

# Electromagnetic Probes in Heavy-Ion Collisions II

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- 1 Electromagnetic probes and vector mesons
  - Relation to chiral symmetry
- 2 Elementary vacuum cross sections: hadrons  $\rightarrow \ell^+\ell^-$ 
  - chiral symmetry constraints
  - Electrodynamics of pions and  $\rho$  mesons (VMD model)
  - Dalitz decays of hadron resonances
- 3 Dileptons in pp and pA collisions at SIS energies
  - The Transport Model GiBUU
  - Baryon-resonance model at SIS energies
  - Dileptons in pp and pNb reactions at HADES
- 4 Conclusions and Outlook

# Why Electromagnetic Probes?

- $\gamma, l^\pm$ : only e. m. interactions
- whole matter evolution

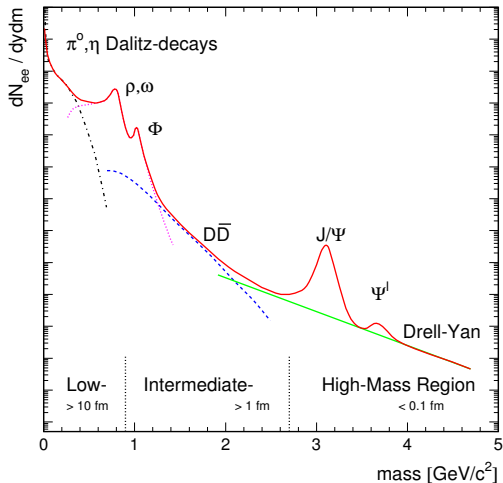
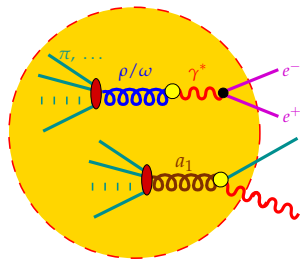


Fig. by A. Drees (from [RW00])

# Vector Mesons and electromagnetic Probes

- **photon** and **dilepton** thermal emission rates given by **same** electromagnetic-current-correlation function ( $J_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f$ )

$$\Pi_{\mu\nu}^<(q) = \int d^4x \exp(iq \cdot x) \langle J_\mu(0) J_\nu(x) \rangle_T = -2n_B(q_0) \Pi_{\mu\nu}^{(\text{ret})}(q)$$

$$q_0 \frac{dN_\gamma}{d^4x d^3\vec{q}} = -\frac{\alpha_{\text{em}}}{2\pi^2} g^{\mu\nu} \text{Im} \Pi_{\mu\nu}^{(\text{ret})}(q) \Big|_{q_0=|\vec{q}|} f_B(p_0)$$

$$\frac{dN_{e^+e^-}}{d^4x d^4k} = -g^{\mu\nu} \frac{\alpha^2}{3q^2 \pi^3} \text{Im} \Pi_{\mu\nu}^{(\text{ret})}(q) \Big|_{q^2=M_{e^+e^-}^2} f_B(p_0)$$

- **Caveat:** NOT manifestly Lorentz covariant  $\Leftrightarrow$  **heat-bath rest frame!**
- to lowest order in  $\alpha$ :  $4\pi\alpha \Pi_{\mu\nu} \simeq \Sigma_{\mu\nu}^{(\gamma)}$
- derivable from underlying thermodynamic potential,  $\Omega$ !

# Vector Mesons and chiral symmetry

- **vector** and **axial-vector** mesons  $\leftrightarrow$  respective current correlators

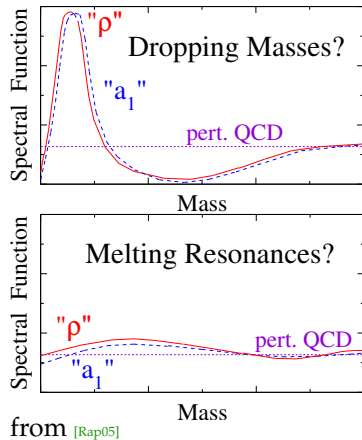
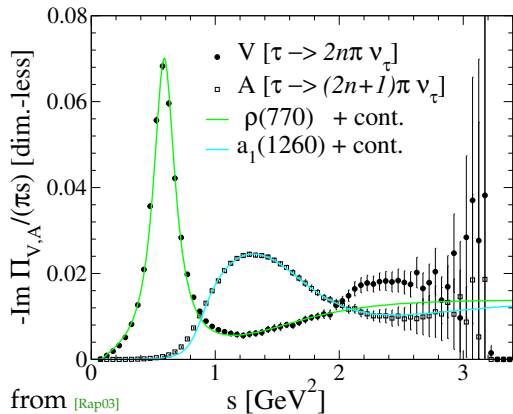
$$\Pi_{V/A}^{\mu\nu}(p) := \int d^4x \exp(ipx) \left\langle J_{V/A}^\nu(0) J_{V/A}^\mu(x) \right\rangle_{\text{ret}}$$

- Ward-Takahashi Identities of  $\chi$  symmetry  $\Rightarrow$  **Weinberg-sum rules**

$$f_\pi^2 = - \int_0^\infty \frac{dp_0^2}{\pi p_0^2} [\text{Im } \Pi_V(p_0, 0) - \text{Im } \Pi_A(p_0, 0)]$$

- spectral functions of vector (e.g.  $\rho$ ) and axial vector (e.g.  $a_1$ ) directly related to **order parameter of chiral symmetry!**

# Vector Mesons and chiral symmetry



# Chiral-symmetry constraints

- different realizations of **chiral symmetry**
- equivalent only on shell (“**low-energy theorems**”)
- model-independent conclusions only in **low-temperature/density limit** (chiral perturbation theory) or from **lattice-QCD calculations**
- QCD sum rules (see Lect. I):  
allow for dropping-mass or melting-resonance scenario
- use **phenomenological hadronic many-body theory** (HMBT) to assess medium modifications of vector mesons
  - build models with **hadrons** as effective degrees of freedom
  - based on (**chiral**) **symmetries**
  - constrained by data on cross sections, branching ratios,... in vacuum
  - in-medium properties assessed by **many-body (thermal) field theory**

# Example: vector-meson dominance model

- early model for **electromagnetic interaction** of charged pions

[Sak60, KLZ67, GS68, Her92, Hee00]

- QED like U(1)-gauge model with massive vector meson for  $\rho_0$  and  $\pi^\pm$
- Stückelberg: introduce auxiliary scalar field for free vector mesons:

$$\mathcal{L}_\rho = -\frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m^2V_\mu V^\mu + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) + m\varphi\partial_\mu V^\mu$$

- gauge invariant under local transformation

$$\delta V_\mu(x) = \partial_\mu\chi(x), \quad \delta\varphi = m\chi(x)$$

- usual way of gauge fixing using gauge condition

$$\partial_\mu V^\mu = -\xi m\varphi$$

- effective Lagrangian of free  $\rho$  meson, Stückelberg and FP ghosts

$$\begin{aligned}\mathcal{L}_{\rho,\text{gf}} = & -\frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{m}{2}V_\mu V^\mu - \frac{1}{2\xi}(\partial_\mu V^\mu)^2 + \frac{1}{2}(\partial_\mu\varphi)(\partial_\mu\varphi) - \frac{\xi m^2}{2}\varphi^2 \\ & + (\partial_\mu\eta^*)(\partial_\mu\eta) - \xi m^2\eta^*\eta\end{aligned}$$



# Example: vector-meson dominance model

- so far: free  $\rho$  meson and free ghosts
- ghosts only relevant for **ideal gas thermodynamics**
  - $V^\mu$ : four bosonic field degrees  
(3 transverse with mass  $m$ , 1 longitudinal with mass  $\sqrt{\xi}m$ )
  - $\varphi$ : 1 bosonic Stückelberg ghost with mass  $\sqrt{\xi}m$
  - $\eta^*, \eta$ : 2 pseudofermionic Faddeev Popov fields with mass  $\sqrt{\xi}m$
  - **in partition sum**: 3 bosons with mass  $m$  + 2 bosons with mass  $\sqrt{\xi}m$  – 2 FP ghosts with mass  $\sqrt{\xi}m \Rightarrow$  effectively three bosons with mass  $m$
  - partition sum independent of gauge parameter,  $\xi$ !
  - $\xi \rightarrow \infty$ : “unitary gauge”  $\rightarrow$  only three bosonic  $\rho$ -degrees of freedom!
- Coupling to pions: **obey gauge invariance!** (like scalar QED)

$$\mathcal{L}_\pi = (D_\mu \pi)^* (D^\mu \pi) - m_\pi^2 |\pi|^2 - \frac{\lambda}{8} |\pi|^4$$

- $D_\mu = \partial_\mu + igV_\mu$ ;  $g$ :  $\rho\pi\pi$  coupling

# Example: vector-meson dominance model

- add photons:  $D_\mu = \partial_\mu + igV_\mu + ieA_\mu$
- Lagrangian for photons: usual gauge fixed QED
- additional direct  $\rho\gamma$  mixing [KLZ67]

$$\mathcal{L}_{\rho\gamma} = -\frac{e}{2g_{\rho\gamma}} V_{\mu\nu} A^{\mu\nu}$$

- classical field equations:  $\Rightarrow$  **electromagnetic current**

$$j_{\text{em}}^\nu = \partial_\mu A^{\mu\nu} = ie \left( 1 - \frac{g}{g_{\rho\gamma}} \right) \pi \overleftrightarrow{D}^\nu \pi^* + \frac{e}{g_{\rho\gamma}} m^2 V^\mu + \frac{e^2}{g_{\rho\gamma}^2} \partial_\mu A^{\mu\nu}$$

- for  $g_{\rho\gamma} = g$ :  $j_{\text{em}}^\nu = \frac{e}{g} m^2 V^\nu + \mathcal{O}(e^2)$ :  $\Rightarrow$  **“vector-meson dominance”**

# Example: vector-meson dominance model

- calculate  $\rho$  selfenergy

$$i\Sigma_{\rho}^{\mu\nu}(p) = \mu \text{ [diagram 1]} + \mu \text{ [diagram 2]}$$

Diagram 1: A loop diagram representing the self-energy of a  $\rho$  meson. It consists of two wavy lines representing  $\rho$  mesons, each with momentum  $p$ . These are connected by a dashed loop representing a lepton  $l$ . The loop momentum is  $l$  on the top arc and  $l+p$  on the bottom arc.

Diagram 2: A tadpole diagram representing the self-energy of a  $\rho$  meson. It consists of a single wavy line representing a  $\rho$  meson with momentum  $\nu$ . A dashed loop representing a lepton  $l$  is attached to the wavy line at a vertex.

- transversality from gauge invariance:

$$\Sigma_{\rho}^{\mu\nu}(q) = \left( q^2 g^{\mu\nu} - q^{\mu} q^{\nu} \right) \tilde{\Sigma}(q^2)$$

- electromagnetic form factor of pions

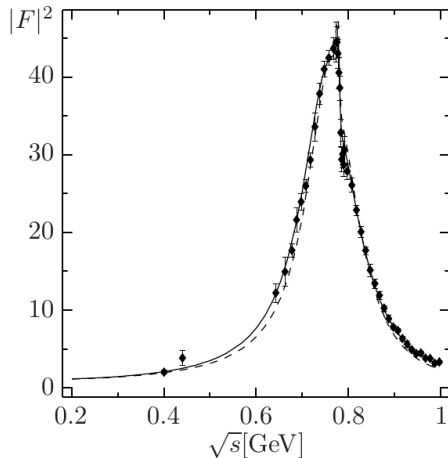
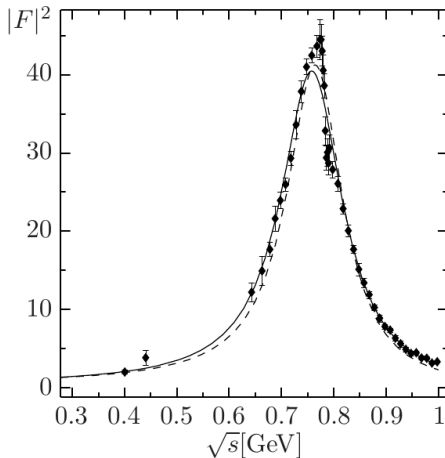
$$F(k^2) = \frac{\text{[diagram 1]}}{\text{[diagram 2]}}$$

Diagram 1 (Numerator): A diagram for the electromagnetic form factor of a pion. On the left, a red line representing a  $\pi^-$  meson and a red line representing a  $\pi^+$  meson meet at a vertex. A green wavy line representing a  $\rho$  meson connects this vertex to another vertex on the right. From the right vertex, two black lines representing electrons ( $e$ ) emerge.

Diagram 2 (Denominator): A diagram representing the pion propagator. It shows a red line for  $\pi^-$  and a red line for  $\pi^+$  meeting at a vertex, which is connected by a black wavy line to another vertex where two black lines for electrons ( $e$ ) emerge.

# Example: vector-meson dominance model

- fit to observables: **em. form factor of  $\pi$**

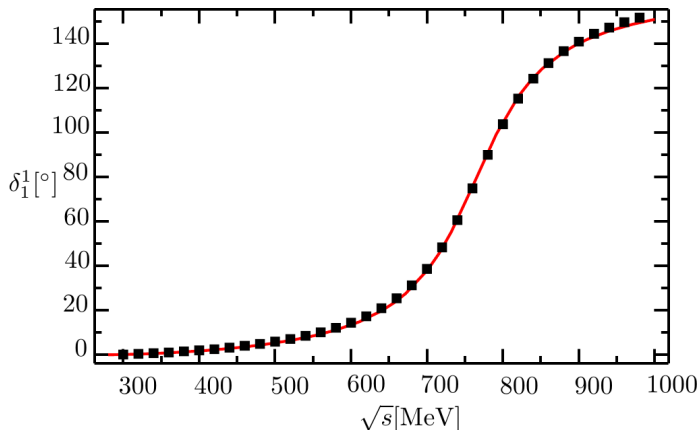


- best fit:  $g = 5.683$ ,  $g_{\rho\gamma} = 5.171$ ,  $m_\rho = 765 \text{ MeV}/c^2$   
strict VMD:  $g = g_{\rho\gamma} = 5.38$ ,  $m_\rho = 770 \text{ MeV}/c^2$   
data: [B<sup>+</sup>85]

# Example: vector-meson dominance model

- $\pi\pi \rightarrow \pi\pi$  phase shift in  $I = 1$  channel

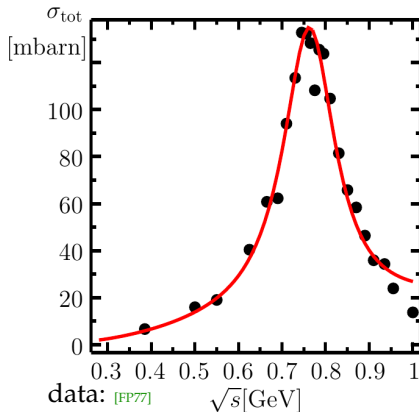
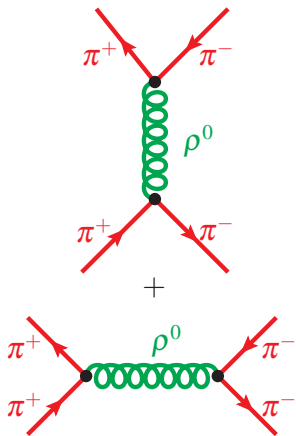
$$\delta_1^1 = \arccos \frac{\text{Re } G_\rho}{|G_\rho|}$$



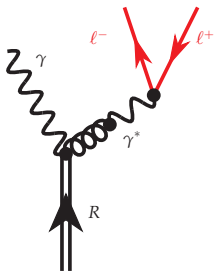
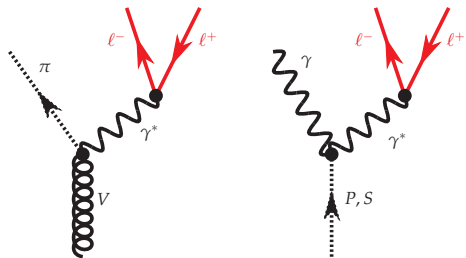
data: [FP77]

# Example: vector-meson dominance model

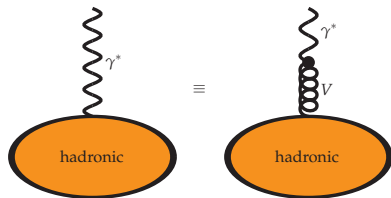
- $\pi\pi \rightarrow \pi\pi$  total cross section

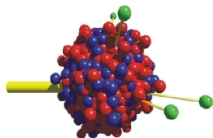


# Dalitz decays



- **Dalitz decay:**  
1 particle  $\rightarrow$  3 particles
- $V: \omega \rightarrow \pi + \gamma^* \rightarrow \pi + e^+ + e^-$
- $P, S:$   
 $\pi, \eta \rightarrow \gamma + \gamma^* \rightarrow \gamma + e^+ + e^-$
- $R$ : Baryon resonances  
 $\Delta, N^* \rightarrow N + V \rightarrow N + \gamma^* \rightarrow N + e^+ + e^-$
- vector-meson dominance





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## GiBUU

The Giessen Boltzmann-Uehling-Uhlenbeck Project

- Boltzmann-Uehling-Uhlenbeck (BUU) framework for hadronic transport
- reaction types:  $pA$ ,  $\pi A$ ,  $\gamma A$ ,  $eA$ ,  $\nu A$ ,  $AA$
- open-source modular Fortran 95/2003 code
- version control via Subversion
- publicly available releases: <http://gibuu.physik.uni-giessen.de>
- Review on hadronic transport (GiBUU): [BCG<sup>+</sup>12]



# The Boltzmann-Uehling-Uhlenbeck Equation

- time evolution of **phase-space distribution functions**

$$[\partial_t + (\vec{\nabla}_p H_i) \cdot \vec{\nabla}_x - (\vec{\nabla}_x H_i) \cdot \vec{\nabla}_p] f_i(t, \vec{x}, \vec{p}) = I_{\text{coll}}[f_1, \dots, f_i, \dots, f_j]$$

- Hamiltonian  $H_i$ 
  - selfconsistent hadronic mean fields, Coulomb potential, “off-shell potential”
- collision term  $I_{\text{coll}}$ 
  - two- and three-body decays/collisions
  - multiple coupled-channel problem
  - resonances described with relativistic Breit-Wigner distribution

$$\mathcal{A}(x, p) = -\frac{1}{\pi} \frac{\text{Im } \Pi}{(p^2 - M^2 - \text{Re } \Pi)^2 + (\text{Im } \Pi)^2}; \quad \text{Im } \Pi = -\sqrt{p^2} \Gamma$$

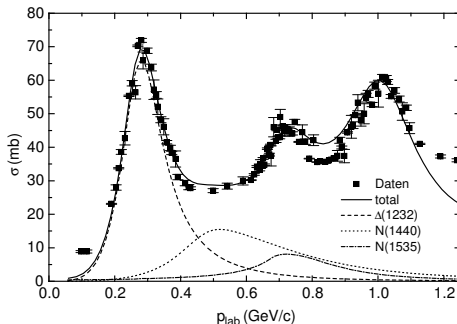
- off-shell propagation: test particles with **off-shell potential**

# Resonance Model

- reactions dominated by resonance scattering:  $ab \rightarrow R \rightarrow cd$
- Breit-Wigner cross-section formula

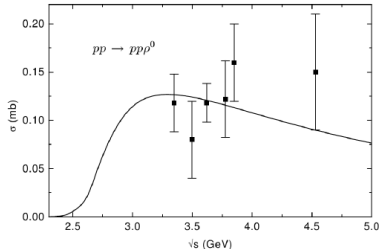
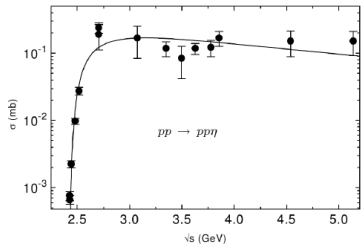
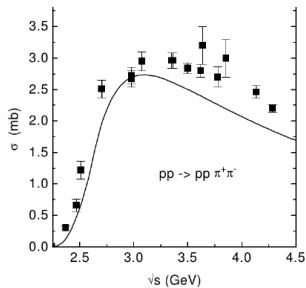
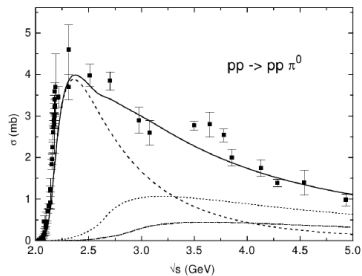
$$\sigma_{ab \rightarrow R \rightarrow cd} = \frac{2s_R + 1}{(2s_a + 1)(2s_b + 1)} \frac{4\pi}{p_{\text{lab}}^2} \frac{s\Gamma_{ab \rightarrow R}\Gamma_{R \rightarrow cd}}{(s - m_R^2)^2 + s\Gamma_{\text{tot}}^2}$$

- applicable for low-energy nuclear reactions  $E_{\text{kin}} \lesssim 1.1 \text{ GeV}$
- example:  $\sigma_{\pi^- p \rightarrow \pi^- p}$  [Teis (PhD thesis 1996), data: Baldini et al, Landolt-Börnstein 12 (1987)]



# Resonance Model

- further cross sections



# Extension to HADES energies

• [WHM12]

• keep same resonances (parameters from Manley analysis)

	rating	$M_0$ [MeV]	$\Gamma_0$ [MeV]	$ \mathcal{M}^2 /16\pi$ [mb GeV <sup>2</sup> ]		branching ratio in %						
				$NR$	$\Delta R$	$\pi N$	$\eta N$	$\pi \Delta$	$\rho N$	$\sigma N$	$\pi N^*(1440)$	$\sigma \Delta$
P <sub>11</sub> (1440)	****	1462	391	70	—	69	—	22 <sub>P</sub>	—	9	—	—
S <sub>11</sub> (1535)	***	1534	151	8	60	51	43	—	2 <sub>S</sub> + 1 <sub>D</sub>	1	2	—
S <sub>11</sub> (1650)	****	1659	173	4	12	89	3	2 <sub>D</sub>	3 <sub>D</sub>	2	1	—
D <sub>13</sub> (1520)	****	1524	124	4	12	59	—	5 <sub>S</sub> + 15 <sub>D</sub>	21 <sub>S</sub>	—	—	—
D <sub>15</sub> (1675)	****	1676	159	17	—	47	—	53 <sub>D</sub>	—	—	—	—
P <sub>13</sub> (1720)	*	1717	383	4	12	13	—	—	87 <sub>P</sub>	—	—	—
F <sub>15</sub> (1680)	****	1684	139	4	12	70	—	10 <sub>P</sub> + 1 <sub>F</sub>	5 <sub>P</sub> + 2 <sub>F</sub>	12	—	—
P <sub>33</sub> (1232)	****	1232	118	OBE	210	100	—	—	—	—	—	—
S <sub>31</sub> (1620)	**	1672	154	7	21	9	—	62 <sub>D</sub>	25 <sub>S</sub> + 4 <sub>D</sub>	—	—	—
D <sub>33</sub> (1700)	*	1762	599	7	21	14	—	74 <sub>S</sub> + 4 <sub>D</sub>	8 <sub>S</sub>	—	—	—
P <sub>31</sub> (1910)	****	1882	239	14	—	23	—	—	—	—	67	10 <sub>P</sub>
P <sub>33</sub> (1600)	***	1706	430	14	—	12	—	68 <sub>P</sub>	—	—	20	—
F <sub>35</sub> (1905)	**	1881	327	7	21	12	—	1 <sub>P</sub>	87 <sub>P</sub>	—	—	—
F <sub>37</sub> (1950)	****	1945	300	14	—	38	—	18 <sub>F</sub>	—	—	—	44 <sub>F</sub>

• production channels in Teis:  $NN \rightarrow N\Delta, NN \rightarrow NN^*, N\Delta^*, NN \rightarrow \Delta\Delta$

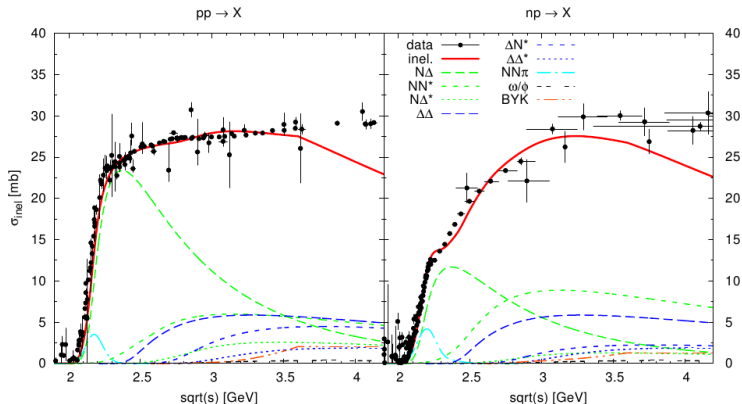
• extension to  $NN \rightarrow \Delta N^*, \Delta\Delta^*, NN \rightarrow NN\pi,$

$NN \rightarrow NN\rho, NN\omega, NN\pi\omega, NN\phi,$

$NN \rightarrow BYK$  ( $B = N, \Delta, Y = \Lambda, \Sigma$ )

# Extension to HADES energies

- good description of total pp, pn (inelastic) cross section

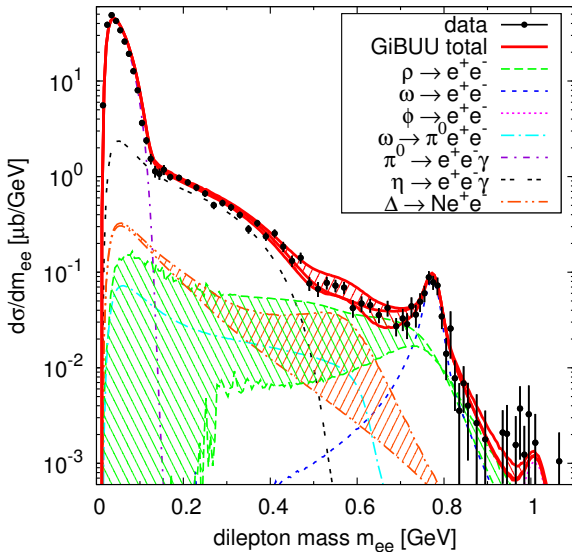


- dilepton sources

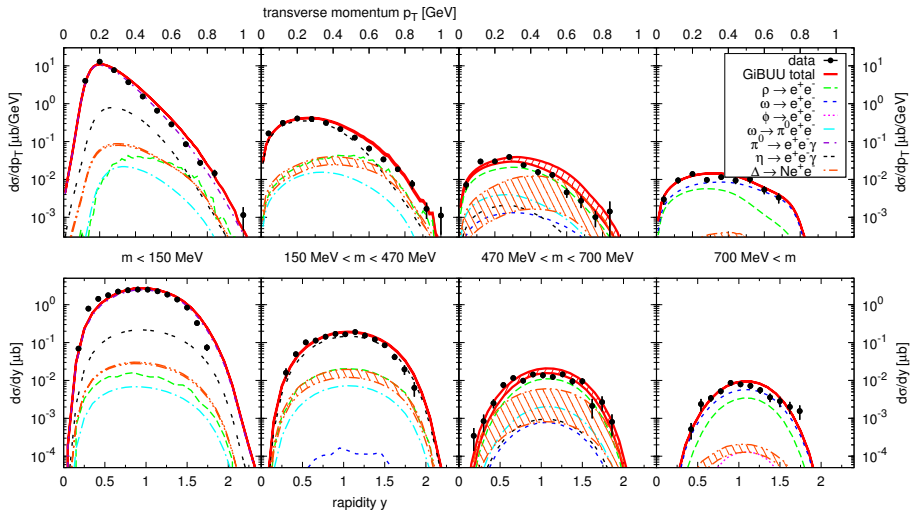
- Dalitz decays:  $\pi^0, \eta \rightarrow \gamma l^+ l^-$ ;  $\omega \rightarrow \pi^0 l^+ l^-$ ,  $\Delta \rightarrow N l^+ l^-$
- $\rho, \omega, \phi \rightarrow l^+ l^-$ : invariant mass  $l^+ l^-$  spectra  $\Rightarrow$   
spectral properties of vector mesons
- for details, see [WHM12]

# p p at HADES ( $E_{\text{kin}} = 3.5 \text{ GeV}$ )

p + p at 3.5 GeV

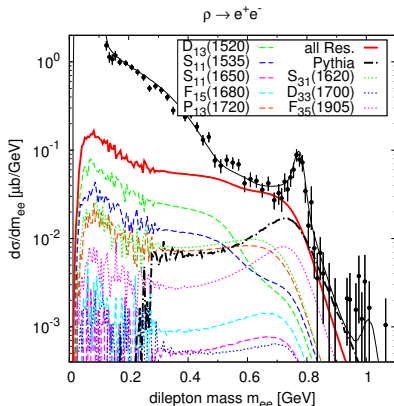
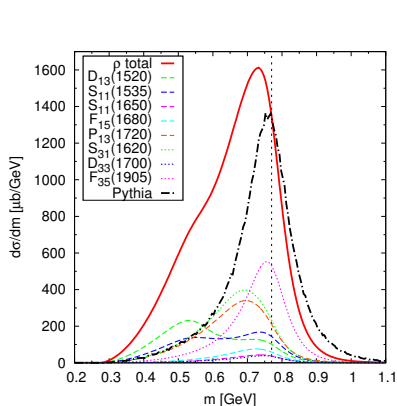


# p p at HADES ( $E_{\text{kin}} = 3.5 \text{ GeV}$ )



# “ $\rho$ meson” in $pp$

- production through hadron resonances  
 $NN \rightarrow NR \rightarrow NN\rho, NN \rightarrow N\Delta \rightarrow NN\pi\rho$

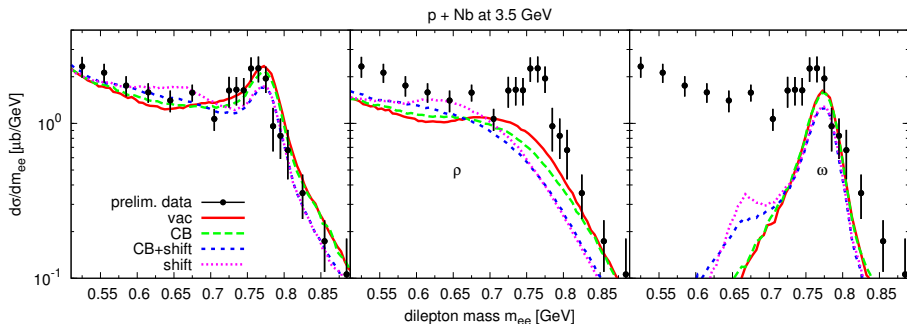


- “ $\rho$ ”-line shape “modified” already in elementary hadronic reactions
- due to production mechanism via resonances



# p Nb at HADES (3.5 GeV)

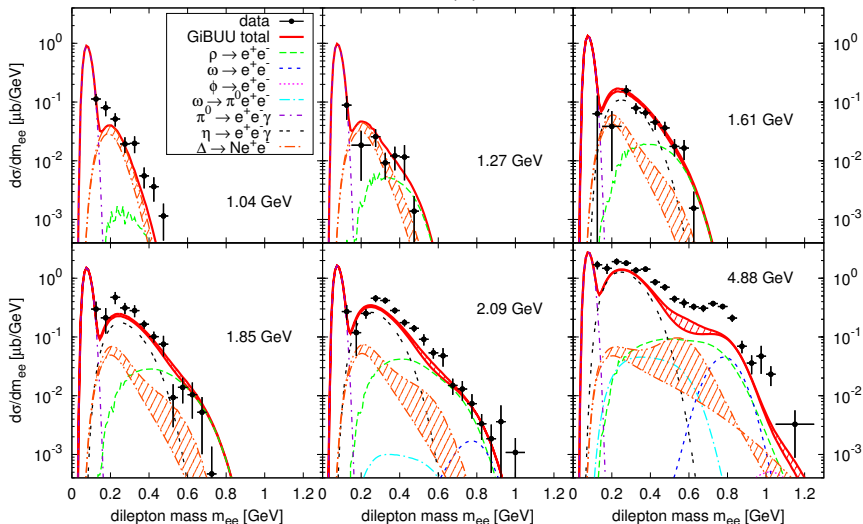
- medium effects built in transport model
  - binding effects, Fermi smearing, Pauli blocking
  - final-state interactions
  - production from secondary collisions
- sensitivity on medium effects of vector-meson spectral functions?



# Comparison to old DLS data (pp)

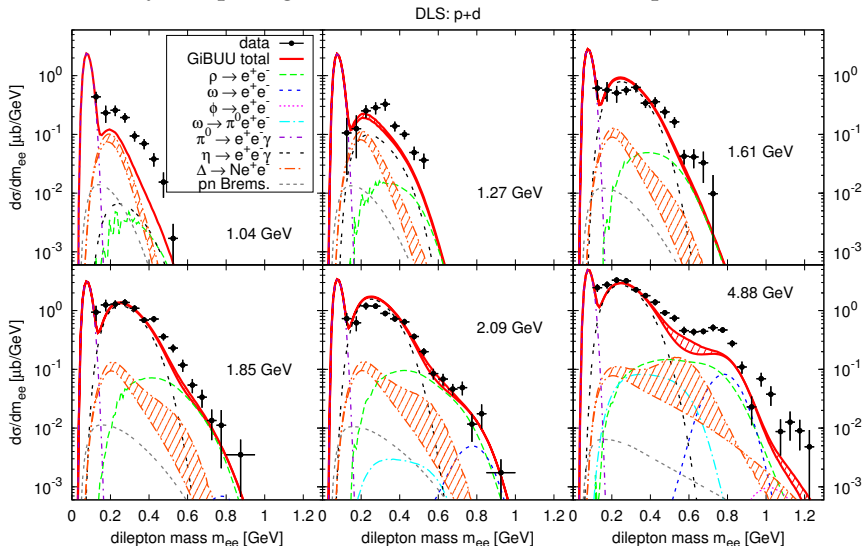
- HADES data consistent with DLS data
- checked by comparing HADES data within DLS acceptance

DLS: p+p



# Comparison to old DLS data (pd)

- HADES data consistent with DLS data
- checked by comparing HADES data within DLS acceptance



# Conclusions and Outlook

- dilepton spectra  $\Leftrightarrow$  in-medium em. current correlator
- effective hadronic models for dilepton sources
  - vector-meson dominance model (VMD)
  - low-mass region  $0 \leq M \lesssim 1 \text{ GeV}$ :  $j_{\text{em}}^{(\text{had})\mu} \propto V^\mu$  ( $V \in \{\rho, \omega, \phi\}$ )
  - direct relation between dilepton signal and VM spectral functions
  - interactions with mesons and baryons
  - models constrained by phenomenology in pp, pn, pA
  - medium modifications predicted by finite-temperature QFT
- Elementary reactions at SIS energies
  - GiBUU for pp, pn with resonance model for all HADES energies
  - pn still a problem?
  - p Nb, AA work in progress

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