Hydrodynamics of heavy-ion collisions

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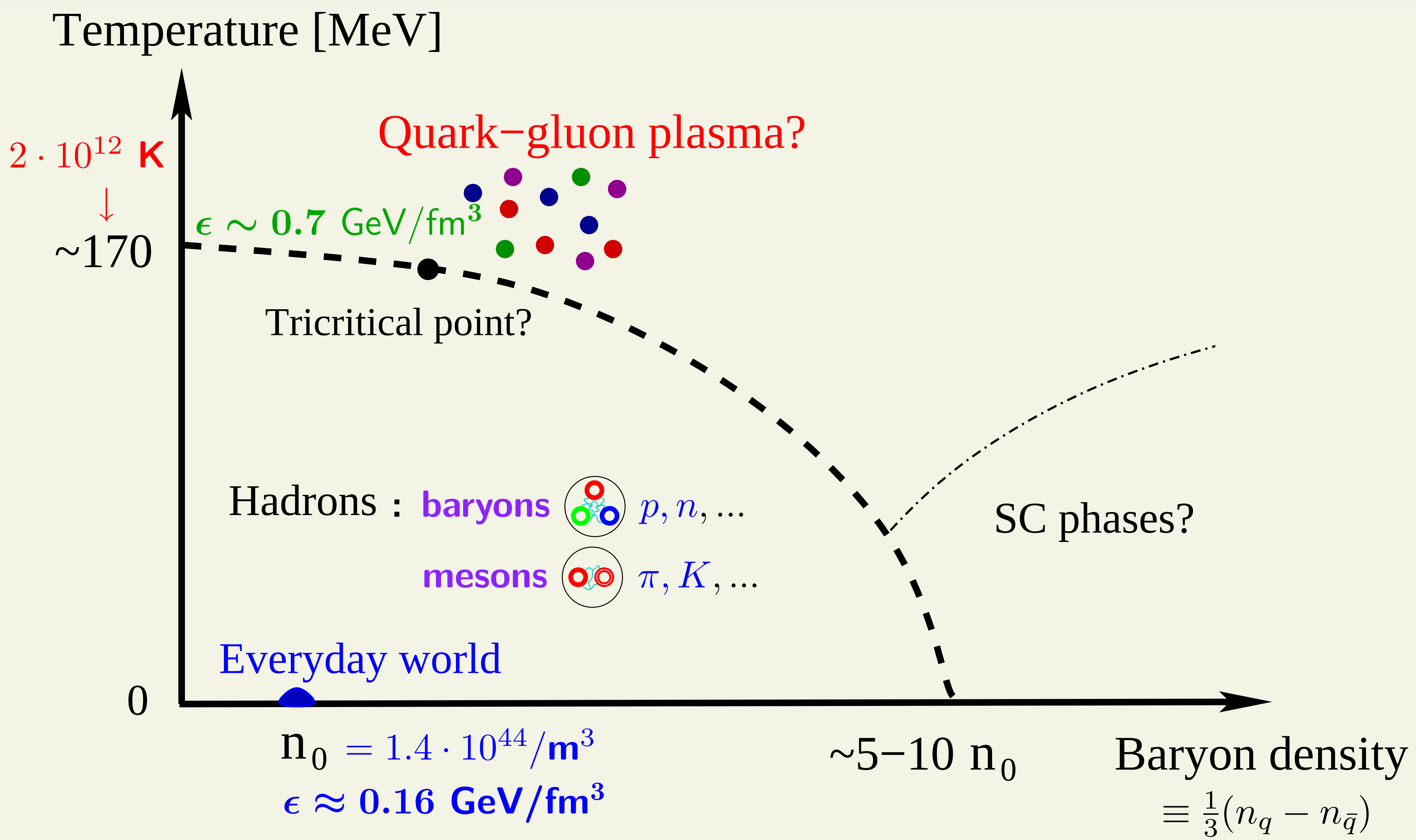
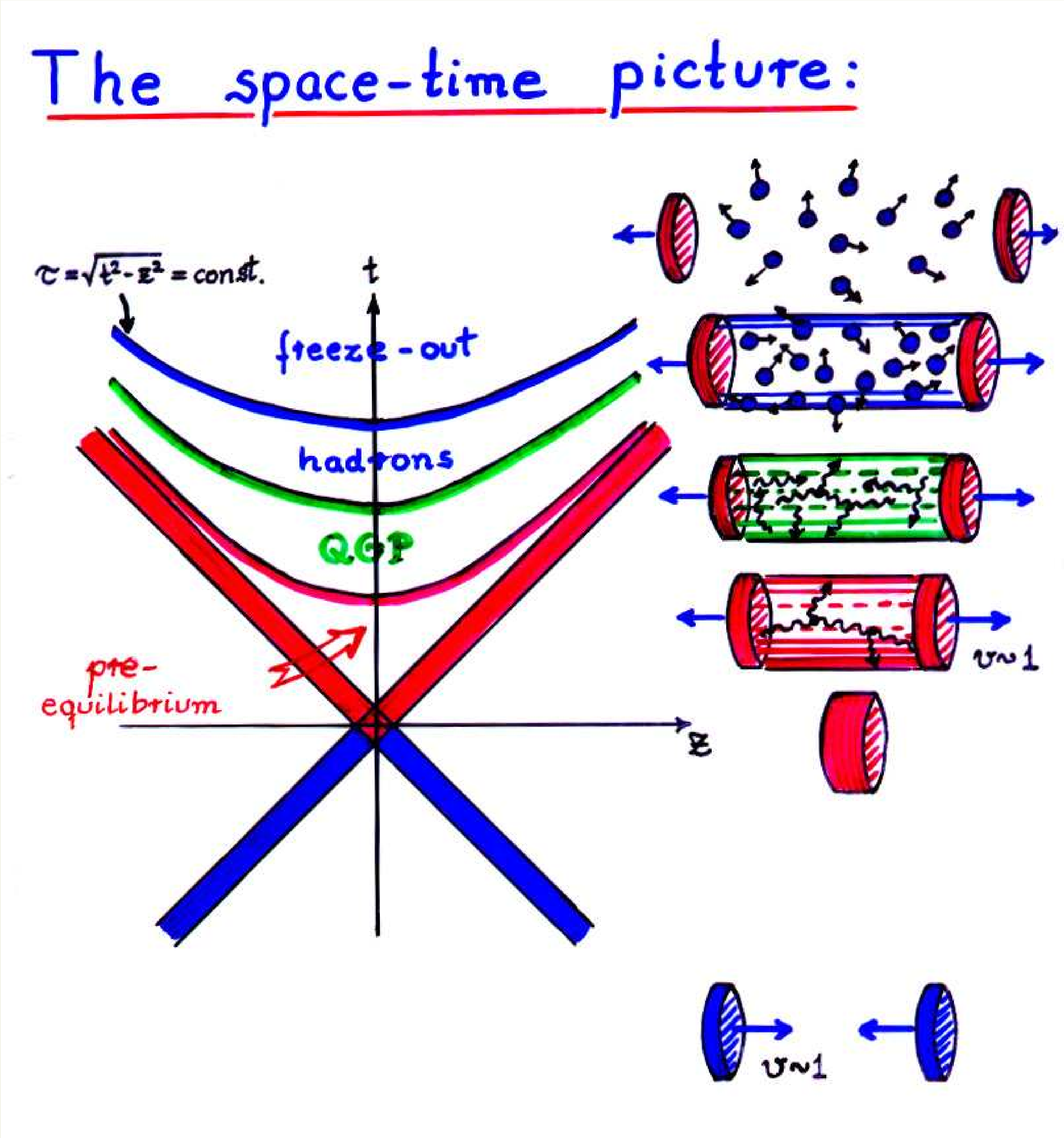
Nuclear phase diagram





Transient matter

- **lifetime**
  $t \sim 10\,\mathrm{fm/c}$
  $\sim 10^{-23}$ **seconds**
- **small size**
  $r \sim 10\,\mathrm{fm}$
  $\sim 10^{-14}\,\mathrm{m}$
- **rapid expansion**

Multiplicity @ LHC
~ 15000

Conservation laws

Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

Conservation of charge:

$$\partial_\mu N^\mu(x) = 0$$

Local conservation of particle number and energy-momentum.

$$\Longleftrightarrow$$ **Hydrodynamics!**

This can be generalized to multicomponent systems and systems with several conserved charges:

$$\partial_\mu N_i^\mu = 0,$$

$i =$ **baryon number, strangeness, charge. . .**

Consider only **baryon number conservation**, $i = B$.

$\Rightarrow$ **5 equations** contain **14 unknowns**!

$\Rightarrow$ **The system of equations does not close.**

$\Rightarrow$ Provide **9 additional equations** or
Eliminate **9 unknowns**.

So **what are the components** of $T^{\mu\nu}$ and N^{μ}?

- N^{μ} and $T^{\mu\nu}$ can be decomposed with respect to **arbitrary, normalized, time-like 4-vector** u^{μ},

$$u_{\mu}u^{\mu} = 1$$

- Define a **projection operator**

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}, \quad \Delta^{\mu\nu}u_{\nu} = 0,$$

which **projects on the 3-space** orthogonal to u^{μ}.

- **Then**

$$\boxed{N^\mu = nu^\mu + \nu^\mu}$$

where

$n = N^\mu u_\mu$ — **is (baryon) charge density in the frame where $u = (1, \mathbf{0})$, local rest frame, LRF**

$\nu^\mu = \Delta^{\mu\nu} N_\nu$ — **is charge flow in LRF,**

and

$$\boxed{T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}}$$

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$ **energy density in LRF**

$P \equiv -\frac{1}{3}\Delta^{\mu\nu} T_{\mu\nu}$ **isotropic pressure in LRF**

$q^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$ **energy flow in LRF**

$\pi^{\mu\nu} \equiv [\frac{1}{2}(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\nu{}_\beta \Delta^\mu{}_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}] T^{\alpha\beta}$

(trace-free) stress tensor in LRF

- **The 14 unknowns in 5 equations:**

$$\left.\begin{array}{cc} N^\mu & 4 \\ T^{\mu\nu} & 10 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{cc} n,\ \epsilon,\ P & 3 \\ q^\mu & 3 \\ \nu^\mu & 3 \\ \pi^{\mu\nu} & 5 \end{array}\right.$$

- So far u^μ is **arbitrary**. It attains a **physical meaning** by relating it to N^μ or $T^{\mu\nu}$:

1. **Eckart frame**:
$$u_E^\mu \equiv \frac{N^\mu}{\sqrt{N_\nu N^\nu}}$$
u^μ is 4-velocity of charge flow, $\nu^\mu = 0$.
The 14 unknowns are n, ϵ, P, q^μ, $\pi^{\mu\nu}, u^\mu$.
2. **Landau frame**:
$$u_L^\mu \equiv \frac{T^{\mu\nu}u_\nu}{\sqrt{u_\alpha T^{\alpha\beta}T_{\beta\gamma}u^\gamma}}$$
u^μ is 4-velocity of energy flow, $q^\mu = 0$.
The 14 unknowns are n, ϵ, P, ν^μ, $\pi^{\mu\nu}, u^\mu$.

- In general, the hydrodynamical equations are **not closed** and **cannot be solved uniquely**.

Ideal hydrodynamics

Suppose particles are in **local thermodynamical equilibrium**, i.e., single particle phase space distribution function is given by:

$$f_i(x,k) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{k_\mu u^\mu(x) - \mu(x)}{T(x)}\right) \pm 1\right]^{-1}$$

where

$T(x)$ and $\mu(x)$:	local temperature and chemical potential
$u(x)^\mu$:	local 4-velocity of fluid flow.

Then kinetic theory definitions give

$$\begin{aligned} N^\mu(x) &\equiv \sum_i q_i \int \frac{\mathrm{d}^3\mathbf{k}}{E}\, k^\mu f_i(x,k) = n(T,\mu)\, u^\mu \\ T^{\mu\nu}(x) &\equiv \sum_i \int \frac{\mathrm{d}^3\mathbf{k}}{E}\, k^\mu k^\nu f_i(x,k) \\ &= (\epsilon(T,\mu) + P(T,\mu))\, u^\mu u^\nu - P(T,\mu)\, g^{\mu\nu} \end{aligned}$$

where

$$n(T,\mu) = \sum_i q_i \int \mathrm{d}^3\mathbf{k}\, f_i(x,E)$$ **is local charge density,**

$$\epsilon(T,\mu) = \sum_i \int \mathrm{d}^3\mathbf{k}\, E f_i(x,E)$$ **is local energy density and**

$$P(T,\mu) = \sum_i \int \mathrm{d}^3\mathbf{k}\, \frac{\mathbf{k}^2}{3E} f_i(x,E)$$ **is local pressure.**

Note! $f(x,E)$ **is distribution in local rest frame:** $u^\mu = (1,\mathbf{0})$.

$\rightarrow$ **Local thermodynamical equilibrium implies there is no viscosity:**

$$\boxed{\nu^\mu = q^\mu = \pi^{\mu\nu} = 0.}$$

Ideal fluid approximation:

$$\begin{aligned} N^\mu &= nu^\mu \\ T^{\mu\nu} &= (\epsilon + P)u^\mu u^\nu - Pg^{\mu\mu} \end{aligned}$$

- **Local equilibrium** $\Rightarrow$ **no viscosity**: $\nu^\mu = q^\mu = \pi^{\mu\nu} = 0$.
- **Now N^μ and $T^{\mu\nu}$ contain 6 unknowns, ϵ, P, n and u^μ, but there are still only 5 equations!**
- **In thermodynamical equilibrium ϵ, P and n are not independent! They are specified by two variables, T and μ.**
- **The equation of state (EoS), $P(T, \mu)$ eliminates one unknown!**
- **Any equation of state of the form**

$$P = P(\epsilon, n)$$

closes the system of hydrodynamic equations and makes it uniquely solvable (given initial conditions).

Remark: $P = P(\epsilon, n)$ **is not a complete equation of state in a thermodynamical sense.**

A complete equation of state allows to compute all thermodynamic variables.

For example, $s = s(\epsilon, n)$**:** $\mathrm{d}s = 1/T\mathrm{d}\epsilon - \mu/T\mathrm{d}n$ **(1st law of thermod.)**

$$\frac{1}{T} = \frac{\partial s}{\partial \epsilon}|_n, \qquad \frac{\mu}{T} = -\frac{\partial s}{\partial n}|_\epsilon, \qquad P = Ts + \mu n - \epsilon$$

$P = P(\epsilon, n)$ **does not work!**

$$\frac{\partial P}{\partial \epsilon}|_n = ? \qquad \frac{\partial P}{\partial n}|_\epsilon = ?$$

However, $P = P(T, \mu)$ **does work!**

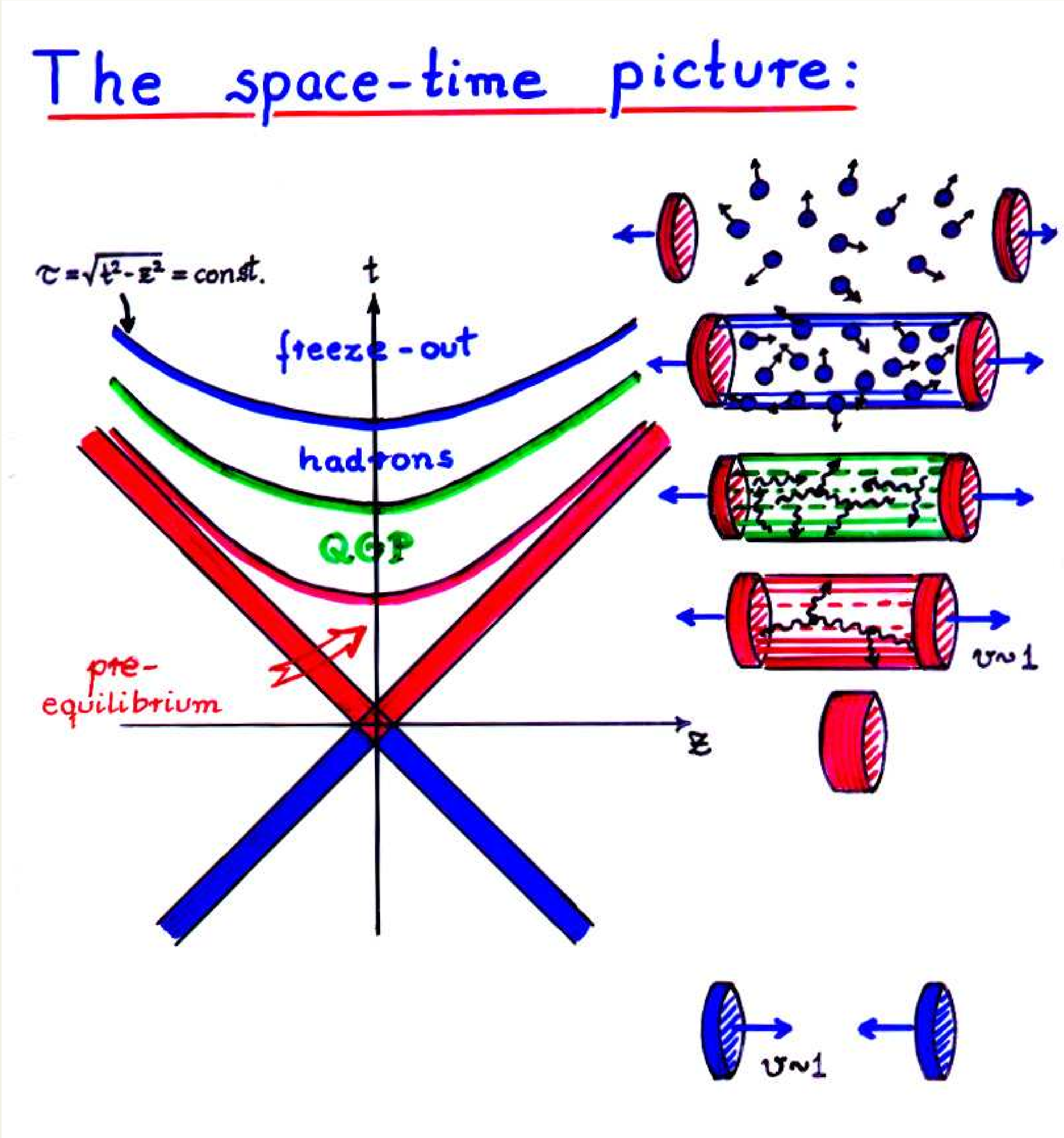
$$\mathrm{d}P = s\mathrm{d}T + n\mathrm{d}\mu \quad \Rightarrow \quad s = \frac{\partial P}{\partial T}|_\mu, \qquad n = \frac{\partial P}{\partial \mu}|_T$$

Entropy in ideal fluid

is conserved!

$$\partial_\mu S^\mu = 0$$

where $S^\mu = su^\mu$**.**


The space-time picture:
$\tau = \sqrt{t^2 - z^2} = const.$
t
freeze-out
hadrons
QGP
pre-equilibrium
z
$v \sim 1$
$v \sim 1$

Usefulness of hydro?

$$\left.\begin{array}{ll}\bullet\ \textbf{Initial state:} & \textbf{unknown}\\ \bullet\ \textbf{Equation of state:} & \textbf{unknown}\\ \bullet\ \textbf{Transport coefficients:} & \textbf{unknown}\\ \bullet\ \textbf{Freeze-out:} & \textbf{unknown}\end{array}\right\} \Rightarrow \quad \textbf{Predictive power?}$$

– *“Hydro doesn’t know where to start nor where to end”* **(M. Prakash)**

Usefulness of hydro?

$$\left.\begin{array}{ll} \bullet\ \textbf{Initial state:} & \textbf{unknown} \\ \bullet\ \textbf{Equation of state:} & \textbf{want to study} \\ \bullet\ \textbf{Transport coefficients:} & \textbf{want to study} \\ \bullet\ \textbf{Freeze-out:} & \textbf{unknown} \end{array}\right\} \Rightarrow \quad \textbf{Predictive power?}$$

$\Longrightarrow$ **Need More Constraints!**

“Hydrodynamical method”

1. Use another model to fix unknowns (and add new assumptions. . .)
   - initial: color glass condensate or pQCD+saturation
   - initial and/or final: hadronic cascade
   - etc.

2. Use data to fix parameters:

Principle		Example @ RHIC
• use one set of data	$\Longleftrightarrow$	$\left.\dfrac{\mathrm{d}N}{\mathrm{d}y\, p_T\, \mathrm{d}p_T}\right\vert_{b=0}$ and $\dfrac{\mathrm{d}N}{\mathrm{d}y}(b)$
• fix parameters to fit it	$\Longleftrightarrow$	$\begin{cases} \epsilon_{0,\max} = 29.6\,\mathsf{GeV/fm}^3 \\ \tau_0 = 0.6\,\mathsf{fm}/c \\ T_{\mathrm{fo}} = 130\,\mathsf{MeV} \end{cases}$
• predict another set of data	$\Longleftrightarrow$	HBT, photons & dileptons, elliptic flow. . .

Equations of motion

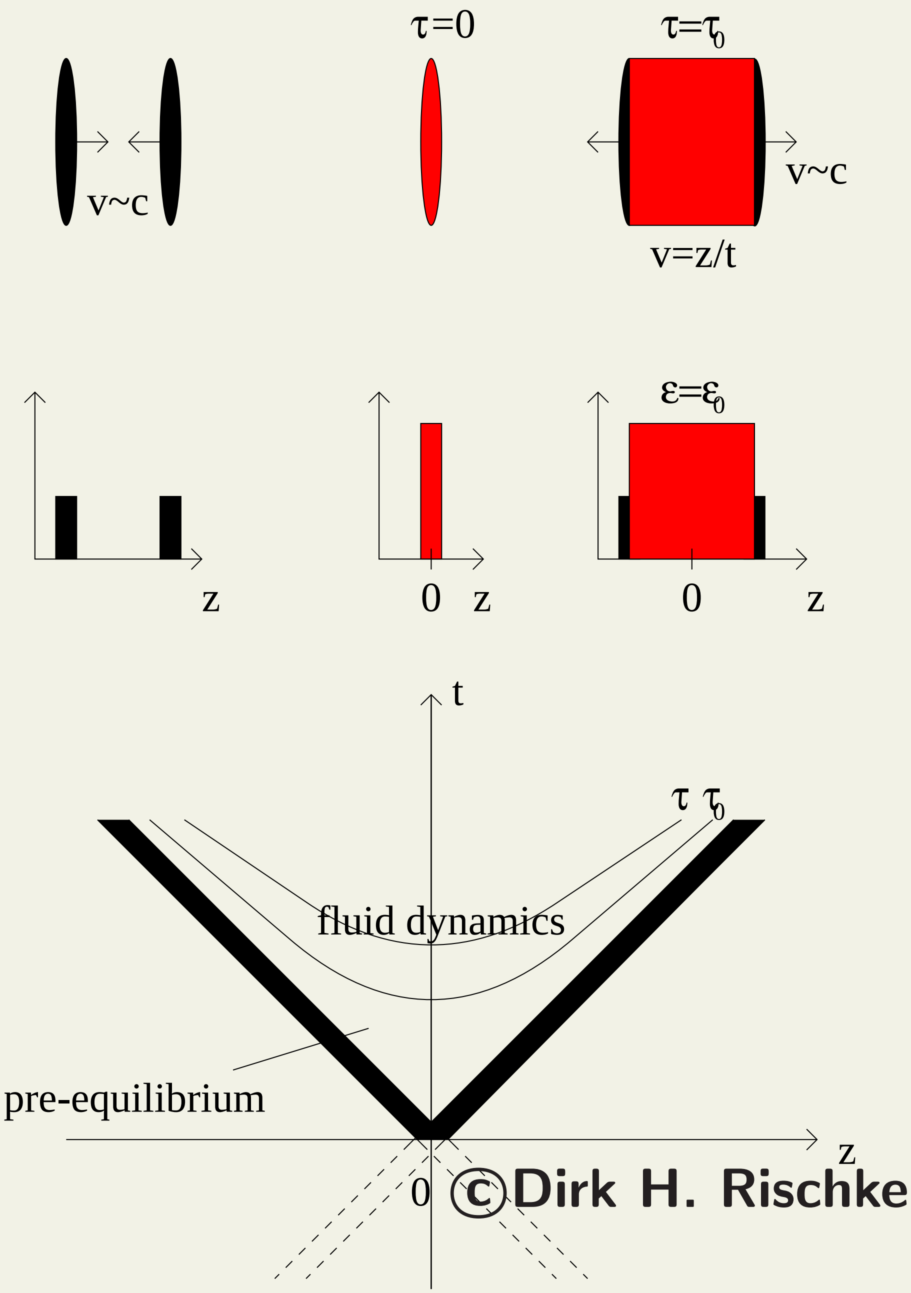
Conservation laws lead to the equations of motion for relativistic fluid:

$$\boxed{\begin{aligned} Dn &= -n\partial_\mu u^\mu \\ D\epsilon &= -(\epsilon + P)\partial_\mu u^\nu \\ (\epsilon + P)Du^\mu &= \nabla^\mu P, \end{aligned}}$$

where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu .$$

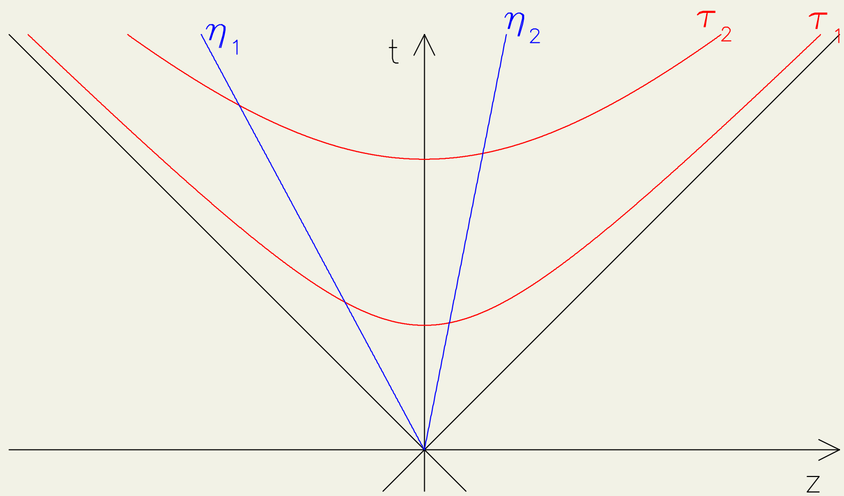
Bjorken hydrodynamics



- **At very large energies, $\gamma \to \infty$ and "Landau thickness" $\to 0$**
- **Lack of longitudinal scale**
  $\Rightarrow$ **scaling flow**

$$v = \frac{z}{t}$$

- **Practical coordinates to describe scaling flow expansion are**



  - **Longitudinal proper time τ:**

$$\boxed{\tau \equiv \sqrt{t^2 - z^2} \quad \Leftrightarrow \quad t = \tau \cosh \eta}$$

  - **Space-time rapidity η:**

$$\boxed{\eta = \frac{1}{2} \ln \frac{t+z}{t-z} \quad \Leftrightarrow \quad z = \tau \sinh \eta}$$

- **Scaling flow** $v = z/t \Rightarrow$ **fluid flow rapidity** $y = \eta$:

$$y = \frac{1}{2}\ln\frac{1+v}{1-v} = \frac{1}{2}\ln\frac{1+z/t}{1-z/t} = \eta$$

- **Ignore transverse expansion:**
  Hydrodynamic equations turn out to be particularly simple:

$$\left.\frac{\partial \epsilon}{\partial \tau}\right|_{\eta} = -\frac{\epsilon + P}{\tau} \qquad (1)$$

$$\left.\frac{\partial P}{\partial \eta}\right|_{\tau} = 0 \qquad (2)$$

$$\left.\frac{\partial n}{\partial \tau}\right|_{\eta} = -\frac{n}{\tau} \qquad (3)$$

- **Eq. (2)** $\Rightarrow$
  - **No force between fluid elements with different** η**!**
  - $P = P(\tau)$, **no** η**-dependence!**

- **Eq. (2) + thermodynamics:**

$$0 = \left.\frac{\partial P}{\partial \eta}\right|_{\tau} = s\left.\frac{\partial T}{\partial \eta}\right|_{\tau} + n\left.\frac{\partial \mu}{\partial \eta}\right|_{\tau}$$

If $n = 0$, $T = T(\tau) \Rightarrow T =$ const. on $\tau =$ const. surface.

- **In general T and ϵ not constant on $\tau =$ const. surface, but usually they are assumed to be**
  $\Rightarrow$ **boost invariance: the system looks the same in all reference frames!**

$$\epsilon = \epsilon(\tau), \qquad n = n(\tau)$$

- **Note that still**

$$\frac{\partial}{\partial \eta} T^{\mu\nu} \neq 0 \neq \frac{\partial}{\partial \eta} u^{\mu}$$

Vector and tensor quantities at finite η Lorentz boosted from values at $\eta = 0$

- **Thermodynamics:**

$$
\begin{aligned}
\mathrm{d}\epsilon &= T\,\mathrm{d}s + \mu\,\mathrm{d}n \\
\epsilon + P &= Ts + \mu n
\end{aligned}
$$

- **Eq. (1):**

$$
\frac{\partial\epsilon}{\partial\tau} + \frac{\epsilon + P}{\tau} = 0
$$

$$
\Rightarrow \quad T\frac{\partial s}{\partial\tau} + \mu\frac{\partial n}{\partial\tau} + T\frac{s}{\tau} + \mu\frac{n}{\tau} = 0
$$

$$
\textbf{(Eq. (3))} \quad \Rightarrow \quad \frac{\partial s}{\partial\tau} + \frac{s}{\tau} = 0
$$

$$
\Rightarrow \quad s(\tau) = s_0\frac{\tau_0}{\tau}
$$

$$
\Rightarrow \quad s\tau = \textbf{const.} \Rightarrow \mathrm{d}S/\mathrm{d}\eta = \textbf{const}
$$

independent of the equation of state!

- **Time evolution of baryon density:**

$$\text{Eq. (3)} \Rightarrow n(\tau) = n_0 \frac{\tau_0}{\tau} \Rightarrow \mathrm{d}N/\mathrm{d}\eta = \textbf{const}$$

also independent of the EoS.

- **Time evolution of energy density and temperature depend on the EoS.**

- **Assume ideal gas equation of state, $P = \frac{1}{3}\epsilon$, $\epsilon \propto T^4$:**

$$\begin{aligned}
\text{Eq. (1)} \quad &\Rightarrow \quad \frac{\partial \epsilon}{\partial \tau} + \frac{4\epsilon}{3\tau} = 0 \\
&\Rightarrow \quad \epsilon(\tau) = \epsilon_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{4}{3}} \\
&\Rightarrow \quad T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{1}{3}}
\end{aligned}$$

- **Note: τ_0: initial time, thermalization time**

Application: Initial energy density estimate

1. <u>“Bjorken estimate”</u>

- At $y = 0$, $E_T = E$
- Thus measuring

$$\left.\frac{\mathrm{d}E_T}{\mathrm{d}y}\right|_{y=0}$$

  gives total energy at $y = 0$.
- Estimate the initial volume:

$$V = A\,\Delta z = \pi R^2 \tau_0\,\Delta\eta$$

- Thus

$$\epsilon = \frac{1}{\pi R^2}\frac{E}{\tau_0 \Delta\eta} = \frac{1}{\pi R^2 \tau_0}\frac{\mathrm{d}E_T}{\mathrm{d}y}$$

- **Take $R = 6.3$ fm and $\tau_0 = 1$ fm/c:**

$$\text{@ SPS:} \quad \frac{\mathrm{d}E_T}{\mathrm{d}y} \approx 400 \text{ GeV} \quad \rightarrow \quad \epsilon \sim 3.2 \text{ GeV/fm}^3$$

$$\text{@ RHIC:} \quad \frac{\mathrm{d}E_T}{\mathrm{d}y} \approx 620 \text{ GeV} \quad \rightarrow \quad \epsilon \sim 5.0 \text{ GeV/fm}^3$$

- **Note that in this approach**

$$\epsilon(\tau) = \epsilon_0 \frac{\tau_0}{\tau}$$

No longitudinal work is done.

- **Pressure does work during expansion, $\mathrm{d}E = -P\mathrm{d}t$:**

$$\frac{\partial \epsilon}{\partial \tau} = \frac{\epsilon + P}{\tau} \Rightarrow \mathrm{d}(\epsilon\tau) = P\mathrm{d}\tau$$

Highly nontrivial

2. Entropy conservation

- **Assume ideal gas of massless particles:**

$$
\begin{aligned}
s &= 4n \Rightarrow \frac{\mathrm{d}S}{\mathrm{d}y} = 4\frac{\mathrm{d}N}{\mathrm{d}y} \\
s &= \frac{4g}{\pi^2}T^3 \\
\epsilon &= \frac{3g}{\pi^2}T^4
\end{aligned}
$$

- **With $s\tau = $ const. these give**

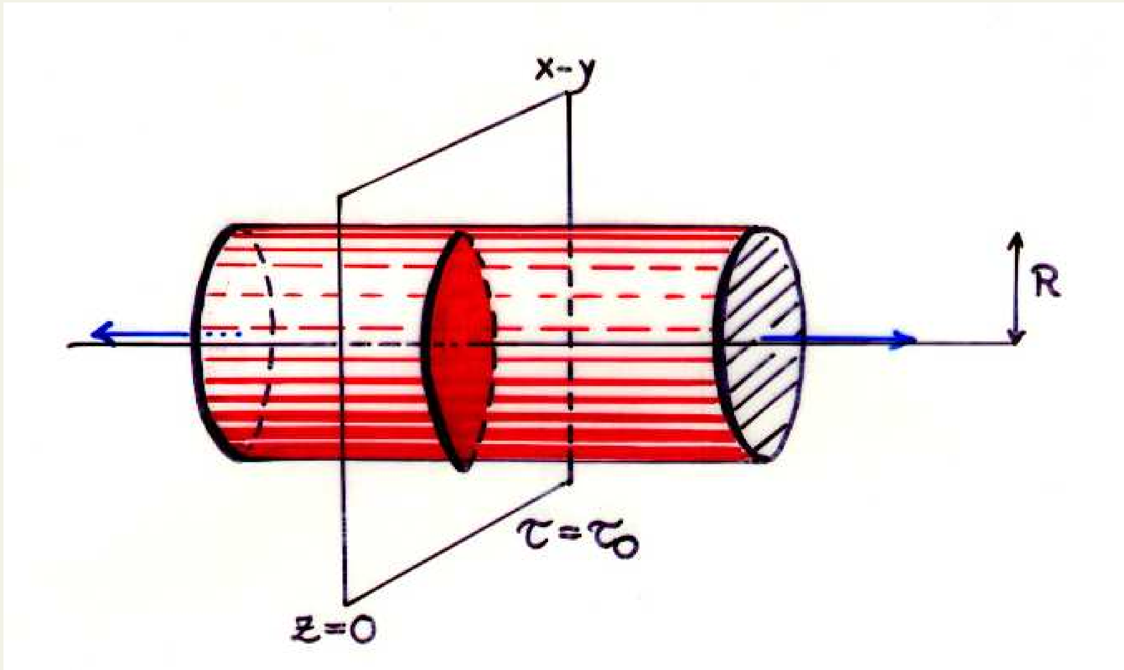
$$
\epsilon_0 = \frac{3}{\pi^{\frac{2}{3}} R^{\frac{8}{3}}\, \tau_0^{\frac{4}{3}}\, g^{\frac{1}{3}}} \left(\frac{\mathrm{d}N}{\mathrm{d}y}\right)^{\frac{4}{3}}
$$

@ RHIC: $\frac{\mathrm{d}N}{\mathrm{d}y} \approx 1000$

$g = 40$ **(2 flavours + gluons)**

$\Rightarrow \epsilon_0 \approx 6.0$ **GeV/fm^3**

Transverse expansion and flow



- Transverse expansion will set in latest at $\tau = R/c_s \approx 10$ fm
- Lifetimes in one dimensional expansion ~ 30 fm
- One dimensional expansion an oversimplification

- 2+1D: longitudinal Bjorken, transverse expansion solved numerically
- 3+1D: expansion in all directions solved numerically

- **Define speed of sound c_s:**

$$c_s^2 = \left.\frac{\partial P}{\partial \epsilon}\right|_{s/n_b}$$

- **large $c_s \Rightarrow$ "stiff EoS"**

- **small $c_s \Rightarrow$ "soft EoS"**

- **For baryon-free matter in rest frame**

$$(\epsilon + P)Du^\mu = \nabla^\mu P \qquad \Longleftrightarrow \qquad \frac{\partial}{\partial \tau} u_\mu = -\frac{c_s^2}{s}\partial_\mu s$$

$\Rightarrow$ **The stiffer the EoS, the larger the acceleration**

Initial conditions

- Initial time from early thermalization argument (+finetuning. . .)
- Total entropy to fit the multiplicity
- Density distribution?
- Multiplicity is proportional to the number of participants
- Glauber model: number of participants/binary collisions

$$N_{part}(b) = \int \mathrm{d}x\mathrm{d}y\, T_A(x + b/2, y)[\ldots$$

where

$$T_A(x,y) = \int_{-\infty}^{\infty} \mathrm{d}z\, \rho(x,y,z) \qquad \text{and} \qquad \rho(x,y,z) = \frac{\rho_0}{1 + e^{\frac{r-R_0}{a}}}$$

are nuclear thickness function and nuclear density distribution

- **"Differential Optical Glauber:"**
  Number of participants per unit area in transverse plane:

$$
\begin{aligned}
n_{\mathrm{WN}}(x,y;b) &= T_A(x+b/2,y)\left[1-\left(1-\frac{\sigma}{B}T_B(x-b/2,y)\right)^B\right] \\
&+ T_B(x-b/2,y)\left[1-\left(1-\frac{\sigma}{A}T_A(x-b/2,y)\right)^A\right]
\end{aligned}
$$

Number of binary collisions per unit area

$$
n_{\mathrm{BC}}(x,y;b) = \sigma_{pp} T_A(x+b/2,y) T_B(x-b/2,y)
$$

- **MC-Glauber:**
  - **sample $\rho(x,y,z)$ to get the positions of nucleons in 2 nuclei**
  - **count # of nucleons closer than $\sqrt{\sigma_{\mathrm{pp}}/\pi}$ in the collision**
  - **this gives n_{WN} and n_{BC}**
  - **repeat to get enough statistics**

Various flavors of Glauber

1. **eWN: energy density** $\epsilon(x, y; b) \propto n_{\mathrm{WN}}$

2. **eBC: energy density** $\epsilon(x, y; b) \propto n_{\mathrm{BC}}$

3. **sWN: entropy density** $s(x, y; b) \propto n_{\mathrm{WN}}$

4. **sBC: entropy density** $s(x, y; b) \propto n_{\mathrm{BC}}$

5. **any combination of these!**

- **multiplicity as function of centrality**

$\Longrightarrow \epsilon(x, y; b) = \kappa \cdot \epsilon_{\mathrm{WN}} + (1-\kappa) \cdot \epsilon_{\mathrm{BC}}$

or $s(x, y; b) = \lambda \cdot s_{\mathrm{WN}} + (1-\lambda) \cdot s_{\mathrm{BC}}$

Equation of state

- **Final state includes π's, K's, nucleons. . .**
  $\Rightarrow$ EoS of interacting hadron gas
  $\Rightarrow$ well approximated by non-interacting gas of hadrons and resonances

$$P(T) = \sum_i \int \mathrm{d}^3p \, \frac{p^2}{3E} \, f(p,T)$$

- **Plasma EoS (=massless parton gas) with proper statistics and $\mu_B \neq 0$:**

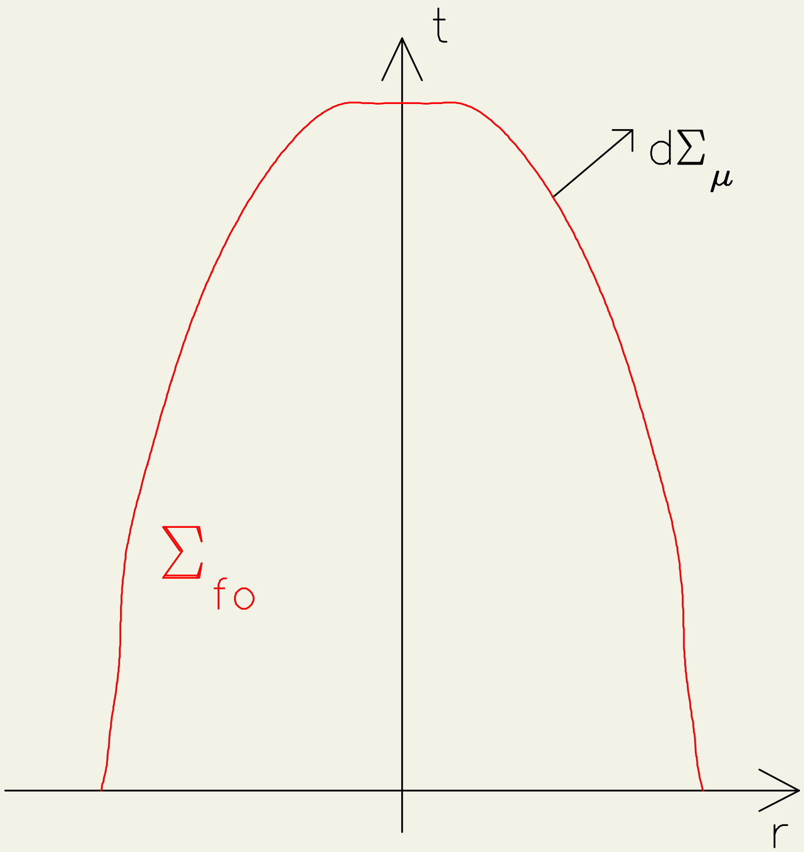
$$P(T,\mu) = \frac{(32+21N_f)\pi^2}{180} T^4 + \frac{1}{9}\mu_B^2 T^2 + \frac{1}{192\pi^2}\mu_B^4 - B$$

  $\Rightarrow$ First order phase transition by Maxwell construction

- **OR parametrized lattice result (only at $\mu_B = 0$):**
  $\Rightarrow$ match your favourite smoothly to HRG

When to end?

- Particles are observed, not fluid
- How and when to convert fluid to particles?
- i.e. how far is hydro valid?



- Kinetic equilibrium requires scattering rate $\gg$ expansion rate
- Scattering rate $\tau_{\rm sc}^{-1} \sim \sigma n \propto \sigma T^3$
- Expansion rate $\theta = \partial_\mu u^\mu$
- Fluid description breaks down when $\tau_{\rm sc}^{-1} \approx \theta$

$\rightarrow$ momentum distributions freeze-out

- $\tau_{\rm sc}^{-1} \propto T^3 \rightarrow$ rapid transition to free streaming
- Approximation: decoupling takes place on constant temperature hypersurface $\Sigma_{\rm fo}$, at $T = T_{\rm fo}$

Cooper-Frye

- Number of particles emitted = Number of particles crossing $\Sigma_{\rm fo}$

$$\Rightarrow \quad N = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, N^\mu$$

- Frozen-out particles do not interact anymore: kinetic theory

$$\Rightarrow \quad N^\mu = \int \frac{\mathrm{d}^3\mathbf{p}}{E} p^\mu f(x, p\cdot u)$$

$$\Rightarrow \quad N = \int \frac{\mathrm{d}^3\mathbf{p}}{E} \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)$$

- Invariant single inclusive momentum spectrum: (Cooper-Frye formula)

$$\boxed{E\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}^3} = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)}$$

Cooper and Frye, PRD 10, 186 (1974)

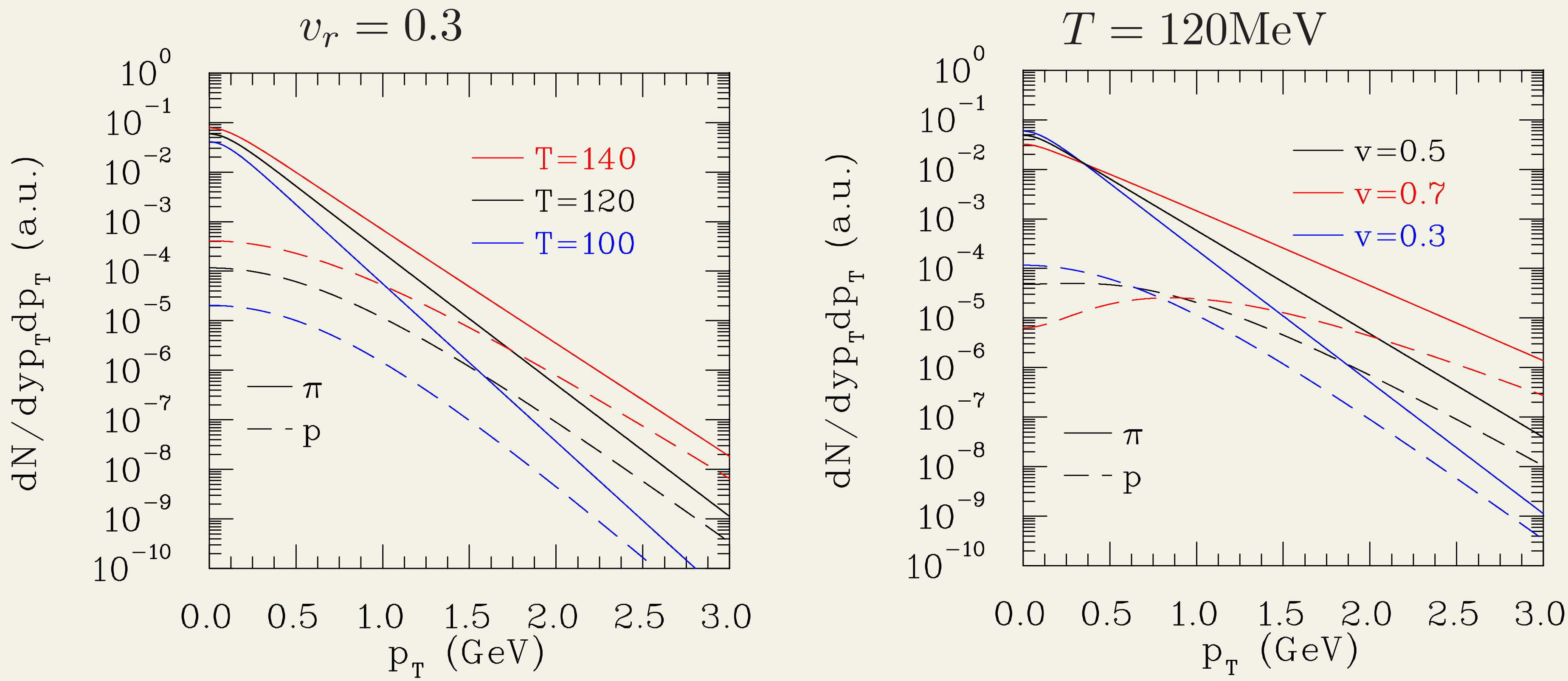
Blast wave

(Siemens and Rasmussen, PRL 42, 880 (1979))

- **Freeze-out surface a thin cylindrical shell radius r, thickness $\mathrm{d}r$, expansion velocity v_r, decoupling time τ_{fo}, boost invariant**
- **Cooper-Frye for Boltzmannions**

$$\frac{\mathrm{d}N}{\mathrm{d}y\, p_T\, \mathrm{d}p_T} = \frac{g}{\pi}\, \tau_{\mathrm{fo}}\, r\, m_T\, \mathrm{I}_0\!\left(\frac{v_r \gamma_r p_T}{T}\right) \mathrm{K}_1\!\left(\frac{\gamma_r m_T}{T}\right)$$

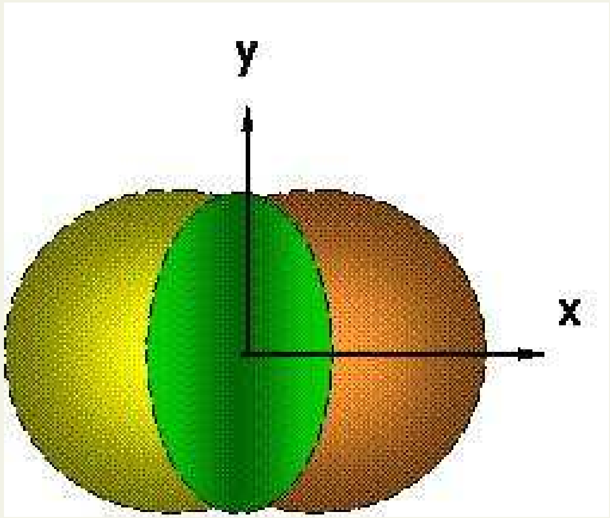
effect of temperature and flow velocity



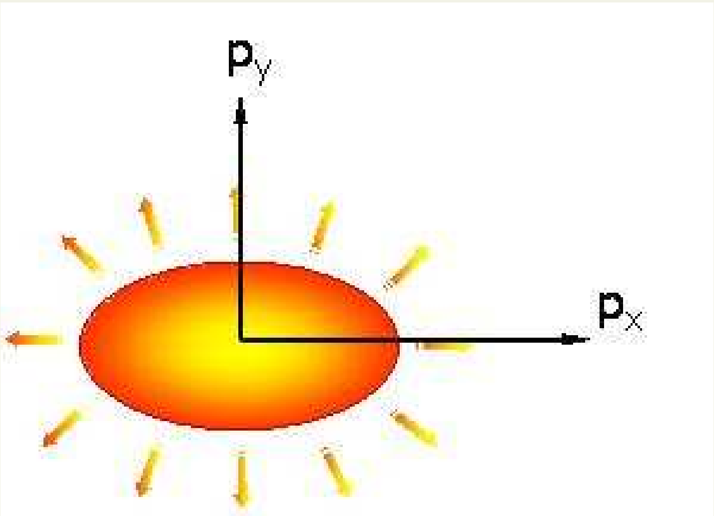
- **The larger the temperature, the flatter the spectra**
- **The larger the velocity, the flatter the spectra ⇒ blueshift**
- **The heavier the particle, the more sensitive it is to flow (shape and slope)**

Elliptic flow v_2

spatial anisotropy $\quad\rightarrow\quad$ final azimuthal momentum anisotropy



$\Longrightarrow$



- **Anisotropy in coordinate space + rescattering**

$\Rightarrow$ **Anisotropy in momentum space**

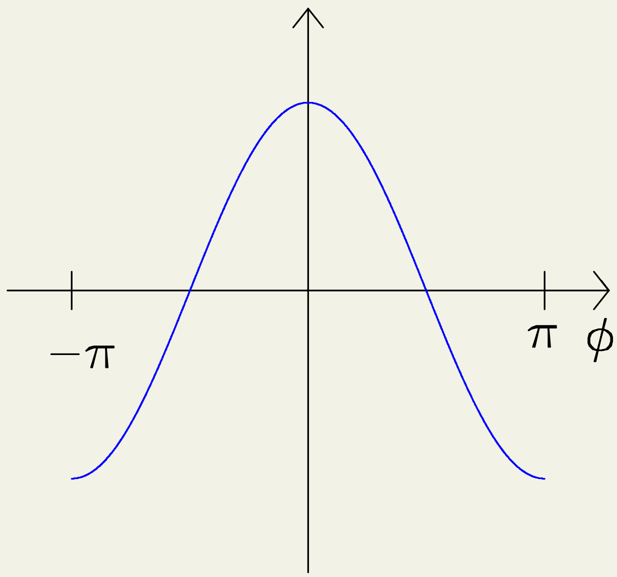
$$\frac{\partial}{\partial \tau} u_x = -\frac{c_s^2}{s}\frac{\partial}{\partial x} s \qquad \text{and} \qquad \frac{\partial}{\partial \tau} u_y = -\frac{c_s^2}{s}\frac{\partial}{\partial y} s$$

Elliptic flow v_2

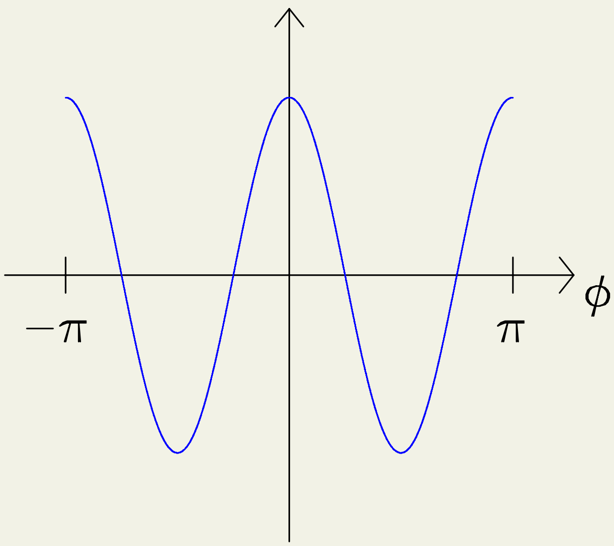
- **Fourier expansion of momentum distribution:**

$$\frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T\, \mathrm{d}\phi} = \frac{1}{2\pi}\frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T}(1 + 2v_1(y,p_T)\cos\phi + 2v_2(y,p_T)\cos 2\phi + \cdots)$$

v_1: **Directed flow: preferred direction**

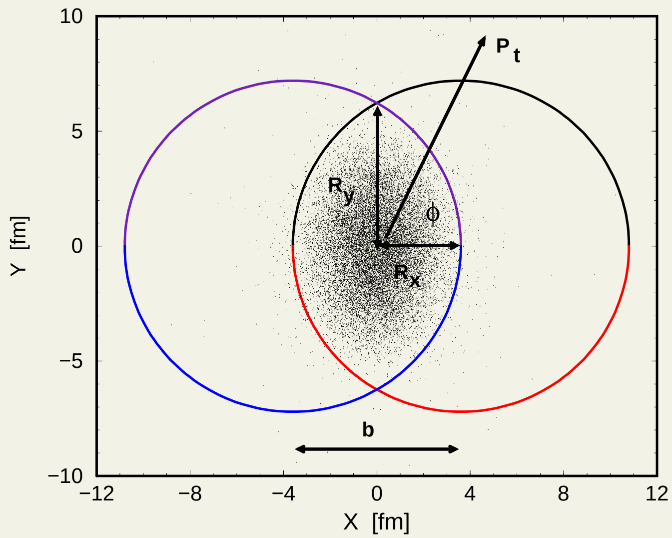


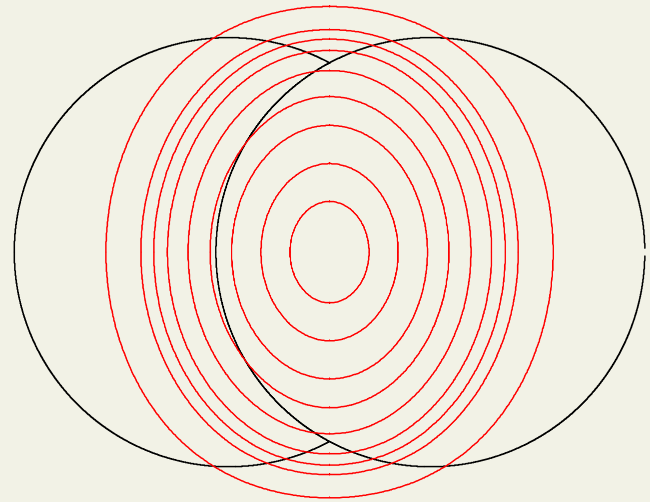
v_2: **Elliptic flow: preferred plane**



sensitive to speed of sound $c_s^2 = \partial p/\partial e$ **and shear viscosity** η

Measures of anisotropy





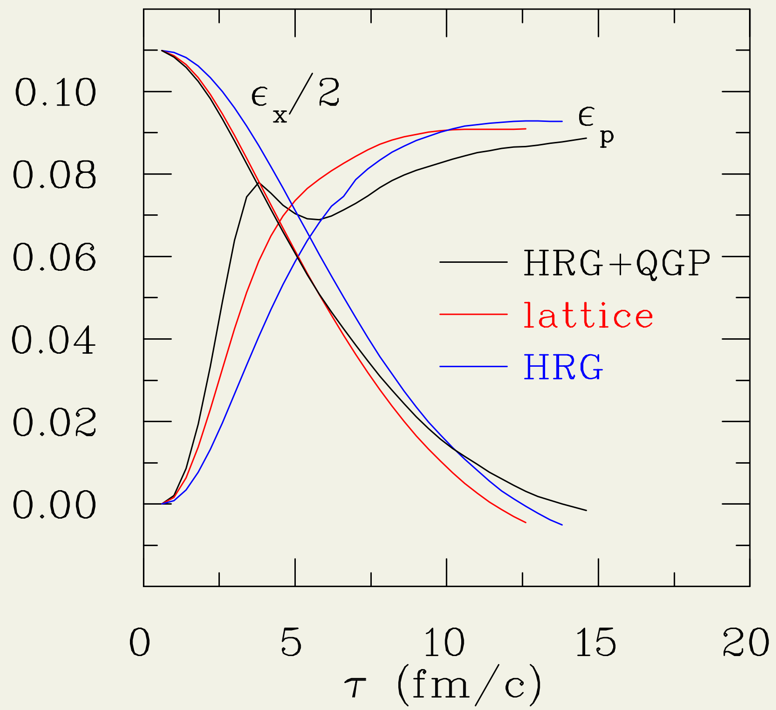
- **Spatial eccentricity**

$$\epsilon_x = \frac{\langle\langle y^2 - x^2 \rangle\rangle}{\langle\langle y^2 + x^2 \rangle\rangle} = \frac{\int \mathrm{d}x\,\mathrm{d}y\,\epsilon \cdot (y^2 - x^2)}{\int \mathrm{d}x\,\mathrm{d}y\,\epsilon \cdot (y^2 - x^2)}$$

- **Momentum anisotropy**

$$\epsilon_p = \frac{\langle T^{xx} - T^{yy} \rangle}{\langle T^{xx} + T^{yy} \rangle} = \frac{\int \mathrm{d}x\,\mathrm{d}y\,T^{xx} - T^{yy}}{\int \mathrm{d}x\,\mathrm{d}y\,T^{xx} - T^{yy}}$$

- **Au+Au @ RHIC, $b = 6$ fm:**



- ϵ_x **decreases during the evolution** $\Rightarrow$ **elliptic flow is self-quenching**

- **Most of** ϵ_p **is built up early in the evolution**

v_2

- **Not only collective but also thermal motion**
- **Elliptic flow v_2 a.k.a. p_T-averaged v_2:**

$$v_2 = \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle} = \frac{\int \mathrm{d}\phi\ \cos(2\phi) \frac{\mathrm{d}N}{\mathrm{d}y\,\mathrm{d}\phi}}{\mathrm{d}N/\mathrm{d}y}$$

- **p_T-differential v_2**

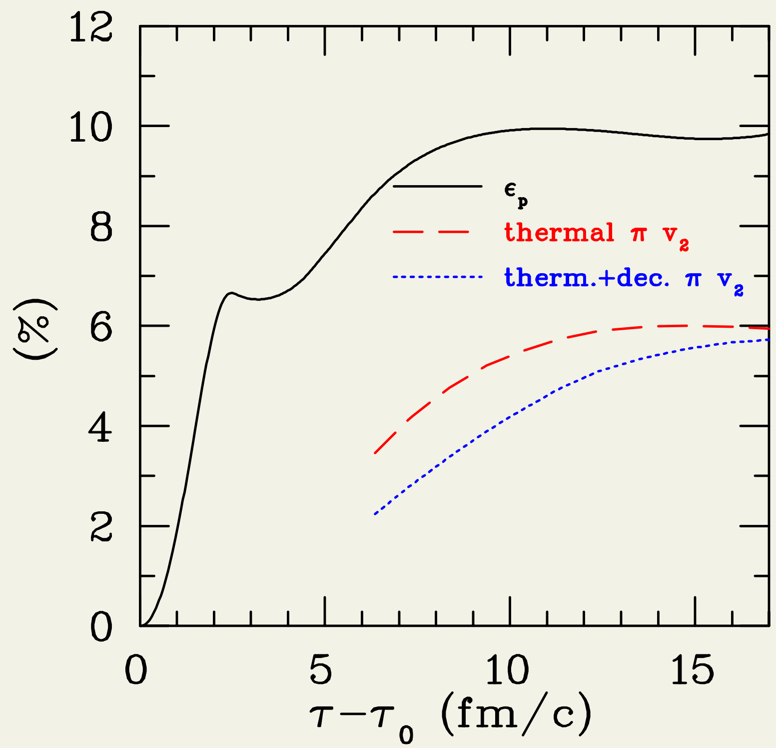
$$v_2(p_T) = \frac{\int \mathrm{d}\phi\ \cos(2\phi) \frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T\, \mathrm{d}\phi}}{\int \mathrm{d}\phi \frac{\mathrm{d}N}{\mathrm{d}y\, p_T \mathrm{d}p_T\, \mathrm{d}\phi}}$$

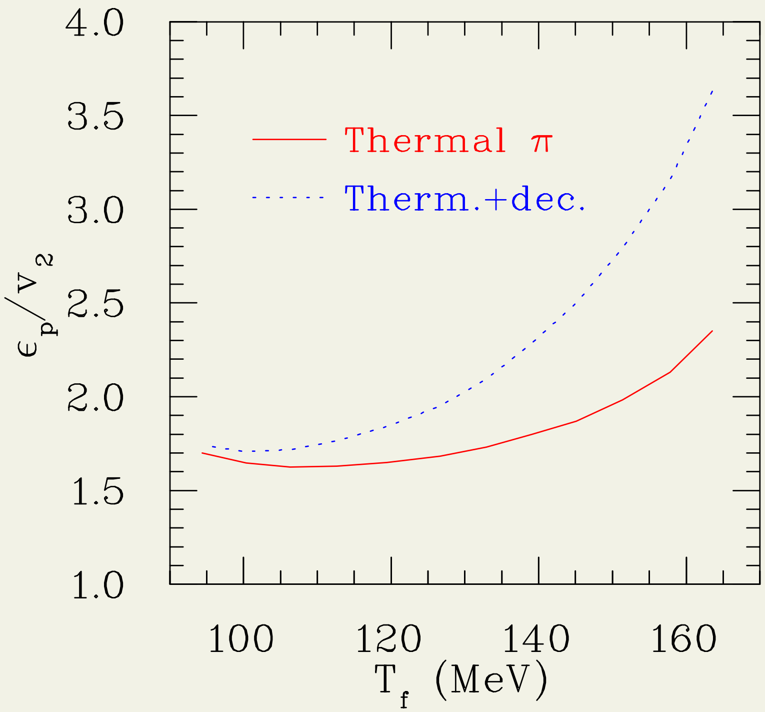
- **If $m_1 > m_2$, $v_2(m_1) > v_2(m_1)$, but $v_2(p_T, m_1) < v_2(p_T, m_2)$!**
- **No contradiction, since**

$$v_2 = \frac{\int \mathrm{d}p_T\, v_2(p_T) \frac{\mathrm{d}N}{\mathrm{d}p_T}}{\int \mathrm{d}p_T\, \frac{\mathrm{d}N}{\mathrm{d}p_T}}$$

ϵ_p vs. v_2

- Au+Au @ RHIC, $b = 7$ fm:

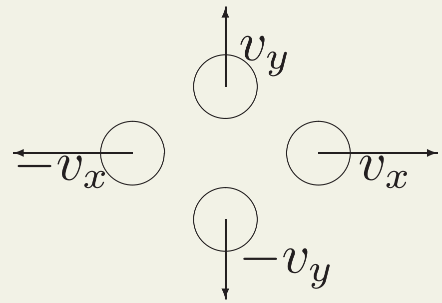




- **NO** clear correspondence
- especially if one includes resonance decays

Why $m_1 < m_2 \Rightarrow v_2(p_T, m_1) > v_2(p_T, m_2)$?

Simple source:



Each element has unit volume, decouples at the same time at the same temperature. Flow velocity in plane is larger than out of plane, $|v_x| > |v_y|$.

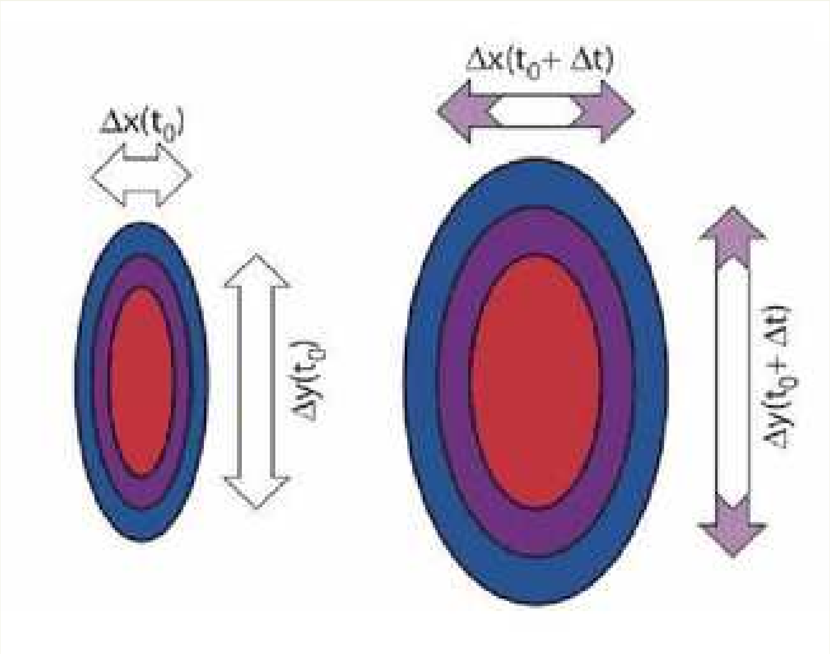
Boltzmann distribution and Cooper-Frye formula:

$$
\begin{aligned}
v_2(p_T) &= \frac{\mathrm{I}_2\left(\frac{\gamma_x v_x p}{T}\right) - e^{\frac{E}{T}(\gamma_x - \gamma_y)}\mathrm{I}_2\left(\frac{\gamma_y v_y p}{T}\right)}{\mathrm{I}_0\left(\frac{\gamma_x v_x p}{T}\right) + e^{\frac{E}{T}(\gamma_x - \gamma_y)}\mathrm{I}_0\left(\frac{\gamma_y v_y p}{T}\right)} \\
&= \frac{C_1 - e^{\lambda\sqrt{m^2+p^2}}C_2}{C_3 + e^{\lambda\sqrt{m^2+p^2}}C_4}
\end{aligned}
$$

mass increases, numerator decreases and denominator increases
$\rightarrow v_2$ decreases

Early thermalization?

- ϵ_p/ϵ_x **almost independent of** b, **i.e. the initial value of** ϵ_x
- **Before thermalization,** $\tau < \tau_0$ **system expands to all directions,** ϵ_x **decreases**



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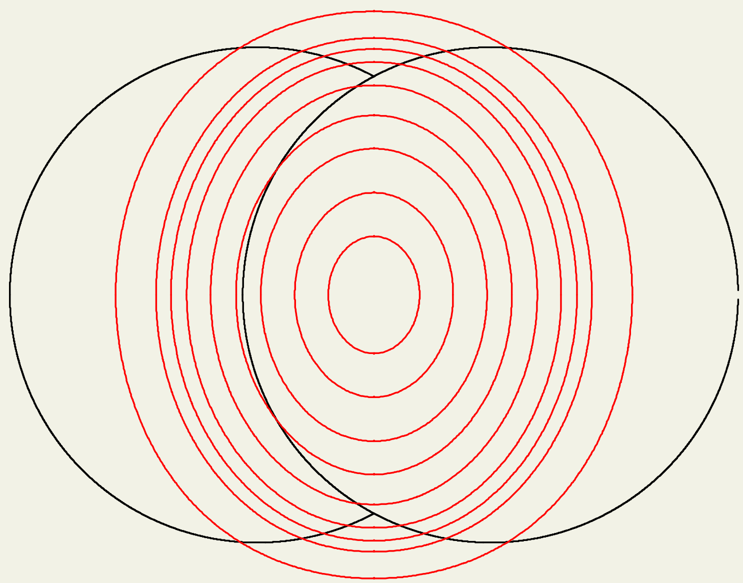
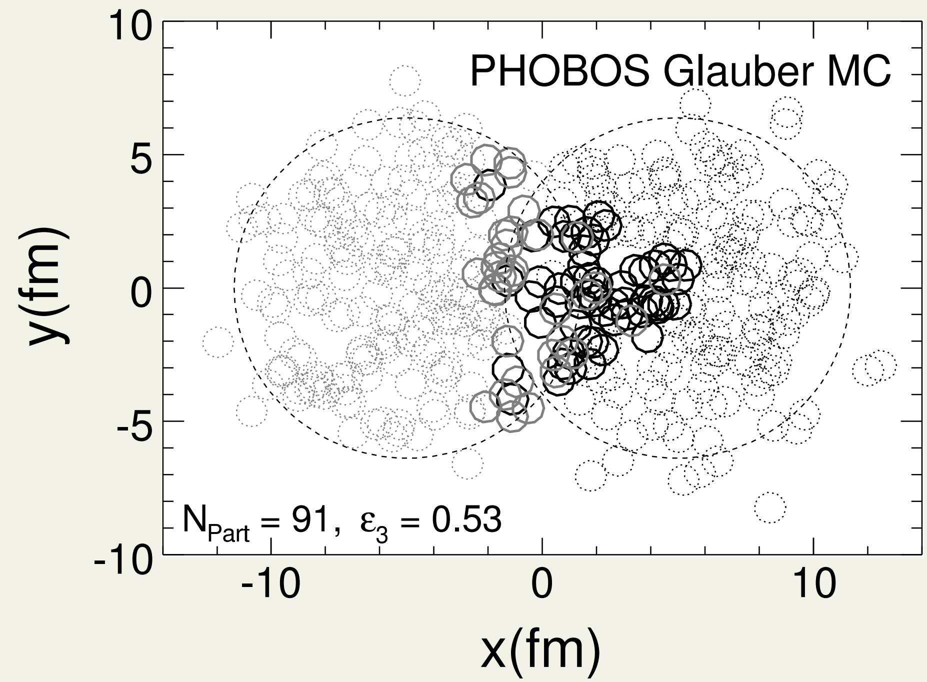
$\Longrightarrow$ **Hydrodynamical evolution must start early or final** v_2 **is too small**
- **We do not know if** v_2 **could build up before thermalization...**

event-by-event

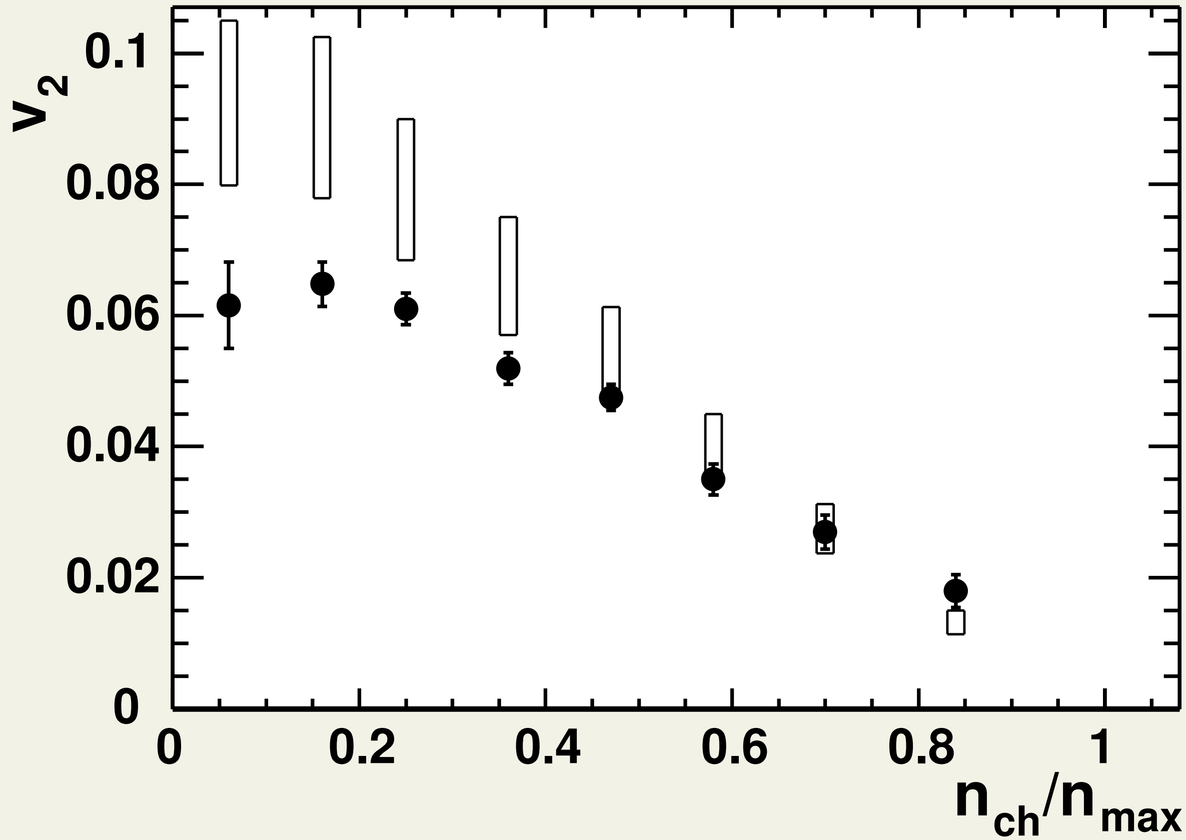




- **shape fluctuates event-by-event**
- **all coefficients v_n finite**

Success of ideal hydrodynamics

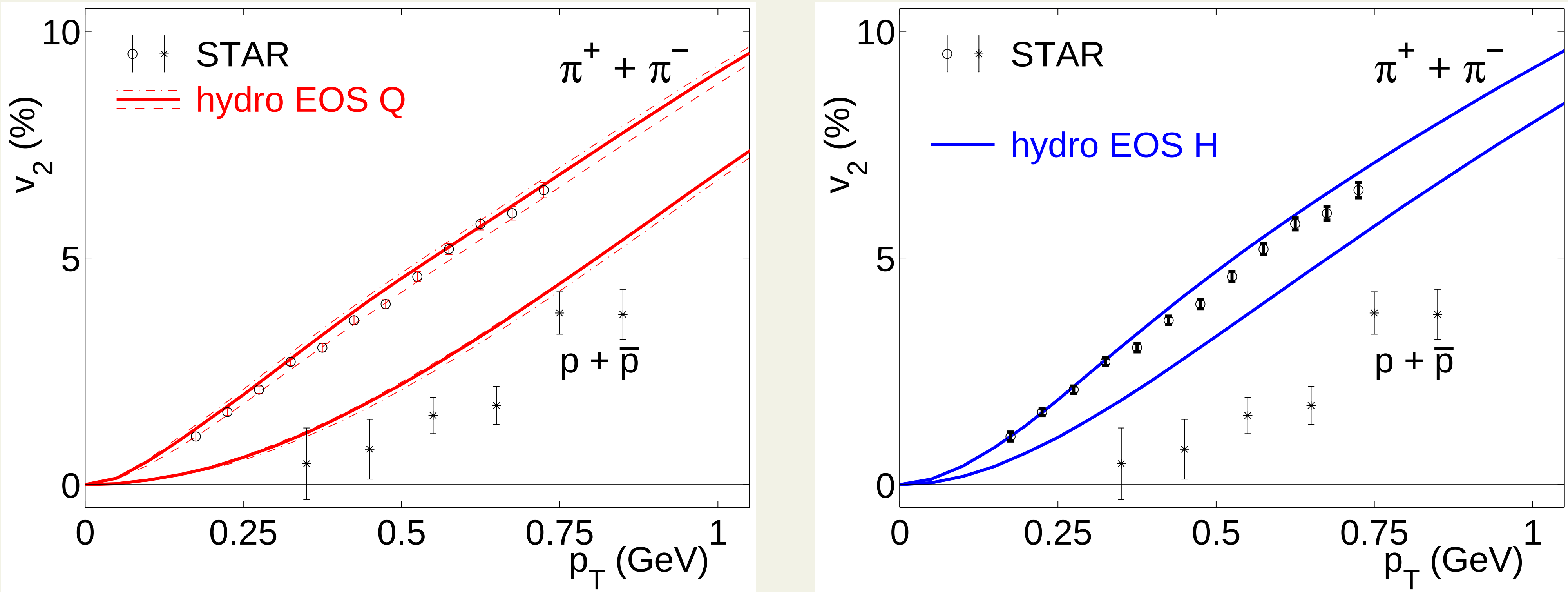
- p_T-averaged v_2 of charged hadrons:



- works beautifully in central and semi-central collisions
- but why is $v_{2,\mathrm{obs}} > v_{2,hydro}$ in most central collisions?

Success of ideal hydrodynamics

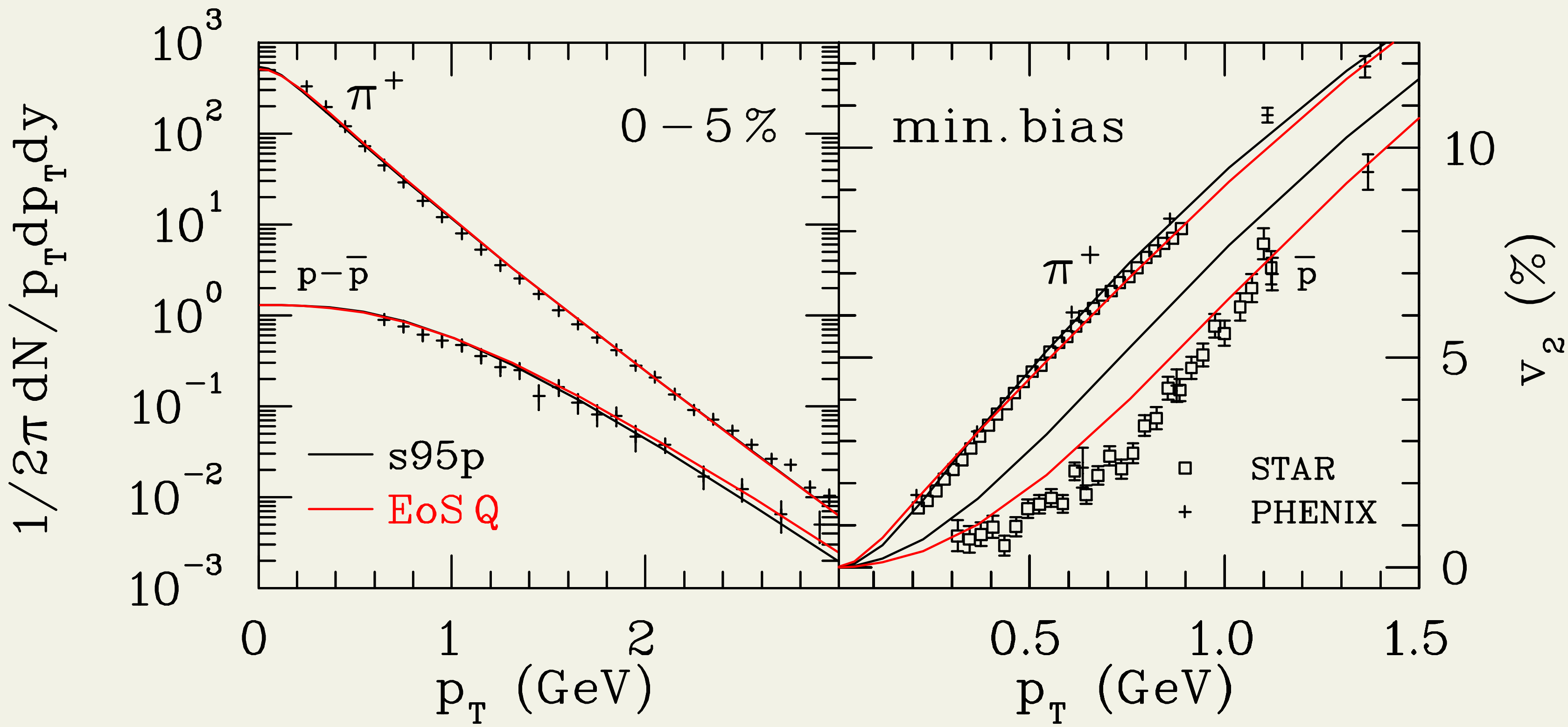
Kolb, Heinz, Huovinen et al ('01) minbias Au+Au at RHIC



not perfect agreement but plasma EoS favored

Lattice EoS

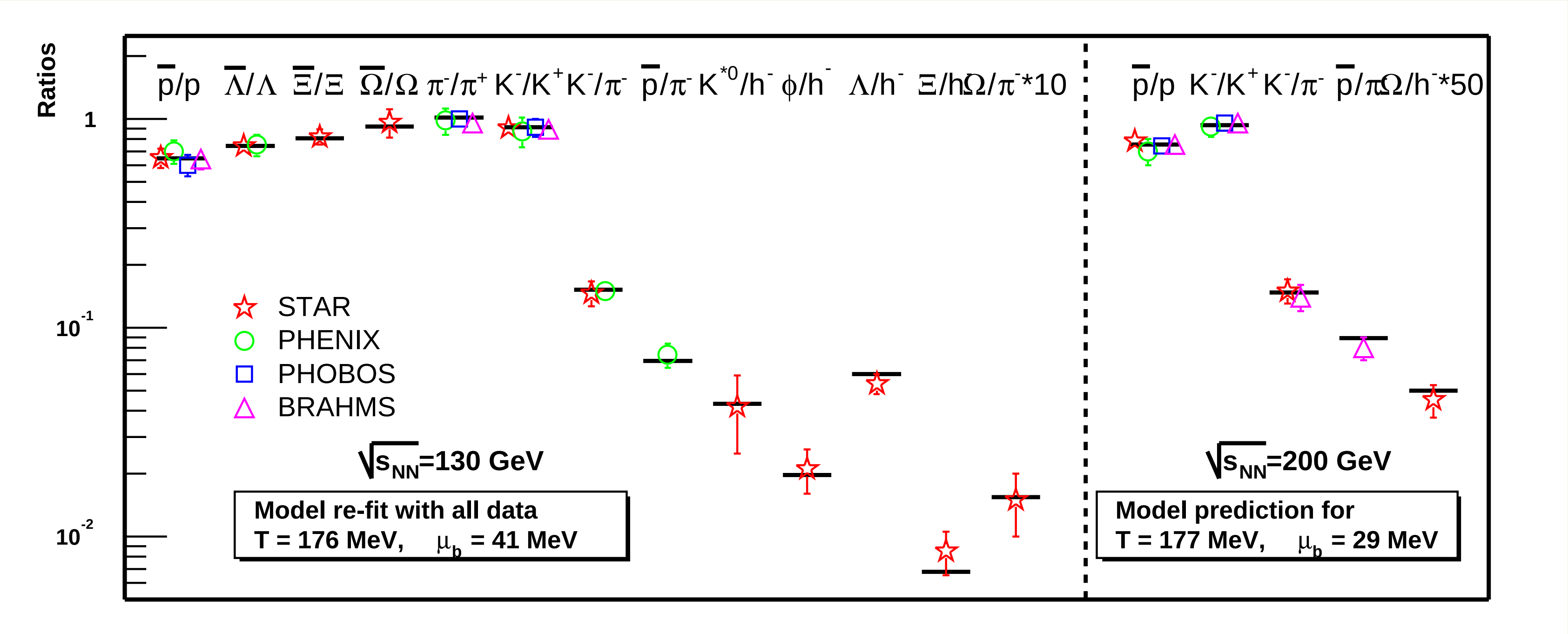
- ideal hydro, Au+Au at $\sqrt{s_{NN}} = 200$ GeV
- chemical equilibrium



- **s95p:** $T_{dec} = 140$ **MeV**
- **EoS Q**: first order phase transition at $T_c = 170$ MeV, $T_{dec} = 125$ **MeV**

Thermal models

- Hadronic phase: ideal gas of massive hadrons and resonances
- in chemical equilibrium



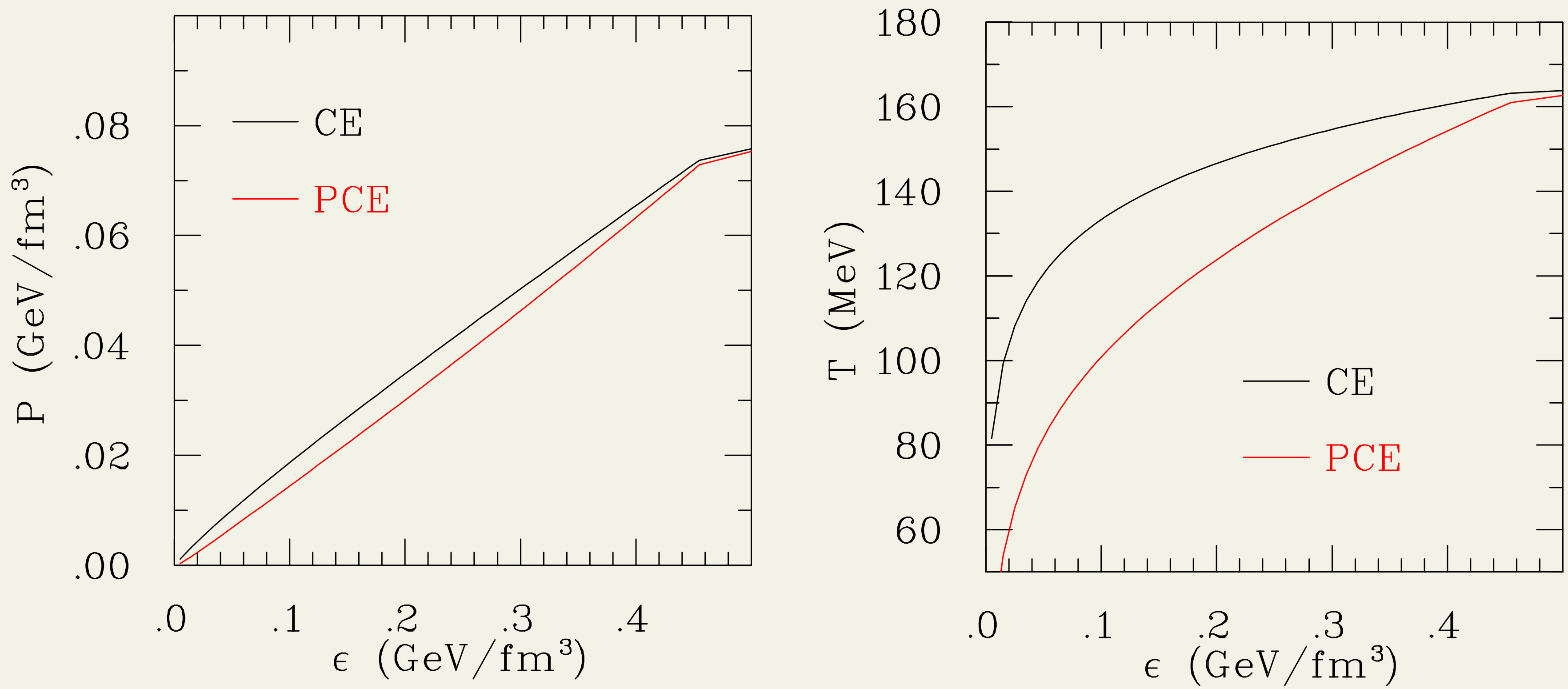
- **Particle ratios** $\Longleftrightarrow$ $T \approx 160$–170 **MeV temperature**

- **Evolution to** $T \approx 100$–120 **MeV temperature**

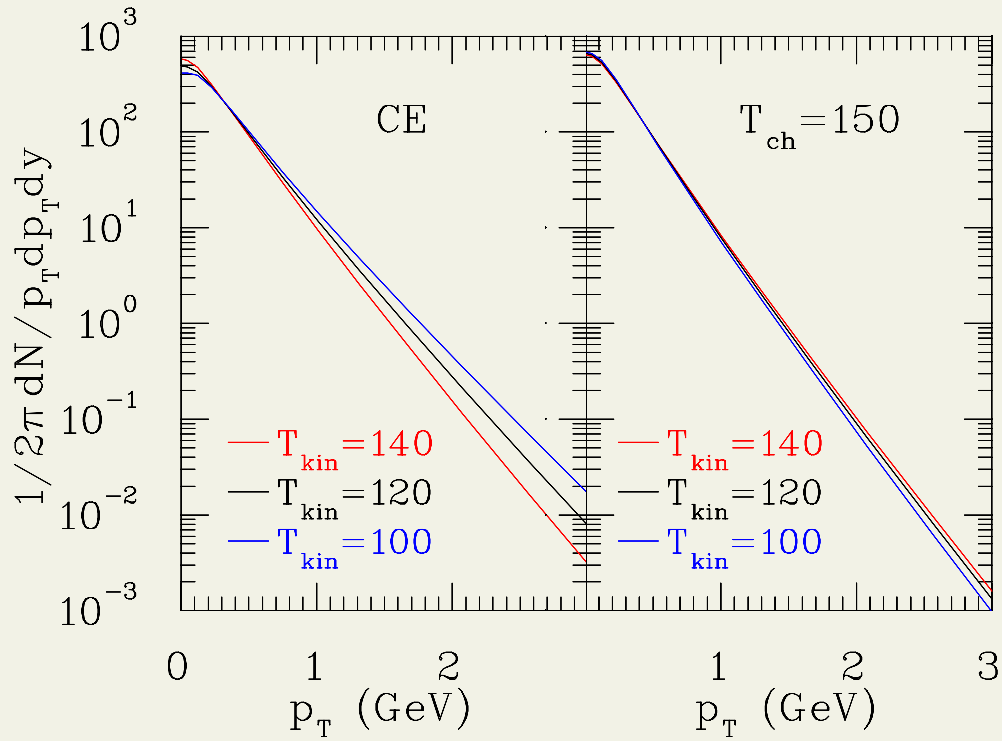
$\Rightarrow$ **In hydro particle ratios become wrong**

Chemical non-equilibrium

- **Treat number of pions, kaons etc. as conserved quantum numbers below T_{ch} (Bebie et al, Nucl.Phys.B378:95-130,1992)**
- $P = P(\epsilon, n_b)$ **changes very little,** **but** $T = T(\epsilon, n_b)$ **changes. . .**



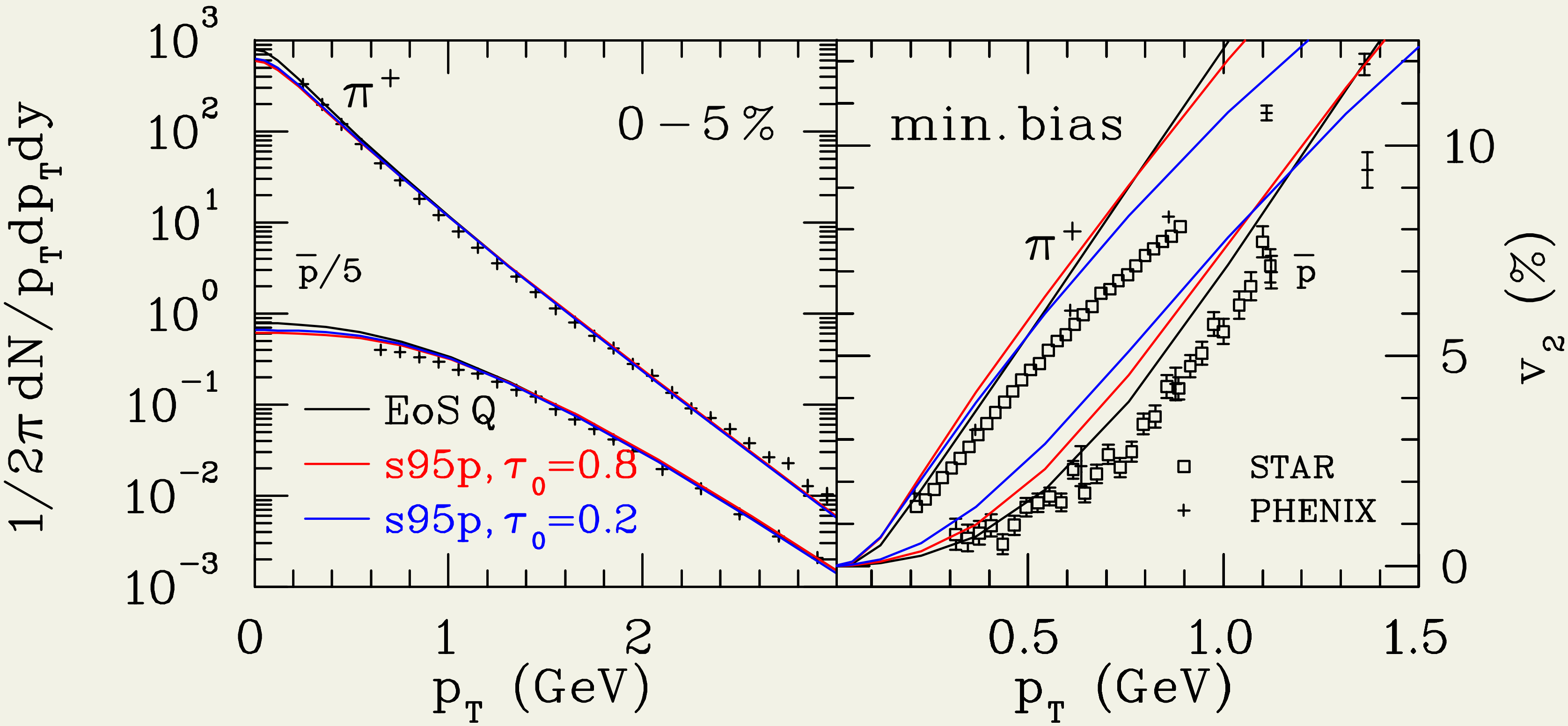
Effect of T_{kin} on pions



- **Longitudinal expansion does work ($p\,\mathrm{d}V$) $\Rightarrow$ $\frac{\mathrm{d}E_T}{\mathrm{d}y}$ decreases**
- **If particle # is conserved, $\langle p_T \rangle$ decreases**
- **In chemical equilibrium mass energy of baryon-antibaryon pairs is converted to kinetic energy $\Rightarrow$ $\langle p_T \rangle$ increases!**

More realistic EoS

- **ideal hydro, Au+Au at** $\sqrt{s_{NN}} = 200$ **GeV**
- $T_{\rm chem} = 150$ **MeV**



- **EoS Q:** $T_{dec} = 120$ **MeV**, $s_{\rm ini} \propto N_{bin}$ $\tau_0 = 0.2$ **fm/**c
- **s95p,** $\tau_0 = 0.8$: $T_{dec} = 120$ **MeV**, $s_{\rm ini} \propto N_{bin}$, $\tau_0 = 0.8$ **fm/**c
- **s95p,** $\tau_0 = 0.2$: $T_{dec} = 120$ **MeV**, $s_{\rm ini} \propto N_{bin} + N_{part}$, $\tau_0 = 0.2$ **fm/**c

Summary

- Hydrodynamics is a useful tool to model collision dynamics
  - approximation at its best
  - but it can reproduce (some of) the data

- we have observed hydrodynamical behaviour at RHIC and LHC