beta K}] \\ &= Z^{-1}\mathrm{Tr}[\psi(\vec{x}, \tau)e^{-\beta K}e^{\beta K}\bar{\psi}(\vec{y}, 0)e^{-\beta K}] \\ &= Z^{-1}\mathrm{Tr}[\psi(\vec{x}, \tau)e^{-\beta K}\bar{\psi}(\vec{y}, \beta)] \\ &= Z^{-1}\mathrm{Tr}[e^{-\beta K}\bar{\psi}(\vec{y}, \beta)\psi(\vec{x}, \tau)] \end{aligned} \tag{118}$$

Field Theory at Finite T and μ

Fermions

$$K = \hat{H} - \mu\hat{Q} \tag{117}$$

$\beta \geqslant \tau$ and $\hat{\rho} = e^{-\beta K}$

$$\begin{aligned}
G_F(\vec{x},\vec{y};\tau,0) &= Z^{-1}\mathrm{Tr}\left[e^{-\beta K}T_\tau\left[\psi(\vec{x},\tau)\bar{\psi}(\vec{y},0)\right]\right] \\
&= Z^{-1}\mathrm{Tr}[\psi(\vec{x},\tau)\bar{\psi}(\vec{y},0)e^{-\beta K}] \\
&= Z^{-1}\mathrm{Tr}[\psi(\vec{x},\tau)e^{-\beta K}e^{\beta K}\bar{\psi}(\vec{y},0)e^{-\beta K}] \\
&= Z^{-1}\mathrm{Tr}[\psi(\vec{x},\tau)e^{-\beta K}\bar{\psi}(\vec{y},\beta)] \\
&= Z^{-1}\mathrm{Tr}[e^{-\beta K}\bar{\psi}(\vec{y},\beta)\psi(\vec{x},\tau)] \\
&= -Z^{-1}\mathrm{Tr}\left[e^{-\beta K}T_\tau\left[\psi(\vec{x},\tau)\bar{\psi}(\vec{y},\beta)\right]\right] = \\
&= -G_F(\vec{x},\vec{y};\tau,\beta)
\end{aligned} \tag{118}$$

Field Theory at Finite T and μ

Fermions

$$K = \hat{H} - \mu\hat{Q} \tag{117}$$

$\beta \geqslant \tau$ and $\hat{\rho} = e^{-\beta K}$

$$\begin{aligned}
G_F(\vec{x},\vec{y};\tau,0) &= Z^{-1}\mathrm{Tr}\left[e^{-\beta K}T_\tau\left[\psi(\vec{x},\tau)\bar{\psi}(\vec{y},0)\right]\right] \\
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&= Z^{-1}\mathrm{Tr}[e^{-\beta K}\bar{\psi}(\vec{y},\beta)\psi(\vec{x},\tau)] \\
&= -Z^{-1}\mathrm{Tr}\left[e^{-\beta K}T_\tau\left[\psi(\vec{x},\tau)\bar{\psi}(\vec{y},\beta)\right]\right] = \\
&= -G_F(\vec{x},\vec{y};\tau,\beta)
\end{aligned} \tag{118}$$

$$\psi(\vec{x},0) = -\psi(\vec{x},\beta), \quad \bar{\psi}(\vec{x},0) = -\bar{\psi}(\vec{x},\beta) \tag{119}$$

Field Theory at Finite T and μ

Fermions

$$G_F(\vec{x}, \vec{y}; \tau, 0) = -G_F(\vec{x}, \vec{y}; \tau, \beta) \tag{120}$$

Field Theory at Finite T and μ

Fermions

$$G_F(\vec{x},\vec{y};\tau,0) = -G_F(\vec{x},\vec{y};\tau,\beta) \tag{120}$$

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Field Theory at Finite T and μ

Fermions

$$G_F(\vec{x}, \vec{y}; \tau, 0) = -G_F(\vec{x}, \vec{y}; \tau, \beta) \tag{120}$$

$$\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta), \quad \bar{\psi}(\vec{x}, 0) = -\bar{\psi}(\vec{x}, \beta) \tag{121}$$

$$\psi_\alpha(\vec{x}, \tau) = \frac{1}{\sqrt{V}} \sum_n \sum_{\vec{p}} e^{i(\vec{p}\vec{x} + \omega_n \tau)} \tilde{\psi}_{\alpha;n}(\vec{p})$$

Field Theory at Finite T and μ

Fermions

$$G_F(\vec{x},\vec{y};\tau,0) = -G_F(\vec{x},\vec{y};\tau,\beta) \tag{120}$$

$$\psi(\vec{x},0) = -\psi(\vec{x},\beta), \quad \bar{\psi}(\vec{x},0) = -\bar{\psi}(\vec{x},\beta) \tag{121}$$

$$\psi_\alpha(\vec{x},\tau) = \frac{1}{\sqrt{V}}\sum_n\sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\tilde{\psi}_{\alpha;n}(\vec{p})$$

$$\omega_n\beta = (2n+1)\pi \tag{122}$$

Field Theory at Finite T and μ

Fermions

$$G_F(\vec{x},\vec{y};\tau,0) = -G_F(\vec{x},\vec{y};\tau,\beta) \tag{120}$$

$$\psi(\vec{x},0) = -\psi(\vec{x},\beta), \quad \bar{\psi}(\vec{x},0) = -\bar{\psi}(\vec{x},\beta) \tag{121}$$

$$\psi_\alpha(\vec{x},\tau) = \frac{1}{\sqrt{V}}\sum_n\sum_{\vec{p}} e^{i(\vec{p}\vec{x}+\omega_n\tau)}\tilde{\psi}_{\alpha;n}(\vec{p})$$

$$\omega_n\beta = (2n+1)\pi \qquad \omega_n = (2n+1)\pi T \tag{122}$$

Matsubara frequencies for fermions

Field Theory at Finite T and μ

Fermions

$$Z = \prod_{n} \prod_{\vec{p}} \prod_{\nu} \int d\tilde{\psi}^{\dagger}_{\nu;n}(\vec{p}) d\tilde{\psi}_{\nu;n}(\vec{p}) e^{W} \qquad (123)$$

Field Theory at Finite T and μ

Fermions

$$Z = \prod_n \prod_{\vec{p}} \prod_\nu \int d\tilde{\psi}^\dagger_{\nu;n}(\vec{p}) d\tilde{\psi}_{\nu;n}(\vec{p}) e^W \tag{123}$$

$$\sum_n \sum_{\vec{p}} i\tilde{\psi}^\dagger_{\nu;n}(\vec{p}) D_{\nu\lambda} \psi_{\lambda;n}(\vec{p}) \tag{124}$$

$$D = -i\beta \left[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma}\vec{p} - m\gamma^0 \right] \tag{125}$$

Field Theory at Finite T and μ

Fermions

$$Z = \prod_n \prod_{\vec{p}} \prod_\nu \int d\tilde{\psi}^\dagger_{\nu;n}(\vec{p}) d\tilde{\psi}_{\nu;n}(\vec{p}) e^W \tag{123}$$

$$\sum_n \sum_{\vec{p}} i\tilde{\psi}^\dagger_{\nu;n}(\vec{p}) D_{\nu\lambda} \psi_{\lambda;n}(\vec{p}) \tag{124}$$

$$D = -i\beta \left[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma}\vec{p} - m\gamma^0 \right] \tag{125}$$

$$Z = \det D, \qquad \ln \det D = \operatorname{Tr} \ln D \tag{126}$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = \sum_{n} \sum_{\vec{p}} \mathrm{Tr}_D \ln D, \qquad \mathrm{Tr}_D \mathcal{O} = \sum_{\nu\lambda} \mathcal{O}_{\nu\lambda} \tag{127}$$

Field Theory at Finite T and μ
Fermions

$$\ln Z = \sum_n \sum_{\vec{p}} \mathrm{Tr}_D \ln D, \qquad \mathrm{Tr}_D \mathcal{O} = \sum_{\nu\lambda} \mathcal{O}_{\nu\lambda} \tag{127}$$

$$\mathrm{Tr}_D \ln \left[-i\beta \left((-i\omega_n + \mu) - \gamma^0 \vec{\gamma}\vec{p} - m\gamma^0 \right) \right]$$

(128)

Field Theory at Finite T and μ

Fermions

$$\ln Z = \sum_n \sum_{\vec{p}} \mathrm{Tr}_D \ln D, \qquad \mathrm{Tr}_D \mathcal{O} = \sum_{\nu\lambda} \mathcal{O}_{\nu\lambda} \tag{127}$$

$$\mathrm{Tr}_D \ln \left[-i\beta \left((-i\omega_n + \mu) - \gamma^0 \vec{\gamma}\vec{p} - m\gamma^0 \right) \right] =$$
$$= \mathrm{Tr}_D \ln \left[-i\beta \left(-i\omega_n + \mu \right) \right] - \mathrm{Tr}_D \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\gamma^0 \vec{\gamma}\vec{p} + m\gamma^0}{-i\omega_n + \mu} \right)^k \tag{128}$$

Field Theory at Finite T and μ

Fermions

$$(\gamma^0\vec{\gamma}\vec{p} \quad + \quad m\gamma^0)(\gamma^0\vec{\gamma}\vec{p} \quad + \quad m\gamma^0) \tag{129}$$

Field Theory at Finite T and μ

Fermions

$$(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0)(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0) = \vec{p}^2 + m^2 \quad (129)$$

Field Theory at Finite T and μ

Fermions

$$(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0)(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0) = \vec{p}^2 + m^2 \qquad (129)$$

$$\mathrm{Tr}_D(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0)^{2k} = 4(\vec{p}^2 + m^2)^k \qquad (130)$$

$$\mathrm{Tr}_D(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0)^{2k+1} = 0, \qquad k = 0, 1, 2, \ldots \qquad (131)$$

Field Theory at Finite T and μ

Fermions

$$(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0)(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0) = \vec{p}^2 + m^2 \quad (129)$$

$$\mathrm{Tr}_D(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0)^{2k} = 4(\vec{p}^2 + m^2)^k \quad (130)$$

$$\mathrm{Tr}_D(\gamma^0\vec{\gamma}\vec{p} + m\gamma^0)^{2k+1} = 0, \qquad k = 0, 1, 2, \dots \quad (131)$$

$$\begin{aligned}
&\mathrm{Tr}_D \ln\left[-i\beta\left((-i\omega_n + \mu) - \gamma^0\vec{\gamma}\vec{p} - m\gamma^0\right)\right] = \\
&= \mathrm{Tr}_D \ln\left[-i\beta\left(-i\omega_n + \mu\right)\right] - 4\sum_{k=1}^{\infty}\frac{1}{2k}\left(\frac{\vec{p}^2 + m^2}{(-i\omega_n + \mu)^2}\right)^k = \\
&= 2\ln\left[\beta^2\left(\omega_n + i\mu\right)^2\right] - 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{\vec{p}^2 + m^2}{(\omega_n + i\mu)^2}\right)^k = \\
&= 2\ln\left[\beta^2\left((\omega_n + i\mu)^2 + \omega^2\right)\right]
\end{aligned} \quad (132)$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = 2\sum_n \sum_{\vec{p}} \ln\left[\beta^2\left((\omega_n + i\mu)^2 + \omega^2\right)\right] = \\ = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2\left(\omega_n^2 + (\omega-\mu)^2\right)\right] + \right. \\ \left. + \ln\left[\beta^2\left(\omega_n^2 + (\omega+\mu)^2\right)\right] \right\} \tag{133}$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = 2\sum_n \sum_{\vec{p}} \ln\left[\beta^2\left((\omega_n + i\mu)^2 + \omega^2\right)\right] = \\ = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2\left(\omega_n^2 + (\omega-\mu)^2\right)\right] + \right. \\ \left. + \ln\left[\beta^2\left(\omega_n^2 + (\omega+\mu)^2\right)\right]\right\} \tag{133}$$

$$\ln\left[\beta^2(\omega_n^2 + (\omega\pm\mu)^2)\right] = \\ = \int_1^{\beta^2(\omega\pm\mu)^2} \frac{dx}{x + (2n+1)^2\pi^2} + \ln\left[1 + (2n+1)^2\pi^2\right] \tag{134}$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = 2\sum_n \sum_{\vec{p}} \ln\left[\beta^2\left((\omega_n + i\mu)^2 + \omega^2\right)\right] = = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2\left(\omega_n^2 + (\omega-\mu)^2\right)\right] + + \ln\left[\beta^2\left(\omega_n^2 + (\omega+\mu)^2\right)\right] \right\} \tag{133}$$

$$\ln\left[\beta^2(\omega_n^2 + (\omega \pm \mu)^2)\right] = = \int_1^{\beta^2(\omega\pm\mu)^2} \frac{dx}{x + (2n+1)^2\pi^2} + \ln\left[1 + (2n+1)^2\pi^2\right] \tag{134}$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = 2\sum_n \sum_{\vec{p}} \ln\left[\beta^2\left((\omega_n + i\mu)^2 + \omega^2\right)\right] = $$
$$= \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2\left(\omega_n^2 + (\omega-\mu)^2\right)\right] + \right. \tag{133}$$
$$\left. + \ln\left[\beta^2\left(\omega_n^2 + (\omega+\mu)^2\right)\right] \right\}$$

$$\ln\left[\beta^2(\omega_n^2 + (\omega\pm\mu)^2)\right] =$$
$$= \int_1^{\beta^2(\omega\pm\mu)^2} \frac{dx}{x + (2n+1)^2\pi^2} + \ln\left[1 + (2n+1)^2\pi^2\right] \tag{134}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{x + (2n+1)^2\pi^2} = ?$$

Field Theory at Finite T and μ

Mathematical Tricks

Theorem

If the function $g(z)$ and has finite number of simple poles ξ_k

$$g(z) = (z - \xi_1)^{-1}(z - \xi_2)^{-1} \cdots (z - \xi_N)^{-1} r(z),$$

where $r(z)$ has no poles, $\xi_k \neq (2n+1)\pi i$ and

$$|g(z)| < \frac{C}{|z|}, \quad |z| \to \infty,$$

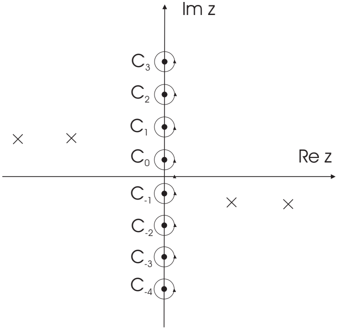
then

$$\sum_{n=-\infty}^{\infty} g((2n+1)\pi i) = -\frac{1}{2} \sum_{k=1}^{N} \tanh\left(\frac{\xi_k}{2}\right) \operatorname*{res}_{z=\xi_k} g(z)$$

Field Theory at Finite T and μ

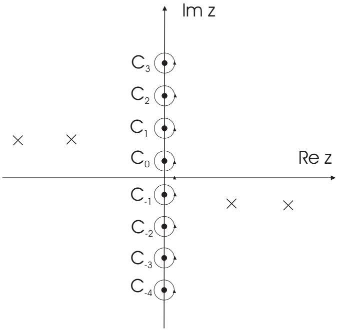
Mathematical Tricks

$$\sum_{n=-\infty}^{\infty} g((2n+1)\pi i) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \oint_{C_n} \frac{dz}{2\pi i} g(z) \tanh\left(\frac{z}{2}\right) \tag{135}$$



Field Theory at Finite T and μ
Mathematical Tricks

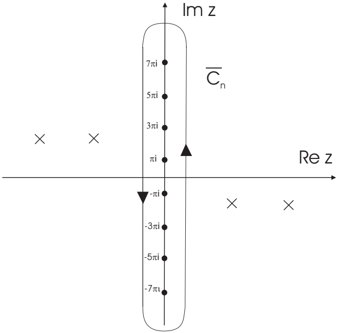
$$\tanh\left(\frac{\epsilon-(2n+1)\pi i}{2}\right)\sim\frac{2}{\epsilon}, \qquad \frac{1}{2}\oint_{C_n}\frac{dz}{2\pi i}\tanh\left(\frac{z}{2}\right)=1 \tag{136}$$



Field Theory at Finite T and μ

Mathematical Tricks

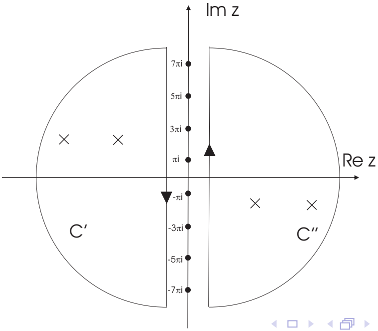
$$\sum_{n=-\infty}^{\infty} g((2n+1)\pi i) = \lim_{n\to\infty} \frac{1}{2} \oint_{\bar{C}_n} \frac{dz}{2\pi i} g(z) \tanh\left(\frac{z}{2}\right) \quad (137)$$



Field Theory at Finite T and μ

Mathematical Tricks

$$\sum_{n=-\infty}^{\infty} g((2n+1)\pi i) = \frac{1}{2}\left(\oint_{C'} \frac{dz}{2\pi i} g(z) \tanh\left(\frac{z}{2}\right) + \right.$$
$$\left. + \oint_{C''} \frac{dz}{2\pi i} g(z) \tanh\left(\frac{z}{2}\right)\right) \quad (138)$$



Field Theory at Finite T and μ

Mathematical Tricks

$$
\begin{aligned}
&\frac{1}{2}\oint_{C'}\frac{dz}{2\pi i}g(z)\tanh\left(\frac{z}{2}\right)+\\
&+\frac{1}{2}\oint_{C''}\frac{dz}{2\pi i}g(z)\tanh\left(\frac{z}{2}\right)=\\
&=-\frac{1}{2}\sum_{n=1}^{N}\tanh\left(\frac{\xi_n}{2}\right)\operatorname*{res}_{z=\xi_n}g(z)
\end{aligned}
\tag{139}
$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2} \tag{140}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2}, \quad z \to (2n+1)\pi i \Longrightarrow \frac{1}{x + (2n+1)^2\pi^2} \tag{140}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2}, \quad z \to (2n+1)\pi i \Longrightarrow \frac{1}{x + (2n+1)^2\pi^2} \tag{140}$$

Poles of $g(z)$

$$\xi_1 = \sqrt{x}, \qquad \xi_2 = -\sqrt{x} \tag{141}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$g(z) = \frac{1}{x - z^2}, \quad z \to (2n+1)\pi i \Longrightarrow \frac{1}{x + (2n+1)^2\pi^2} \tag{140}$$

Poles of $g(z)$

$$\xi_1 = \sqrt{x}, \qquad \xi_2 = -\sqrt{x} \tag{141}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2n+1)^2\pi^2} = = -\frac{1}{2}\left(\tanh\left(\frac{\sqrt{x}}{2}\right) \operatorname*{res}_{z=\sqrt{x}} \frac{1}{x - z^2} + + \tanh\left(\frac{-\sqrt{x}}{2}\right) \operatorname*{res}_{z=-\sqrt{x}} \frac{1}{x - z^2}\right) \tag{142}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x - z^2} = \mp\frac{1}{2\sqrt{x}} \tag{143}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x - z^2} = \mp \frac{1}{2\sqrt{x}} \tag{143}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2n+1)^2\pi^2} = \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right) \tag{144}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x - z^2} = \mp \frac{1}{2\sqrt{x}} \tag{143}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2n+1)^2\pi^2} = \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right) \tag{144}$$

$$\int_1^{\beta^2(\omega\pm\mu)^2} \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right) dx \tag{145}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x-z^2} = \mp\frac{1}{2\sqrt{x}} \tag{143}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x+(2n+1)^2\pi^2} = \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right) \tag{144}$$

$$\int_1^{\beta^2(\omega\pm\mu)^2} \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right) dx = \\ = 2\ln\cosh\left(\frac{\beta(\omega\pm\mu)}{2}\right) - 2\ln\cosh 1 \tag{145}$$

Field Theory at Finite T and μ

Mathematical Tricks

$$\operatorname*{res}_{z=\pm\sqrt{x}} \frac{1}{x - z^2} = \mp \frac{1}{2\sqrt{x}} \tag{143}$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2n+1)^2\pi^2} = \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right) \tag{144}$$

$$\int_1^{\beta^2(\omega\pm\mu)^2} \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right) dx = \\ = 2\ln\cosh\left(\frac{\beta(\omega\pm\mu)}{2}\right) - 2\ln\cosh 1 \tag{145}$$

Field Theory at Finite T and μ

Fermions

$$\ln\cosh\left(\frac{\beta(\omega \pm \mu)}{2}\right) \tag{146}$$

Field Theory at Finite T and μ

Fermions

$$\ln\cosh\left(\frac{\beta(\omega\pm\mu)}{2}\right) = \ln\left(e^{\frac{\beta(\omega\pm\mu)}{2}} + e^{\frac{\beta(\omega\pm\mu)}{2}}\right) - \ln 2 \tag{146}$$

Field Theory at Finite T and μ

Fermions

$$\ln\cosh\left(\frac{\beta(\omega\pm\mu)}{2}\right) = \ln\left(e^{\frac{\beta(\omega\pm\mu)}{2}} + e^{\frac{\beta(\omega\pm\mu)}{2}}\right) - \ln 2 = \\ = \ln\left[e^{-\frac{\beta(\omega\pm\mu)}{2}}\left(e^{\beta(\omega\pm\mu)}+1\right)\right] - \ln 2 \tag{146}$$

Field Theory at Finite T and μ

Fermions

$$
\begin{aligned}
\ln\cosh\left(\frac{\beta(\omega\pm\mu)}{2}\right) &= \ln\left(e^{\frac{\beta(\omega\pm\mu)}{2}}+e^{\frac{\beta(\omega\pm\mu)}{2}}\right)-\ln 2 = \\
&= \ln\left[e^{-\frac{\beta(\omega\pm\mu)}{2}}\left(e^{\beta(\omega\pm\mu)}+1\right)\right]-\ln 2 \\
&= -\frac{\beta(\omega\pm\mu)}{2}+\ln\left(e^{\beta(\omega\pm\mu)}+1\right)-\ln 2
\end{aligned}
\qquad (146)
$$

Field Theory at Finite T and μ

Fermions

$$
\begin{aligned}
\ln\cosh\left(\frac{\beta(\omega\pm\mu)}{2}\right) &= \ln\left(e^{\frac{\beta(\omega\pm\mu)}{2}} + e^{\frac{\beta(\omega\pm\mu)}{2}}\right) - \ln 2 = \\
&= \ln\left[e^{-\frac{\beta(\omega\pm\mu)}{2}}\left(e^{\beta(\omega\pm\mu)} + 1\right)\right] - \ln 2 \\
&= -\frac{\beta(\omega\pm\mu)}{2} + \ln\left(e^{\beta(\omega\pm\mu)} + 1\right) - \ln 2
\end{aligned}
\tag{146}
$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = \sum_n \sum_{\vec{p}} \Big\{ \ln \left[\beta^2(\omega_n^2 + (\omega - \mu)^2)\right] + $$

$$+ \ln \left[\beta^2(\omega_n^2 + (\omega + \mu)^2)\right] \Big\}$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2(\omega_n^2 + (\omega - \mu)^2)\right] + \right.$$

$$\left. + \ln\left[\beta^2(\omega_n^2 + (\omega + \mu)^2)\right] \right\}$$

$$\ln\left[\beta^2(\omega_n^2 + (\omega \pm \mu)^2)\right] = \int_1^{\beta^2(\omega\pm\mu)^2} \frac{dx}{x + (2n+1)^2\pi^2} + const$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = \sum_n \sum_{\vec{p}} \Big\{ \ln\left[\beta^2(\omega_n^2 + (\omega-\mu)^2)\right] + \\ + \ln\left[\beta^2(\omega_n^2 + (\omega+\mu)^2)\right] \Big\}$$

$$\ln\left[\beta^2(\omega_n^2 + (\omega\pm\mu)^2)\right] = \int_1^{\beta^2(\omega\pm\mu)^2} \frac{dx}{x + (2n+1)^2\pi^2} + const$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x+(2n+1)^2\pi^2} = \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right)$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2(\omega_n^2 + (\omega-\mu)^2)\right] + \right.$$

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$$\ln\left[\beta^2(\omega_n^2 + (\omega\pm\mu)^2)\right] = \int_1^{\beta^2(\omega\pm\mu)^2} \frac{dx}{x + (2n+1)^2\pi^2} + const$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2n+1)^2\pi^2} = \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right)$$

$$\int_1^{\beta^2(\omega\pm\mu)^2} \tanh\left(\frac{\sqrt{x}}{2}\right)\frac{dx}{2\sqrt{x}} = -\beta(\omega\pm\mu) + 2\ln\left(e^{\beta(\omega\pm\mu)} + 1\right) + const$$

Field Theory at Finite T and μ

Fermions

$$\ln Z = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2(\omega_n^2 + (\omega-\mu)^2)\right] + \right.$$

$$\left. + \ln\left[\beta^2(\omega_n^2 + (\omega+\mu)^2)\right] \right\}$$

$$\ln\left[\beta^2(\omega_n^2 + (\omega\pm\mu)^2)\right] = \int_1^{\beta^2(\omega\pm\mu)^2} \frac{dx}{x + (2n+1)^2\pi^2} + const$$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{x + (2n+1)^2\pi^2} = \frac{1}{2\sqrt{x}} \tanh\left(\frac{\sqrt{x}}{2}\right)$$

$$\int_1^{\beta^2(\omega\pm\mu)^2} \tanh\left(\frac{\sqrt{x}}{2}\right)\frac{dx}{2\sqrt{x}} = -\beta(\omega\pm\mu) + 2\ln\left(e^{\beta(\omega\pm\mu)} + 1\right) + const$$

$$\ln Z = \sum_{\vec{p}} \left[\beta\omega - \ln\left(e^{\beta(\omega+\mu)} + 1\right) - \ln\left(e^{\beta(\omega-\mu)} + 1\right)\right] \quad (147)$$

Field Theory at Finite T and μ

Fermion

$$\sum_{\vec{p}} \tag{148}$$

Field Theory at Finite T and μ

Fermion

$$\sum_{\vec{p}} \rightarrow V \int \frac{d^3p}{(2\pi)^3} \tag{148}$$

Field Theory at Finite T and μ

Fermion

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^3} \tag{148}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega - \ln\left(e^{\beta(\omega-\mu)} + 1\right) - \ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{149}$$

Field Theory at Finite T and μ

Fermion

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^3} \tag{148}$$

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$$\ln Z = -\beta\Omega$$

Field Theory at Finite T and μ

Fermion

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^3} \tag{148}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega - \ln\left(e^{\beta(\omega-\mu)} + 1\right) - \ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{149}$$

$$\ln Z = -\beta\Omega$$

$$\Omega = V \int \frac{d^3p}{(2\pi)^3} \left[-\omega + T\ln\left(e^{\beta(\omega-\mu)} + 1\right) + T\ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{150}$$

Field Theory at Finite T and μ

Fermion

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^3} \tag{148}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega - \ln\left(e^{\beta(\omega-\mu)} + 1\right) - \ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{149}$$

$$\ln Z = -\beta\Omega$$

$$\Omega = V \int \frac{d^3p}{(2\pi)^3} \left[-\omega + T\ln\left(e^{\beta(\omega-\mu)} + 1\right) + T\ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{150}$$

$$P = \int \frac{d^3p}{(2\pi)^3} \left[\omega - T\ln\left(e^{\beta(\omega-\mu)} + 1\right) - T\ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{151}$$

Field Theory at Finite T and μ

Fermion

$$\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^3} \tag{148}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega - \ln\left(e^{\beta(\omega-\mu)} + 1\right) - \ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{149}$$

$$\ln Z = -\beta\Omega$$

$$\Omega = V \int \frac{d^3p}{(2\pi)^3} \left[-\omega + T\ln\left(e^{\beta(\omega-\mu)} + 1\right) + T\ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{150}$$

$$P = \int \frac{d^3p}{(2\pi)^3} \left[\omega - T\ln\left(e^{\beta(\omega-\mu)} + 1\right) - T\ln\left(e^{\beta(\omega+\mu)} + 1\right)\right] \tag{151}$$

Contents

Feynman Rules

Partition function for interacting scalar bosons

$$Z[J] = \int \mathcal{D}\phi \; e^{W[\phi]+J\phi} \tag{152}$$

Feynman Rules

Partition function for interacting scalar bosons

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$$\langle\langle \phi^n \rangle\rangle = Z^{-1} \frac{\delta^n Z}{\delta J^n}\bigg|_{J=0} \tag{153}$$

Feynman Rules

Partition function for interacting scalar bosons

$$Z[J] = \int \mathcal{D}\phi \; e^{W[\phi]+J\phi} \tag{152}$$

$$\langle\langle\phi^n\rangle\rangle = Z^{-1}\frac{\delta^n Z}{\delta J^n}\bigg|_{J=0} \tag{153}$$

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Feynman Rules

Partition function for interacting scalar bosons

$$Z[J] = \int \mathcal{D}\phi \; e^{W[\phi]+J\phi} \tag{152}$$

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$$W[\phi] = -\frac{1}{2}\phi D \phi - V[\phi] \tag{154}$$

$$Z[J] = e^{-V[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D\phi + J\phi} \tag{155}$$

Feynman Rules

Partition function for interacting scalar bosons

$$Z[J] = \int \mathcal{D}\phi \; e^{W[\phi]+J\phi} \tag{152}$$

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$$Z[J] = e^{-V[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D\phi + J\phi} \tag{155}$$

$\phi \rightarrow \phi + D^{-1}J$

$$Z[J] = e^{-V[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D\phi + \frac{1}{2}JD^{-1}J} \tag{156}$$

Feynman Rules

Partition function for interacting scalar bosons

$$Z[J] = \int \mathcal{D}\phi \; e^{W[\phi]+J\phi} \tag{152}$$

$$\langle\langle \phi^n \rangle\rangle = Z^{-1} \frac{\delta^n Z}{\delta J^n}\bigg|_{J=0} \tag{153}$$

$$W[\phi] = -\frac{1}{2}\phi D\phi - V[\phi] \tag{154}$$

$$Z[J] = e^{-V[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D\phi + J\phi} \tag{155}$$

$\phi \to \phi + D^{-1}J$

$$Z[J] = e^{-V[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi \; e^{-\frac{1}{2}\phi D\phi + \frac{1}{2}JD^{-1}J} \tag{156}$$

$$Z[J] = e^{-\beta\Omega_0} e^{-V[\frac{\delta}{\delta J}]} e^{\frac{1}{2}JD^{-1}J} \tag{157}$$

Feynman Rules
Boson Propagator

Scalar boson

$$\phi D\phi \equiv \int d\tau_1 \int d\tau_2 \int d^3x_1 d^3x_2 \phi(\tau_1, \vec{x}_1) \times \times D(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2)\phi(\tau_2, \vec{x}_2) \quad (158)$$

Feynman Rules

Boson Propagator

Scalar boson

$$\phi D\phi \equiv \int d\tau_1 \int d\tau_2 \int d^3x_1 d^3x_2 \phi(\tau_1, \vec{x}_1) \times \times D(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2)\phi(\tau_2, \vec{x}_2) \quad (158)$$

$$D(\tau, \vec{x}) = \beta^{-1} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{i(\omega_n \tau + \vec{p}\vec{x})} D(\omega_n, \vec{p}) \quad (159)$$

Feynman Rules
Boson Propagator

Scalar boson

$$\phi D\phi \equiv \int d\tau_1 \int d\tau_2 \int d^3x_1 d^3x_2 \phi(\tau_1, \vec{x}_1) \times \times D(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2)\phi(\tau_2, \vec{x}_2) \quad (158)$$

$$D(\tau, \vec{x}) = \beta^{-1} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{i(\omega_n \tau + \vec{p}\vec{x})} D(\omega_n, \vec{p}) \quad (159)$$

Boson propagator

$$\mathcal{G}_B(\omega_n, \vec{p}) = \frac{1}{D(\omega_n, \vec{p})} = \frac{1}{\omega_n^2 + \vec{p}^2 + m^2} \quad (160)$$

Feynman Rules

Partition function for interacting charged scalar bosons

$$Z[J] = \int \mathcal{D}\phi \mathcal{D}\phi^* \; e^{W[\phi,\phi^*]+J^*\phi+J\phi^*} \qquad (161)$$

Feynman Rules

Partition function for interacting charged scalar bosons

$$Z[J] = \int \mathcal{D}\phi \mathcal{D}\phi^* \; e^{W[\phi,\phi^*]+J^*\phi+J\phi^*} \tag{161}$$

$$\langle\langle \phi^n \phi^{*m} \rangle\rangle = Z^{-1} \frac{\delta^{n+m} Z}{\delta J^{*n} \delta J^m}\Big|_{J=J^*=0} \tag{162}$$

Feynman Rules

Partition function for interacting charged scalar bosons

$$Z[J] = \int \mathcal{D}\phi \mathcal{D}\phi^* \; e^{W[\phi,\phi^*]+J^*\phi+J\phi^*} \tag{161}$$

$$\langle\langle \phi^n \phi^{*m} \rangle\rangle = Z^{-1} \frac{\delta^{n+m} Z}{\delta J^{*n} \delta J^m} |_{J=J^*=0} \tag{162}$$

$$W[\phi] = -\phi^* D\phi - V[\phi, \phi^*] \tag{163}$$

Feynman Rules

Partition function for interacting charged scalar bosons

$$Z[J] = \int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{W[\phi,\phi^*]+J^*\phi+J\phi^*} \tag{161}$$

$$\langle\langle\phi^n\phi^{*m}\rangle\rangle = Z^{-1}\frac{\delta^{n+m}Z}{\delta J^{*n}\delta J^m}\Big|_{J=J^*=0} \tag{162}$$

$$W[\phi] = -\phi^* D\phi - V[\phi,\phi^*] \tag{163}$$

$$Z[J] = e^{-V[\frac{\delta}{\delta J^*},\frac{\delta}{\delta J}]}\int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{-\phi^* D\phi+J^*\phi+J\phi^*} \tag{164}$$

Feynman Rules

Partition function for interacting charged scalar bosons

$$Z[J] = \int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{W[\phi,\phi^*]+J^*\phi+J\phi^*} \tag{161}$$

$$\langle\langle \phi^n \phi^{*m} \rangle\rangle = Z^{-1} \frac{\delta^{n+m} Z}{\delta J^{*n} \delta J^m}|_{J=J^*=0} \tag{162}$$

$$W[\phi] = -\phi^* D\phi - V[\phi, \phi^*] \tag{163}$$

$$Z[J] = e^{-V\left[\frac{\delta}{\delta J^*}, \frac{\delta}{\delta J}\right]} \int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{-\phi^* D\phi + J^*\phi + J\phi^*} \tag{164}$$

$\phi \to \phi + D^{-1}J, \phi^* \to \phi^* + J^* D^{-1}$

$$Z[J] = e^{-V\left[\frac{\delta}{\delta J}, \frac{\delta}{\delta J^*}\right]} \int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{-\phi^* D\phi + J^* D^{-1} J} \tag{165}$$

Feynman Rules

Partition function for interacting charged scalar bosons

$$Z[J] = \int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{W[\phi,\phi^*]+J^*\phi+J\phi^*} \tag{161}$$

$$\langle\langle \phi^n \phi^{*m} \rangle\rangle = Z^{-1} \frac{\delta^{n+m} Z}{\delta J^{*n} \delta J^m} |_{J=J^*=0} \tag{162}$$

$$W[\phi] = -\phi^* D\phi - V[\phi, \phi^*] \tag{163}$$

$$Z[J] = e^{-V\left[\frac{\delta}{\delta J^*}, \frac{\delta}{\delta J}\right]} \int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{-\phi^* D\phi + J^*\phi + J\phi^*} \tag{164}$$

$\phi \to \phi + D^{-1}J, \phi^* \to \phi^* + J^* D^{-1}$

$$Z[J] = e^{-V\left[\frac{\delta}{\delta J}, \frac{\delta}{\delta J^*}\right]} \int \mathcal{D}\phi\mathcal{D}\phi^* \; e^{-\phi^* D\phi + J^* D^{-1} J} \tag{165}$$

$$Z[J] = e^{-\beta\Omega_0} e^{-V\left[\frac{\delta}{\delta J} \frac{\delta}{\delta J^*}\right]} e^{J^* D^{-1} J} \tag{166}$$

Feynman Rules
Boson Propagator

Charged scalar boson

$$\phi^* D\phi \equiv \int d\tau_1 \int d\tau_2 \int d^3x_1 d^3x_2 \phi^*(\tau_1, \vec{x}_1) \times \times D(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2)\phi(\tau_2, \vec{x}_2) \quad (167)$$

Feynman Rules

Boson Propagator

Charged scalar boson

$$\phi^* D\phi \equiv \int d\tau_1 \int d\tau_2 \int d^3x_1 d^3x_2 \phi^*(\tau_1, \vec{x}_1)\times \times D(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2)\phi(\tau_2, \vec{x}_2) \quad (167)$$

$$D(\tau, \vec{x}) = \beta^{-1} \sum_n \int \frac{d^3}{(2\pi)^3} e^{i(\omega_n\tau + \vec{p}\vec{x})} D(\omega_n, \vec{p}) \quad (168)$$

Boson propagator

$$\mathcal{G}_B(\omega_n, \vec{p}) = \frac{1}{D(\omega_n, \vec{p})} = \frac{1}{(\omega_n + i\mu)^2 + \vec{p}^2 + m^2} \quad (169)$$

Contents

Feynman Rules

Partition function for interacting fermions

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{W[\bar{\psi},\psi]+\bar{\eta}\psi+\bar{\psi}\eta} \qquad (170)$$

Feynman Rules

Partition function for interacting fermions

$$Z[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{W[\bar{\psi},\psi]+\bar{\eta}\psi+\bar{\psi}\eta} \tag{170}$$

$$\langle\langle \psi^n \bar{\psi}^m \rangle\rangle = Z^{-1} \frac{\overrightarrow{\delta}^n}{\delta\bar{\eta}} \frac{\overleftarrow{\delta}^m}{\delta\eta}\bigg|_{\eta=\bar{\eta}=0} Z \tag{171}$$

$$\left\{\frac{\overrightarrow{\delta}}{\delta\eta},\frac{\overrightarrow{\delta}}{\delta\eta}\right\} = \left\{\frac{\overleftarrow{\delta}}{\delta\eta},\frac{\overleftarrow{\delta}}{\delta\eta}\right\} = \left\{\frac{\overrightarrow{\delta}}{\delta\eta},\frac{\overleftarrow{\delta}}{\delta\eta}\right\} = \left\{\frac{\overleftarrow{\delta}}{\delta\eta},\frac{\overrightarrow{\delta}}{\delta\eta}\right\} = 0$$

$$\left\{\frac{\overrightarrow{\delta}}{\delta\bar{\eta}},\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}\right\} = \left\{\frac{\overleftarrow{\delta}}{\delta\bar{\eta}},\frac{\overleftarrow{\delta}}{\delta\bar{\eta}}\right\} = \left\{\frac{\overrightarrow{\delta}}{\delta\bar{\eta}},\frac{\overleftarrow{\delta}}{\delta\bar{\eta}}\right\} == \left\{\frac{\overleftarrow{\delta}}{\delta\bar{\eta}},\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}\right\} = 0$$

$$\left\{\frac{\overrightarrow{\delta}}{\delta\eta},\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}\right\} = \left\{\frac{\overleftarrow{\delta}}{\delta\eta},\frac{\overleftarrow{\delta}}{\delta\bar{\eta}}\right\} = \left\{\frac{\overrightarrow{\delta}}{\delta\eta},\frac{\overleftarrow{\delta}}{\delta\bar{\eta}}\right\} = \left\{\frac{\overleftarrow{\delta}}{\delta\eta},\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}\right\} = 0$$

Feynman Rules

Derivatives

$$\frac{\overrightarrow{\delta}}{\delta\eta(x)}\eta(y) = \delta(x-y), \qquad \eta(y)\frac{\overleftarrow{\delta}}{\delta\eta(x)} = \delta(x-y)$$

$$\frac{\overrightarrow{\delta}}{\delta\bar\eta(x)}\bar\eta(y) = \delta(x-y), \qquad \bar\eta(y)\frac{\overleftarrow{\bar\delta}}{\delta\bar\eta(x)} = \delta(x-y)$$

$$\frac{\overrightarrow{\delta}}{\delta\eta(x)}\bar\eta(y) = -\bar\eta(y)\frac{\overrightarrow{\delta}}{\delta\eta(x)}, \qquad \frac{\overrightarrow{\bar\delta}}{\delta\bar\eta(x)}\eta(y) = -\eta(y)\frac{\overrightarrow{\bar\delta}}{\delta\bar\eta(x)}$$

$$\bar\eta(y)\frac{\overleftarrow{\delta}}{\delta\eta(x)} = -\frac{\overleftarrow{\delta}}{\delta\eta(x)}\bar\eta(y), \qquad \eta(y)\frac{\overleftarrow{\bar\delta}}{\delta\bar\eta(x)} = -\frac{\overleftarrow{\bar\delta}}{\delta\bar\eta(x)}\eta(y)$$

Feynman Rules

Partition function for interacting fermions

$$W[\phi] = -\psi D\psi - V[\bar{\psi}, \psi] \qquad (172)$$

Feynman Rules

Partition function for interacting fermions

$$W[\phi] = -\psi D\psi - V[\bar{\psi}, \psi] \tag{172}$$

$$Z[\bar{\eta}, \eta] = e^{-V\left[\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}, \frac{\overleftarrow{\delta}}{\delta\eta}\right]} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-\bar{\psi}D\psi + \bar{\eta}\psi + \bar{\psi}\eta} \tag{173}$$

Feynman Rules

Partition function for interacting fermions

$$W[\phi] = -\psi D\psi - V[\bar{\psi}, \psi] \tag{172}$$

$$Z[\bar{\eta}, \eta] = e^{-V\left[\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}, \frac{\overleftarrow{\delta}}{\delta\eta}\right]} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-\bar{\psi}D\psi+\bar{\eta}\psi+\bar{\psi}\eta} \tag{173}$$

$$\psi \to \psi + D^{-1}\eta, \qquad \bar{\psi} \to \bar{\psi} + \bar{\eta}D^{-1}$$

$$Z[\bar{\eta}, \eta] = e^{-V\left[\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}, \frac{\overleftarrow{\delta}}{\delta\eta}\right]} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-\bar{\psi}D\psi+\bar{\eta}D^{-1}\eta} \tag{174}$$

Feynman Rules

Partition function for interacting fermions

$$W[\phi] = -\psi D\psi - V[\bar{\psi}, \psi] \tag{172}$$

$$Z[\bar{\eta}, \eta] = e^{-V\left[\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}, \frac{\overleftarrow{\delta}}{\delta\eta}\right]} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-\bar{\psi}D\psi + \bar{\eta}\psi + \bar{\psi}\eta} \tag{173}$$

$$\psi \to \psi + D^{-1}\eta, \qquad \bar{\psi} \to \bar{\psi} + \bar{\eta}D^{-1}$$

$$Z[\bar{\eta}, \eta] = e^{-V\left[\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}, \frac{\overleftarrow{\delta}}{\delta\eta}\right]} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-\bar{\psi}D\psi + \bar{\eta}D^{-1}\eta} \tag{174}$$

$$Z[\bar{\eta}, \eta] = e^{-\beta\Omega_0} e^{-V\left[\frac{\overrightarrow{\delta}}{\delta\bar{\eta}}, \frac{\overleftarrow{\delta}}{\delta\eta}\right]} e^{\bar{\eta}D^{-1}\eta} \tag{175}$$

Feynman Rules

Fermion Propagator

$$\bar{\psi} D \psi \equiv \int d\tau_1 \int d\tau_2 \int d^3x_1 d^3x_2 \bar{\psi}(\tau_1, \vec{x}_1) \times \\ \times \not{D}(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2) \psi(\tau_2, \vec{x}_2) \quad (176)$$

Feynman Rules

Fermion Propagator

$$\bar{\psi} D \psi \equiv \int d\tau_1 \int d\tau_2 \int d^3x_1 d^3x_2 \bar{\psi}(\tau_1, \vec{x}_1) \times \\ \times \not{D}(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2) \psi(\tau_2, \vec{x}_2) \quad (176)$$

$$\not{D}(\tau, \vec{x}) = \beta^{-1} \sum_n \int \frac{d^3p}{(2\pi)^3} e^{i(\omega_n \tau + \vec{p}\vec{x})} \; \not{D}(\omega_n, \vec{p}) \quad (177)$$

Fermion propagator

$$\mathcal{G}_F(\omega_n, \vec{p}) = \frac{1}{\not{D}(\omega_n, \vec{p})} = \frac{1}{(-i\omega_n + \mu)\gamma_0 - \vec{\gamma}\vec{p} - m} \quad (178)$$

Contents

Feynman Rules

Loop Calculation

Example: the sigma model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial_\mu \sigma - M^2 \sigma^2) + \bar{\psi}(\not{\partial} - m)\psi + g\sigma\bar{\psi}\psi \qquad (179)$$

Feynman Rules

Loop Calculation

Example: the sigma model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial_\mu \sigma - M^2 \sigma^2) + \bar{\psi}(\not{\partial} - m)\psi + g\sigma\bar{\psi}\psi \quad (179)$$

$$Z = \int \mathcal{D}\sigma \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\int d\tau \int d^3x \mathcal{L}} \quad (180)$$

Feynman Rules
Loop Calculation

Example: the sigma model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma\partial_\mu\sigma - M^2\sigma^2) + \bar{\psi}(\not{\partial} - m)\psi + g\sigma\bar{\psi}\psi \qquad (179)$$

$$Z = \int \mathcal{D}\sigma\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\int d\tau \int d^3x \mathcal{L}} \qquad (180)$$

$$Z = \int \mathcal{D}\sigma e^{-\beta\Omega_F + \frac{1}{2}\int d\tau \int d^3x(\partial_\mu\sigma\partial_\mu\sigma - M^2\sigma^2)} \qquad (181)$$

Feynman Rules
Loop Calculation

Example: the sigma model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial_\mu \sigma - M^2\sigma^2) + \bar{\psi}(\not{\partial} - m)\psi + g\sigma\bar{\psi}\psi \quad (179)$$

$$Z = \int \mathcal{D}\sigma \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\int d\tau \int d^3x \mathcal{L}} \quad (180)$$

$$Z = \int \mathcal{D}\sigma e^{-\beta\Omega_F + \frac{1}{2}\int d\tau \int d^3x (\partial_\mu\sigma\partial_\mu\sigma - M^2\sigma^2)} \quad (181)$$

$$\Omega_F(T, \mu|\sigma) = -T \ln Z_F[\sigma] \quad (182)$$

Feynman Rules
Loop Calculation

Example: the sigma model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma \partial_\mu \sigma - M^2 \sigma^2) + \bar{\psi}(\not{\partial} - m)\psi + g\sigma\bar{\psi}\psi \tag{179}$$

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$$Z = \int \mathcal{D}\sigma e^{-\beta\Omega_F + \frac{1}{2}\int d\tau \int d^3x (\partial_\mu\sigma\partial_\mu\sigma - M^2\sigma^2)} \tag{181}$$

$$\Omega_F(T, \mu|\sigma) = -T \ln Z_F[\sigma] \tag{182}$$

$$\Omega_F(T, \mu|\sigma) = \Omega_F^{(0)}(T, \mu) + \Omega_F^{(2)}(T, \mu)\sigma^2 + \dots \tag{183}$$

Feynman Rules
Loop Calculation

Example: the sigma model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma\partial_\mu\sigma - M^2\sigma^2) + \bar{\psi}(\not\partial - m)\psi + g\sigma\bar{\psi}\psi \quad (179)$$

$$Z = \int \mathcal{D}\sigma\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\int d\tau \int d^3x \mathcal{L}} \quad (180)$$

$$Z = \int \mathcal{D}\sigma e^{-\beta\Omega_F + \frac{1}{2}\int d\tau \int d^3x(\partial_\mu\sigma\partial_\mu\sigma - M^2\sigma^2)} \quad (181)$$

$$\Omega_F(T,\mu|\sigma) = -T\ln Z_F[\sigma] \quad (182)$$

$$\Omega_F(T,\mu|\sigma) = \Omega_F^{(0)}(T,\mu) + \Omega_F^{(2)}(T,\mu)\sigma^2 + \dots \quad (183)$$

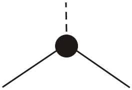
$$\Omega_F^{(2)}(T,\mu) = ? \quad (184)$$

Feynman Rules

$$\mathcal{G}_B(\omega_n, \vec{p}) = (\omega_n^2 + \vec{p}^2 + m^2)^{-1}$$
$$\omega_n = 2n\pi T$$

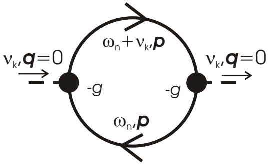
$$\mathcal{G}_F(\omega_n, \vec{p}) =$$
$$= [(-i\omega_n + \mu)\gamma_0 - \vec{\gamma}\vec{p} - m]^{-1}$$
$$\omega_n = (2n+1)\pi T$$

$$-g$$



Feynman Rules
Loop Calculation

The contribution to $\Omega^{(2)}$



$$(-1)\beta^{-1}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}_D\left[\frac{1}{(-i\omega_n-i\nu_k+\mu)\gamma_0-\vec{\gamma}\vec{p}-m}\times\right.$$
$$\left.\times\frac{1}{(-i\omega_n+\mu)\gamma_0-\vec{\gamma}\vec{p}-m}\right] \quad (185)$$

Feynman Rules
Loop Calculation

The contribution to $\Omega^{(2)}$

$$\mathrm{Tr}_D\left[\frac{1}{(-i\omega_n - i\nu_k + \mu)\gamma_0 - \vec{\gamma}\vec{p} - m}\cdot\frac{1}{(-i\omega_n + \mu)\gamma_0 - \vec{\gamma}\vec{p} - m}\right] = $$
$$= \frac{\mathrm{Tr}_D\left[((-i\omega_n - i\nu_k + \mu)\gamma_0 - \vec{\gamma}\vec{p} + m)\,((-i\omega_n + \mu)\gamma_0 - \vec{\gamma}\vec{p} + m)\right]}{((-i\omega_n - i\nu_k + \mu)^2 - \omega^2)\,((-i\omega_n + \mu)^2 - \omega^2)} \tag{186}$$

$$\omega^2 = \vec{p}^2 + m^2$$

Feynman Rules

Loop Calculation

$$\mathrm{Tr}_D\left[\left((-i\omega_n - i\nu_k + \mu)\gamma_0 - \vec{\gamma}\vec{p} + m\right)\left((-i\omega_n + \mu)\gamma_0 - \vec{\gamma}\vec{p} + m\right)\right] = \\ = 4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2) \quad (187)$$

Feynman Rules
Loop Calculation

$$\mathrm{Tr}_D\left[\left((-i\omega_n - i\nu_k + \mu)\gamma_0 - \vec{\gamma}\vec{p} + m\right)\left((-i\omega_n + \mu)\gamma_0 - \vec{\gamma}\vec{p} + m\right)\right] = \\ = 4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2) \quad (187)$$

$$(-1)T\int\frac{d^3p}{(2\pi)^3}\sum_{n=-\infty}^{\infty}\times \\ \times\frac{4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2)}{\left((-i\omega_n - i\nu_k + \mu)^2 - \omega^2\right)\left((-i\omega_n + \mu)^2 - \omega^2\right)} \quad (188)$$

Feynman Rules
Loop Calculation

$$\sum_{n=-\infty}^{\infty} \frac{4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2)}{\left((\omega_n + \nu_k + i\mu)^2 - \omega^2\right)\left((\omega_n + i\mu)^2 - \omega^2\right)} \quad (189)$$

Feynman Rules
Loop Calculation

$$\sum_{n=-\infty}^{\infty} \frac{4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2)}{\left((\omega_n + \nu_k + i\mu)^2 - \omega^2\right)\left((\omega_n + i\mu)^2 - \omega^2\right)} \tag{189}$$

$$\sum_{n=-\infty}^{\infty} g((2n+1)\pi T) = -\frac{1}{2T}\sum_{k=1}^{N} \tanh\left(\frac{\xi_k}{2T}\right) \operatorname*{res}_{z=\xi_k} g(z) \tag{190}$$

Feynman Rules

Loop Calculation

$$\sum_{n=-\infty}^{\infty} \frac{4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2)}{\left((\omega_n + \nu_k + i\mu)^2 - \omega^2\right)\left((\omega_n + i\mu)^2 - \omega^2\right)} \quad (189)$$

$$\sum_{n=-\infty}^{\infty} g((2n+1)\pi T) = -\frac{1}{2T} \sum_{k=1}^{N} \tanh\left(\frac{\xi_k}{2T}\right) \operatorname*{res}_{z=\xi_k} g(z) \quad (190)$$

$\omega_n = (2n+1)\pi T$

Feynman Rules

Loop Calculation

$$\sum_{n=-\infty}^{\infty} \frac{4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2)}{((\omega_n + \nu_k + i\mu)^2 + \omega^2)\left((-iz + \mu)^2 + \omega^2\right)} =$$

$$-\frac{1}{2T} \sum_{k=1}^{4} \tanh\left(\frac{\xi_k}{2}\right) \times \tag{191}$$

$$\operatorname*{res}_{z=\xi_k} \frac{4((-i\omega_n - i\nu_k + \mu)(-i\omega_n + \mu) - \omega^2 + 2m^2)}{((z - i\nu_k + \mu)^2 - \omega^2)\left((z + \mu)^2 + \omega^2\right)}$$

Poles

$$\begin{aligned} \xi_1 &= i\nu_k - \mu - \omega, & \xi_2 &= i\nu_k - \mu + \omega \\ \xi_3 &= -\mu - \omega, & \xi_4 &= -\mu + \omega \end{aligned} \tag{192}$$

Feynman Rules

Loop Calculation

$$\operatorname*{res}_{\xi_1} \frac{((z - i\nu_k + \mu)(z+\mu) + \omega^2)}{((z - i\nu_k + \mu)^2 - \omega^2)\left((z+\mu)^2 + \omega^2\right)} = \\ = \frac{1}{2\omega} \frac{i\nu_k\omega + 2m^2}{(\omega - i\nu_k)^2 - \omega^2} \quad (193)$$

$$\operatorname*{res}_{\xi_2} \frac{((z - i\nu_k + \mu)(z+\mu) + \omega^2)}{((z - i\nu_k + \mu)^2 - \omega^2)\left((z+\mu)^2 + \omega^2\right)} = \\ = -\frac{1}{2\omega} \frac{i\nu_k\omega + 2m^2}{(\omega + i\nu_k)^2 - \omega^2} \quad (194)$$

Feynman Rules

Loop Calculation

$$\operatorname*{res}_{\xi_3} \frac{((z - i\nu_k + \mu)(z + \mu) + \omega^2)}{((z - i\nu_k + \mu)^2 - \omega^2)\,((z + \mu)^2 + \omega^2)} = \\ = -\frac{1}{2\omega} \frac{-i\nu_k\omega + 2m^2}{(\omega + i\nu_k)^2 - \omega^2} \quad (195)$$

$$\operatorname*{res}_{\xi_4} \frac{((z - i\nu_k + \mu)(z + \mu) + \omega^2)}{((z - i\nu_k + \mu)^2 - \omega^2)\,((z + \mu)^2 + \omega^2)} = \\ = \frac{1}{2\omega} \frac{-i\nu_k\omega + 2m^2}{(\omega - i\nu_k)^2 - \omega^2} \quad (196)$$

Feynman Rules

Loop Calculation

$$\sum_n (\ldots) =$$

$$= -\frac{1}{T\omega}\tanh\left(\frac{-\omega-\mu+i\nu_k}{2T}\right)\frac{i\nu_k\omega+2m^2}{(\omega-i\nu_k)^2-\omega^2} -$$
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Feynman Rules
Loop Calculation

$$\tanh(x + in\pi) = \tanh(x) \tag{198}$$

Feynman Rules
Loop Calculation

$$\tanh(x + in\pi) = \tanh(x) \Longrightarrow$$

$$\Longrightarrow \tanh\left(\frac{\omega \pm \mu \pm i\nu_k}{2T}\right) = \tanh\left(\frac{\omega \pm \mu \pm i\nu_k}{2T}\right) \qquad (198)$$

$$\nu_k = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots$$

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$$\Omega^{(2)} = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3\omega}\left(\tanh\left(\frac{\omega+\mu}{2T}\right) + \right.$$

$$\left. + \tanh\left(\frac{\omega-\mu}{2T}\right)\right)\frac{\omega^2+m^2}{(4\omega^2+\nu_k^2)} \quad (200)$$

Feynman Rules

One-loop fermion diagram

$$\tanh\left(\frac{x}{2T}\right) = \frac{e^{x/2T} - e^{-x/2T}}{e^{x/2T} + e^{-x/2T}} = \\ = 1 - \frac{2}{e^{x/T}+1} = 1 - 2f(x) \quad (201)$$

$$f(x) = \frac{1}{\exp(x/T)+1} \quad (202)$$

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Feynman Rules

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$$\Omega^{(2)} = \int \frac{d^3p}{(2\pi)^3\omega(4\omega^2+\nu_k^2)} \frac{\omega^2+m^2}{}\left(1 - f(\omega-\mu) - f(\omega+\mu)\right) \quad (204)$$