Lectures on Spinodal Instabilities in Phase Transitions: Problems

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I-1

Thermodynamic limit: Consider uniform matter in a volume V; its total energy is E and it contains a total of N "particles". In the thermodynamics limit, $V \to \infty$, the physical properties depend only on the intensive quantities $\varepsilon \equiv E/V$ (the energy density) and $\rho \equiv E/V$ (the number density); so the entropy is $S(V, E, N) = V\sigma(\varepsilon, \rho)$ where $\sigma \equiv S/V$ is the entropy density.

1. Show that $1/T = \partial_E S(V, E, N)_{VN}$ is given by $\beta(\varepsilon, \rho) = \partial_\varepsilon \sigma(\varepsilon, \rho)$ and the the quantity $-\mu/T = \partial_N S(V, E, N)_{VE}$ is given by $\alpha(\varepsilon, \rho) = \partial_\rho \sigma(\varepsilon, \rho)$.

2. Show that $p/T = \partial_V S(V, E, N)_{EN}$ is given by $\pi(\varepsilon, \rho) = \sigma(\varepsilon, \rho) - \beta(\varepsilon, \rho)\varepsilon - \alpha(\varepsilon, \rho)\rho$.

I-2

Canonical scenario: In the canonical scenario the independent variable are ρ and T and the basic function is the free energy density $f = \varepsilon - T\sigma$. Show that the chemical potential and the entropy density can be obtained as $\mu_T(\rho) = \partial_\rho f_T(\rho)$ and $\sigma_T(\rho) = -\partial_T f_T(\rho)$; show furthermore that the pressure is $p_T(\rho) = \rho^2 \partial_\rho (f_T(\rho)/\rho)$.

I-3

Phase coexistence: Assume that the free energy density $f_T(\rho)$ is locally concave. Then there must exist two densities, ρ_1 and ρ_2, at which the tangents to $f_T(\rho)$ coincide.

1. Show that then the chemical potentials at ρ_1 and ρ_2 are equal.

2. Show that the corresponding two pressures are also equal.

I-4

Interface profile: Consider two coexisting phases of bulk matter having a common interface. The interface profile $\rho(x)$ is determined by the equation $C\partial_x^2 \rho(x) = \mu_T(\rho(x)) - \mu_0$. This equation is mathematically equivalent to that governing the motion $\xi(t)$ of a particle in a potential, $M\partial_t^2 \xi = \partial_\xi U(\xi)$, identifying x with the 'time' t and ρ with the 'position' ξ; C is then the 'mass' M, while $-\Delta f_T(\rho) = f_T^M(\rho) - f_T(\rho)$ is the 'potential' $U(\xi)$. Show that the limiting behavior is $\xi \to \rho_1$ and $\partial_t \xi \to 0$ for $t \to -\infty$ and $\xi \to \rho_2$ and $\partial_t \xi \to 0$ for $t \to +\infty$. Furthermore, show that energy conservation yields the relation $\frac{1}{2}C(\partial_x \rho)^2 = \Delta f_T(\rho(x))$.

Isotropic flow in $\boldsymbol{N}$ dimensions: If the spatial variation of the viscosity coefficients η (shear) and ζ (bulk) may be ignored, the Euler equation becomes $\boldsymbol{\nabla}\cdot\boldsymbol{T} = \boldsymbol{\nabla}p - \eta\boldsymbol{\Delta}\boldsymbol{v} - [\frac{1}{3}\eta+\zeta]\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})$, where $\boldsymbol{T}(\boldsymbol{r},t)$ is the spatial part of the stress tensor $T^{\mu\nu}$ and $\boldsymbol{v}(\boldsymbol{r},t)$ is the local flow velocity. II-1

1. Show that for an isotropic expansion $[\rho(\boldsymbol{r}) = \rho(r)$ and $\boldsymbol{v}(\boldsymbol{r}) = v(r)\hat{\boldsymbol{r}}]$ the dissipative term in the Euler equation contains η and ζ only in the combination $\xi = \frac{4}{3}\eta + \zeta$, in any spatial dimension N.

2. For such isotropic flows in N dimensions, determine the limiting velocity profile $v(r)$ for which the dissipative term in the Euler equation vanishes.

II-2

Sound speeds: Verify these expressions for the isentropic and isothermal sound speeds:

1. $v_s^2 \;\equiv\; (\rho/h)(\partial p/\partial\rho)_s \;=\; -(T/h)[h^2\sigma_{\varepsilon\varepsilon} + 2h\rho\sigma_{\varepsilon\rho} + \rho^2\sigma_{\rho\rho}]$, where $s = \sigma/\rho$,

2. $v_T^2 \;\equiv\; (\rho/h)(\partial p/\partial\rho)_T \;=\; -(T/h)(\rho T/\sigma_{\varepsilon\varepsilon})[\sigma_{\varepsilon\varepsilon}\sigma_{\rho\rho} - \rho\sigma_{\varepsilon\rho}^2]$, with $\sigma_\varepsilon \equiv \partial_\varepsilon\sigma$, *et cetera.*