Statistical Model for the QCD Phase Diagram - III -

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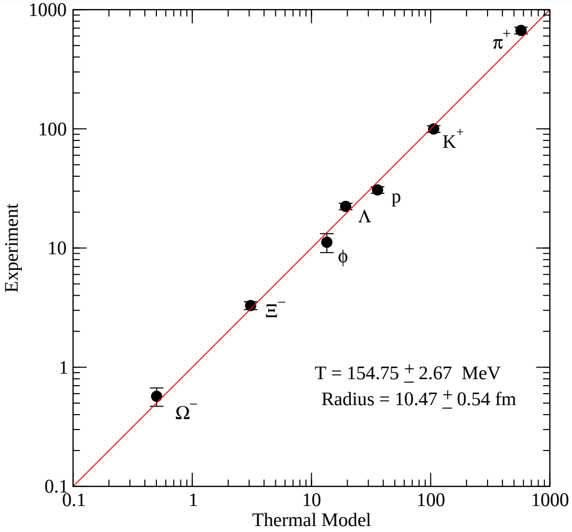
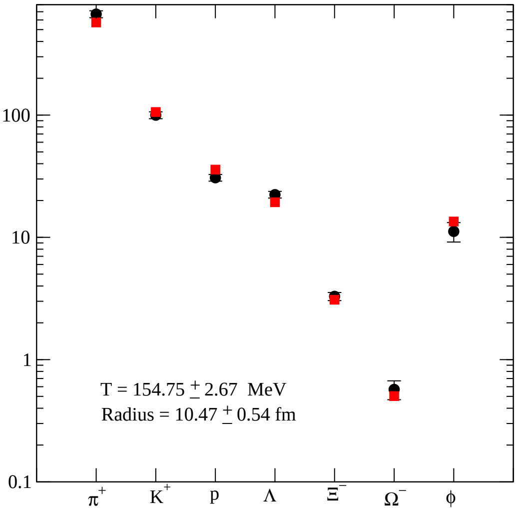
Outline


1000
100
10
1
0.1
Experiment
0.1
1
10
100
1000
Thermal Model
π^+
K^+
p
Λ
ϕ
Ξ^-
Ω^-
T = 154.75 ± 2.67 MeV
Radius = 10.47 ± 0.54 fm


100
10
1
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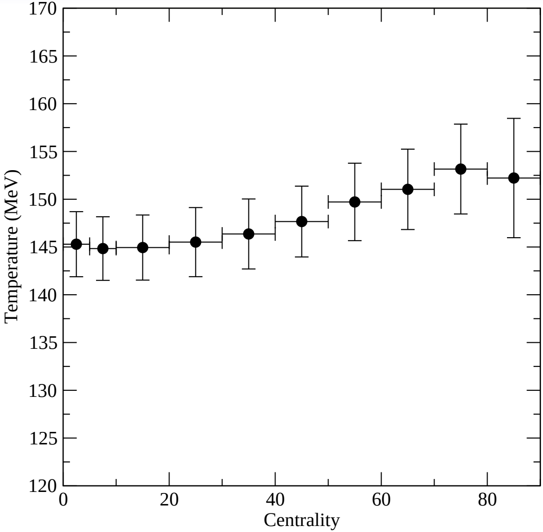
Temperature (MeV)
170
165
160
155
150
145
140
135
130
125
120
0
20
40
60
80
Centrality

Excluded Volume Corrections.

Why do we need them?
The pressure at zero temperature for a single degree of freedom is given by

$$P = \int_0^{\mu} \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E}$$

For massless particles, or at very high chemical potential (high density)

$$\begin{aligned} P &= \int_0^{\mu} p^2 dp \frac{4\pi}{2\pi^2} \\ &= \frac{1}{2\pi^2} \mu^4 \end{aligned}$$

which leads to:

$$P(\text{quarks}) = 2x2x3x \frac{1}{2\pi^2} \left(\frac{\mu}{3}\right)^4 - B$$

and

$$P(\text{nucleons}) = 2x2x \frac{1}{2\pi^2} (\mu)^4$$

i.e

$$P(\text{quarks}) < P(\text{nucleons})$$

and the system reverts back to the nucleon phase at very high densities.

Nucleon Phase -> Quark Phase -> Nucleon Phase

Excluded volume corrections prevent this from happening. This has been implemented in all the thermal model codes.

Relation between grand canonical and canonical ensembles:

$$Z_{GC}(T,V,\mu) = \sum_{N=0}^{\infty} e^{\frac{\mu N}{T}} Z_C(T,V,N)$$

Relation between grand canonical and pressure ensembles:

$$Z_p(T,P,\mu) = \int_0^{\infty} dV\; e^{\frac{PV}{T}} Z_{GC}(T,V,\mu)$$

Excluded Volume Corrections.

$$
\begin{aligned}
Z_{GC} &= \exp\left\{ V \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}+\frac{\mu}{T}} \right\} \\
&= \sum_{N=0}^{\infty} \frac{V^N}{N!} e^{\mu N/T} \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N
\end{aligned}
$$

with excluded volume corrections

$$
\begin{aligned}
Z_{GC} \rightarrow & \sum_{N=0}^{\infty} \frac{(V - V_0 N)^N}{N!} e^{\mu N/T} \\
& \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N \theta(V - V_0 N)
\end{aligned}
$$

Excluded Volume Corrections.

It is more convenient to consider these corrections in the pressure ensemble:

$$Z_p \equiv \int_0^\infty dV\ e^{-PV/T} \sum_{N=0}^{\infty} \frac{V^N}{N!} e^{\mu N/T} \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N$$

$$Z_p \rightarrow \sum_{N=0}^{\infty} \int_0^\infty dV\ e^{-PV/T} \frac{(V - V_0 N)^N}{N!} e^{\mu N/T} \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N \theta(V - V_0 N)$$

introduce $x \equiv V - V_0 N$.

Excluded Volume Corrections.

$$
\begin{aligned}
Z_p &= \sum_{N=0}^{\infty} \int_0^{\infty} dx\ e^{-Px/T} \\
&\quad \frac{x^N}{N!} e^{-PV_0 N/T} e^{\mu N/T} \\
&\quad \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N
\end{aligned}
$$

a new variable $\bar{\mu} \equiv \mu - PV_0$.
This is reminiscent of Randrup's shift in the chemical potential.

Excluded Volume Corrections.

$$Z_p \rightarrow \sum_{N=0}^{\infty} \int_0^{\infty} dx \; e^{-Px/T} \frac{x^N}{N!} e^{\bar{\mu}N/T} \left[\int \frac{d^3p}{(2\pi)^3} e^{-\frac{E}{T}} \right]^N$$

which is the original partition function with the he replacement

$$\boxed{\bar{\mu} = \mu - P\ V_0}$$

D.H. Rischke, M.I. Gorenstein, H. Stöcker, W. Greiner, ZfP C51, 485 (1991). J. C., M.I. Gorenstein, J. Stålnacke and E. Suhonen P. S. 48 277-280 (1993).

Excluded Volume Corrections.

The particle number density now becomes:

$$
\begin{aligned}
n &= \frac{T}{V}\frac{\partial}{\partial\mu}\ln\ Z \\
&= \frac{T}{V}\frac{\partial\bar{\mu}}{\partial\mu}\frac{\partial}{\partial\bar{\mu}}\ln\ Z \\
&= \frac{\partial\bar{\mu}}{\partial\mu}n_0 \\
&= [1 - V_0\ n]\,n_0
\end{aligned}
$$

$$\boxed{n = \frac{n_0}{1 + V_0 n_0}}$$

Effects Cancel Out in Ratios.

J.C., K. Redlich, H. Satz, E. Suhonen, ZfP C33, 151, (1986)

D.H. Rischke, M.I. Gorenstein, H. Stöcker, W. Greiner, ZfP C51, 485 (1991).

Canonical Corrections.

Exact Strangeness Conservation.

For a small system at low temperatures ($T \approx 50$ MeV), e.g. at GSI canonical corrections are necessary.
Instead of

$$N_K \approx \exp -M_K/T$$

one gets

$$N_K \approx \exp -2M_K/T$$

Extra suppression is due to strangeness conservation and lack of a large heat bath. This correction disappears quickly at higher energies and is already small at AGS energies.
First proposed by R. Hagedorn.

Exact Strangeness Conservation.

$$Z = \mathrm{Tr}\, e^{-\frac{H}{T} + \frac{\mu N}{T}}$$

Insert a Kronecker delta in the trace:

$$\sum_i n_i(S=1) + 2\sum_j n_j(S=2) + 3\sum_k n_k(3) =$$

$$\sum_i \bar{n}_i(S=-1) + 2\sum_j \bar{n}_j(S=-2) + 3\sum_k \bar{n}_k(S=-3)$$

and rewrite it as

$$\delta\left(\sum_i n_i(S=1) + ..., \sum_i \bar{n}_i(S=-1) + ...\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi$$

$$\exp\left(i\phi\sum_i n_i(S=1) + .. - i\phi\sum_i \bar{n}_i(S=-1)\right)$$

Exact Strangeness Conservation.

$$
\begin{aligned}
Z &= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left\{ Z_1 e^{i\phi} + Z_{-1} e^{-i\phi} \right\} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left\{ \sqrt{Z_1 Z_{-1}} \left[\sqrt{\frac{Z_1}{Z_{-1}}} e^{i\phi} + \sqrt{\frac{Z_{-1}}{Z_1}} e^{-i\phi} \right] \right\}
\end{aligned}
$$

Z_1: sum of all particles with strangeness 1, e.g. K^+
Z_{-1}: sum of all particles with strangeness -1, e.g. Λ

Exact Strangeness Conservation.

Use

$$\exp\left\{\frac{x}{2}\left(t+\frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} I_m(x)t^m$$

to obtain

$$Z = \frac{1}{2\pi}\int_0^{2\pi} e^{ip\phi} \sum_{p=-\infty}^{\infty} I_p(x_1)y_1^p$$

where

$$y_1 = \sqrt{\frac{Z_1}{Z_{-1}}} \qquad x_1 = 2\sqrt{Z_1 Z_{-1}}$$

$$Z = I_0(x_1)$$

Exact Strangeness Conservation.

In more detail, e.g. the multiplicity of K^+

$$N_{K^+} = \frac{T}{Z}\frac{\partial I_0(x_1)}{\partial \mu_{K^+}}\bigg|_{\mu_{K^+}=0}$$

Use

$$\frac{d}{dz}I_0(z) = I_1(z)$$

Exact Strangeness Conservation.

$$
\begin{aligned}
N_{K^+} &= \frac{T}{Z}\frac{\partial}{\partial \mu_{K^+}} I_0(x_1) \\
&= \frac{T}{I_0(x_1)} I_1(x_1) \frac{\partial x_1}{\partial \mu_{K^+}} \\
&= \frac{T}{I_0(x_1)} I_1(x_1) \frac{\partial 2\sqrt{Z_1 Z_{-1}}}{\partial \mu_{K^+}} \\
&= \frac{I_1(x_1)}{I_0(x_1)} \sqrt{\frac{Z_{-1}}{Z_1}} N^0_{K^+}
\end{aligned}
$$

where $N^0_{K^+}$ refers to the "unmodified" kaon multiplicity.

Exact Strangeness Conservation.

In the small volume limit this becomes

$$\lim_{z\to 0} I_0(z) = 1$$

and

$$\lim_{z\to 0} I_1(z) = \frac{z}{2}$$

$$\begin{aligned} \lim_{V\to 0} &= N^0_{K^+} Z_{-1} \\ \lim &= N^0_{K^+} Z_{-1} \\ &= N^0_{K^+} \left[N^0_{K^-} + N^0_{\Lambda} + \cdots \right] \end{aligned}$$

i.e., the particle multiplicity is

- proportional to V^2, and not V^1.
- proportional to $\exp(-2m_K/T)$ or to $\exp(-(m_K + m_\Lambda)/T)$ and not simply $\exp(-m_K/T)$, i.e. there is additional suppression of strange particles.

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Exact Strangeness Conservation

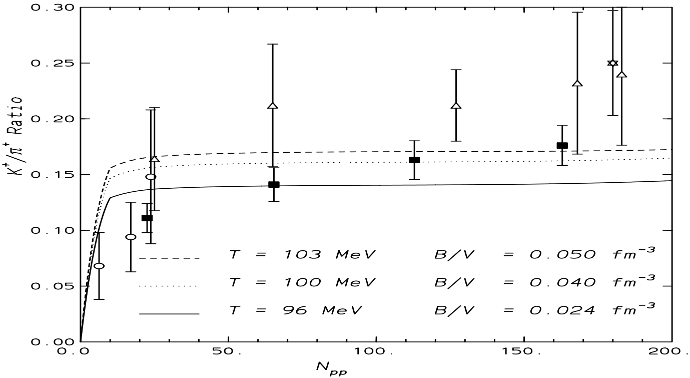

Figure 1

CANONICAL CORRECTIONS
are
IRRELEVANT.
Except at very low energies.

Transverse Momentum Distribution

STAR collaboration, B.I. Abelev at al.
arXiv:nucl-ex/0607033; Phys. Rev. C75, 064901 (2007)
PHENIX collaboration, A. Adare et al.
Phys. Rev. **C83**, 064903 (2011)
ALICE collaboration, K. Aamodt et al.
arXiv:1101.4110 [hep-ex]
CMS collaboration, V. Khachatryan et al.
arXiv: 1102.4282 [hep-ex]
ATLAS collaboration, G. Aad et al.
New J. Phys. **13** (2011) 053033.

All use the Tsallis distribution for $p-p$ collisions.

Tsallis Distribution

Possible generalization of Boltzmann-Gibbs statistics

Constantino Tsallis
Rio de janeiro, CBPF
J. Stat. Phys. 52 (1988) 479-487

Citations: 1 389
However:
Citations in HEP: 403

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MINISTÉRIO DA CIÊNCIA E TECNOLOGIA

CNPq

CBPF

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Notas de Física

CBPF-NF-062/87

POSSIBLE GENERALIZATION OF BOLTZMANN-GIBBS STATISTICS

by

Constantino TSALLIS

RIO DE JANEIRO
1987

CBPF-NF-062/87

Multifractal concepts and structures are quickly acquiring importance in many active areas (e.g., non-linear dynamical systems, growth models, commensurate/incommensurate structures). This is due to their utility as well as to their elegance. Within this framework, the quantity which is normally scaled is p_i^q, where p_i is the probability associated to an event and q any real number [1]. We shall use this quantity to generalize the standard expression of the entropy S in information theory, namely $S = -k\sum_{i=1}^{W} p_i \ln p_i$, where $W \in \mathbb{N}$ is the total number of possible (microscopic) configurations and $\{p_i\}$ the associated probabilities. We *postulate* for the entropy

$$S_q \equiv k\,\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \qquad (q \in \mathbb{R}) \qquad (1)$$

where k is a conventional positive constant and $\sum_{i=1}^{W} p_i = 1$. We immediately verify that

$$S_1 \equiv \lim_{q\to 1} S_q = k \lim_{q\to 1} \frac{1-\sum_{i=1}^{W} p_i e^{(q-1)\ln p_i}}{q-1} = -k\sum_{i=1}^{W} p_i \ln p_i \qquad (1')$$

where we have used the replica-trick type of expansion. We illustrate definition (1) in Fig. 1. S_q may be rewritten as follows:

$$S_q = \frac{k}{q-1}\sum_{i=1}^{W} p_i\left(1-p_i^{q-1}\right) \qquad (2)$$

Transverse Momentum Distribution

STAR, PHENIX, ALICE, CMS, ATLAS use:

$$\frac{\mathrm{d}^2 N}{\mathrm{d}p_\mathrm{t}\mathrm{d}y} = p_\mathrm{t} \times \frac{\mathrm{d}N}{\mathrm{d}y} \frac{(n-1)(n-2)}{nC(nC + m_0(n-2))} \left(1 + \frac{m_\mathrm{t} - m_0}{nC}\right)^{-n}$$

Direct connection with Tsallis distribution.

In the Tsallis distribution the total number of particles is given by:

$$N = gV \int \frac{d^3p}{(2\pi)^3} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{q/(1-q)} ,$$

The corresponding momentum distribution is given by

$$E\frac{dN}{d^3p} = gVE\frac{1}{(2\pi)^3} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{q/(1-q)} ,$$

which, in terms of the rapidity and transverse mass variables, becomes (for $\mu = 0$)

$$\left.\frac{d^2N}{dp_t\, dy}\right|_{y=0} = gV\frac{p_t m_t}{(2\pi)^2} \left[1 + (q-1)\frac{m_t}{T}\right]^{q/(1-q)} ,$$

J.C. and D. Worku, J. Phys. G **G39** (2012) 025006; arXiv:1203.4343[hep-ph].

Rewrite the Tsallis distribution using

$$[1+(q-1)x]^{1/(1-q)} = \exp\left(\frac{1}{1-q}\ln\left[1+(q-1)x\right]\right),$$

and consider the limit $q \to 1$

$$\begin{aligned}\lim_{q\to 1} \quad & [1+(q-1)x]^{1/(1-q)} \\ &= \exp\frac{1}{(1-q)}(q-1)x \\ &= \exp(-x), \qquad (1)\end{aligned}$$

The Tsallis distribution reduces to the Boltzmann distribution in the limit where $q \to 1$

$$\lim_{q\to 1}\frac{d^2N}{dp_t\, dy} = gV\frac{p_t m_t \cosh y}{(2\pi)^2}\exp\left(-\frac{m_t\cosh y - \mu}{T}\right). \qquad (2)$$

In all cases q is close to one, typically between 1.05 and 1.2.

Comparison of Tsallis with STAR, ALICE, CMS distributions

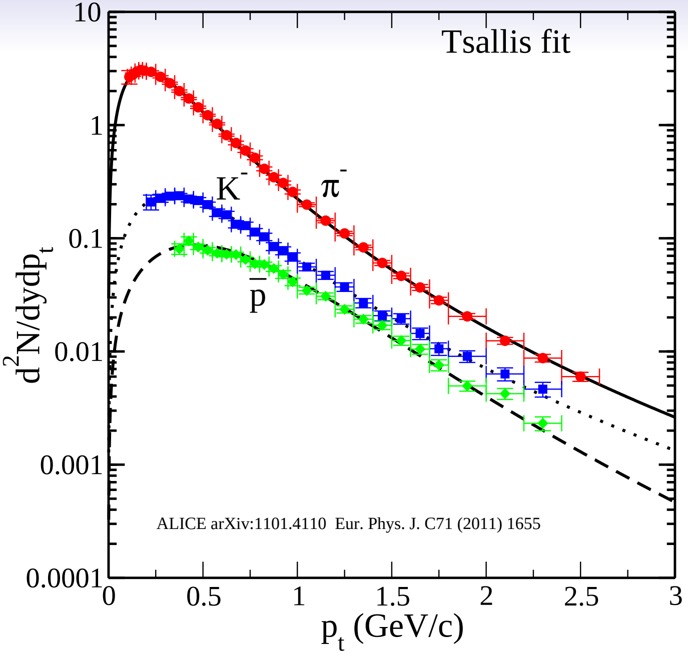
$$\frac{\mathrm{d}^2 N}{\mathrm{d}p_\mathrm{t}\mathrm{d}y} = gV\frac{p_t m_t}{(2\pi)^2}\left[1+(q-1)\frac{m_t}{T}\right]^{q/(1-q)}, \tag{3}$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}p_\mathrm{t}\mathrm{d}y} = p_\mathrm{t}\times\frac{\mathrm{d}N}{\mathrm{d}y}\frac{(n-1)(n-2)}{nC(nC+m_0(n-2))}\left[1+\frac{m_\mathrm{t}-m_0}{nC}\right]^{-n} \tag{4}$$

$$n \rightarrow \frac{q}{q-1}$$

$$nC \rightarrow \tfrac{T}{q-1}\tfrac{m_t-m_0}{m_t}$$

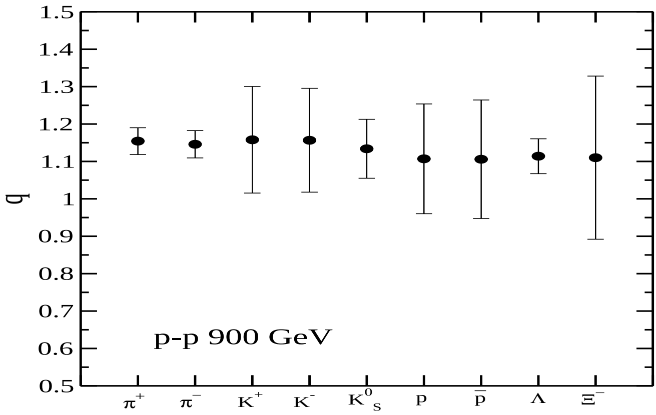
Only a factor of m_T differs! However, m_0 shouldn't appear as it destroys m_T scaling.


Tsallis fit
K^-
π^-
$\bar{p}$
$d^2N/dydp_t$
p_t (GeV/c)
ALICE arXiv:1101.4110 Eur. Phys. J. C71 (2011) 1655
10
1
0.1
0.01
0.001
0.0001
0
0.5
1
1.5
2
2.5
3

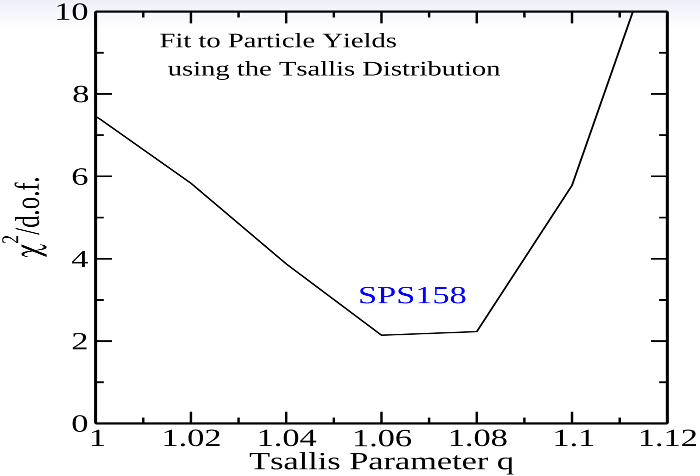
$p-p$ 900 GeV		
Particle	q	T
π^+	1.154 $\pm$0.036	0.0682 $\pm$0.0026
π^-	1.146 $\pm$0.036	0.0704 $\pm$ 0.0027
K^+	1.158 $\pm$0.142	0.0690 $\pm$0.0223
K^-	1.157 $\pm$0.139	0.0681 $\pm$ 0.0217
K_S^0	1.134 $\pm$0.079	0.0923 $\pm$0.0139
p	1.107 $\pm$0.147	0.0730 $\pm$ 0.0425
$\bar{p}$	1.106 $\pm$0.158	0.0764 $\pm$0.0464
Λ	1.114 $\pm$0.047	0.0698 $\pm$ 0.0148
Ξ^-	1.110 $\pm$0.218	0.0440 $\pm$ 0.0752

Table: Fitted values of the T and q parameters measured in $p-p$ collisions by the ALICE and CMS collaborations using the Tsallis form for the momentum distribution.

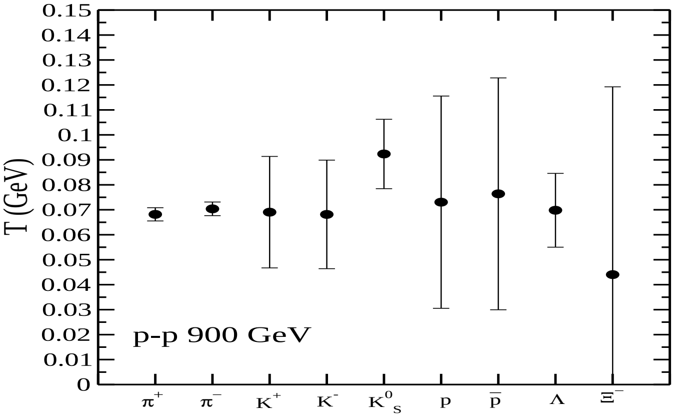
$p-p$ 900 GeV		
Particle	T Tsallis vs C ALICE (MeV)	q
π^+	70 (126)	1.147
K^+	70 (160)	1.156
p	73 (196)	1.110



Values of the Tsallis parameter q for different species of hadrons.



J. C., G. Hamar, P. Levai, S. Wheaton
Journal of Physics **G 36** (2009) 064018.



Values of the Tsallis temperature T for different species of hadrons.

J.C. and D. Worku e-Print: arXiv:1110.5526 [hep-ph]