Color Superconductivity in Quark Matter

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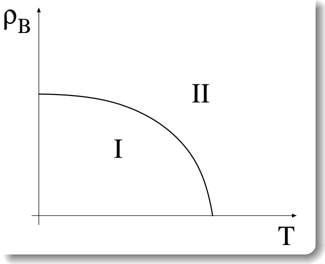
"Dense Matter In Heavy Ion Collisions and Astrophysics",
JINR Dubna (Russia), August 21 – September 1, 2006.

Overview: The QCD phase diagram

- early conjecture:



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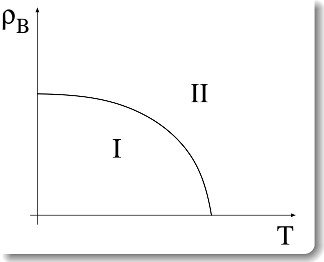

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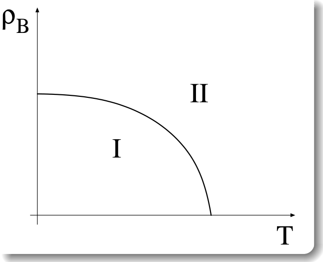

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"The true phase diagram may actually be substantially more complex . . . "

- Collins & Perry, PRL (1975)

  "Also we might expect superfluidity or superconductivity."

Color superconducting phases

- early work: Barrois (1977); Frautschi (1978); Bailin & Love (1984)
- “rediscovery”: Alford, Rajagopal, Wilczek, PLB (1998); Rapp, Schäfer, Shuryak, Velkovsky, PRL (1998).

$$\Delta \sim 100 \text{ MeV} \quad \rightarrow \quad T_c \sim 50 \text{ MeV}$$

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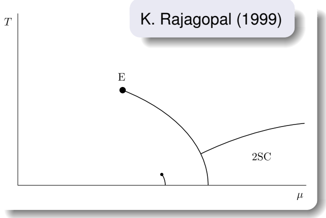

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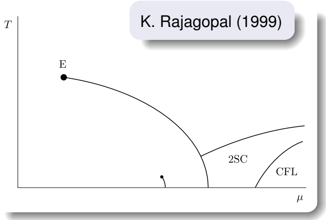

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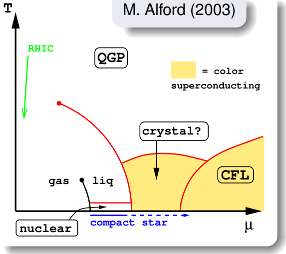

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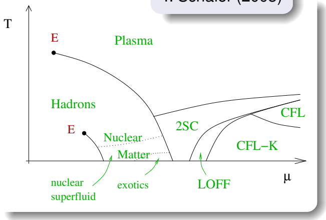

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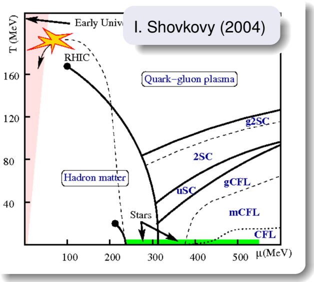

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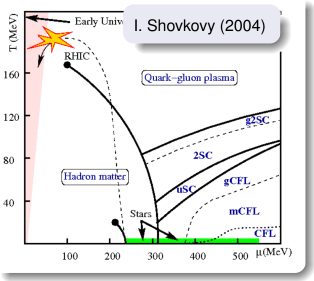

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Outline

1. overview: phase diagrams ✓
2. diquark condensates
3. 2-flavor color superconductivity
4. gap equations
5. 3-flavor color superconductivity
6. realistic masses
7. neutral matter
8. unconventional pairing (spin 1, gapless, LOFF, gluonic phase)
9. discussion

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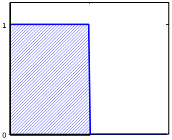
Cooper instabilities

- ideal Fermi gas:
  - ➜ pair creation @ Fermi surface with no free energy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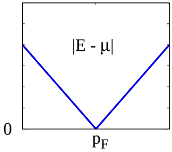
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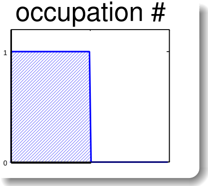
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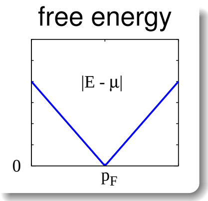
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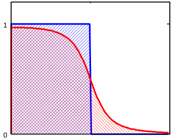

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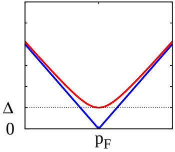

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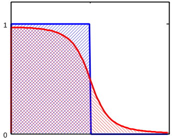

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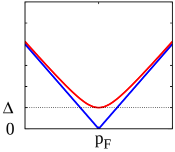

- QCD: attractive qq interaction ➜ diquark condensates

Field operators

- quark field operator: $q(x) = \begin{pmatrix} q_1(x) \\ \vdots \\ q_{4N_fN_c}(x) \end{pmatrix}$
    - 4 Dirac × N_f flavor × N_c color components
    - annihilates a quark or creates an antiquark

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- adjoint operator: $q^\dagger$, $\bar{q} = q^\dagger \gamma^0$
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- examples:
  - "chiral condensate": $\langle \bar{q} q \rangle$
  - quark number density: $\langle \bar{q} \gamma^0 q \rangle = \langle q^\dagger q \rangle$
  - electric charge density:

$$\langle \bar{q}\, \hat{Q} \gamma^0\, q \rangle = \tfrac{2}{3} n_u - \tfrac{1}{3} n_d - \tfrac{1}{3} n_s, \qquad \hat{Q} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}_f$$

  - color charge densities

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- Bogoliubov rotation:

$$|g.s.\rangle \;=\; \prod_{\vec{p},s,c,c'} \left[\cos\theta_s^b(\vec{p}) + \varepsilon_{3cc'}\, e^{i\xi_s^b(\vec{p})}\,\sin\theta_s^b(\vec{p})\, b^\dagger(\vec{p},s,u,c)\, b^\dagger(-\vec{p},s,d,c')\right]$$

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Diquark condensates

- diquark condensates: $\langle q^T \, \hat{\mathcal{O}} \, q \rangle$
- Pauli principle: $q_i q_j = -q_j q_i$

$$\Rightarrow \; q^T \, \hat{\mathcal{O}} \, q \; = \; q_i \, \hat{\mathcal{O}}_{ij} \, q_j \; = \; -q_j \, \hat{\mathcal{O}}_{ij} \, q_i \; = \; -q_j \, \hat{\mathcal{O}}^T_{ji} \, q_i \; = \; -q^T \, \hat{\mathcal{O}}^T \, q$$

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➜ $\hat{\mathcal{O}}$ must be totally antisymmetric: $\hat{\mathcal{O}}^T = -\hat{\mathcal{O}}$

Operators in flavor and color space

- Pauli matrices (for two flavors):

$$\underbrace{\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},}_{\text{symmetric triplet}} \quad \underbrace{\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\text{antisymm. singlet}}$$

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- Gell-Mann matrices (for three flavors or colors):

$$\underbrace{\mathbb{1}, \quad \lambda_1, \quad \lambda_3, \quad \lambda_4, \quad \lambda_6, \quad \lambda_8,}_{\text{symmetric sextet}} \qquad \underbrace{\lambda_2, \quad \lambda_5, \quad \lambda_7}_{\text{antisymmetric antitriplet}}$$

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  - antitriplet: The vector $\langle q^T \begin{pmatrix} \lambda_7 \\ -\lambda_5 \\ \lambda_2 \end{pmatrix} q \rangle$ transforms like an antiquark $\bar{q} = \begin{pmatrix} \bar{r} \\ \bar{g} \\ \bar{b} \end{pmatrix}$ under $SU(3)_c$.

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- antisymmetric:
  - $C\gamma_5$ (scalar)
  - C (pseudoscalar)
  - $C\gamma^\mu\gamma_5$ (vector)

- symmetric:
  - $C\gamma^\mu$ (axial vector)
  - $C\sigma^{\mu\nu}$ (tensor)

Combined operators

	symmetric	antisymmetric
Dirac	$C\gamma^\mu$, $C\sigma^{\mu\nu}$ A T	C, $C\gamma_5$, $C\gamma_5\gamma^\mu$ P S V
$U(2)$	$\underbrace{\mathbb{1}, \tau_1, \tau_3}_{3}$	$\underbrace{\tau_2}_{1}$
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$$\langle q^T C\gamma_5 \tau_2 \lambda_2 q\rangle \sim \underbrace{(\uparrow\downarrow - \downarrow\uparrow)}_{\text{spin}} \otimes \underbrace{(ud - du)}_{\text{flavor}} \otimes \underbrace{(r\,g - g\,r)}_{\text{color}}$$

Symmetry properties: color

- only red and green quarks are paired:

$$\langle q^T \lambda_2 q\rangle \sim \langle rg - gr\rangle \hat{=} \langle \bar{b}\rangle$$ “antiblue”

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➜ equivalent to the "simple" ansatz

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but: There is a conserved *"modified baryon number"*:

$$\tilde{B} \quad q \;\rightarrow\; e^{i\alpha(1-\sqrt{3}\lambda_8)} q \quad \Rightarrow \quad \delta \;\rightarrow\; \delta$$

Symmetry properties: global symmetries

- $\delta \;\equiv\; \langle q^T\, C\gamma_5\, \tau_2\, \lambda_2\, q\rangle$
- baryon number:

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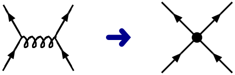
- chiral symmetry:
  - $SU(2)_V$: $\quad q \;\rightarrow\; e^{i\theta_a \frac{\tau_a}{2}} q \quad\Rightarrow\quad \delta \;\rightarrow\; \delta$
  - $SU(2)_A$: $\quad q \;\rightarrow\; e^{i\theta_a \frac{\tau_a}{2}\gamma_5} q \quad\Rightarrow\quad \delta \;\rightarrow\; \delta$ $\quad$ conserved

Outline

Model interaction

- NJL-type models
    - replace gluon exchange by point interactions:



$$\mathcal{L}_{int}(x) \overset{e.g.}{=} -g\,(\bar{q}(x)\,\gamma^\mu\lambda_a\,q(x))^2$$

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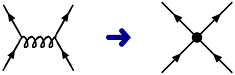

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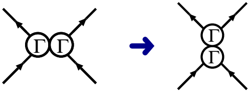
- Fierz transformation
  - identically rewrite particle-antiparticle interactions as particle particle interactions:



$$(\bar{q}\,\Gamma^{(I)}\,q)^2 \;=\; \sum_D d^I_D\,(\bar{q}\Gamma^{(D)}C\bar{q}^T)(q^T C\Gamma^{(D)}q)$$

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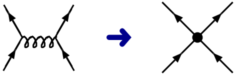

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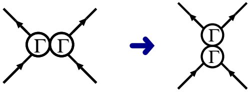

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- toy model:

$$\mathcal{L}_{int} = H\sum_{A=2,5,7}(\bar{q}\,i\gamma_5\tau_2\lambda_A\,C\bar{q}^T)(q^T C\,i\gamma_5\tau_2\lambda_A\,q)$$

Nambu-Gorkov formalism

- interaction Lagrangian:

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- artificially double # d.o.f. *(Nambu-Gorkov spinors)*:

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} q \\ C\bar{q}^T \end{pmatrix}, \qquad \bar{\Psi} = \frac{1}{\sqrt{2}} \left(\bar{q}, q^T C\right)$$

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➜ $$\mathcal{L}_{int} = 4H \sum_{A=2,5,7} \bar{\Psi} \begin{pmatrix} 0 & i\gamma_5\tau_2\lambda_A \\ 0 & 0 \end{pmatrix} \Psi\ \bar{\Psi} \begin{pmatrix} 0 & 0 \\ i\gamma_5\tau_2\lambda_A & 0 \end{pmatrix} \Psi$$

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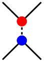
➜ vertices:



$$= 4Hi\,\Gamma_A^{\uparrow} \otimes \Gamma_A^{\downarrow} = 4Hi \begin{pmatrix} 0 & i\gamma_5\tau_2\lambda_A \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ i\gamma_5\tau_2\lambda_A & 0 \end{pmatrix}$$

Nambu-Gorkov propagator

- kinetic term + chemical potential:

$$\mathcal{L}_{kin} + \mu\, q^\dagger q \;=\; \bar{q}(i\partial\!\!\!/ + \mu\gamma^0)q$$

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$$\begin{aligned}\mathcal{L}_{kin} + \mu\, q^\dagger q &= \bar{q}(i\partial\!\!\!/ + \mu\gamma^0)q \\ &= \frac{1}{2}\left[\, \bar{q}(i\partial\!\!\!/ + \mu\gamma^0)q \;-\; q^T C(i\overleftarrow{\partial\!\!\!/} + \mu\gamma^0)C\bar{q}^T \,\right]\end{aligned}$$

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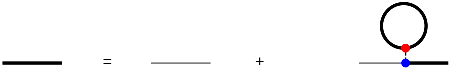
➜ inverse bare propagator in momentum space:

$$S_0^{-1}(p) = \begin{pmatrix} p\!\!\!/ + \mu\gamma^0 & 0 \\ 0 & p\!\!\!/ - \mu\gamma^0 \end{pmatrix}$$

Mean-field propagator

- dressed propagator (Hartree approximation):



$$iS(p) = iS_0(p) + iS_0(p)\,(-i\Sigma)\,iS(p)$$

$$\Leftrightarrow\ S^{-1}(p) = S_0^{-1}(p) - \Sigma$$

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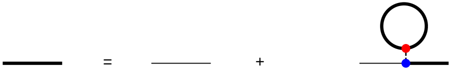

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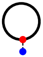
- self-energy:



$$-i\Sigma = \bigcirc = 4iH\sum_A\Big\{\ \Gamma_A^{\uparrow}\int\frac{d^4k}{(2\pi)^4}\,(-\tfrac{1}{2})\mathrm{Tr}[\Gamma_A^{\downarrow}\,iS(k)] + \Gamma_A^{\downarrow}\int\frac{d^4k}{(2\pi)^4}\,(-\tfrac{1}{2})\mathrm{Tr}[\Gamma_A^{\uparrow}\,iS(k)]\Big\}$$

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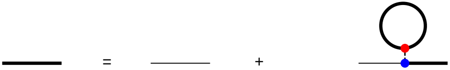

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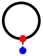
$$-i\Sigma \;=$$



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➜ selfconsistency problem!

Gap equation

- selfconsistency problem: $S^{-1} = S_0^{-1} - \Sigma[S]$

Gap equation

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- ansatz:

$$\Sigma = \begin{pmatrix} 0 & -\Delta\,\gamma_5\tau_2\lambda_2 \\ \Delta^*\,\gamma_5\tau_2\lambda_2 & 0 \end{pmatrix} \Rightarrow S^{-1} = \begin{pmatrix} \not{p} + \mu\gamma^0 & \Delta\,\gamma_5\tau_2\lambda_2 \\ -\Delta^*\,\gamma_5\tau_2\lambda_2 & \not{p} - \mu\gamma^0 \end{pmatrix}$$

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- result:

$$\Delta = 16H\,\Delta\,i\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k_0^2-\omega_-^2}+\frac{1}{k_0^2-\omega_+^2}\right)$$

  "gap equation"

  quasiparticle dispersion laws: $\omega_\mp = \sqrt{(|\vec{k}| \mp \mu)^2 + |\Delta|^2}$

Propagator

- dressed propagator: $S = \begin{pmatrix} \not{p} + \mu\gamma^0 & \Delta\,\gamma_5\tau_2\lambda_2 \\ -\Delta^*\,\gamma_5\tau_2\lambda_2 & \not{p} - \mu\gamma^0 \end{pmatrix}^{-1}$
    - dimension: $2 \times 4 \times N_f \times N_c$
      ➜ 48×48 matrix for $N_f = 2$, $N_c = 3$
    - inversion straight forward, but some work required . . .

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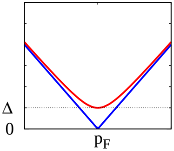
  - inversion straight forward, but some work required ...
- diagonalization:

$$S(p^0,\vec{p}) = U(\vec{p}) \begin{pmatrix} \frac{1}{p^0 - \lambda_1(\vec{p})} & & \\ & \ddots & \\ & & \frac{1}{p^0 - \lambda_{48}(\vec{p})} \end{pmatrix} U^\dagger(\vec{p})\,\gamma^0$$

  - $U(\vec{p})$ = unitary matrix, does not depend on p^0 !

Dispersion relations

free energy



- 48 eigenvalues

  $=$ 24 quasiparticle dispersion relations:

  - $\omega_-(\vec{p}) = \sqrt{(|\vec{p}| - \mu)^2 + |\Delta|^2}$ (8-fold) red and green quarks
  - $\omega_+(\vec{p}) = \sqrt{(|\vec{p}| + \mu)^2 + |\Delta|^2}$ (8-fold) " antiquarks
  - $\epsilon_-(\vec{p}) = \big|\, |\vec{p}| - \mu \,\big|$ (4-fold) blue quarks
  - $\epsilon_+(\vec{p}) = \big|\, |\vec{p}| + \mu \,\big|$ (4-fold) " antiquarks

- $+$ 24 quasiholes: $-\omega_\mp(\vec{p}), \; -\epsilon_\mp(\vec{p})$

Gap equation: solutions

- gap equation: $\Delta = 16H\,\Delta\, i \int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k_0^2-\omega_-^2} + \frac{1}{k_0^2-\omega_+^2}\right)$

Gap equation: solutions

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- turning out the sum (see Yudichev's lecture), $T \to 0$:

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➜ nontrivial solutions always exist for $H > 0$!

Linearized Lagrangian

- back to our Lagrangian:

$$\hat{\mathcal{L}} = \bar{q}(i\partial\!\!\!/ + \mu\gamma^0)q + H \sum_{A=2,5,7} (\bar{q}\, i\gamma_5\tau_2\lambda_A\, C\bar{q}^T)(q^T C\, i\gamma_5\tau_2\lambda_A\, q)$$

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- linearization:

$$x = \langle x\rangle + \delta x, \qquad x^\dagger = \langle x\rangle^* + \delta x^\dagger$$

$$\Rightarrow\; x^\dagger x \;\approx\; |\langle x\rangle|^2 + \langle x\rangle^*\delta x + \langle x\rangle\delta x^\dagger \;=\; \langle x\rangle^* x + \langle x\rangle x^\dagger - |\langle x\rangle|^2$$

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- linearize $\mathcal{L}_{int}$ around $\Delta = -2H\langle q^T C\,\gamma_5\tau_2\lambda_A\, q\rangle$

Linearized Lagrangian

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- linearize $\mathcal{L}_{int}$ around $\Delta = -2H\langle q^T C\,\gamma_5\tau_2\lambda_A\,q\rangle$
- result, using Nambu-Gorkov spinors:

$$\mathcal{L}_{MF} \;=\; \bar{\Psi}\begin{pmatrix} i\partial\!\!\!/ + \mu\gamma^0 & \Delta\,\gamma_5\tau_2\lambda_2 \\ -\Delta^*\,\gamma_5\tau_2\lambda_2 & -i\overleftarrow{\partial\!\!\!/} - \mu\gamma^0 \end{pmatrix}\Psi \;-\; \frac{|\Delta|^2}{4H}$$

Thermodynamic potential

- (grand canonical) thermodynamic potential:

$$\Omega(T,\mu) \;=\; -\tfrac{T}{V}\ln\mathcal{Z} \;=\; -\tfrac{T}{V}\ln\,\mathbf{Tr}\,\exp\left(-\tfrac{1}{T}\int d^3x\,(\mathcal{H}-\mu q^\dagger q)\right)$$

Thermodynamic potential

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$$\Omega(T,\mu) \;=\; -\frac{T}{V}\ln\mathcal{Z} \;=\; -\frac{T}{V}\ln\mathbf{Tr}\,\exp\left(-\frac{1}{T}\int d^3x\,(\mathcal{H}-\mu q^\dagger q)\right)$$

- mean-field Lagrangian:

$$\mathcal{L}_{MF} \;=\; \bar{\Psi}S^{-1}\Psi \;-\; \frac{|\Delta|^2}{4H} \;=\; \mathcal{T}-\mathcal{V}$$

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- general result:

$$\Omega(T,\mu) \;=\; -T\sum_n\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\,\mathrm{Tr}\,\ln\left(\frac{1}{T}\,S^{-1}(i\omega_n,\vec{k})\right) \;+\; \mathcal{V}$$

  - $\ln A = \ln\left((1-(1-A)\right) = \sum_{n=1}^{\infty}\frac{1}{n}(1-A)^n$
  - useful formula: $\mathrm{Tr}\,\ln A = \ln\mathrm{Det}\,A$

Thermodynamic potential

- result after Matsubara summation:

$$\begin{aligned}\Omega(T,\mu) \;=\; -\int \frac{d^3p}{(2\pi)^3}\Big\{ &\;\; 8\left(\frac{\omega_-}{2} + T\,\ln(1+e^{-\omega_-/T})\right. \\ &\qquad \left. + \frac{\omega_+}{2} + T\,\ln(1+e^{-\omega_+/T})\right) \\ &+4\left(\frac{\epsilon_-}{2} + T\,\ln(1+e^{-\epsilon_-/T})\right. \\ &\qquad \left. + \frac{\epsilon_+}{2} + T\,\ln(1+e^{-\epsilon_+/T})\right)\Big\} \\ &+ \frac{|\Delta|^2}{4H}\end{aligned}$$

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$$\rightarrow\; \Delta \;=\; 4H\,T\sum_n \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\,\mathrm{Tr}\left[\,iS\,\Gamma_2^{\downarrow}\,\right]$$ gap equation!

Thermodynamic quantities

- standard thermodynamic relations:
  - pressure: $p = -\Omega$
  - density: $n = -\frac{\partial\Omega}{\partial\mu}$
  - entropy density: $s = -\frac{\partial\Omega}{\partial T}$
  - energy density: $\varepsilon = -p + Ts + \mu n$

Density

- example: density

$$n = T\sum_n \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\,\mathrm{Tr}\,[\,S\,\Gamma^0\,], \qquad \Gamma^0 = \frac{\partial S^{-1}}{\partial\mu} = \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix}$$

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$$\begin{aligned} n &= \int \frac{d^3p}{(2\pi)^3} \Big\{ \, 4\Big(\frac{\partial\omega_-}{\partial\mu} + \frac{\partial\omega_+}{\partial\mu}\Big) + 2\Big(\frac{\partial\epsilon_-}{\partial\mu} + \frac{\partial\epsilon_+}{\partial\mu}\Big) \Big\} \\ &\equiv \int \frac{d^3p}{(2\pi)^3} \Big\{ \, 8f_\Delta(p) + 4f_0(p) \, \Big\} \end{aligned}$$

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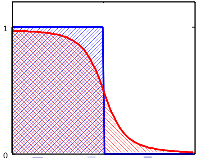
➜ occupation functions:

$$f_\Delta(p) = \tfrac{1}{2}\big(\tfrac{\partial \omega_-}{\partial \mu} + \tfrac{\partial \omega_+}{\partial \mu}\big) = \tfrac{1}{2}\big(\tfrac{\mu - p}{\omega_-} + \tfrac{\mu + p}{\omega_+}\big)$$

$$f_0(p) = \tfrac{1}{2}\big(\tfrac{\partial \epsilon_-}{\partial \mu} + \tfrac{\partial \epsilon_+}{\partial \mu}\big) = \theta(\mu - p)$$



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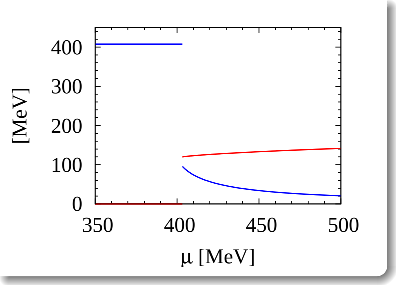
Numerical results

- solutions of the gap equations for

  M and Δ:



- first:

  Berges & Ragagopal, NPB (1999)

- here:

  model with 6 different condensates

  M.B, Hošek, Oertel, PRD (2002)

- simultaneous first-order chiral ($M \downarrow$) and superconducting ($\Delta \uparrow$) phase transitions.

Outline

Reminder

- diquark condensate: $\langle q^T \, \hat{\mathcal{O}} \, q \rangle$
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- two flavors: $\tau_A = \tau_2 \quad \Rightarrow \quad \vec{s} = (s_{22}, s_{25}, s_{27})$
  - can always be rotated into “antiblue” direction:

$$\vec{s} \,\rightarrow\, \vec{s}\,' \;=\; \vec{s}\,U \;=\; (s'_{22}, 0, 0), \quad U \in SU(3)_c$$

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    - In general, that's all we can do . . .

Idealized case

- three degenerate flavors: $M_u = M_d = M_s$
  - ➜ $SU(3)_f$-symmetric
  - ➜ $(s_{AA'})$ can always be diagonalized by combined color and flavor rotations:

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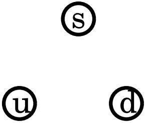
- eight possible phases:

normal quark matter (NQ)

$s_{22} = s_{55} = s_{77} = 0$



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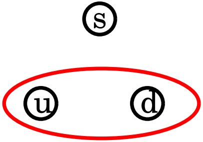
2SC phase

$s_{22} \neq 0, \quad s_{55} = s_{77} = 0$

+ two more phases of this kind



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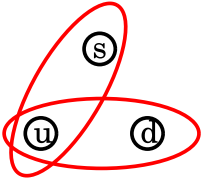

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CFL phase

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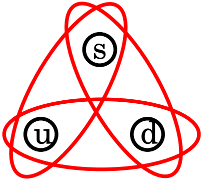

Color-flavor locking

- CFL pairing pattern (idealized case): $s_{22} = s_{55} = s_{77}$

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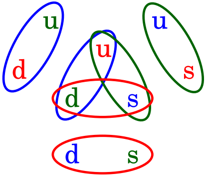

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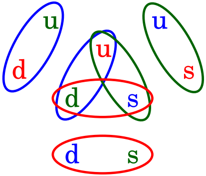

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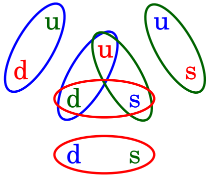

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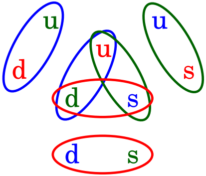

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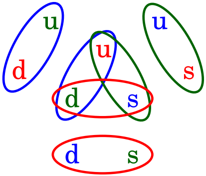

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  - baryon #: broken ➜ 1 scalar Goldstone boson

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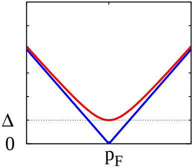

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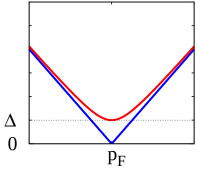
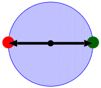
- Cooper pairs in BCS theory:
  - opposite momenta

free energy





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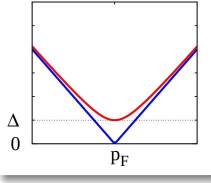

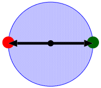

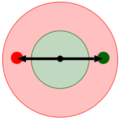

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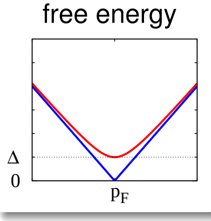

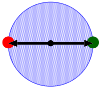

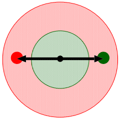

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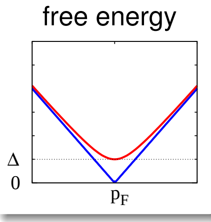

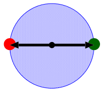

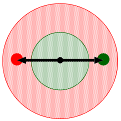

Which phase is favored?

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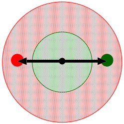

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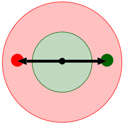

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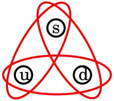

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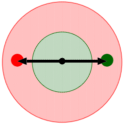

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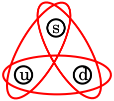

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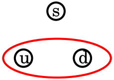
$$M_s \gtrsim \mu \gg M_{u,d} \;\; \rightarrow \;\; p_F^s \ll p_F^d \approx p_F^u \;\; \rightarrow \;\; \text{2SC}$$



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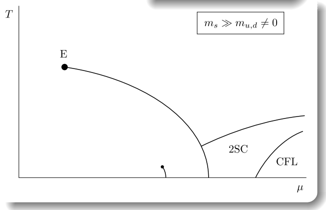
K. Rajagopal (1999)



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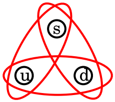

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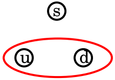

3-flavor NJL model

- Lagrangian: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\bar{q}q} + \mathcal{L}_{qq}$
  - free part: $\mathcal{L}_0 = \bar{q}(i\partial\!\!\!/ - \hat{m})q \; , \qquad \hat{m} = diag_f(m_u, m_d, m_s)$
  - quark-antiquark interaction:

$$\mathcal{L}_{\bar{q}q} = G\left\{(\bar{q}\tau^a q)^2 + (\bar{q}i\gamma_5\tau^a q)^2\right\}$$
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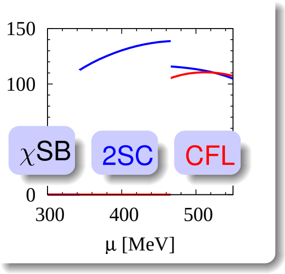
- mean-field approximation:
  - $\bar{q}q$-condensates: $\langle\bar{u}u\rangle$, $\langle\bar{d}d\rangle$, $\langle\bar{s}s\rangle$ ↔ *dynamical masses*
  - qq-condensates: $\langle ud\rangle$, $\langle us\rangle$, $\langle ds\rangle$ ↔ *diquark gaps*

Results for $T = 0$

- “realistic” parameters
- isospin symmetry

Δ_{ud}, $\Delta_{us} = \Delta_{us}$



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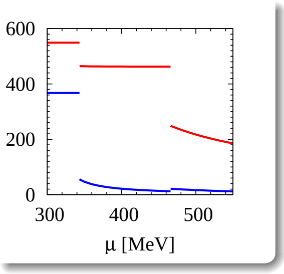

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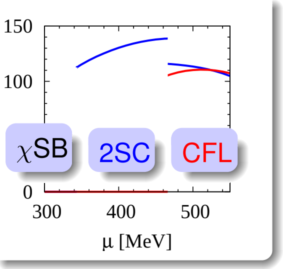

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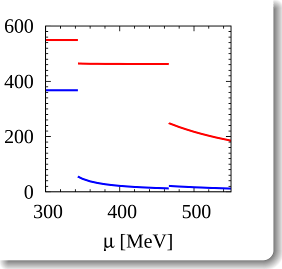
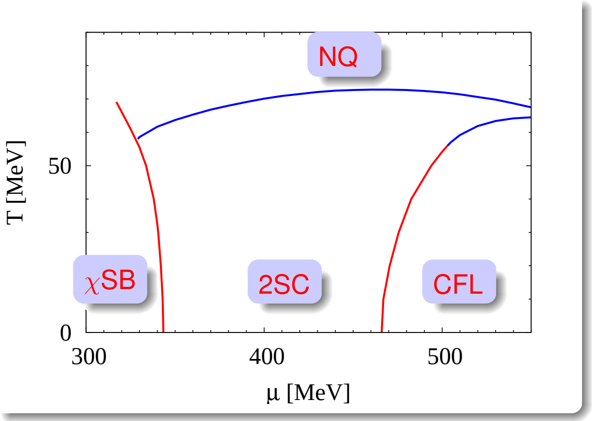


- dynamical quark masses
  - ➜ density dependent, discontinuous functions!

Phase diagram



phase transitions

- 1st order
- 2nd order

Outline

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- four conserved charges; densities: $n_r, \; n_g, \; n_b, \; n_Q$

$$\Leftrightarrow \; n = n_r + n_g + n_b, \quad n_3 = n_r - n_g, \quad n_8 = \frac{1}{\sqrt{3}}(n_r + n_g - 2n_b), \quad n_Q$$

→ four independent chemical potentials: $\mu, \; \mu_3, \; \mu_8, \; \mu_Q$

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but one should always check . . .

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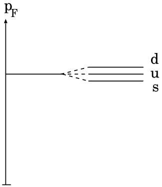

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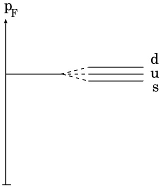

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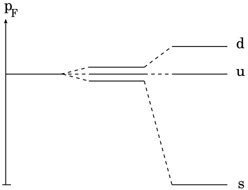
M. Alford, K. Rajagopal, JHEP (2002)



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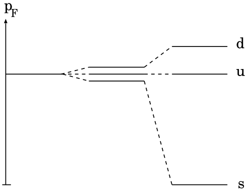

- strong coupling: 2SC possible !

Gapless color superconductors

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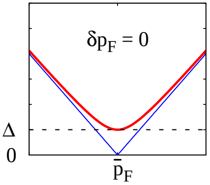
free energy



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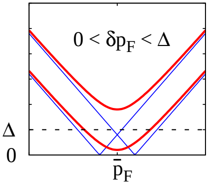

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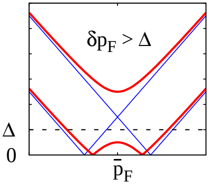

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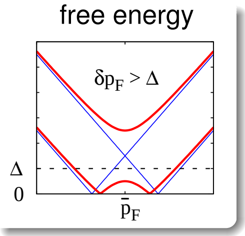
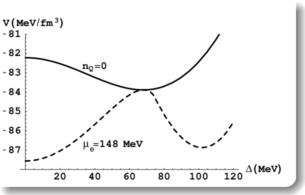


- *gapless 2SC phase* (g2SC)

I. Shovkovy, M. Huang, PLB (2003).

  - unstable at fixed μ_Q
  - can be most favored *neutral homogeneous* solution



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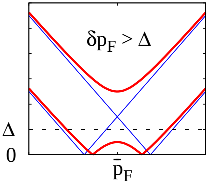
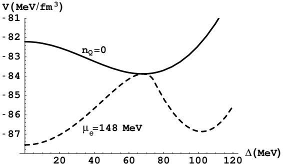


- *gapless 2SC phase* (g2SC)

I. Shovkovy, M. Huang, PLB (2003).

  - unstable at fixed μ_Q
  - can be most favored *neutral homogeneous* solution



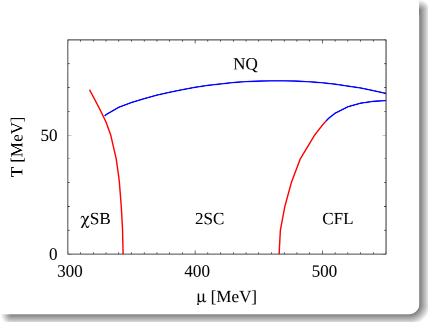
- similar solutions for CFL, uSC, etc.

Alford, Kouvaris, Rajagopal, PRL (2004).

Model calculations

- NJL model *without* imposing neutrality

M.B., M. Oertel, NPA (2002); M. Oertel, M.B., hep-ph/0202098.



- quark phases at T=0:

$$(\chi SB \rightarrow)\ \mathit{2SC} \rightarrow \mathit{CFL}$$

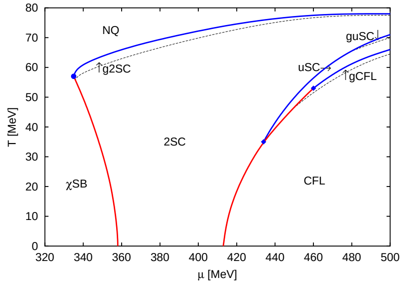
also: Gastineau, Nebauer, Aichelin, PRC(2002).

Model calculations

- NJL model *with* neutrality constraints

Rüster, Werth, M.B., Shovkovy, Rischke, PRD (2005)

- "strong diquark coupling": $H = G$



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also: Blaschke, Fredrikson, Grigorian, Öztaş, Sandin, PRD (2005); Abuki & Kunihiro, NPA (2006).

Model calculations

- NJL model *with* neutrality constraints

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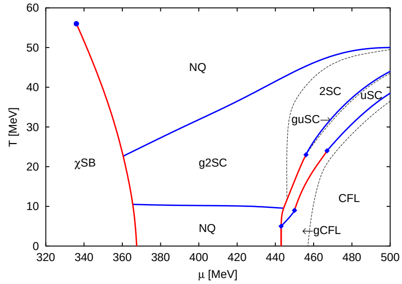
- “intermediate diquark coupling”: $H = 0.75\,G$



- quark phases at T=0:
  - “strong coupling”:

    2SC → *CFL*
  - “intermediate coupling”:

    normal → *gCFL* → *CFL*
    - no 2SC!
    - gapless phases

also: Blaschke, Fredrikson, Grigorian, Öztaş, Sandin, PRD (2005); Abuki & Kunihiro, NPA (2006).

CFL + Goldstone phases

- CFL: chiral symmetry broken ➜ Goldstone bosons
  - “π”, “K”, “η” (by quantum numbers), but mainly diquarks
  - very light: $m \sim \mathcal{O}(10\ \mathrm{MeV})$ (see also contribution by V. Werth)

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  - very light: $m \sim \mathcal{O}(10\ \mathrm{MeV})$ (see also contribution by V. Werth)
- stress imposed by M_s ➜ K^0 condensation

T. Schäfer, PRL (2000); Bedaque & Schäfer, NPA (2002).

  - heuristic argument: $p_F^s = \sqrt{\mu^2 - M_s^2} \simeq \mu - \frac{M_s^2}{2\mu}$

➜ effective strangeness chemical potential: $\mu_s \simeq \frac{M_s^2}{2\mu}$

➜ K^0 condensation if $\mu_s > m_{K^0}$

NJL-model description

- axial transformations: $q \longrightarrow \exp(i\theta_a \frac{\tau_a}{2}\gamma_5)\, q$
  - ➜ $\langle \psi^T\, C\gamma_5\, \tau_A\, \lambda_{A'}\, \psi \rangle \quad \longrightarrow \quad \langle \psi^T\, C\, \tau_{A''}\, \lambda_{A'}\, \psi \rangle$
  - ➜ include pseudoscalar diquark condensates

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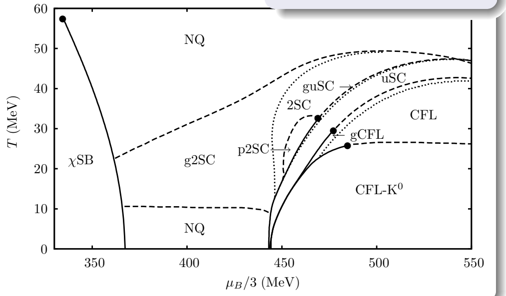
- phase diagram:

H. Warringa, hep-ph/0606063



Neutrinos

- Proto-neutron stars:

  neutrinos trapped during the first few seconds

  → lepton # conserved

  → more electrons:

  $$\mu_e = \mu_d - \mu_u + \mu_\nu$$

  → favors 2SC,

  suppresses CFL

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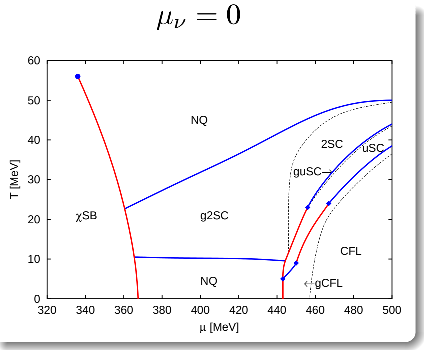
$\mu_\nu = 0$

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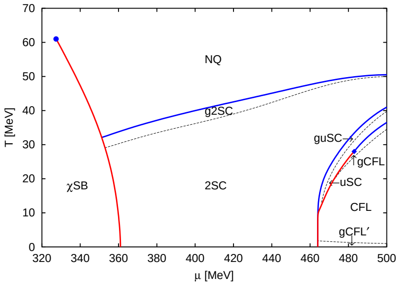
$\mu_\nu = 200$ MeV



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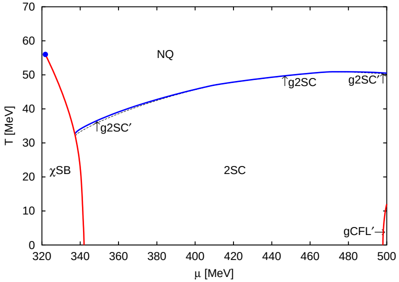

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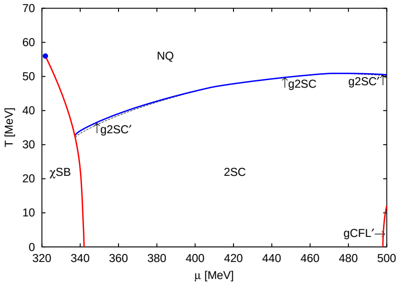
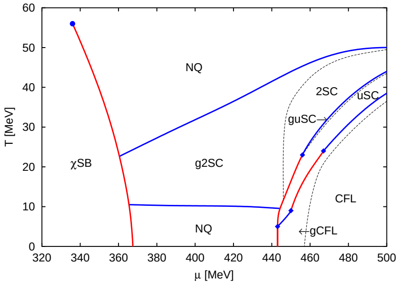


- consequences for the evolution of the star ?

Neutrinos

$$\mu_\nu = 0$$

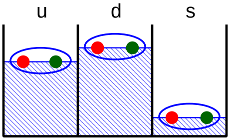


Outline

Spin-1 condensates

- alternative: equal-flavor pairing
  - Pauli principle:

    spin symmetric ➜ spin 1



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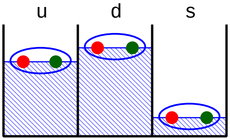

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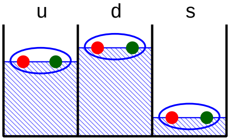
  - all fermion modes gapped
  - smallest gap ~ 100 keV



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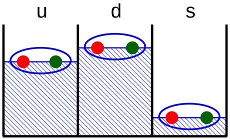

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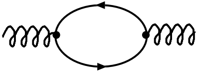
Schmitt, Shovkovy, Wang, PRD (2006).

➜ cooling properties rather different from 2SC and CFL !

Chromomagnetic instabilities

- Meissner effect:

$$m_{M,a}^2 = -\frac{1}{2} \lim_{\vec{p}\to 0} \left(g_{ij} + \frac{p_i p_j}{p^2}\right) \Pi_{aa}^{ij}(0,\vec{p})$$



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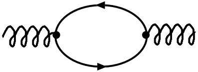

$$m^2_{M,a} = -\frac{1}{2} \lim_{\vec{p}\to 0} (g_{ij} + \frac{p_i p_j}{p^2})\, \Pi^{ij}_{aa}(0,\vec{p})$$

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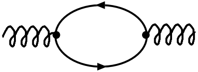
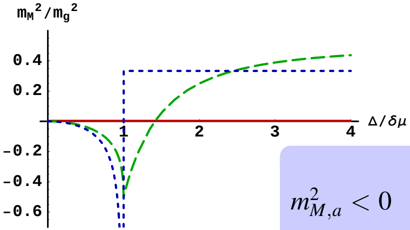


- problem: can become *imaginary* ⟶ unstability !
  - e.g., (g)2SC: Huang & Shovkovy, PRD (2004)

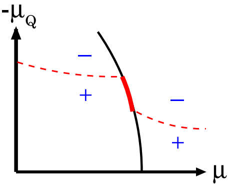


$$m^2_{M,a} < 0 \quad \text{if} \quad \delta p_F > \begin{cases} \frac{\Delta}{\sqrt{2}} & a = 4,\ldots,7 \\ \Delta & a = 8 \end{cases}$$

Mixed quark phases

- basic principle:
    - 1st component positive
    - 2nd component negative
    - globally neutral

Glendenning, PRD (1992)



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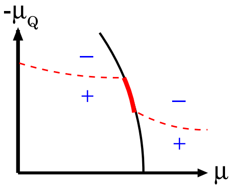

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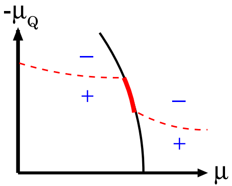

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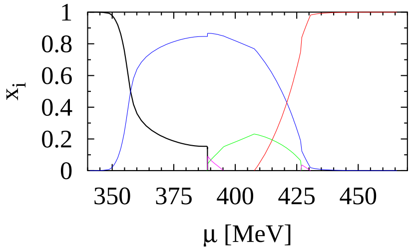
- mixed phases:
    - 2, 3, or 4 components
    - neutral along 1-dimensional lines

Mixed quark phases: NJL-model results

F. Neumann, M.B., M. Oertel, NPA (2003)

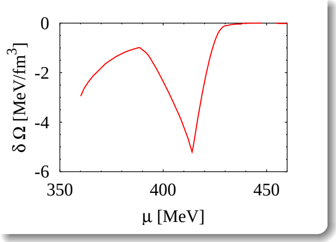
- composition: N 2SC $2SC_{us}$ sSC CFL



- 9 different mixed phases
- 2-, 3-, and 4-component systems
- “exotic” components: sSC, $2SC_{us}$

Mixed quark phases: stability

- bulk free energy gain relative to homogeneous phases:



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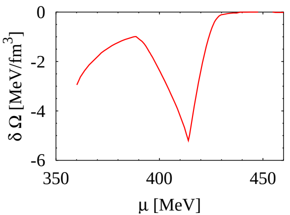

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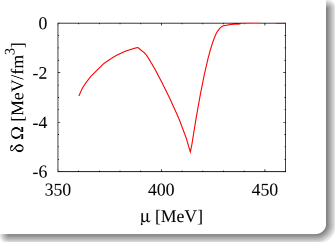

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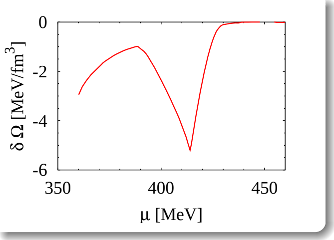

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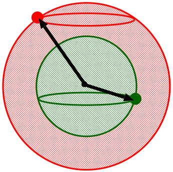
- structure size ~ 10 fm ➜ crystal ?

Crystalline phases

Larkin, Ovchinnikov; Fulde, Ferrel (1964);
Alford, Bowers, Rajagopal, PRD (2001)

- "LOFF" phase:
  - pairs with total momentum $\vec{P} \neq \vec{0}$
  - $p_F^a \neq p_F^b$ no problem
  - less phase space
  - unisotropic: $\langle q(\vec{x})q(\vec{x})\rangle \sim \Delta\, e^{2i\vec{P}\cdot\vec{x}}$
  - gauge equivalent to constant gluon background $\vec{A}^{(8)}$



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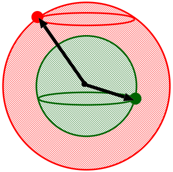

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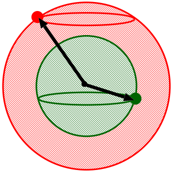

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- (g)2SC:

Giannakis and Ren, PLB (2005)

  - favored for $\delta p_F > \Delta$ ➜ cures $a = 8$, but *not* $a = 4, \ldots, 7$

Gluonic phase

Gorbar, Hashimoto, Miransky, PLB (2006)

- non-zero timelike and spacelike gluon condensates:

$$\mu_8 \sim \langle A_0^{(8)} \rangle, \quad \mu_3 \sim \langle A_0^{(3)} \rangle, \quad B \sim \langle A_3^{(6)} \rangle, \quad C \sim \langle A_3^{(1)} \rangle$$

→ non-zero chromo-electric fields: $F_{03}^{(2)}$, $F_{03}^{(7)}$

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- Ginzburg-Landau analysis near $\delta p_F = \frac{\Delta}{\sqrt{2}}$:
  - cures the problem for $a = 4, \ldots, 7$!

“gauged NJL model”

- Lagrangian (more than QCD . . .):

$$\mathcal{L} = \bar{q} i \not{D} q + H(\bar{q} i \gamma_5 \tau_2 \lambda_A C \bar{q}^T)(q^T C i \gamma_5 b \tau_2 \lambda_A C \bar{q}) - \frac{1}{4} F^{(a)}_{\mu\nu} F^{(a)\,\mu\nu}$$

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- recent investigation: Kiriyama, Rischke, Shovkovy, hep-ph (2005)
    - neglect μ_8, μ_3, and C
    - subtraction: $\tilde{\Omega} = \Omega(\mu_u, \mu_d, \Delta, B) - \Omega(0, 0, 0, B)$

➜ reasonable results, but not quite satisfactory . . .

Outline

Theoretical approach

- NJL models
  - dynamically generated masses and gaps
  - simple interaction: allows for tackling complex problems
  - model parameters largely undetermined
  - mostly restricted to mean field

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    - simple interaction: allows for tackling complex problems
    - model parameters largely undetermined
    - mostly restricted to mean field
- “model independent” analyses, effective theories
    - systematic expansions in Δ/μ, M_s/μ, etc.
    - (unknown) details of the interaction not important
    - expansion parameters not necessarily small
    - cannot predict $\Delta(\mu)$, $M_s(\mu)$, etc.

Theoretical approache (contd.)

- QCD at weak coupling

→ D. Rischke, Prog. Part. Nucl. Phys. (2004)

  - $\mu \gg \Lambda_{QCD}$ ➜ $\alpha_s(\mu) \ll 1$

    ➜ systematic expansion (gluon exchange)

Theoretical approache (contd.)

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  - $\mu \gg \Lambda_{QCD}$ ➜ $\alpha_s(\mu) \ll 1$

    ➜ systematic expansion (gluon exchange)

  - optimistic estimate: $\mu > 1.5$ GeV → $\rho_B > 175\ \rho_0$

    (Rajagopal and Shuster, PRD (2000): $\mu \gg 10^5$ GeV !!!)

    ➜ not applicable to neutron stars

Dyson-Schwinger approach

Nickel, Alkofer, Wambach, PRD (2006)

- QCD Dyson-Schwinger equation for the quark propagator:

  - vertex function from DSE studies in vacuum
  - gluon propagator from DSE + particle-hole corrections

Dyson-Schwinger approach

Nickel, Alkofer, Wambach, PRD (2006)

- QCD Dyson-Schwinger equation for the quark propagator:

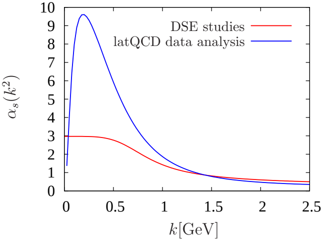
  - vertex function from DSE studies in vacuum
  - gluon propagator from DSE + particle-hole corrections

- features:
  - weak coupling limit for very large densities
  - contact to lattice results in vacuum
  - no parameter
  - quite involved . . .

Dyson-Schwinger approach: results

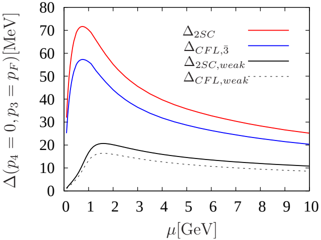
vertex function:

$$\alpha_s(k^2) \propto Z_g(k^2)\Gamma(k^2)$$



Fischer, Alkofer (2003);
Bhagwat et al. (2003)

pairing gaps:



Nickel, Alkofer, Wambach, PRD (2006)

- large gaps at moderate densities!

  (3 times larger than extrapolated weak coupling results)

Empirical information from compact stars

- maximum mass, mass-radius relation
  - ➜ equation of state

Alford, Braby, Paris, Reddy

- cooling
  - ➜ ungapped quarks, specific heat
  - ➜ no normal quark matter, no 2SC (?)

Grigorian, Blaschke, Voskresensky

- direct neutrinos
  - ➜ phase transitions during the first seconds

Carter & Reddy

- rotation frequency
  - moment of inertia ➜ phase transitions
  - core glitches ➜ crystalline phases (?)
  - viscosity (r-mode instabilities)
    - ➜ no pure CFL stars (?)

Glendenning & Weber

Madsen

- magnetic fields

Literature

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7. many others: Nardulli (2002), Ren (2004), Huang (2005), . . .