Multiplicity fluctuations in statistical models

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Outline

Statistical Ensembles

Grand Canonical

V, T, μ

$\langle E \rangle, \langle Q \rangle, \langle \vec{P} \rangle = \vec{0}$

Canonical

V, T, Q

$\langle E \rangle, \langle \vec{P} \rangle = \vec{0}$

Microcanonical

V, E, Q

$\vec{P} = \vec{0}$

Thermodynamic Limit : $\langle N \rangle \rightarrow \infty, V \rightarrow \infty, \frac{\langle N \rangle}{V} = const$

Why has there been so little interest so far?

Experiment

Experimentally this means event by event analysis of data

- Very precise determination of collision centrality
- Good understanding of acceptance and resolution

Why has there been so little interest so far?

Thermal Model

- Statistical Ensembles are equivalent under thermodynamic limit
- Grand Canonical Ensemble often sufficient
- Canonical effects only become important when only few particles are produced
- Quantum statistics effects are generally small (10% for pions)

But: all this changes when one is interested in multiplicity fluctuations

Grand Canonical Boltzmann Pion Gas

Single particle partition function
$$z = \frac{gV}{2\pi^2}\int dp\, p^2 e^{-\frac{E}{T}} = \frac{gV}{2\pi^2}\, m^2 T\, K_2\left(\frac{m}{T}\right)$$

System partition function
$$Z^{GC} = \exp\left(z\, e^{\frac{\mu}{T}}\, \lambda_+ + z\, e^{-\frac{\mu}{T}}\, \lambda_-\right)$$

Expectation values
$$\langle N_+\rangle = \frac{1}{Z^{GC}}\left(\lambda_+ \frac{\partial}{\partial \lambda_+}\right) Z^{GC}\Big|_{\lambda_\pm = 1} = z\, e^{\frac{\mu}{T}}$$
$$\langle N_+^2\rangle = \frac{1}{Z^{GC}}\left(\lambda_+ \frac{\partial}{\partial \lambda_+}\right)^2 Z^{GC}\Big|_{\lambda_\pm = 1} = \left(z\, e^{\frac{\mu}{T}}\right)^2 + z\, e^{\frac{\mu}{T}}$$

Scaled variance
$$\omega = \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle} = 1$$ Poisson !

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Canonical Boltzmann Pion Gas

System partition function $$Z^Q = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-iQ\phi} \exp\left(z\, e^{i\phi}\lambda_+ + z\, e^{-i\phi}\lambda_-\right)\Big|_{\lambda_\pm=1} = I_Q(2z)$$

Expectation values
$$\langle N_+\rangle = \frac{1}{Z^Q}\left(\lambda_+ \frac{\partial}{\partial\lambda_+}\right) Z^Q\Big|_{\lambda_\pm=1} = \frac{I_{Q-1}(2z)}{I_Q(2z)}\, z$$

$$\langle N_+^2\rangle = \frac{1}{Z^Q}\left(\lambda_+ \frac{\partial}{\partial\lambda_+}\right)^2 Z^Q\Big|_{\lambda_\pm=1} = \frac{I_{Q-2}(2z)}{I_Q(2z)}\, z^2 + \frac{I_{Q-1}(2z)}{I_Q(2z)}\, z$$

Scaled variance $$\omega_+ = \frac{\langle N_+^2\rangle - \langle N_+\rangle^2}{\langle N_+\rangle} \underset{V\to\infty}{\to} \frac{1}{2}$$ Not 1!

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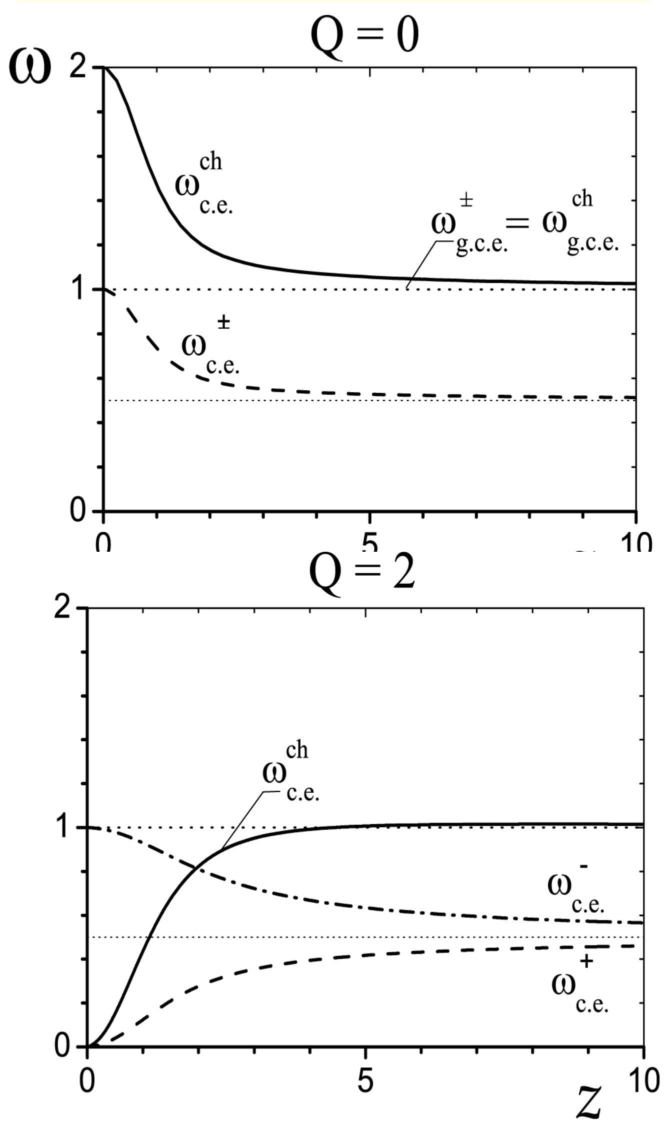
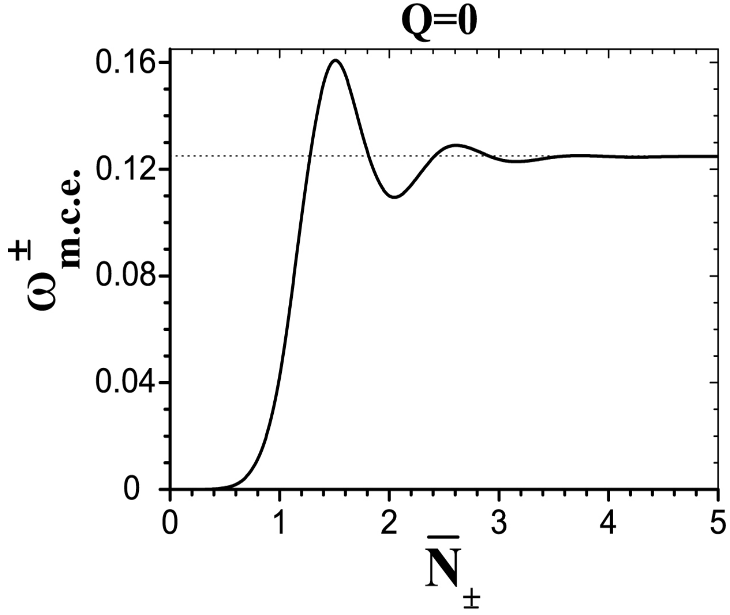
Finite Volume Pion gas

(Boltzmann)

Canonical Ensemble



Microcanonical Ensemble
massless gas



Very fast convergence to asymptotic values for a neutral sytem !

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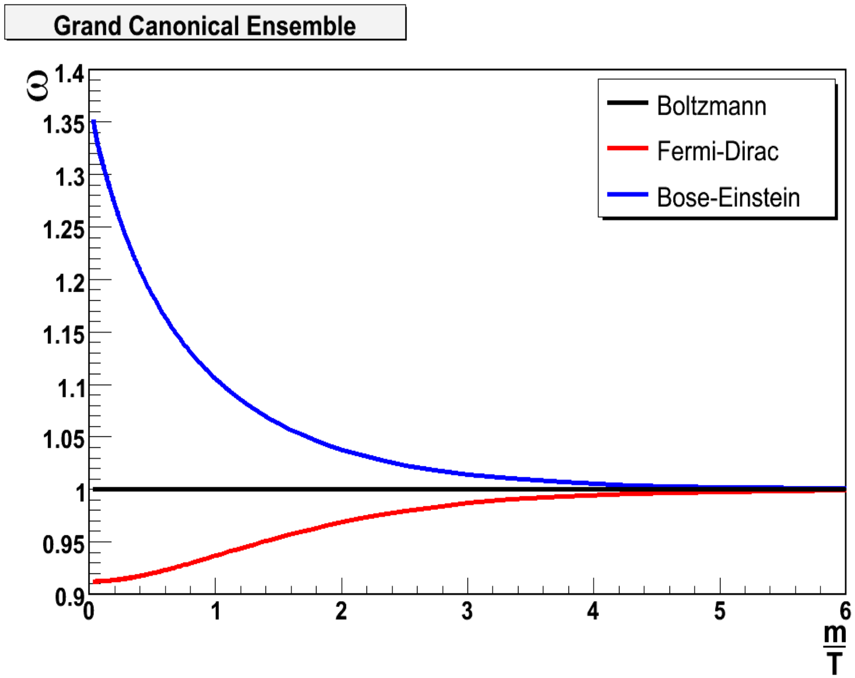
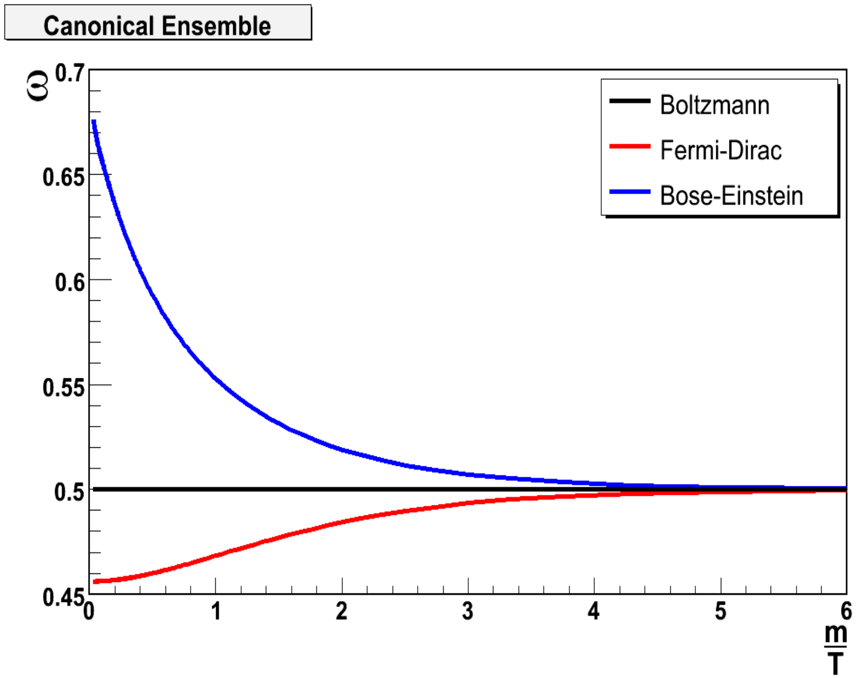
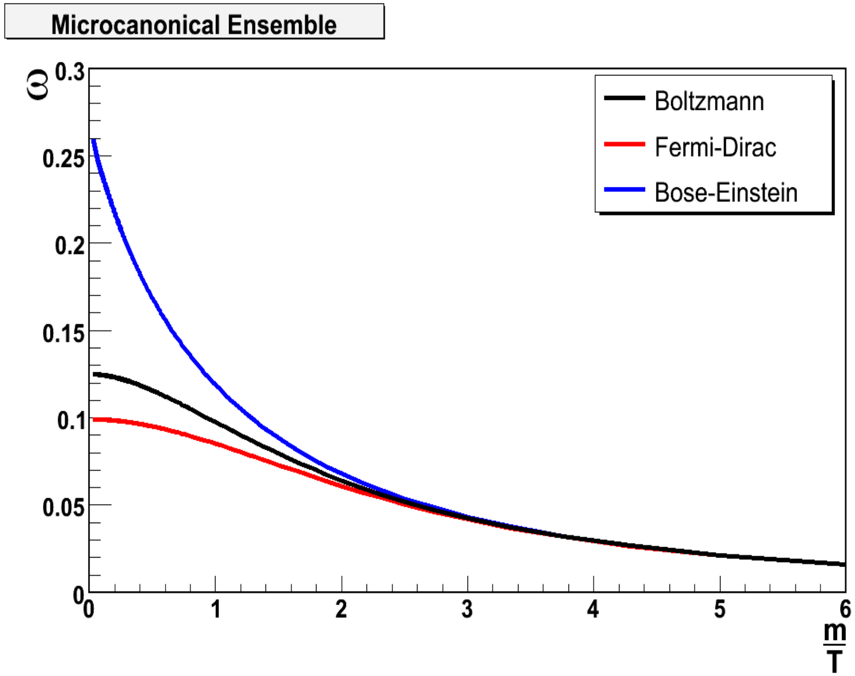
Thermodynamic Limit Pion Gas







π^+ π^- Gas

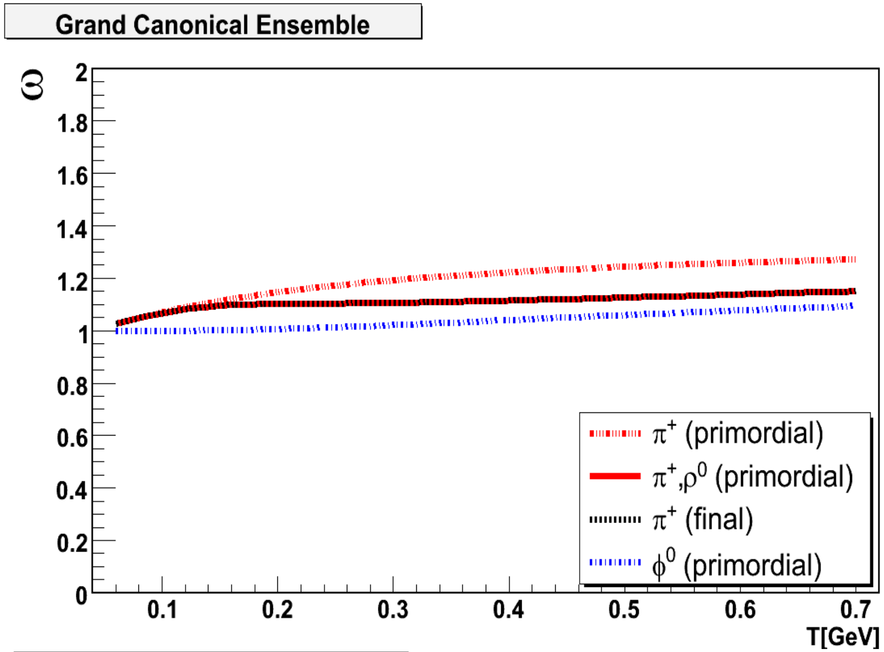
$Q=0 \rightarrow \mu=0$ ⇒ $\omega^+=\omega^-$

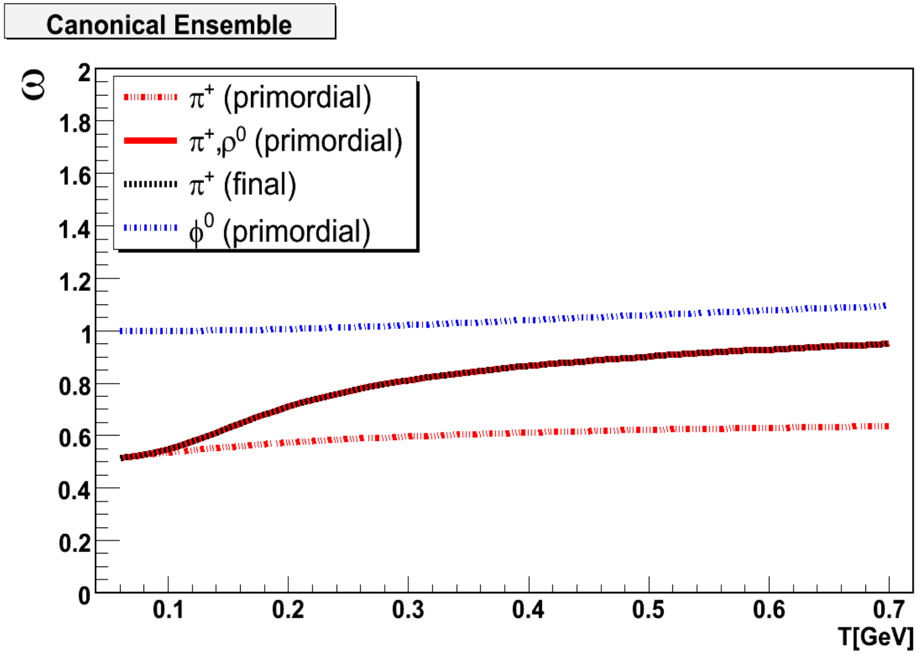
Quantum effects can be quite large, even in a neutral system!

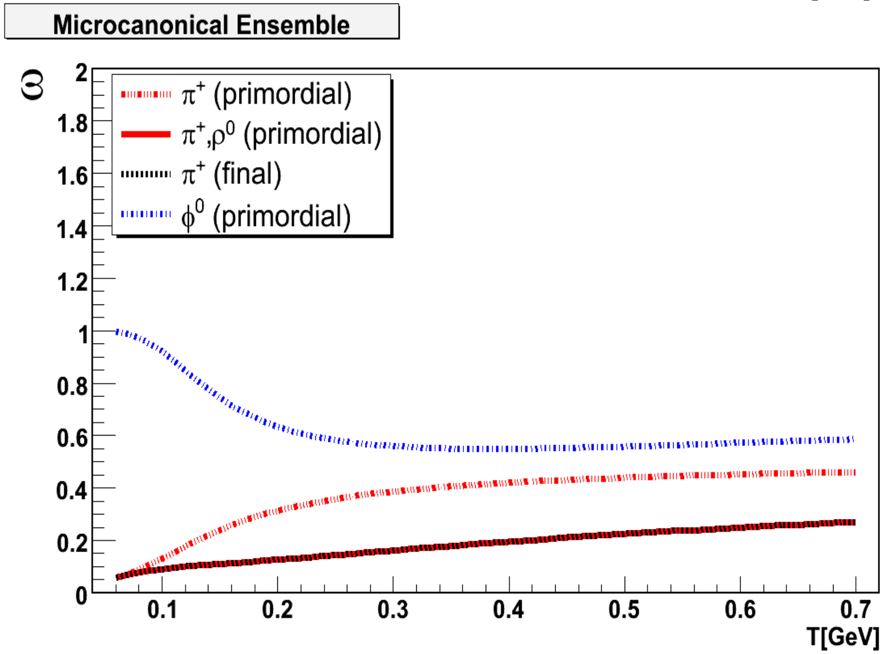
nucl-th/0505069

Pion and Rho0 Gas

Thermodynamic Limit







$\pi^+ \quad \pi^- \quad \rho^0$ Gas

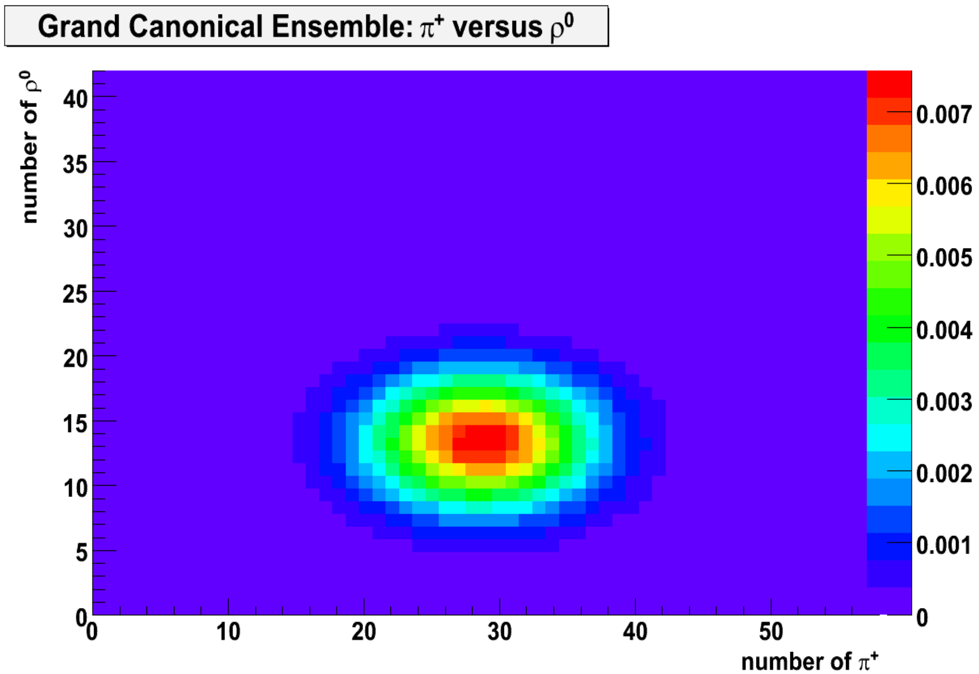
$\rho^0 \rightarrow \pi^+ + \pi^-$

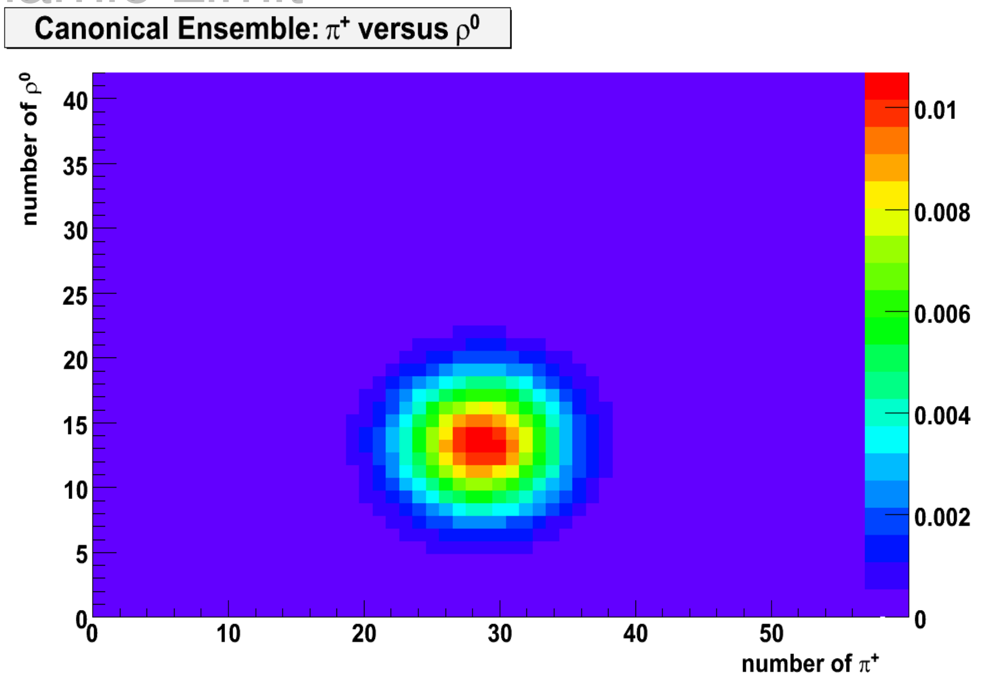
$Q=0 \rightarrow \mu=0$ ⇒ $\omega^+ = \omega^-$

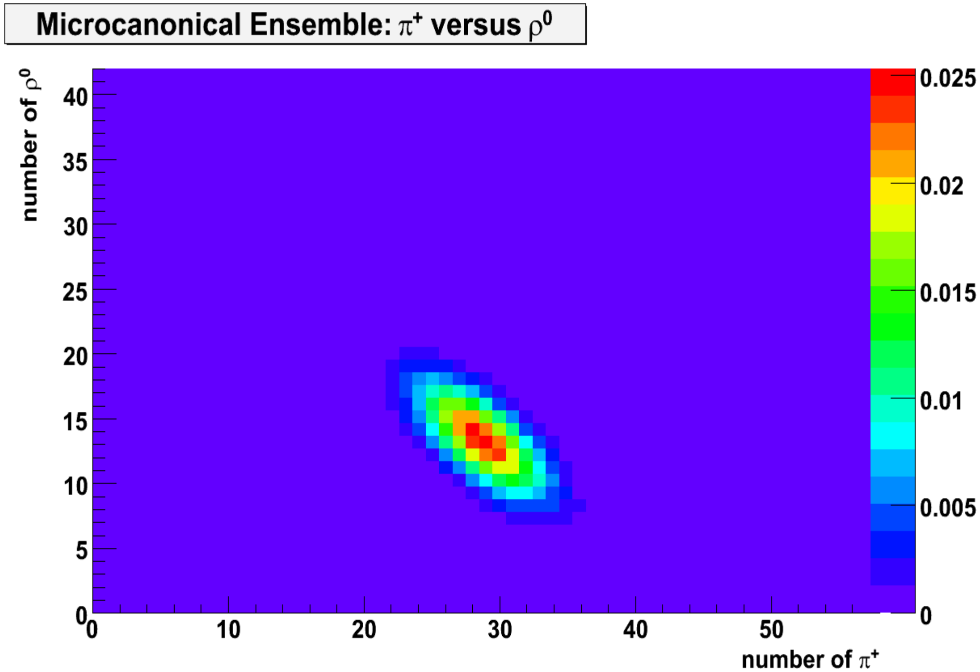
Particle decay can have almost surprising effects !

Pion and Rho0 Gas

Thermodynamic Limit





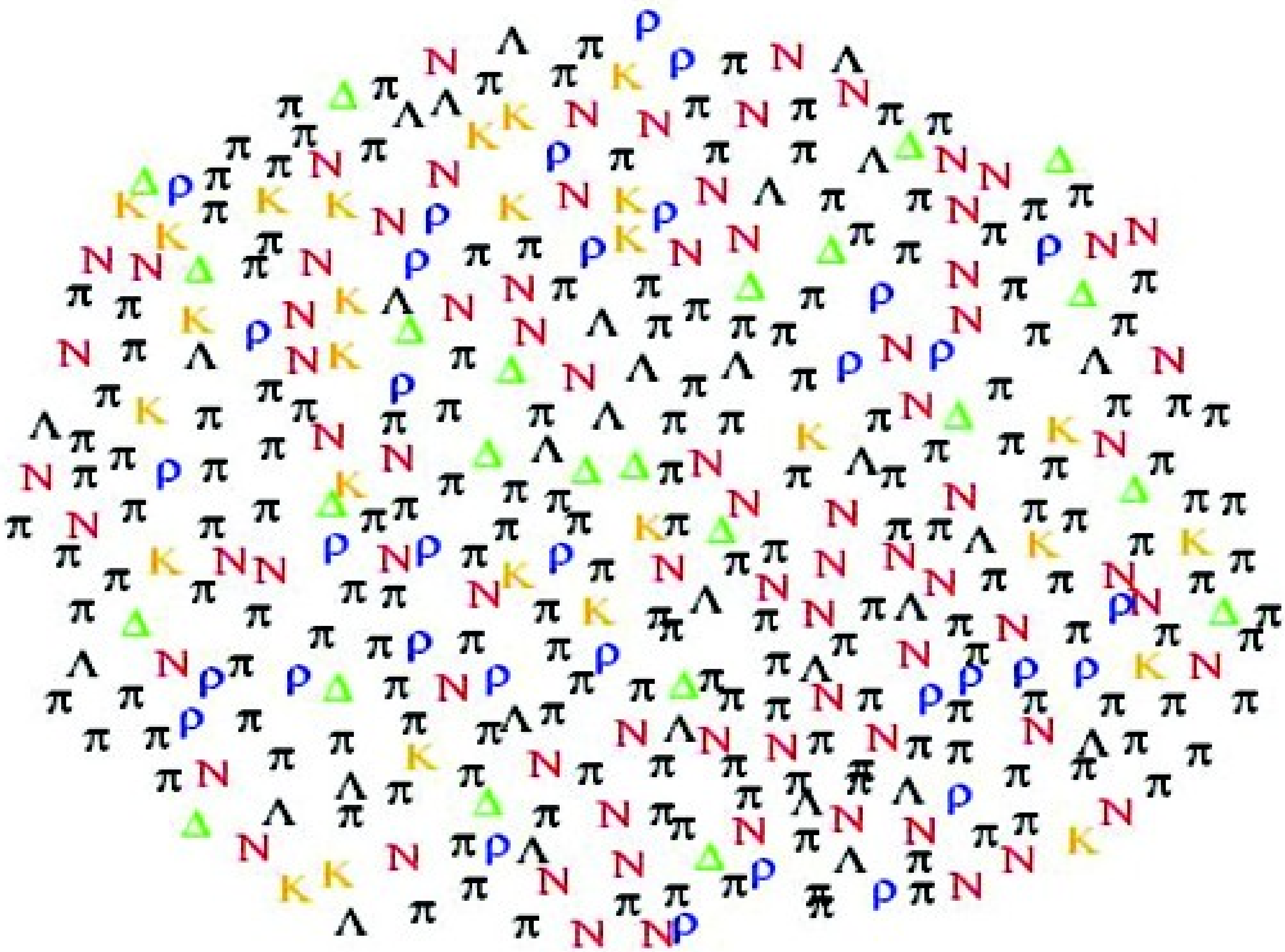


$$T = 160 \ MeV$$

$$r = 2.0 \ fm$$

Use T-limit approximation

Hadron Resonance Gas



Hadron Resonance Gas

Canonical Partition Function

$$Z^{Q^j} = \left[\prod_{j=1}^{3} \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} e^{-iQ^j \phi_j} \right] Z^{GC}(\{\lambda_l\})$$

$$\langle N_k^n \rangle = \frac{1}{Z^{Q^j}} \left(\lambda_k \frac{\partial}{\partial \lambda_k} \right)^n Z^{Q^j}$$

$$Z^{GC}(\{\lambda_l\}) = \exp\left[\sum_l z_l(\lambda_l) \right]$$

$$\lambda_l = \exp\left(i \sum_{j=1}^{3} q_l^j \phi_j \right)$$

$$z_l(\lambda_l) = \frac{g_l V}{2\pi^2} \int_0^\infty p^2 dp \ln\left(1 \pm \exp\left[-\frac{E_l}{T} \right] \lambda_l \right)^{\pm 1}$$

$$q_l^j = (b_l, s_l, q_l)$$

- No practical analytical solution is known
- Only in Boltzmann approximation is an analytical reduction of integrals possible
- heavily osscilating integrant makes numerical evaluation expensive

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Central Limit Theorem Expansion

$$Z^{Q^j} \approx \prod_{j=1}^{3} \left[\int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} e^{-i Q^j \phi_j} \right] \exp\left[V \sum_{n=0}^{\infty} \frac{\kappa_n^{j_1 \ldots j_n}}{n!} \phi_{j_1} \ldots \phi_{j_n} \right]$$

Cumulant tensor $\kappa_n^{j_1 \ldots j_n} = (-i)^n \frac{\partial^n \Psi(\vec{\phi})}{\partial \phi_{j_1} \ldots \partial \phi_{j_n}}$

$\Psi(\vec{\phi}) = \sum_l \frac{z_l(\vec{\phi})}{V}$ Generating function

3-dim Gaussian with first correction term $O(V^{-1/2})$

$$Z^{Q^j} \approx \frac{Z^{GCE}}{(2\pi V)^{2/3} \det\sigma} \exp\left[\frac{-\xi^j \xi_j}{2} \right] \left(1 + O\left(V^{-1/2}\right)\right)$$

where $\sigma \equiv \kappa_2^{1/2}$,and $\xi^j = \left(Q^k - V\kappa_1^k\right) \left(\sigma^{-1}\right)_k^{\ j} V^{-1/2}$

Find physical fugacities from $\frac{\partial Z^{\vec{Q}}}{\partial \vec{Q}} = \vec{0}$

$$\left(\kappa_1^B, \kappa_1^S, \kappa_1^Q\right) \underset{V \to \infty}{\rightarrow} \left(\rho_B, \rho_S, \rho_Q\right)$$

Central Limit Theorem Expansion

Canonical Normalization $P\left(V\kappa_1^j\right) = \frac{Z^{V\kappa_1^j}}{Z^{GC}} \approx \frac{1}{\left(2\pi V\right)^{3/2}\ det\,\sigma}$

4-dim Gaussian $P\left(Q^j, N_k\right) = \frac{Z^{Q^j, N_k}}{Z^{GC}} \approx \frac{1}{\left(2\pi V\right)^{4/2}\ det\,\tilde{\sigma}} \exp\left[-\frac{\tilde{\xi}^j\ \tilde{\xi}_j}{2}\right]$

1-dim Gaussian $P_{V\kappa_1^j}\left(N_k\right) = \frac{Z^{V\kappa_1^j, N_k}}{Z^{V\kappa_1^j}} \approx \frac{det\,\sigma}{\left(2\pi V\right)^{1/2}\ det\,\tilde{\sigma}} \exp\left[-\frac{\tilde{\xi}^j\ \tilde{\xi}_j}{2}\right]$

With variance $D^2 = \frac{V\,det\,\tilde{\sigma}^{\,2}}{det\,\sigma^2}$

And scaled variance $\omega_k = \frac{det\,\sigma_k^2}{det\,\sigma^2\,\rho_k} = \frac{det\,\tilde{\kappa}_2}{det\,\kappa_2\ \ \kappa_1^k}$

Finite volume corrections can be done by Gram Charlier Expansion

Particle Decay

Central Limit Theorem Expansion

$$z_l = \frac{g_l}{2\pi^2} \int p^2 dp \ln\left(1 \pm e^{-\beta(E_l-\mu_l)} e^{i q_l^j \Phi_j} \left[\sum_{n_k=0}^{C_{l,k}} \Gamma_{l,q}^{n_k} e^{i n_k \Phi_k} \right]\right)^{\pm 1}$$

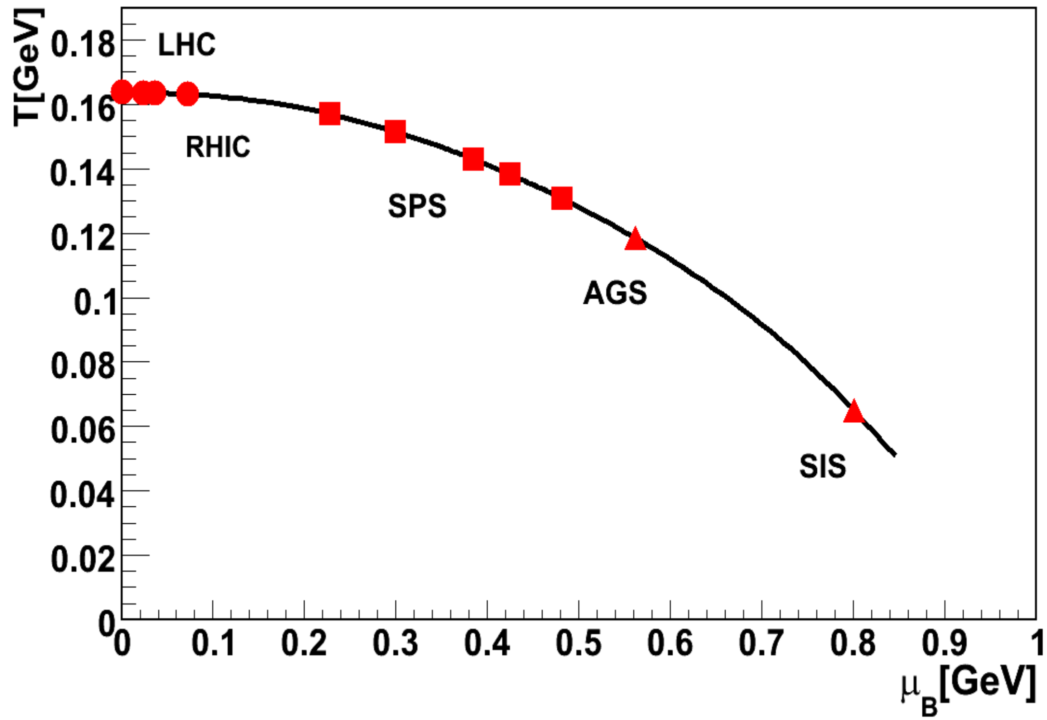
$$\Gamma_{l,q}^{n_k} = \sum_{c_k=n_k}^{C_{l_k}} q^{n_k} (1-q)^{c_k-n_k} \binom{c_k}{n_k} \Gamma_l^{c_k}$$

$$\sum_{n_k=0}^{C_{j_k}} \Gamma_{l,q}^{n_k} = 1$$

q	Detection Probability
c_k	multiplicity of particle k in this channel
$\Gamma_l^{c_k}$	branching ratio of resonance l into c_k particles k
n_k	number of detected particles
$\Gamma_{j,q}^{n_k}$	corrected branching ratio

Chemical Freeze-Out Line

Constant average energy per partilce



Phys.Rev.Lett. 81 (1998) 5284-5286

Phys.Rev. C73 (2006) 044905

Nucl.Phys. A 697 (2002) 902-912

$$\frac{\langle E \rangle}{\langle N \rangle} \sim 1\text{GeV} \qquad Pb-Pb$$

$$\rho_B \Rightarrow \mu_B$$

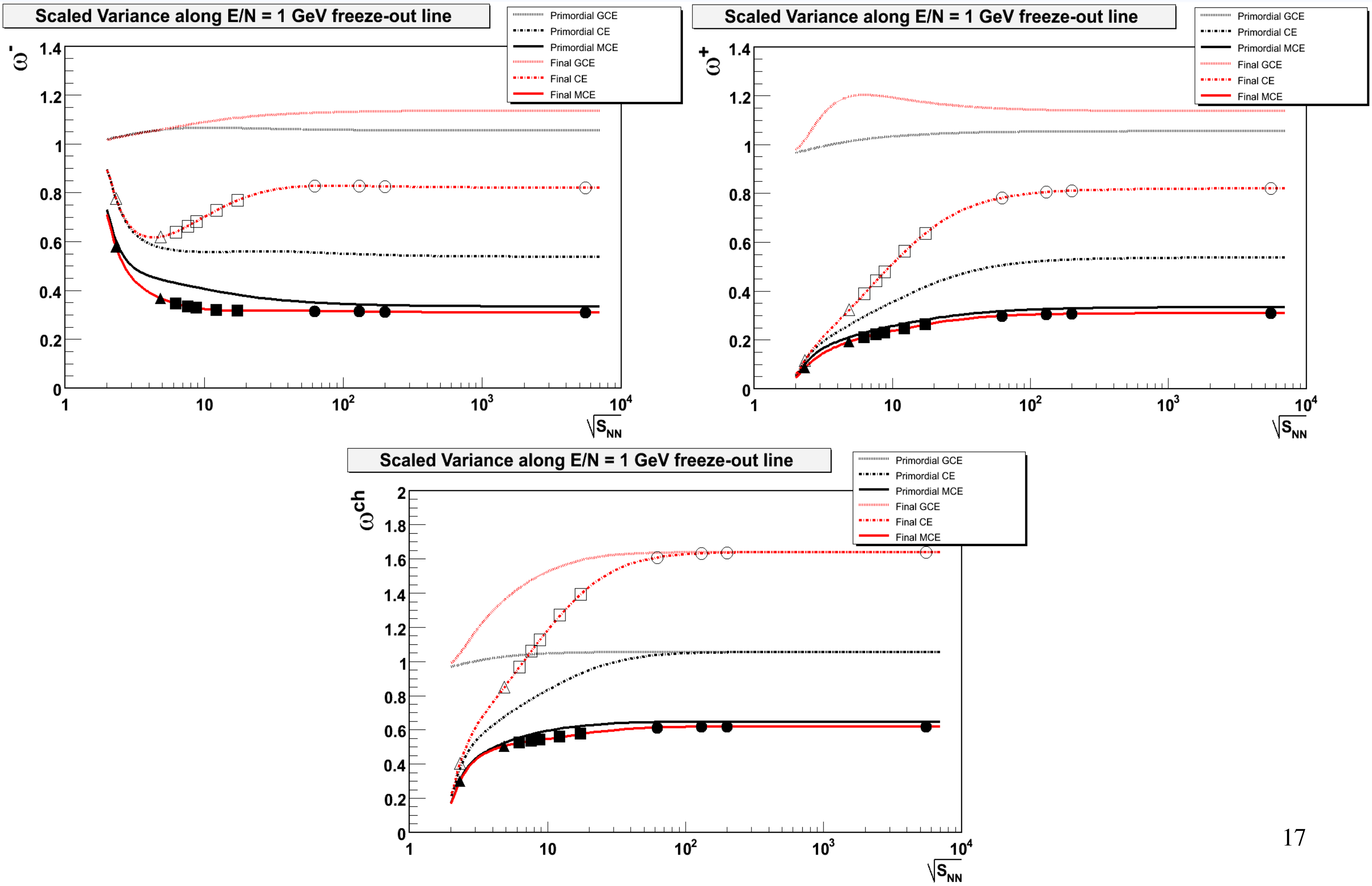
$$\rho_S \Rightarrow \mu_S \qquad \rho_S = 0$$

$$\rho_Q \Rightarrow \mu_Q \qquad \rho_Q = 0.4\,\rho_B$$

$$\gamma_S \simeq 1 - 0.396 \ \exp\left(-1.23 \ \frac{T}{\mu_B}\right)$$

$$\mu_B\left(\sqrt{S_{NN}}\right) \simeq \frac{1.27\,GeV}{1 + \sqrt{S_{NN}}/4.3\text{GeV}}$$

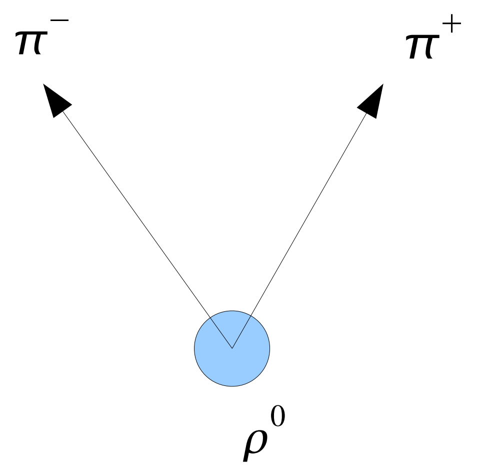
Scaled Variance



Experimental Acceptance

$$\omega^{acc} = 1 - p + p\,\omega^{4\pi}$$

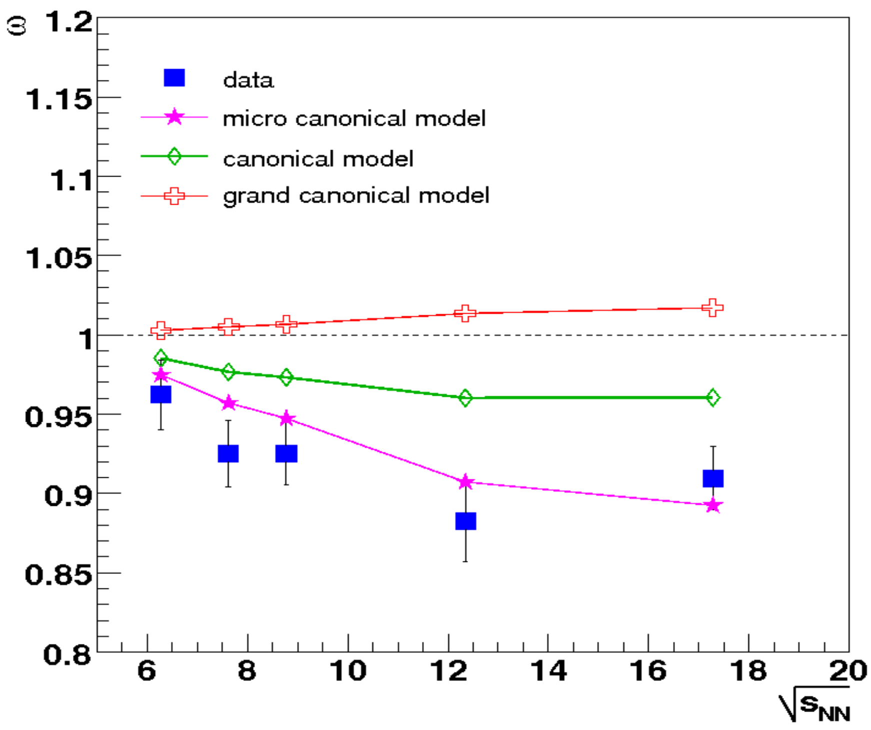
Only valid if a random sample of final particles is detected



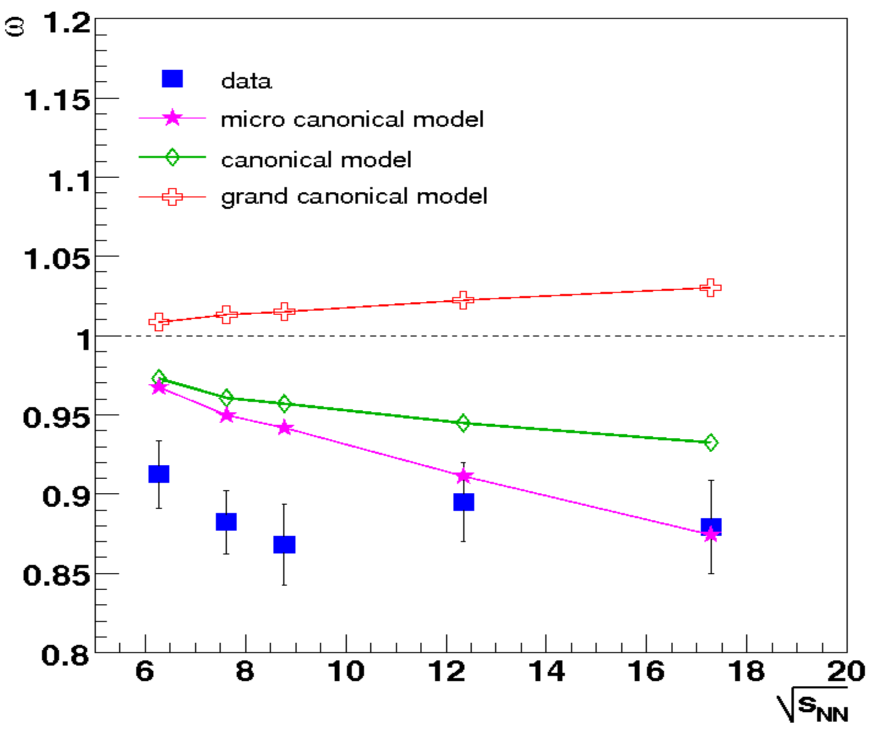
- But decay products are correlated in momentum space.
- Due to an insufficient amount of rescattering after chemical freeze-out decay products do not re-thermalize.
- Simple acceptance scaling could work best for negatively charged particles as decay channels seldom contain two negative particles.
- „Detection“ has a major and non-trivial effect.

Comparison with NA49-Data

negatively charged hadrons



positively charged hadrons



Summary

We have a good understanding of statistical fluctuations for finite system sizes as well as in the thermodynamic limit.

Monte Carlo techniques allow for simulation of experimental acceptance

We cannot yet include:

- Dynamical effects
- Phase Transition

We have not yet included:

- Fluctuation of thermal parameters

Finite Volume Hadron Gas

$T \approx 160\,MeV$
$\gamma_S = 1.0$

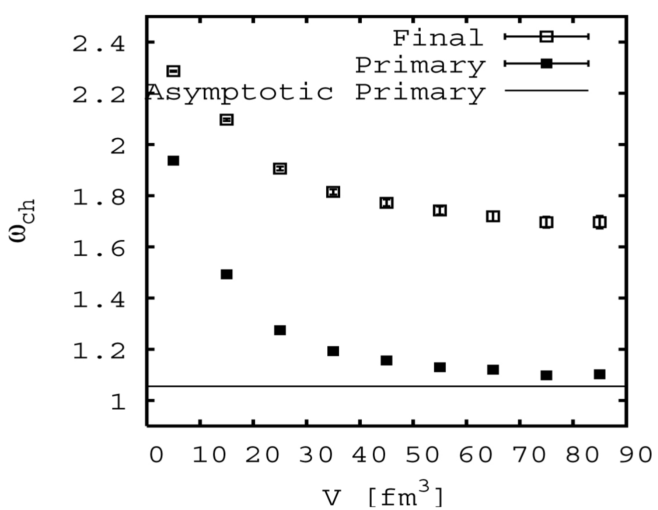
$V = 90\text{fm}^3 \Rightarrow r \approx 2.8\text{fm}$

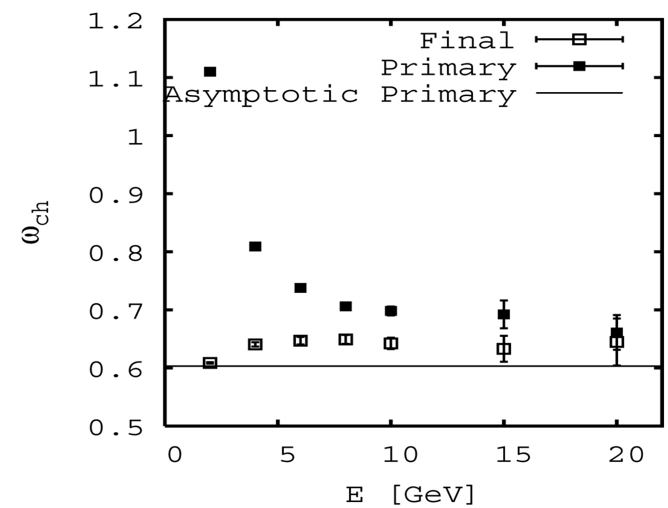
$E = 20\text{GeV}\,(\rho_E = 0.4\text{GeV}) \Rightarrow r \approx 2.3\ fm$

CE

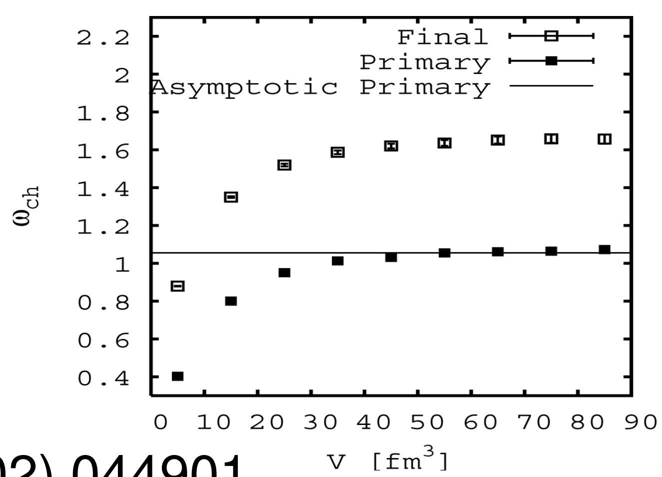
MCE

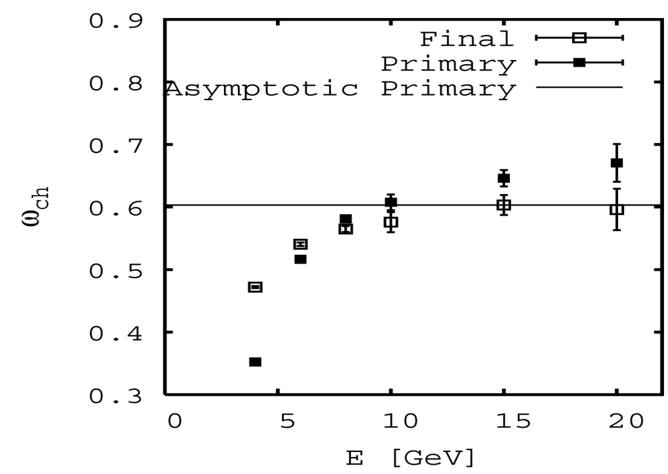
$\{B, S, Q\} = \{0,0,0\}$





$\{B, S, Q\} = \{2,0,2\}$





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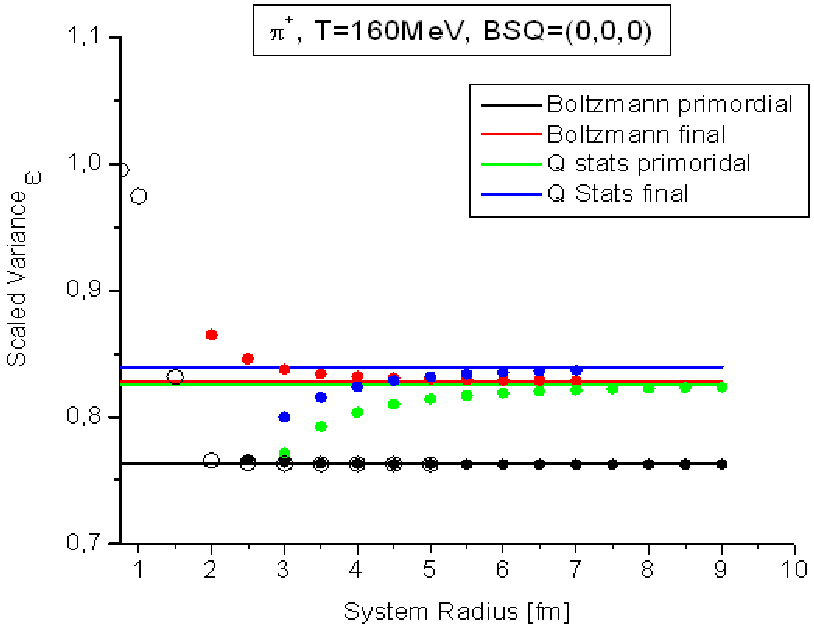
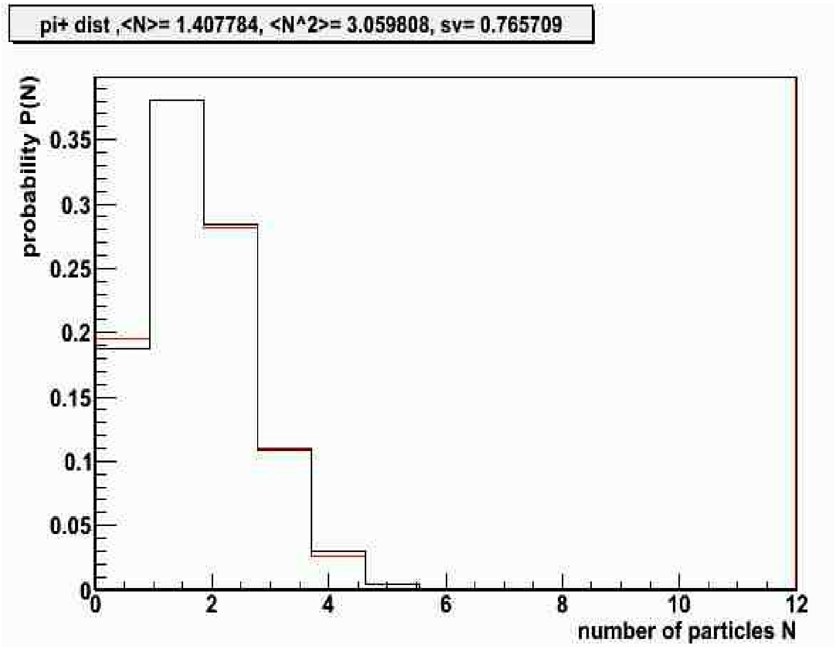
Finite Volume Corrections

Gram Charlier Expansion





$$P\left(N_j\right)=\frac{z_j^{N_j}}{N_j!}\frac{Z_{j\,excl}^{\vec{Q}-N_j\vec{Q}_j}}{Z^{\vec{Q}}}$$

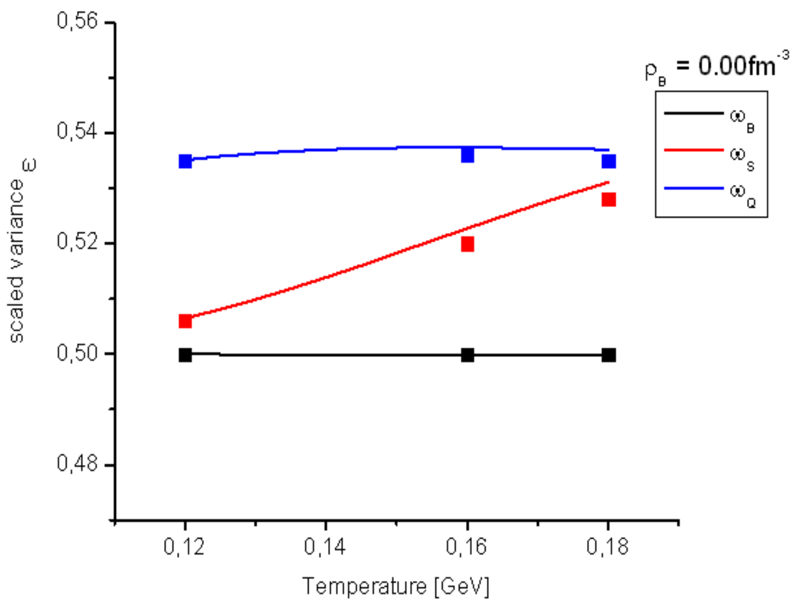
Primordial Boltzmann

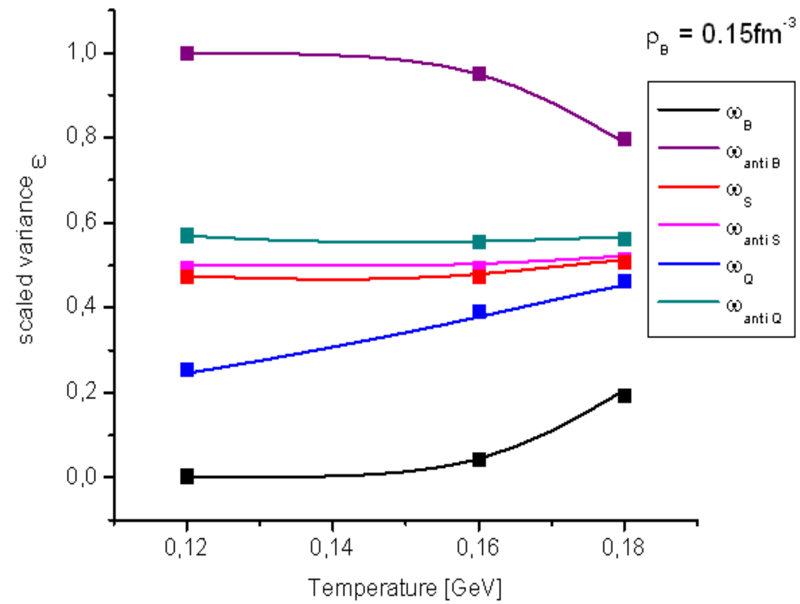
$T=0.160\text{GeV}$ $\quad\gamma_S=1.0$

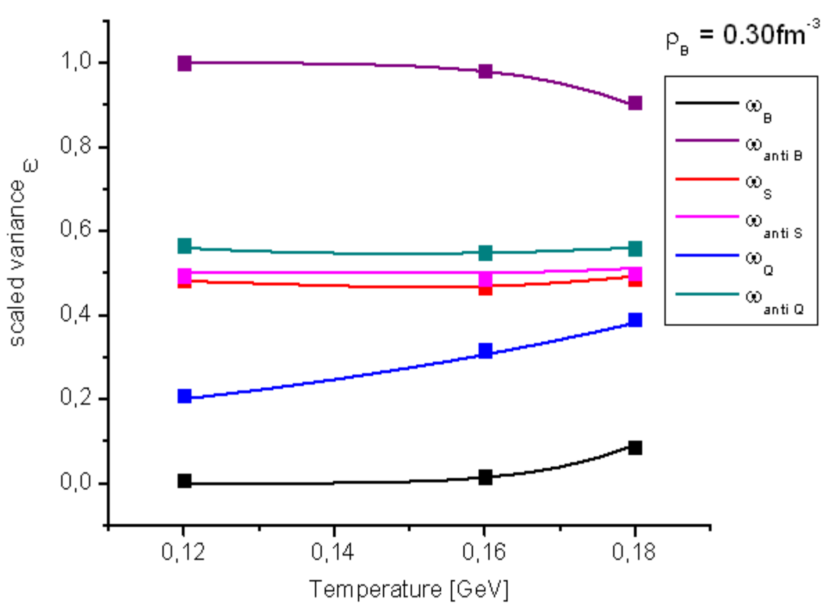
$\left(B,S,Q\right)=\left(0,0,0\right)$ $\quad r=2.0\ fm$

π^+	*approx*	*exact*
Norm	0.9954	1.0
$\langle N\rangle$	1.420	1.408
ω	0.7667	0.7657

Comparison of Methods







Comparison is half hearted, since different particle tables were used!

Microscopic correlator method and Central limit theorem expansion agree „on the dot“. (same table!)

Techniques

Microscopic correlaor (GCE and CE)

$$\langle n_{p,k}\rangle = \frac{1}{\exp\left[\beta\left(E_k-\mu_k\right)\right]-\gamma_k}$$

$$(\Delta p)^3 \frac{gV}{(2\pi)^3} \gg 1$$

$$v_{p,k}^2 = \langle \Delta n_{p,k}^2\rangle = \langle\left(n_{p,k}-\langle n_{p,k}\rangle\right)\rangle = \langle n_{p,k}\rangle\left(1+\gamma_k\langle n_{p,k}\rangle\right)$$

γ_k { +1 Bosons, -1 Fermions, 0 Boltzmann }

$$\langle \Delta n_{p,k}\Delta n_{q,l}\rangle = \underbrace{v_{p,k}^2\delta_{p,q}\delta_{k,l}}_{GCE\ correlator} - v_{p,k}^2 v_{q,l}^2 \frac{q_k q_l}{\sum_{p,k} v_{p,k}^2 q_k^2}$$

$$\mu_k = q_k\mu \qquad E_k=\sqrt{p^2+m_k^2}$$

In a canonical ensemble the variation needs to vanish $\Delta Q=\sum_{p,k} q_k \Delta n_{p,k}=0$

$$\langle(\Delta N_k)^2\rangle = \sum_{p,q,l}\langle \Delta n_{p,k}\Delta n_{q,l}\rangle$$

$$\langle N_k\rangle = \sum_p \langle n_{p,k}\rangle$$

→

$$\omega_k \equiv \frac{\langle(\Delta N_k)^2\rangle}{\langle N_k\rangle}$$

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Particle Decay

Microscopic Correlator

$$G \equiv \prod_R \left(\sum_r b_r^R \prod_i \lambda_i^{n_{i,r}} \right)^{N_R}$$

$$\langle N_i^k \rangle_R = \left[\lambda_i \frac{\partial}{\partial \lambda_i} \right]^k G$$

b_r^R	Branching ration of channel r of resonance R
$n_{i,r}$	multiplicity if species i in channel r
$\langle n_i \rangle_R$	average multplicity of i from decay of R
$\langle N_R \rangle$	average primordial density of R

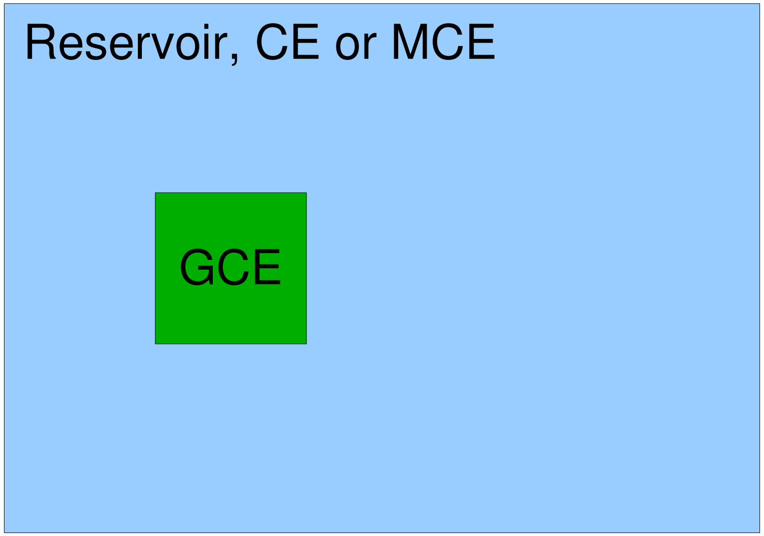
$$\langle \Delta N_i \Delta N_j \rangle = \langle \Delta N_i^* \Delta N_j^* \rangle + \sum_R \left[\langle \Delta N_R^2 \rangle \langle n_i \rangle_R \langle n_j \rangle_R + \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R \right]$$

GCE

$$\langle \Delta N_i \Delta N_j \rangle = \langle \Delta N_i^* \Delta N_j^* \rangle + \sum_R \left[\langle \Delta N_R^2 \rangle \langle n_i \rangle_R \langle n_j \rangle_R + \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R \right]$$
$$+ \sum_R \langle \Delta N_i^* \Delta N_R \rangle \langle n_j \rangle_R + \sum_R \langle \Delta N_j^* \Delta N_R \rangle \langle n_i \rangle_R + \sum_{R \neq R'} \langle \Delta N_R \Delta N_{R'} \rangle \langle n_i \rangle_R \langle n_j \rangle_{R'}$$

CE

Pseudo intensive quantities



GCE is defined as a small subsystem of a large reservoir

$$D^2 = \langle N^2 \rangle - \langle N \rangle^2 \neq \sum_{k=1}^{N} \langle N_k^2 \rangle - \langle N_k \rangle^2$$

Variance is an extensive quantity, but not additive

$$D^2 = \langle (\Delta N)^2 \rangle = \sum_{k=1}^{M} \sum_{l=1}^{M} \langle \Delta N_k \Delta N_l \rangle$$

Correlators vanish only in GC

$$\omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle}$$

Hence scaled variance is not intensive, since different in different ensembles, but pseudo intensive

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