HIC-for-FAIR School and Workshop on

Dense QCD phases in Heavy-Ion Collisions

JINR Dubna, August 21- September 4, 2010

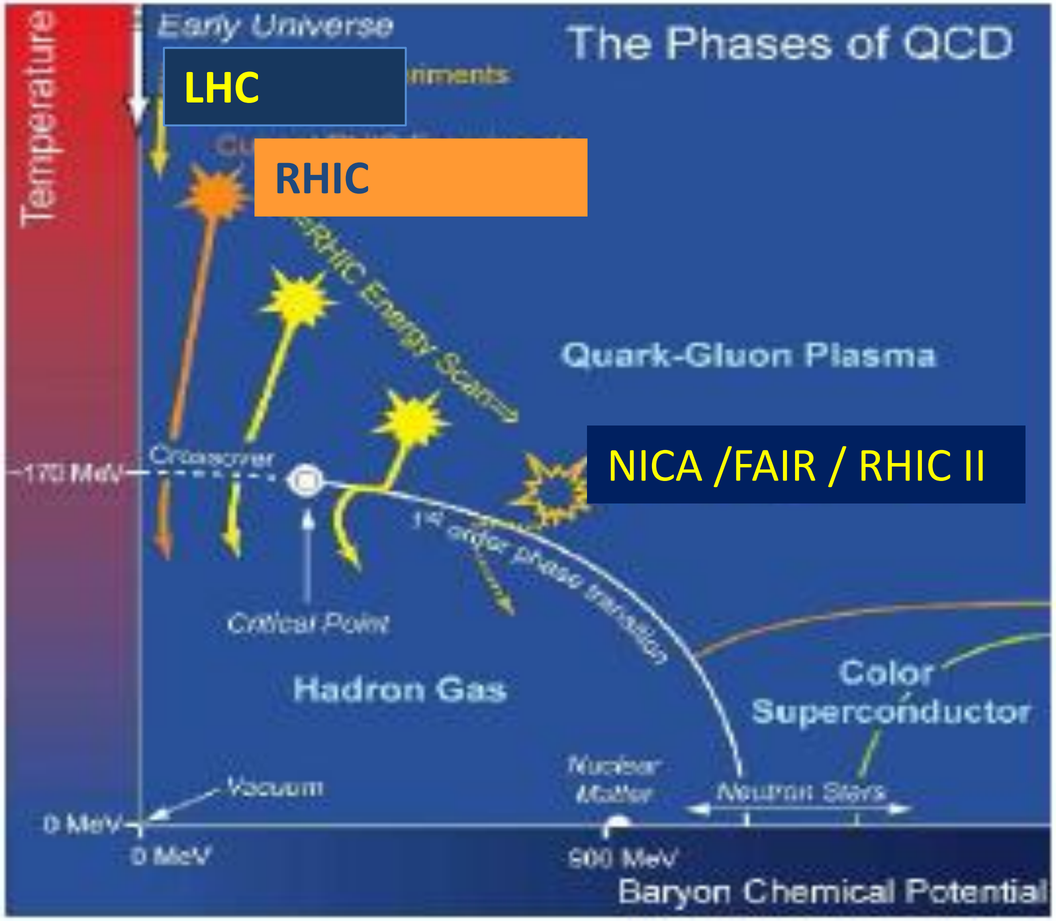
Phases of QCD and critical point from the lattice

Maria Paola Lombardo

The QCD Phase Diagram

First proposal:

Cabibbo and **Parisi, 1975**

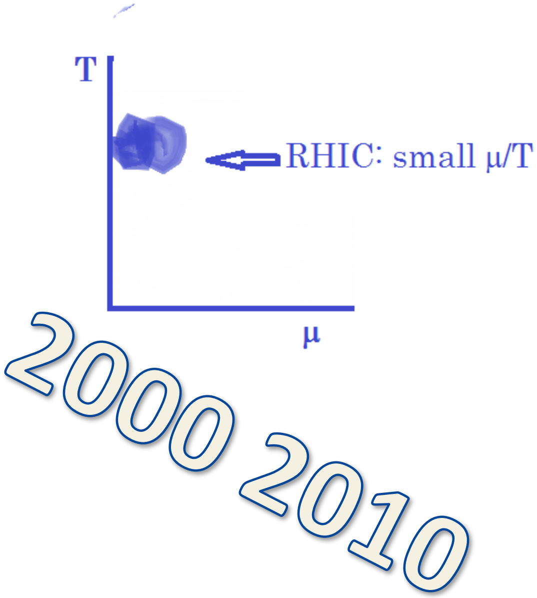


US NSAC Long RangePlan (adapted)

Nicola Cabibbo

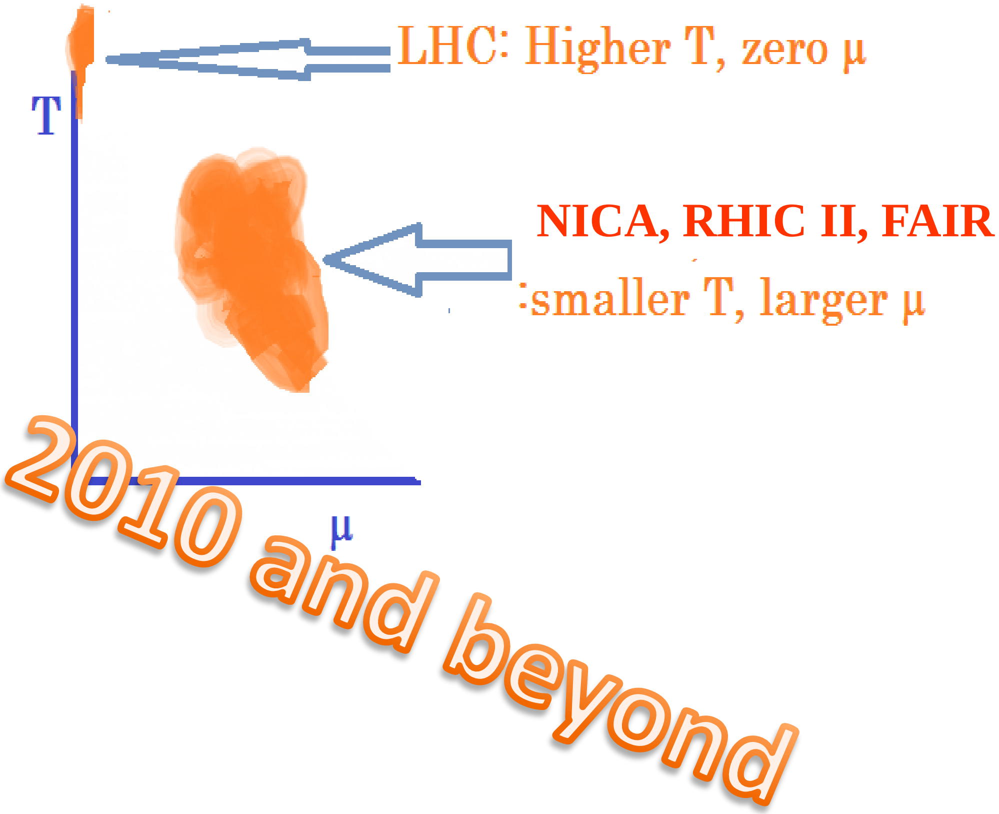
10 April 1935 16 August 2010

Phases of QCD: shift of the focus at the turn of the decade



RHIC: small μ/T

2000 2010



LHC: Higher T, zero μ

NICA, RHIC II, FAIR

smaller T, larger μ

2010 and beyond

THE THEORETICAL APPARATUS: QCD, THE FIELD THEORY OF STRONG INTERACTIONS

$$\mathcal{L} = \mathcal{L}_{YM} + \bar{\psi}(i\gamma_\mu D_\mu + m + \mu\gamma_0)\psi$$

LATTICE QCD ALLOWS FIRST PRINCIPLES CALCULATIONS FROM THE QCD LAGRANGIAN

$$\mathcal{L} = \mathcal{L}_{YM} + \bar{\psi}(i\gamma_\mu D_\mu + m + \mu\gamma_0)\psi$$

We can tune physical parameter, as in real experiments: baryon chemical potential, temperature, isospin chemical potential, strangeness,..

We can also play with number of color and number of flavor.

We can address phenomenological issues as well as theoretical questions.

Outline

Lattice discretization, continuum limit

Application I : The (pseudo)critical temperature

Importance sampling and the sign problem

Application II: The (pseudo)critical line

Application III: The Endpoint of QCD

Discussion: Better control over the endpoint?

The phase diagram for a complex chemical potential, and the sQGP

Application IV: Quarkyonic phase

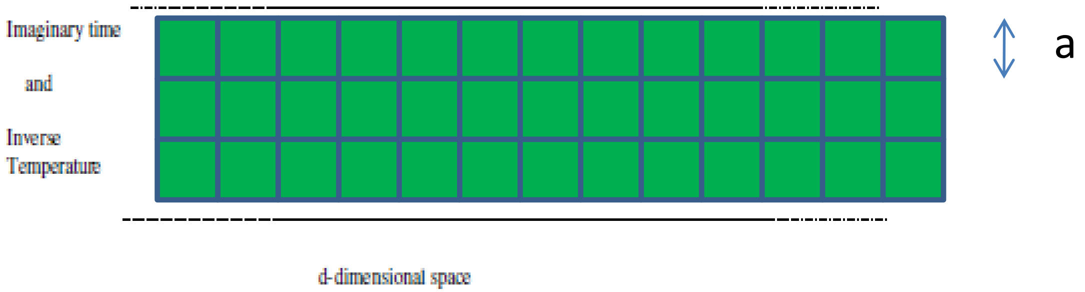
Discussion: Lattice analysis of the freezout region?

Mesoscopic analysis of the phase diagram: towards the solution of the sign problem?

LATTICE DISCRETIZATION

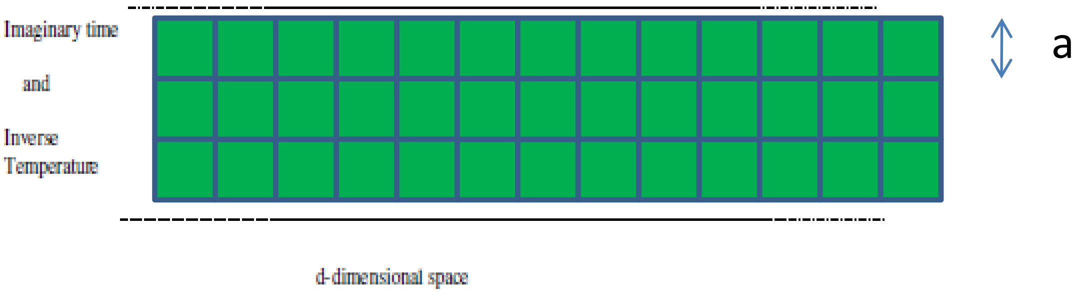
Lattice QCD Thermodynamics at a Glance

$$
\begin{aligned}
\mathcal{L}_{QCD} &= \boxed{6/g^2} \,\mathrm{Tr}\, U_{n,\mu} U_{n+\hat{\mu},\nu} U^{\dagger}_{n+\hat{\nu},\mu} U^{\dagger}_{n,\nu} \\
&+ \sum_{i=1}^{3} (\bar{\psi}_x \gamma_i U_i(x) \psi_{x+\hat{i}} - \bar{\psi}_{x+\hat{i}} \gamma_i U_i^{\dagger}(x) \psi_x) \\
&+ \bar{\psi}_x \gamma_0 e^{\mu} U_0(x) \psi_{x+\hat{0}} - \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu} U_0^{\dagger}(x) \psi_x \\
&+ m\bar{\psi}\psi
\end{aligned}
$$



LATTICE DISCRETIZATION:A **CONTROLLED** APPROXIMATION

$a = A\, e^{-K/g^2}$ at small g



Thanks to QCD asymptotic freedom

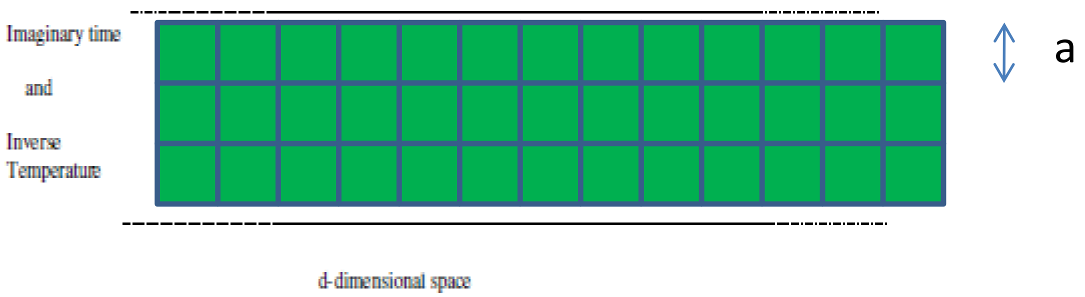
LATTICE DISCRETIZATION:A **CONTROLLED** APPROXIMATION

$$a = \boxed{A}\, e^{\boxed{-K/g^2}}$$

at small g

Needs lattice

From perturbation theory



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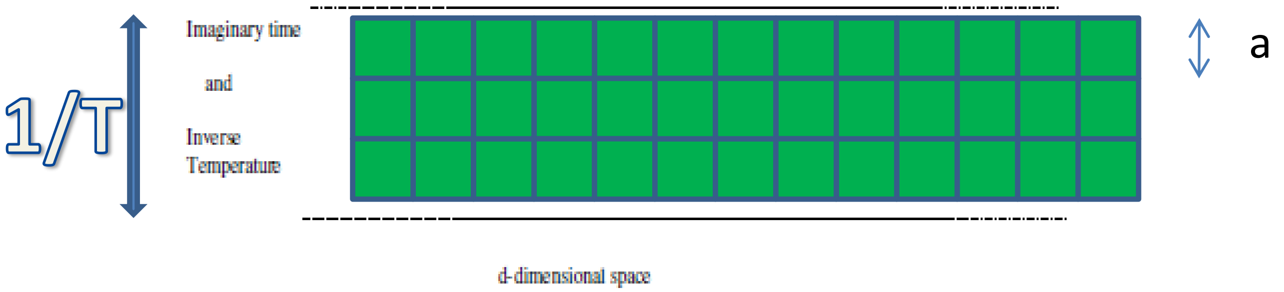


Finite temperature continuum limit: need constant 1/T and smaller a:
Increase number of points in time directions

More on The Lattice (digression)

Path integral is a regulated on a four dimensional lattice

- *Gauge fields: link variables $U_\mu(x)$ for parallel trasport of field $\mathcal{A}$ from x to $x + \hat{\mu}a$*

  $x \longrightarrow x + \hat{\mu}a$
  $U_\mu(x)$

$$U_{x,\mu} = \mathrm{P}\exp\left(ig \int_x^{x+\hat{\mu}a} dx^\mu \, A_\mu(x) \right)$$

- *Gauge invariants and Yang Mill Action:*

$$\begin{aligned} W^{(1,1)}_{n,\mu\nu} &= 1 - \frac{1}{3}\mathrm{Re}\, \square_{n,\mu\nu} \\ &= \mathrm{Re}\,\mathrm{Tr}\, U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu} \\ &= \frac{g^2 a^4}{2} F^a_{\mu\nu} F^a_{\mu\nu} + \mathcal{O}(a^6) \end{aligned}$$

- Lattice Yang Mill Action

$$\beta S_G = \beta \sum_{\substack{n \\ 0 \le \mu < \nu \le 3}} W^{(1,1)}_{n,\mu\nu} \rightarrow \int d^4x \, \mathcal{L}_{YM} + \mathcal{O}(a^2)$$

$\beta = 6/g(a)^2$.

- Lattice fermions

  Simply:

  $\psi(x) \to \psi(n)$!

  $\partial_\mu \psi_f(x) = (\psi(n+\hat{\mu}) - \psi_(n-\hat{\mu}))/2a,$

  [doubling problem and chiral symmetry: staggered fermions, Wilson fermions, chiral fermions]

$\mu\bar{\psi}\gamma_0\psi$ on the lattice
Naive discretisation:

$$\begin{aligned}\phi_{LATT}(n_1, n_2, n_3, n_4) &= \phi(n_1 a, n_2 a, n_3 a, n_4 a) \\ \Delta_\mu \phi_{LATT}(n_1, n_2, n_3, n_4) &= \\ (\phi(n_1 a, (n_\mu + 1)a, n_3 a, n_4 a) &- \phi(n_1 a, (n_\mu a, n_3 a, n_4 a))/2a\end{aligned}$$

Problems with free fermions: the internal energy ϵ diverges in the continuum limit $a \to 0$

$$L = \bar{\psi}_x \gamma_\mu \psi_{x+\mu a} + m\bar{\psi}_x \psi_x + \mu \bar{\psi}_x \gamma_0 \psi_x$$

$$\epsilon \propto \frac{\mu^2}{a^2} \to_{a\to 0} \infty$$

Elegant solution : μ is an external field in the 0th direction

$$\bar{\psi}\gamma_\mu A_\mu \psi \longleftrightarrow i\mu\bar{\psi}\gamma_0\psi$$

- *External fields live on lattice link. (cfr. electrodynamics: $A \to \theta = e^(iA)$)*

- $L(\mu) = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+\hat{0}} - \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu a} \psi_x$

- *Simple intepretation*
  - *Time Forward propagation enhanced by* $e^{\mu a}$
  - *Time Backward propagation discouraged by* $e^{-\mu a}$

  Particles–antiparticle asymmetry!

- $\lim_{a \to 0} J_0 = -\partial_\mu L = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+\hat{0}} + \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu a} \psi_x = \mu \bar{\psi} \gamma_0 \psi$

Via an unitary transformation for the field

$$L(\mu) = L(0)$$

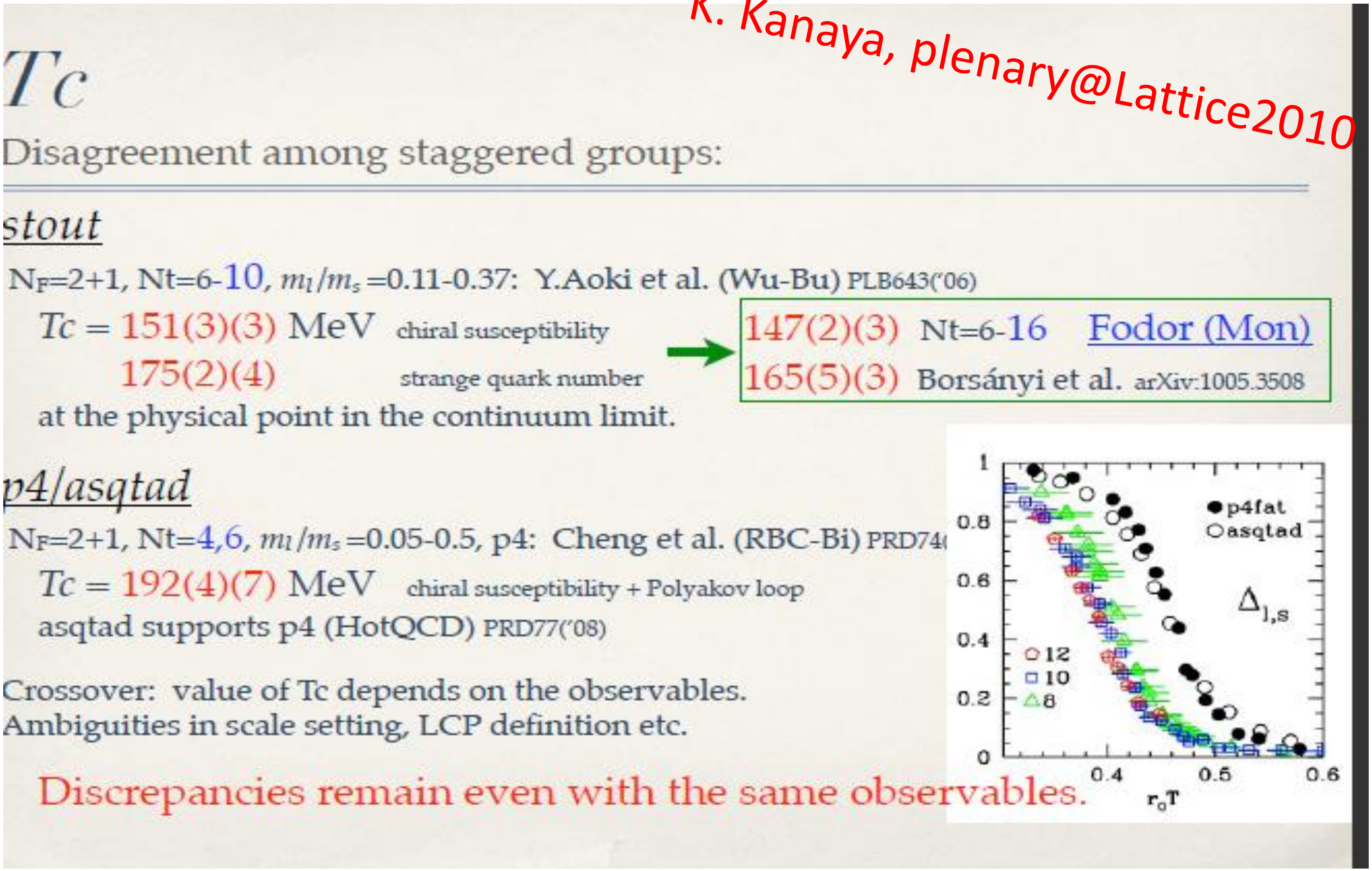
+ boundary conditions

Explicit dependence on fugacity

Application I

THE (PSEUDO)CRITICAL TEMPERATURE

K. Kanaya, plenary@Lattice2010



Tc

Disagreement among staggered groups:

stout

N_F=2+1, Nt=6-10, m_l/m_s=0.11-0.37: Y.Aoki et al. (Wu-Bu) PLB643('06)

Tc = 151(3)(3) MeV chiral susceptibility

175(2)(4) strange quark number

at the physical point in the continuum limit.

→ 147(2)(3) Nt=6-16 Fodor (Mon)

165(5)(3) Borsányi et al. arXiv:1005.3508

p4/asqtad

N_F=2+1, Nt=4,6, m_l/m_s=0.05-0.5, p4: Cheng et al. (RBC-Bi) PRD74

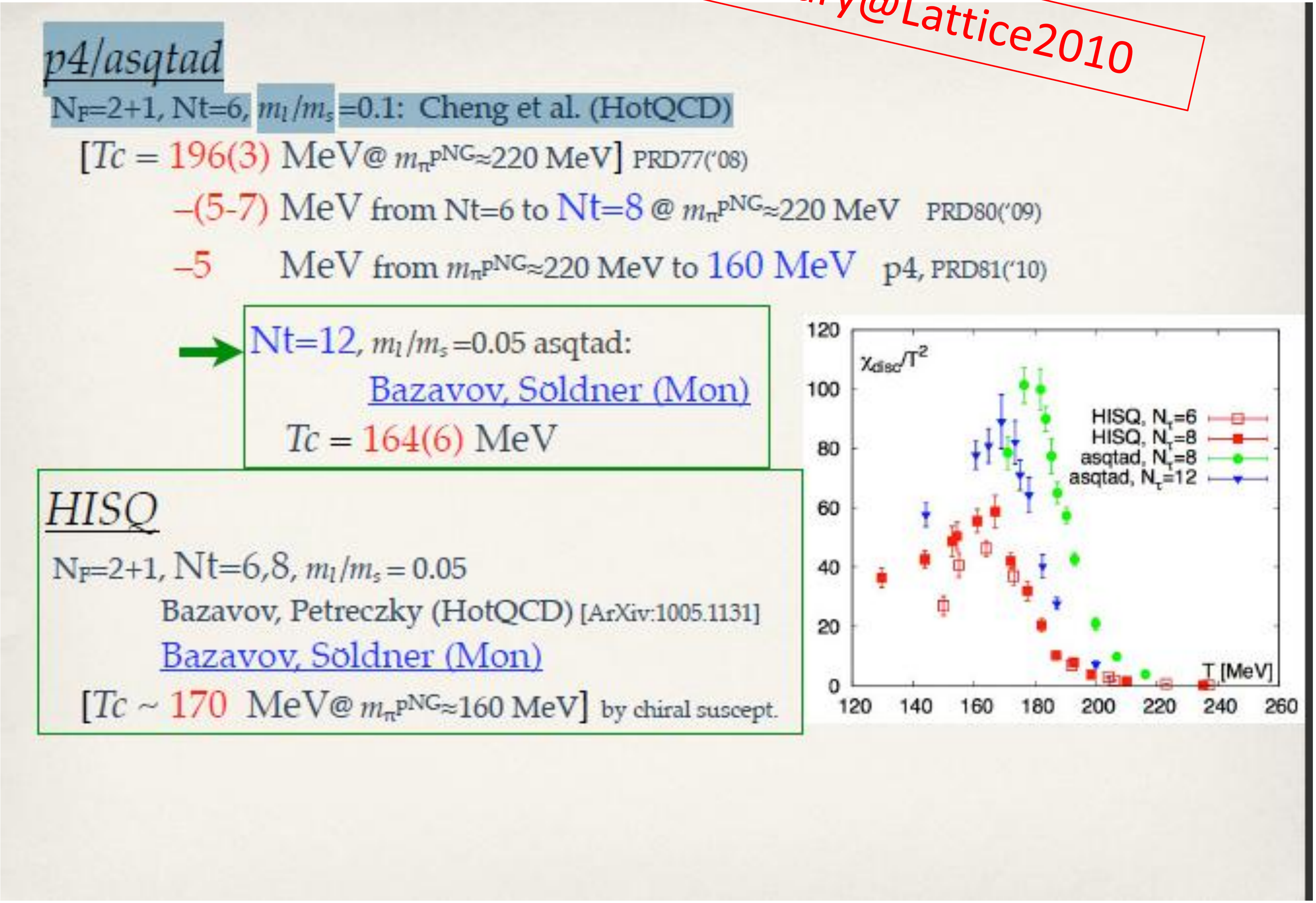
Tc = 192(4)(7) MeV chiral susceptibility + Polyakov loop

asqtad supports p4 (HotQCD) PRD77('08)

Crossover: value of Tc depends on the observables.
Ambiguities in scale setting, LCP definition etc.

Discrepancies remain even with the same observables.

K. Kanaya, plenary@Lattice2010



p4/asqtad

N_F=2+1, Nt=6, m_l/m_s=0.1: Cheng et al. (HotQCD)

[Tc = 196(3) MeV@ m_π^{PNG}≈220 MeV] PRD77('08)

–(5-7) MeV from Nt=6 to Nt=8 @ m_π^{PNG}≈220 MeV PRD80('09)

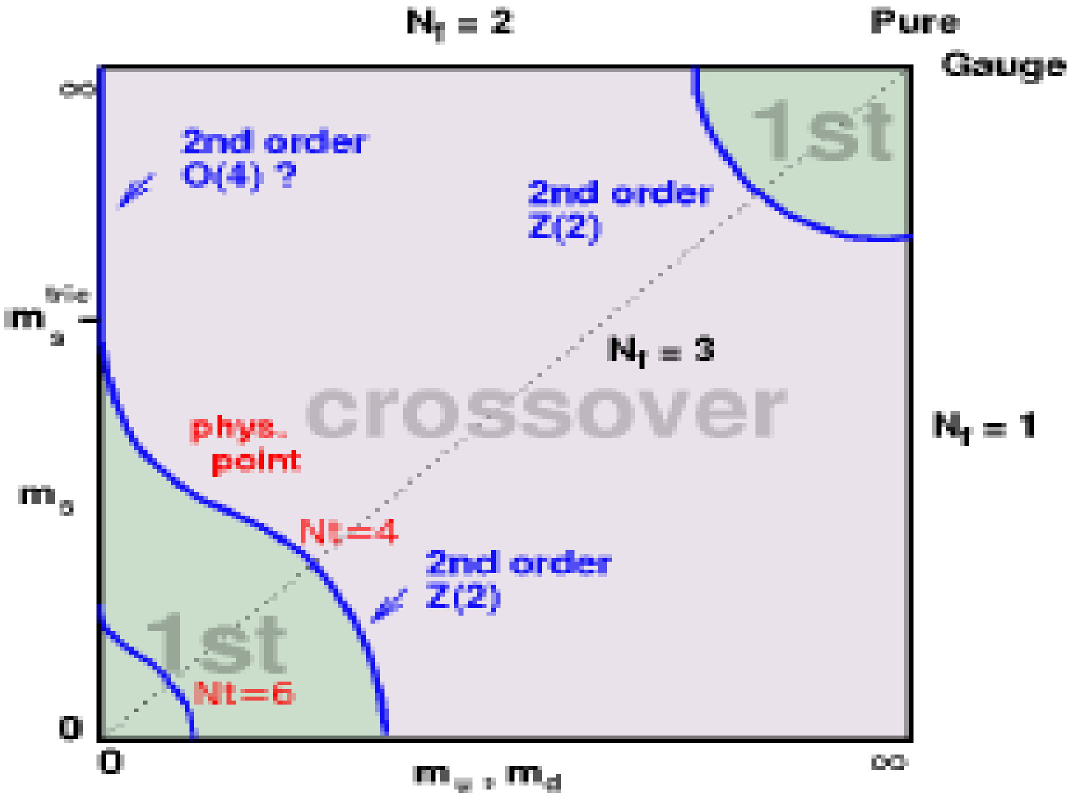
–5 MeV from m_π^{PNG}≈220 MeV to 160 MeV p4, PRD81('10)

Nt=12, m_l/m_s=0.05 asqtad:
Bazavov, Söldner (Mon)
Tc = 164(6) MeV

HISQ

N_F=2+1, Nt=6,8, m_l/m_s = 0.05
Bazavov, Petreczky (HotQCD) [ArXiv:1005.1131]
Bazavov, Söldner (Mon)
[Tc ~ 170 MeV@ m_π^{PNG}≈160 MeV] by chiral suscept.

The Universality issue and the continuum



Ongoing activities

Nf=2
Use different discretizations: Staggered, Wilson, Domain walls

Shape of the critical line -> later

THE IMPORTANCE SAMPLING AND THE SIGN PROBLEM

COMPUTATIONAL SCHEMES

$$\mathcal{Z} = \int d\phi d\bar{\psi} dU e^{-S(\phi,\bar{\psi},U)}; S(\phi,\bar{\psi},U) = \int_0^{1/T} dt \int d^d x \mathcal{L}(\phi,\bar{\psi}U)$$

$$\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \bar{\psi}(i\gamma_\mu D_\mu + m)\psi + \mu\bar{\psi}\gamma_0\psi$$

Two options:

1. Integrate out gluons first:

$$\mathcal{Z}(T,\mu,\bar{\psi},\psi,U) \simeq \mathcal{Z}(T,\mu,\bar{\psi},\psi) \rightarrow$$

   effective approximate fermion models

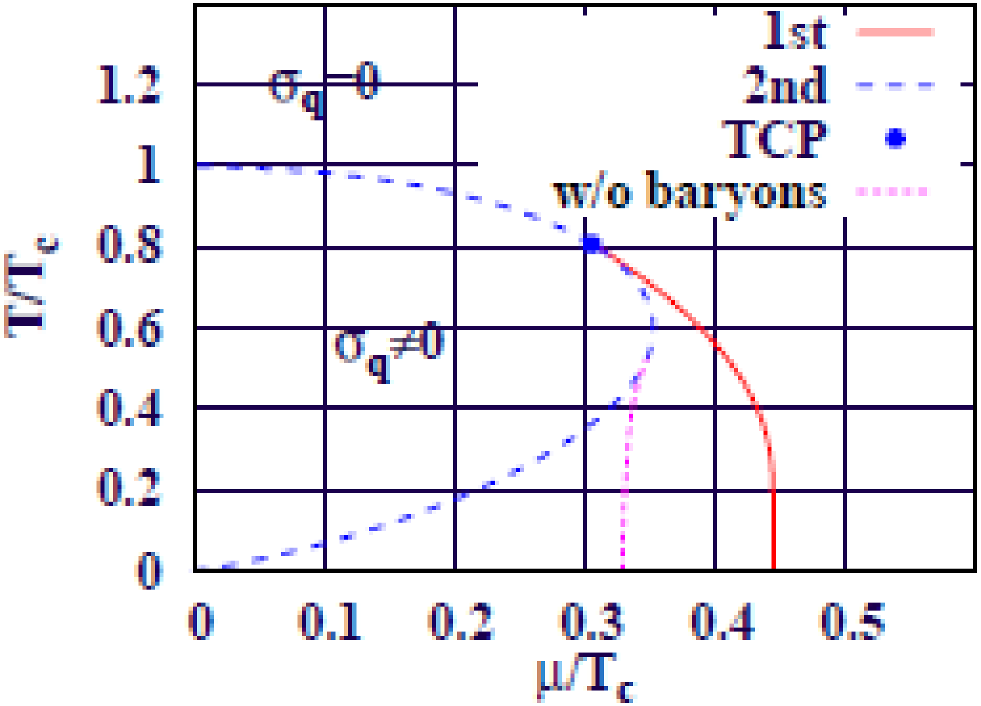
2. Integrate out fermions exactly as S is bilinear in $\psi, \bar{\psi}$
   $S = S_{YM}(U) + \bar{\psi}M(U)\psi$

$$\mathcal{Z}(T,\mu,U) = \int dU e^{-(S_{YM}(U) - log(detM))} \rightarrow$$

   starting point for numerical calculations

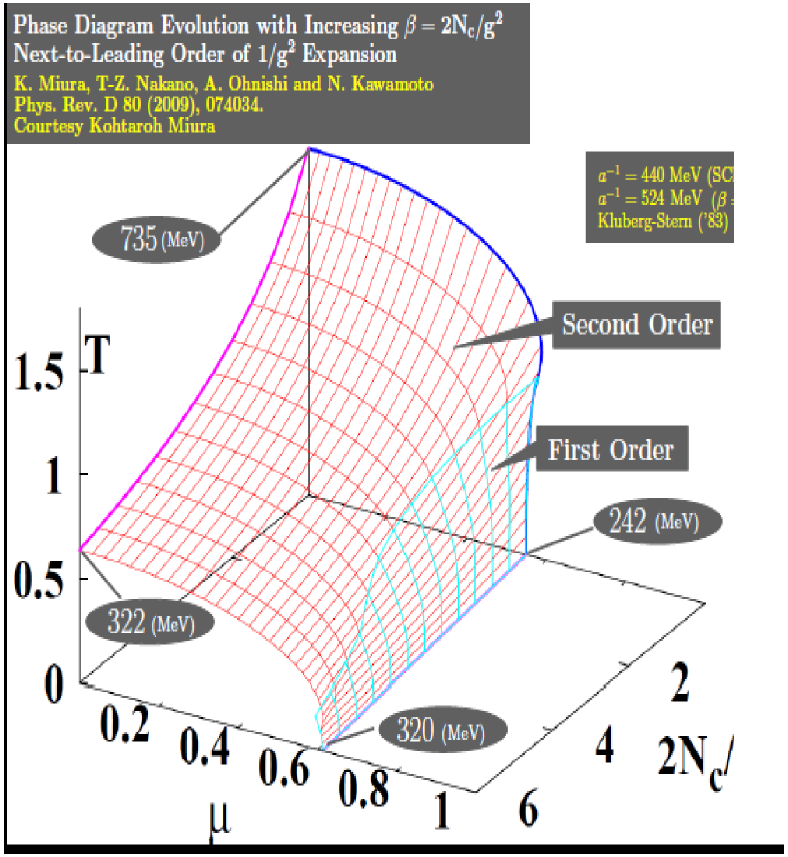
Option1: The strong coupling expansion

A long history..



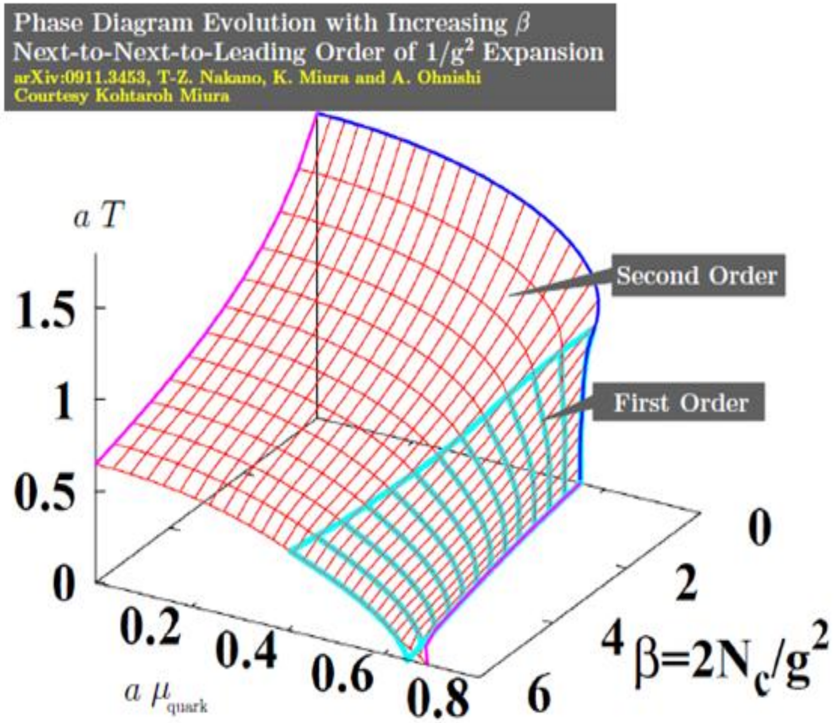
Kawamoto,
Miura, Onishi
2007

The Strong Coupling Expansion
approaching the continuum limit



Phase Diagram Evolution with Increasing $\beta = 2N_c/g^2$
Next-to-Leading Order of $1/g^2$ Expansion
K. Miura, T-Z. Nakano, A. Ohnishi and N. Kawamoto
Phys. Rev. D 80 (2009), 074034.
Courtesy Kohtaroh Miura

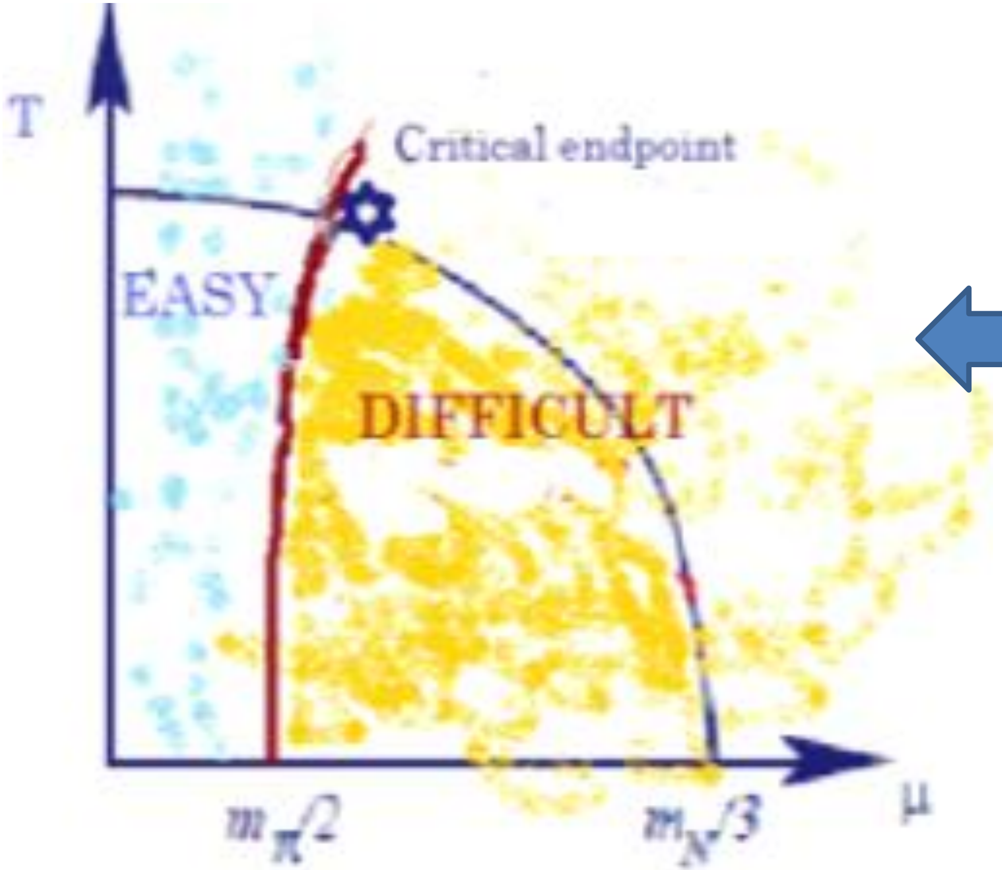
$a^{-1} = 440$ MeV (SC
$a^{-1} = 524$ MeV (β
Kluberg-Stern ('83)



Phase Diagram Evolution with Increasing β
Next-to-Next-to-Leading Order of $1/g^2$ Expansion
arXiv:0911.3453, T-Z. Nakano, K. Miura and A. Ohnishi
Courtesy Kohtaroh Miura

Option 2 : Integrate over fermions and ..

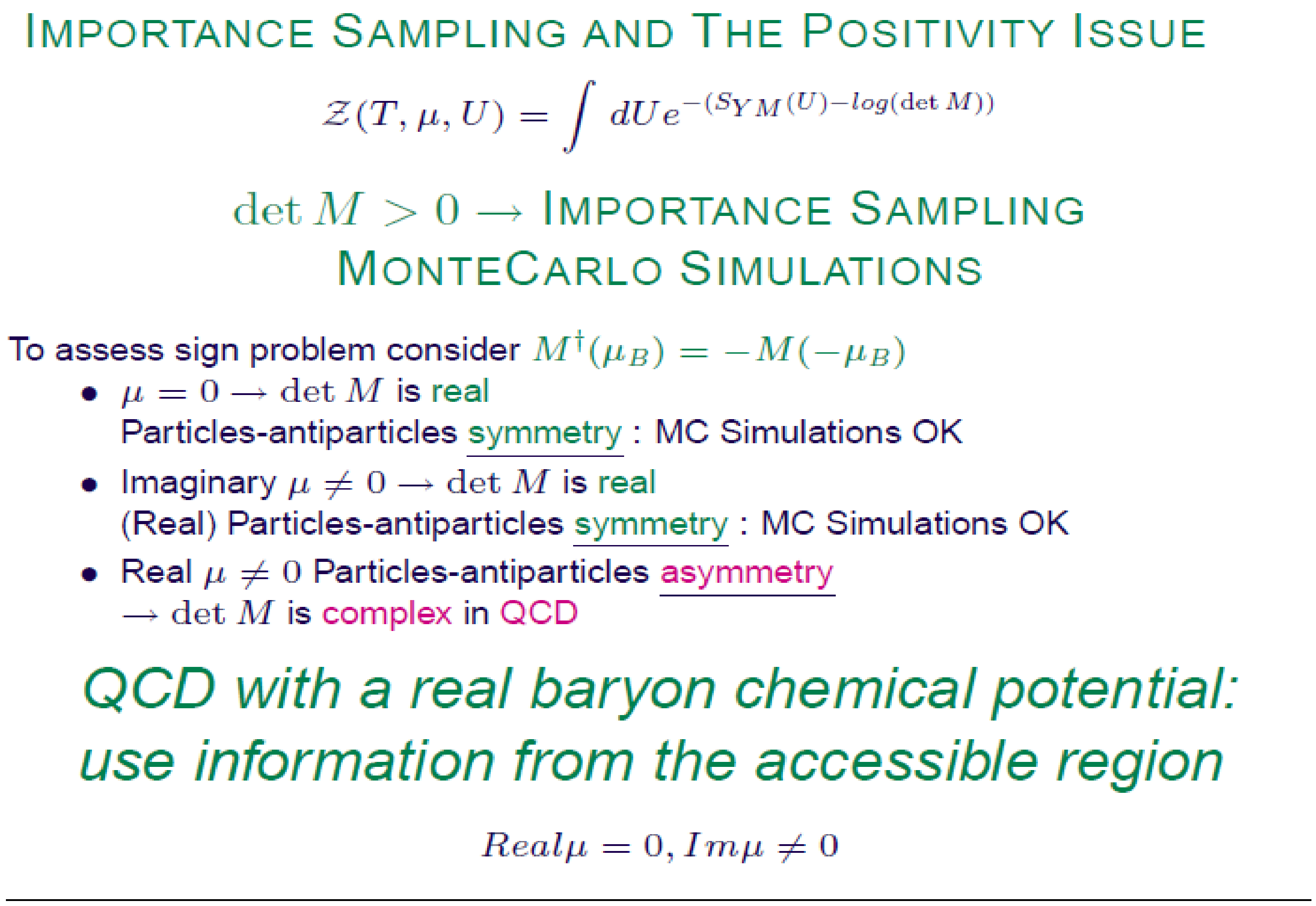
The $m_\pi/2$ barrier



Summary
Of our
efforts!!

(details at
the end)

THE CHALLENGE



IMPORTANCE SAMPLING AND THE POSITIVITY ISSUE

$$\mathcal{Z}(T,\mu,U) = \int dU e^{-(S_{YM}(U) - log(\det M))}$$

$\det M > 0 \longrightarrow$ IMPORTANCE SAMPLING MONTECARLO SIMULATIONS

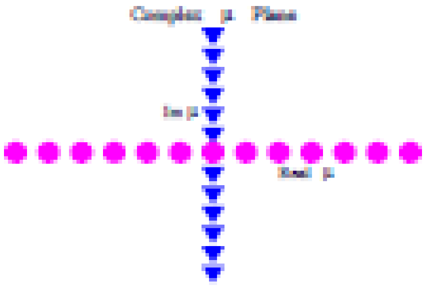
To assess sign problem consider $M^{\dagger}(\mu_B) = -M(-\mu_B)$

- $\mu = 0 \longrightarrow \det M$ is real
  Particles-antiparticles symmetry : MC Simulations OK
- Imaginary $\mu \neq 0 \longrightarrow \det M$ is real
  (Real) Particles-antiparticles symmetry : MC Simulations OK
- Real $\mu \neq 0$ Particles-antiparticles asymmetry
  $\longrightarrow \det M$ is complex in QCD

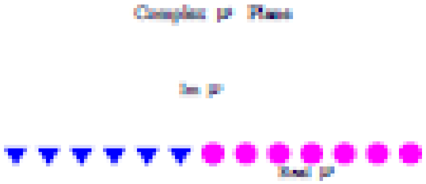
QCD with a real baryon chemical potential: use information from the accessible region

$$Real\mu = 0, Im\mu \neq 0$$

Because of the QCD symmetries, the complex μ_B plane

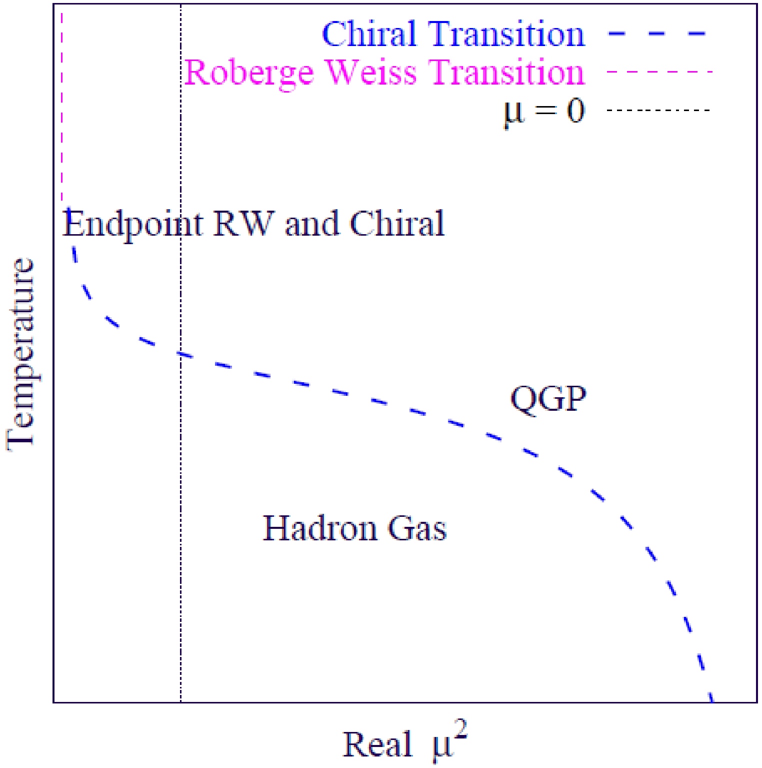


can be mapped onto the complex μ_B^2 plane



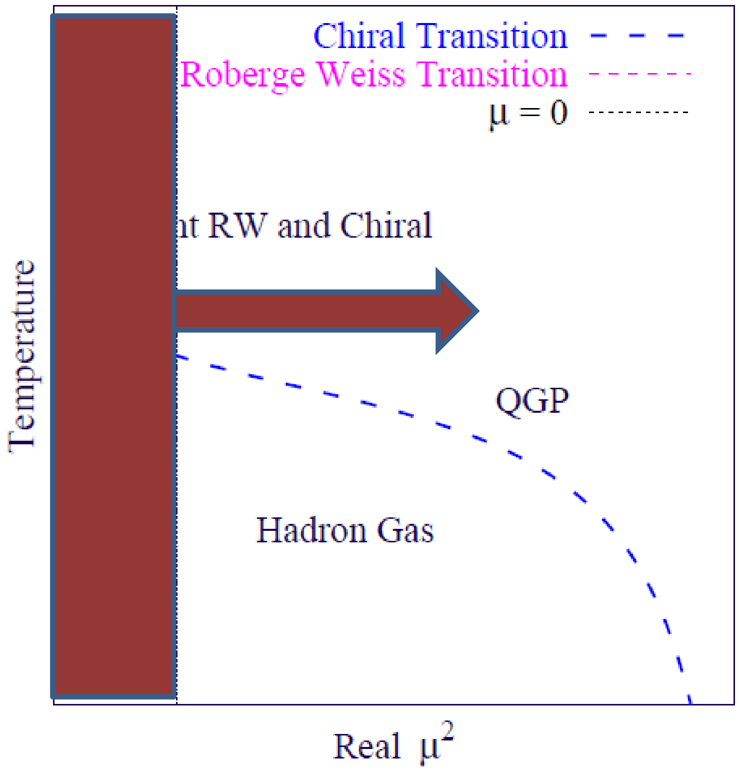
Useful to consider the QCD phase diagram in the temperature,

μ ^2 plane



Imaginary chemical potential

- **Imaginary chemical potential: main problem, control over analytic continuation**

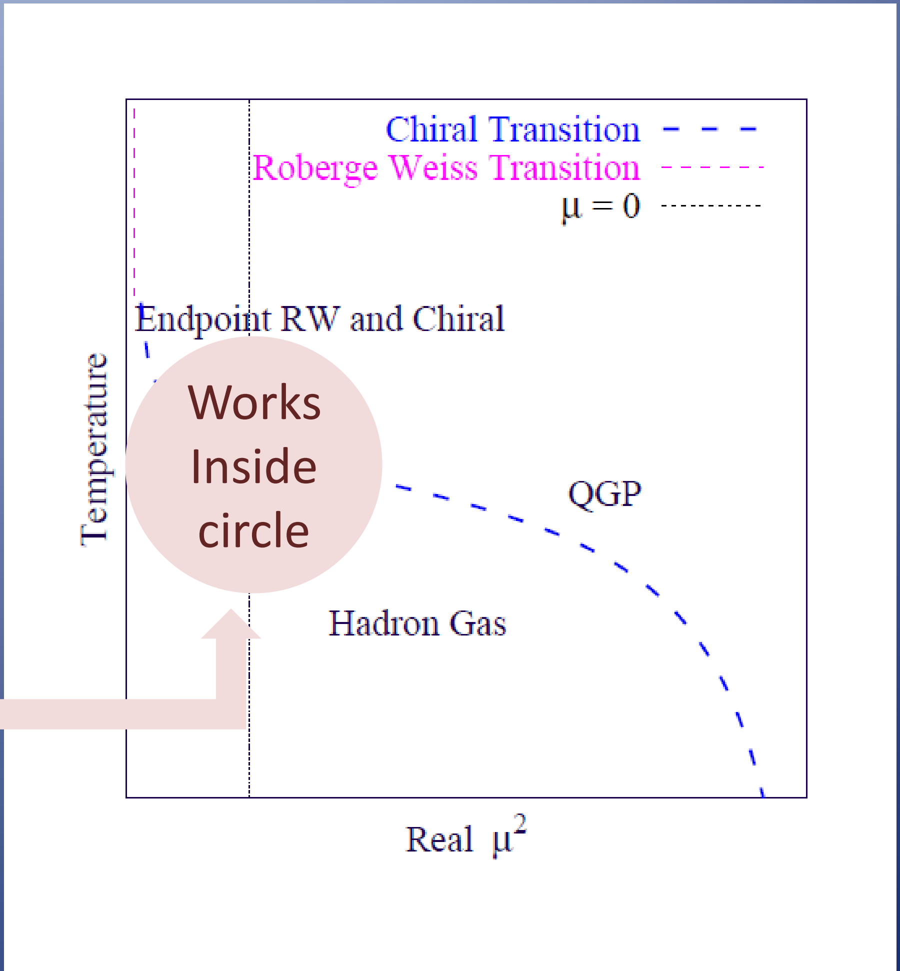


Different strategies for analytic continuation

- Taylor is good when approaching the free field limit
- Critical parametrization approprate in the sQGP?
- Fourier analysis natural in the hadron resonance gas region
- Pade’ approximants effective alternative

Taylor expansion

- **Taylor expansion: main problem, control of the convergence**



Thermodynamics and Taylor expansion

$$\mathcal{Z} = \int \mathcal{D}U (\mathrm{det}M(m_u,\mu_u))^{N_f/4}(\mathrm{det}M(m_d,\mu_d))^{N_f/4}(\mathrm{det}M(m_s,\mu_s))^{N_f/4} e^{-S_g}$$

$$\overset{(2+1)}{=} \int \mathcal{D}U (\mathrm{det}M(m_q,\mu_q))^{1/2}(\mathrm{det}M(m_s,\mu_s))^{1/4} e^{-S_g}$$

$$\mu_q = \mu_u = \mu_d$$

On the lattice at imaginary chemical potential

$$\begin{array}{llll} U_t & \rightarrow & e^{ia\mu_I} U_t & \textit{forward temporal link} \\ U_t^\dagger & \rightarrow & e^{-ia\mu_I} U_t^\dagger & \textit{backward temporal link} \end{array}$$

$\Longrightarrow$ *detM* *real and positive !*

$$\frac{p}{T^4} = \frac{1}{VT^3}\ln\mathcal{Z} \quad \text{pressure}$$

$$\frac{n_i}{T^3} = \frac{1}{VT^2}\frac{\partial\ln\mathcal{Z}}{\partial\mu_i} \quad \text{quark density}$$

$$\frac{p}{T^4}(\hat{\mu}) = \sum_{k,l,n} c_{kln} \quad (\hat{\mu}_u - \hat{\mu}_0)^k(\hat{\mu}_d - \hat{\mu}_0)^l(\hat{\mu}_s - \hat{\mu}_0)^n$$

where $\hat{\mu} = \frac{\mu}{T}$

$$c_{kln} = \frac{1}{k!l!n!}\frac{\partial^k}{\partial\hat{\mu}_u{}^k}\frac{\partial^l}{\partial\hat{\mu}_d{}^l}\frac{\partial^n}{\partial\hat{\mu}_s{}^n}\left(\frac{p}{T^4}\right)$$

$$\xrightarrow{\textit{on the lattice}} c_{kln} = \frac{1}{k!l!n!}\frac{N_\tau^{3-k-l-n}}{N_\sigma^3}\frac{\partial^k}{\partial\mu_u^k}\frac{\partial^l}{\partial\mu_d^l}\frac{\partial^n}{\partial\mu_s^n}(\ln Z)$$

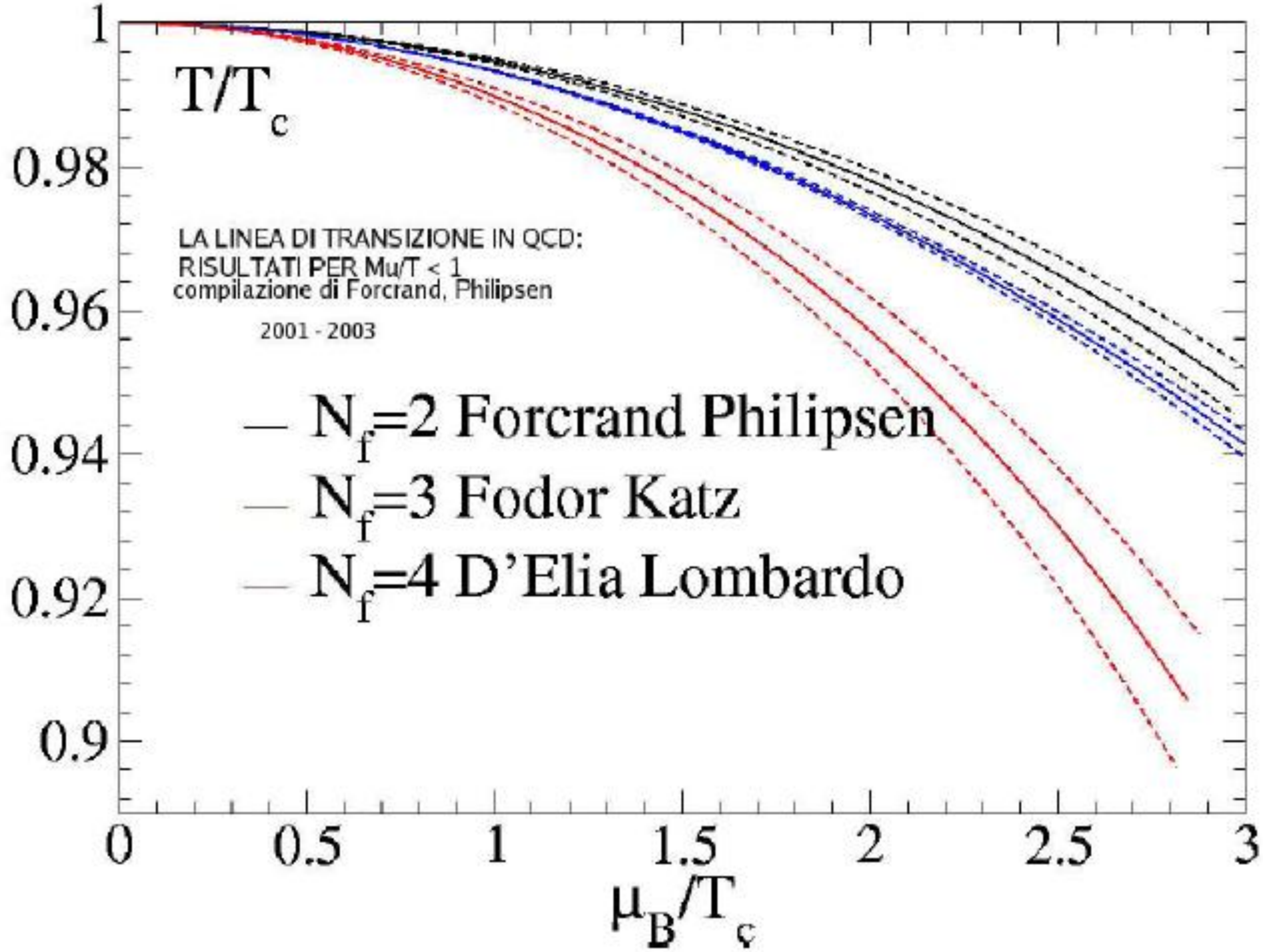
[S. Gottlieb,W. Liu,D. Toussaint,R.L. Renken,R.L. Sugar,Phys.Rev.D38(1988)2888]

[R.V. Gavai and S. Gupta,Phys.Rev.D68(2003)034506]

[C.R. Allton *et al.*,Phys.Rev.D68(2003)014507]

Application II

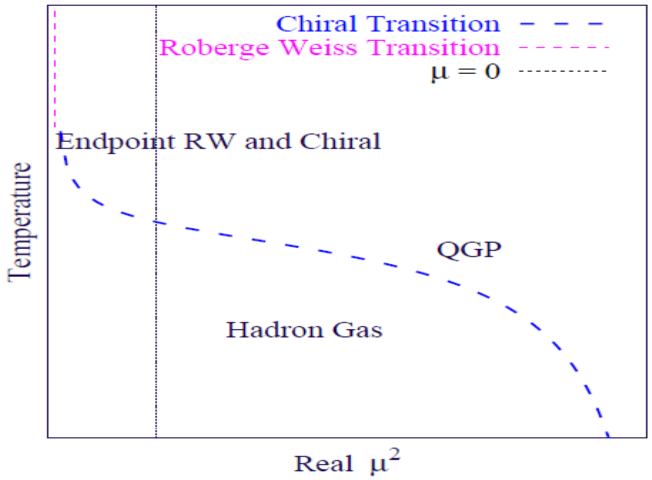
THE PSEUDOCRITICAL LINE



LA LINEA DI TRANSIZIONE IN QCD:
RISULTATI PER Mu/T < 1
compilazione di Forcrand, Philipsen

2001 - 2003

The pseudocritical line



Coefficient K in the Taylor expansion of the transition line, from $N_t = 4$
Compilation by Owe Philipsen, 2008

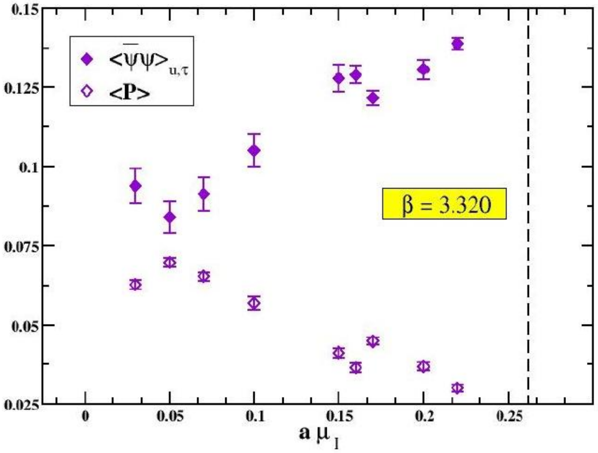
$$\frac{T_c(\mu)}{T_c(0)} = 1 - K(N_f, m_f)\left(\frac{\mu}{\pi T}\right)^2 + \mathcal{O}\left(\left(\frac{\mu}{\pi T}\right)^4\right) \quad .$$

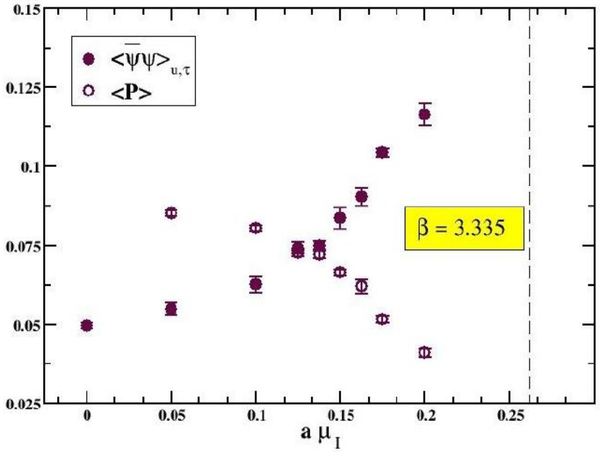
N_f	am	N_s	K	Action	β-Function	Method
2	0.1	16	0.69(35)	p4	non-pert.	Taylor+Rew.
	0.025	6,8	0.500(34)	stag.	2-loop pert.	Imag.
3	0.1	16	0.247(59)	p4	non-pert.	Taylor+Rew.
	0.026	8,12,16	0.667(6)	stag.	2-loop pert.	Imag.
	0.005	16	1.13(45)	p4	non-pert.	Taylor+Rew.
4	0.05	16	0.93(9)	stag.	2-loop pert.	Imag.
2+1	0.0092,0.25	6-12	0.284(9)	stag.	non-pert.	Rew.

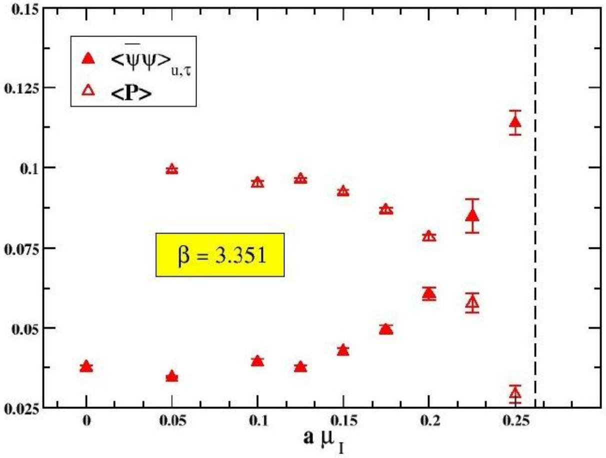
More control:
Taylor + Im mu

Falcone,
Laermann,
MpL

$$\langle \hat{\psi}\psi\rangle_{q,\tau} \equiv \frac{1}{4}\frac{1}{N_\sigma^3 N_t}\langle TrM^{-1}\rangle_\tau\,, \quad q = u, l, s$$

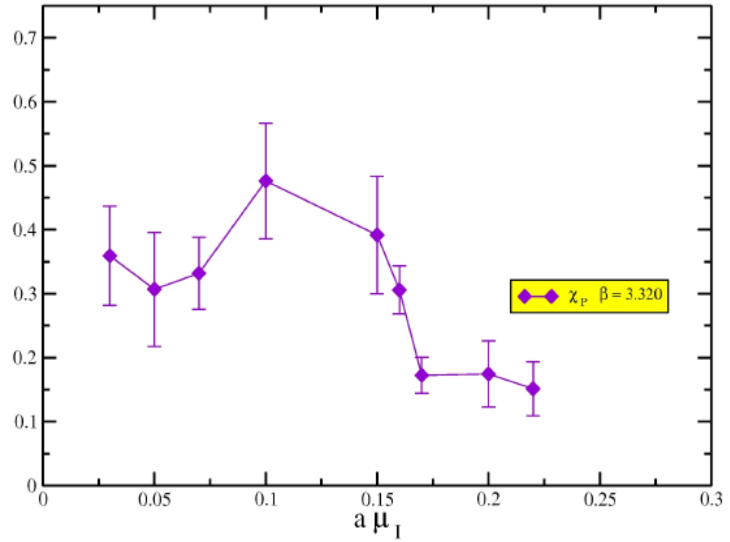




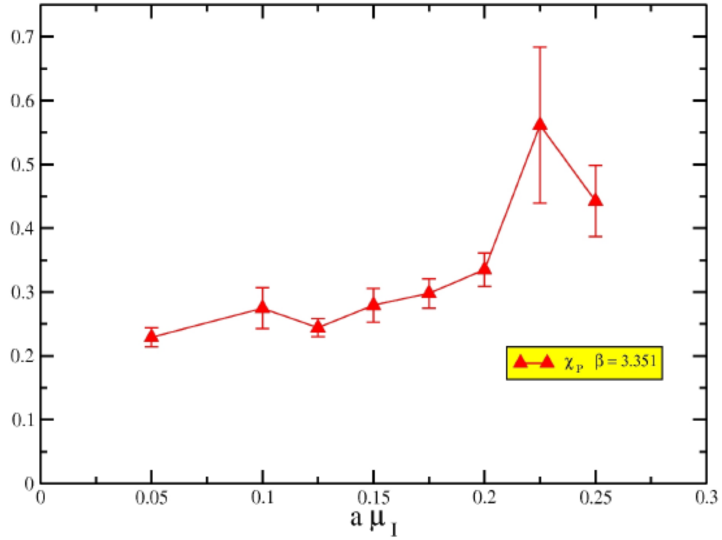


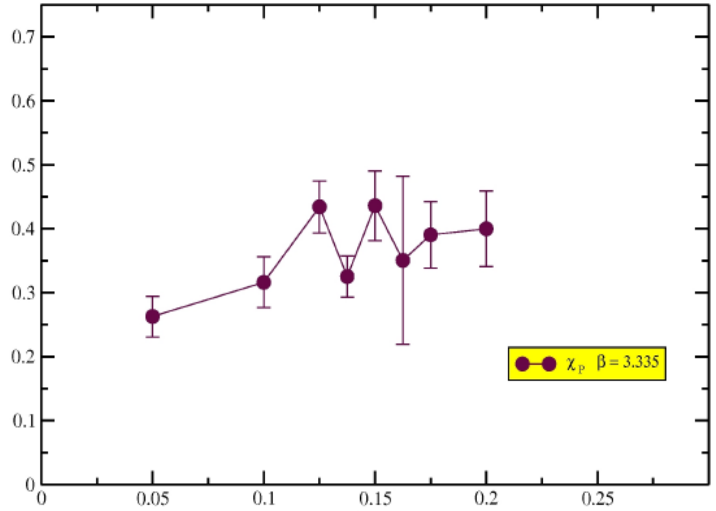
Falcone, Laermann, MpL

Susceptibility of the Polyakov loop



$$\chi_P = N_s^3 \left(\langle P^2 \rangle - \langle P \rangle^2 \right)$$



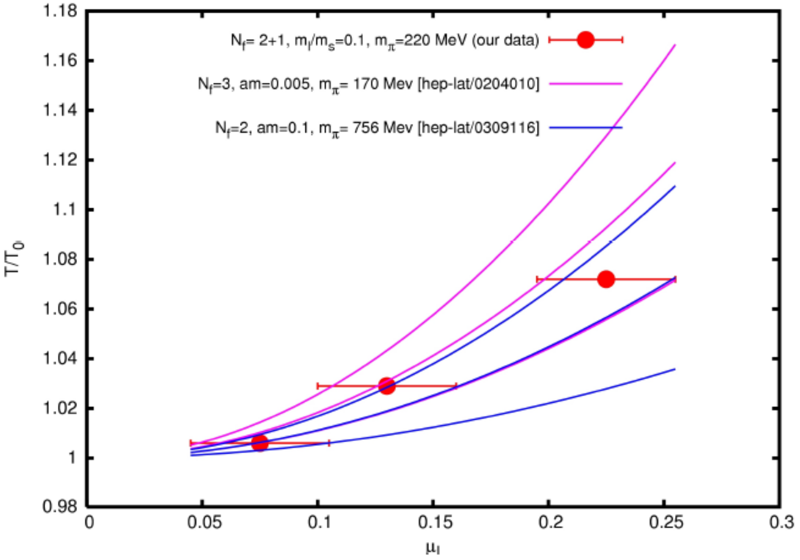


The pseudocritical line

[O. Philpsen,Prog.Theor.Phys.Suppl.174(2008)206]

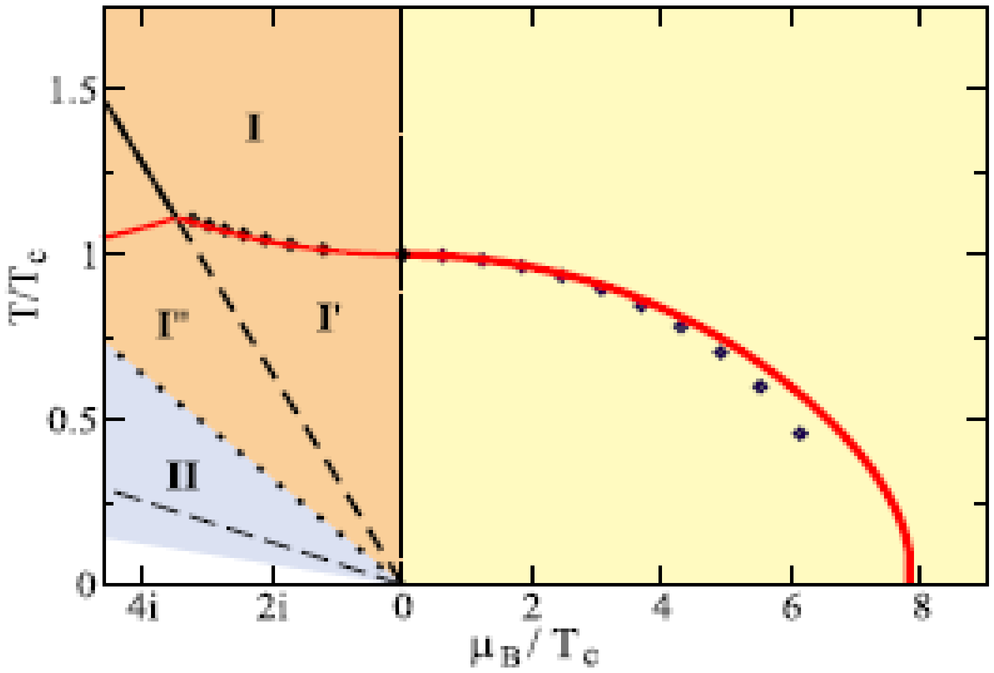
$$\frac{T_c(\mu)}{T_c(0)} = 1 - t_2(N_f, m_f)\left(\frac{\mu}{\pi T}\right)^2 + \mathcal{O}\left(\left(\frac{\mu}{\pi T}\right)^4\right)$$

$$T_c(0) \simeq 204 MeV$$



[hep-lat/0204010]: C.R Allton *et all*, Phys.RevD66(2002)074507 $t_2 = 0.69(35)$
[hep-lat/0309116]: C.R Allton *et all*, Nucl.Phys.Proc.Suppl.129(2004)614 $t_2 = 1.13(45)$

The pseudocritical line



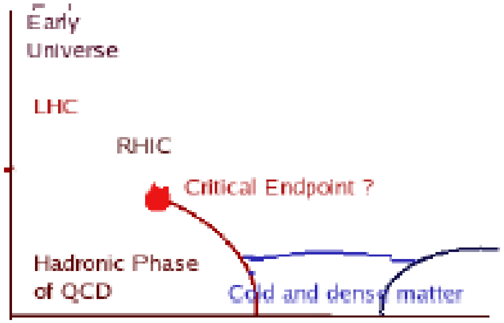
Kämpfer, Bluhm QM08

Data from M. D'Elia, MpL 2004
Analytic continuation can be extended at lower T via Pade' (MpL 2005) or phenomenological models (Kampfer Bluhm 2008)

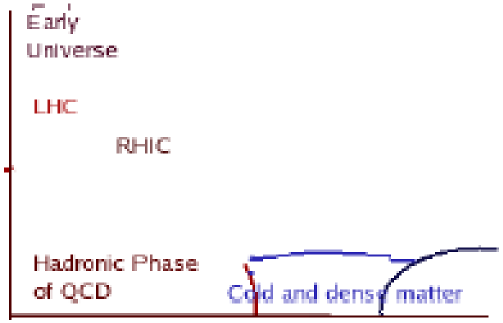
Application III

CRITICAL ENDPOINT

THE CRITICAL ENDPOINT

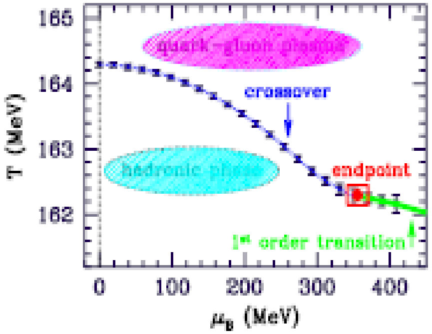


OR



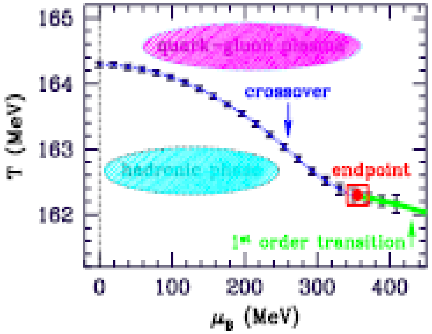
BOTH SCENARIO ARE COMPATIBLE
WITH MODEL CALCULATIONS AND UNIVERSALITY

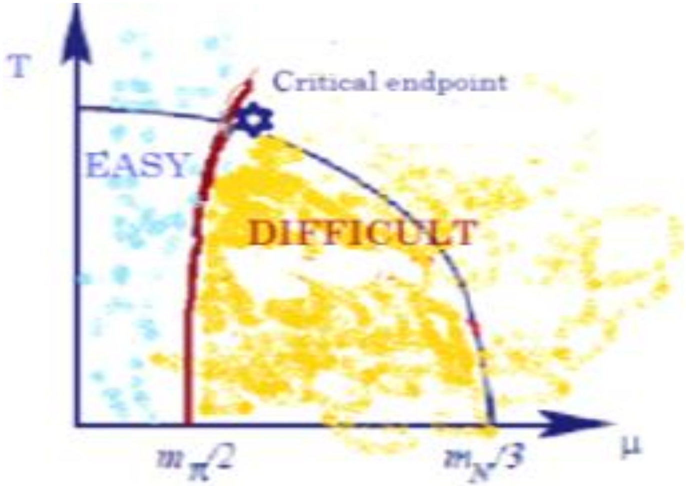
STRATEGY 0 : FODOR KATZ , REWEIGHTING FROM $\mu = 0$



CRITICISM:
Critical point is close to the phase quenched threshold where reweighting fails at T=0
HOWEVER :
Important contribution from the phase does not necessarily hamper reweighting : overlap might still be large or correlation with the phase might be small.
Splittorff, Verbaarschot, MpL , in progress

STRATEGY 0 : FODOR KATZ , REWEIGHTING FROM $\mu = 0$





CRITICISM:

Critical point is close to the phase quenched threshold where reweighting fails at T=0

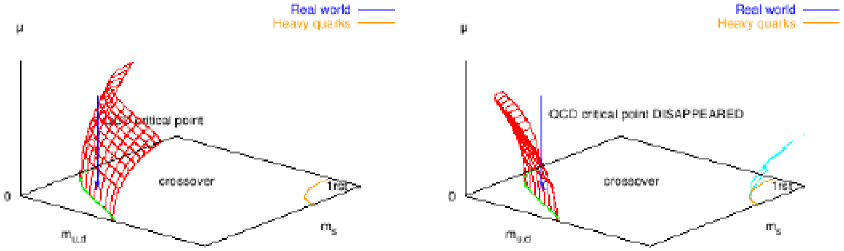
HOWEVER :

Important contribution from the phase does not necessarily hamper reweighting : overlap might still be large or correlation with the phase might be small.

Splittorff, Verbaarschot, MpL , in progress

CHALLENGING THE ENDPOINT

STRATEGY I : FORCRAND-PHILIPSEN



Scenario I or Scenario II ? To decide, measure slope K in

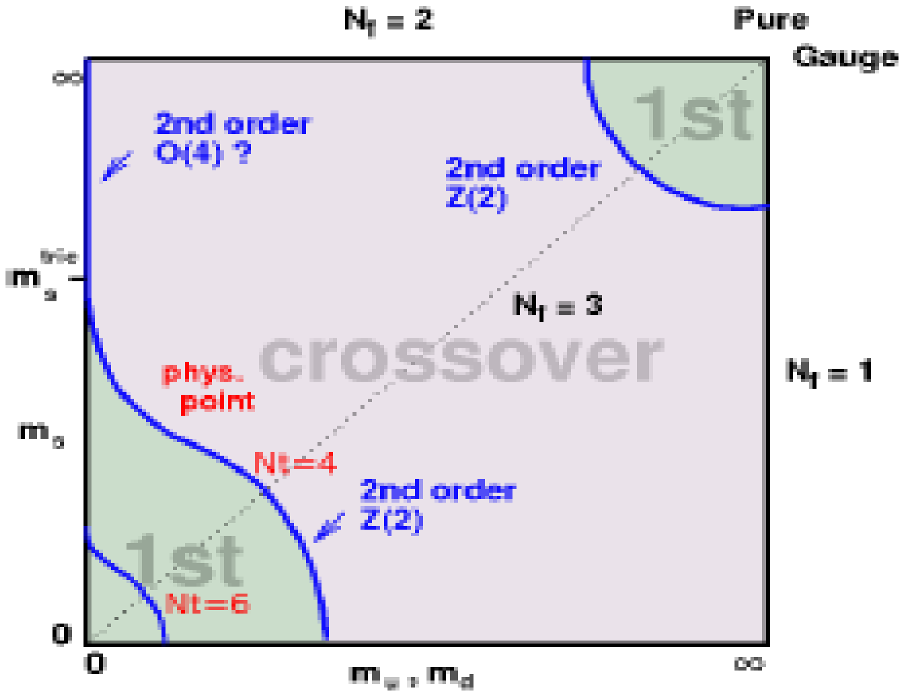
$$\frac{m_c(\mu)}{m_c(0)} = 1 + K\left(\frac{\mu}{T}\right)^2 + \ldots$$

$K > 0$: Scenario I , critical endpoint at small μ_B
$K < 0$: Scenario II, NO critical endpoint at small μ_B

CURRENT RESULTS SUGGEST NO CRITICAL ENDPOINT FOR $\mu_B < 600 MeV$

NB: assume that endpoint is part of the critical surface at m=0

Towards the continuum



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RESCUING THE ENDPOINT

Strategy II : Gavai and Gupta, Bielefeld-RBC

Series expansion for the pressure:

$$P(T,\mu_B) = P(T) + \frac{1}{2}\chi_B^{(2)}(T)\mu_B^2 + \frac{1}{4!}\chi_B^{(4)}(T)\mu_B^4 + \frac{1}{6!}\chi_B^{(6)}(T)\mu_B^6 + \frac{1}{8!}\chi_B^{(8)}(T)\mu_B^8 + \cdots,$$

The quark number susceptibility has the expansion

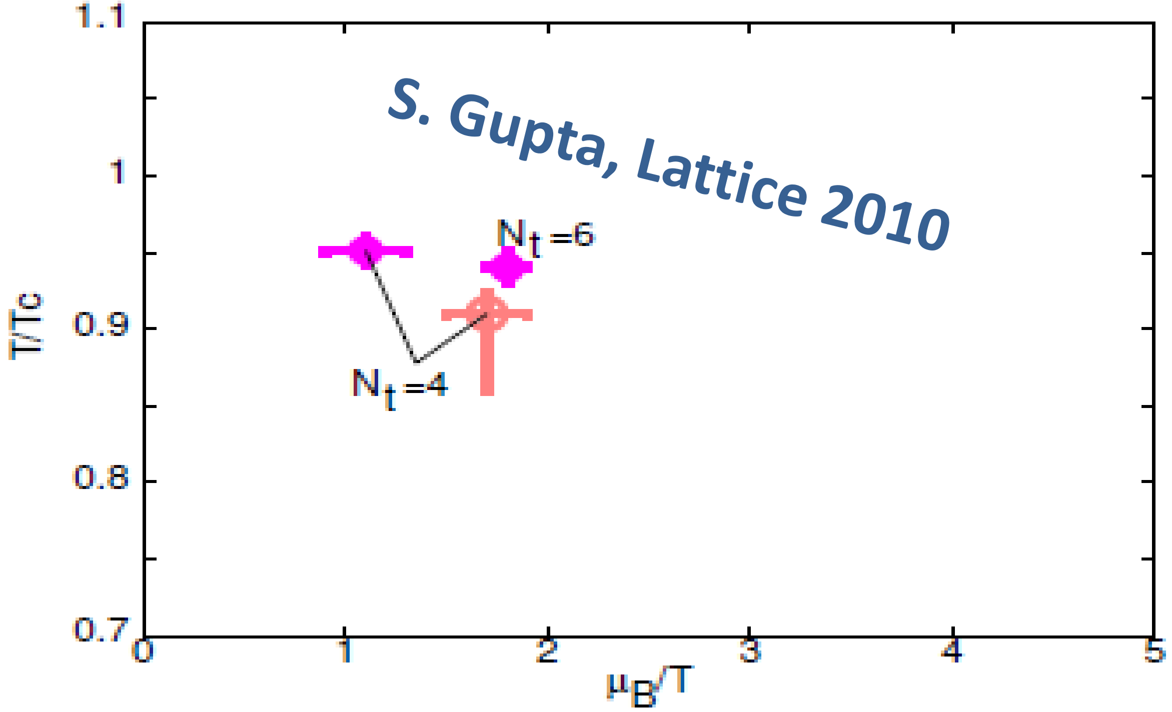
$$\chi_B(T,\mu_B) = \chi_B^{(2)}(T) + \frac{1}{2}\chi_B^{(4)}(T)\mu_B^2 + \frac{1}{4!}\chi_B^{(6)}(T)\mu_B^4 + \frac{1}{6!}\chi_B^{(8)}(T)\mu_B^6 + \cdots.$$

This series is expected to diverge at the QCD critical end point. Radius of convergence is

$$\lim_{n\rightarrow\infty}\mu_*^{(n)} = \sqrt{\frac{1}{n(n-1)}\frac{\chi_B^{(n+2)}}{\chi_B^{(n)}}}.$$

The endpoint is the first singularity in the complex μ plane occurring at real μ.
Coefficients should be all positive at large n

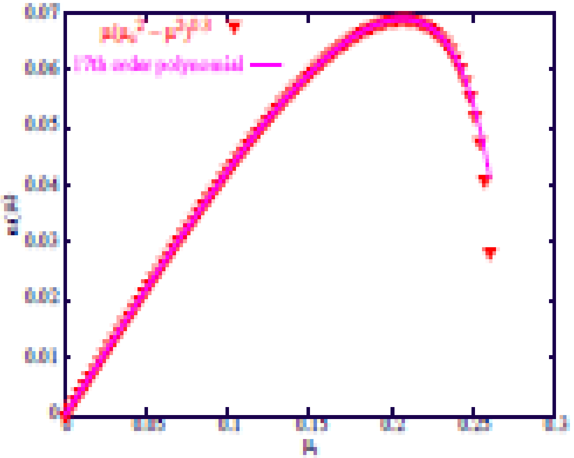
Cutoff dependence and the effect of strange quarks

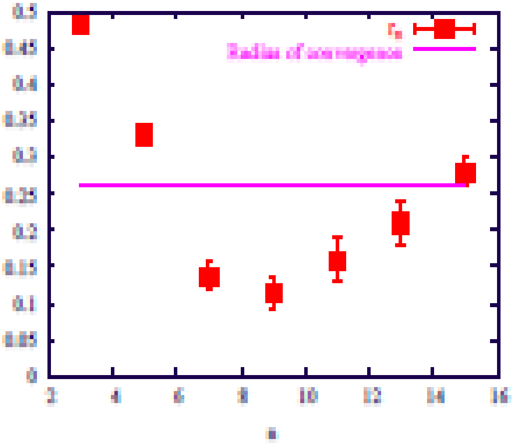


Staggered: $N_f = 2$, $m_\pi = 230$ MeV, $LT \geq 4$ Gavai, SG, 0806.2233
P4: $N_f = 2 + 1$, $m_\pi = 220$ MeV, $LT = 4$ Schmidt, 2010

Taylor expanding the numerical result
at imaginary μ
M. D'Elia, F. Di Renzo, MpL 2007

And computing the radius of convergenc





CAVEAT : the correct result might need many orders

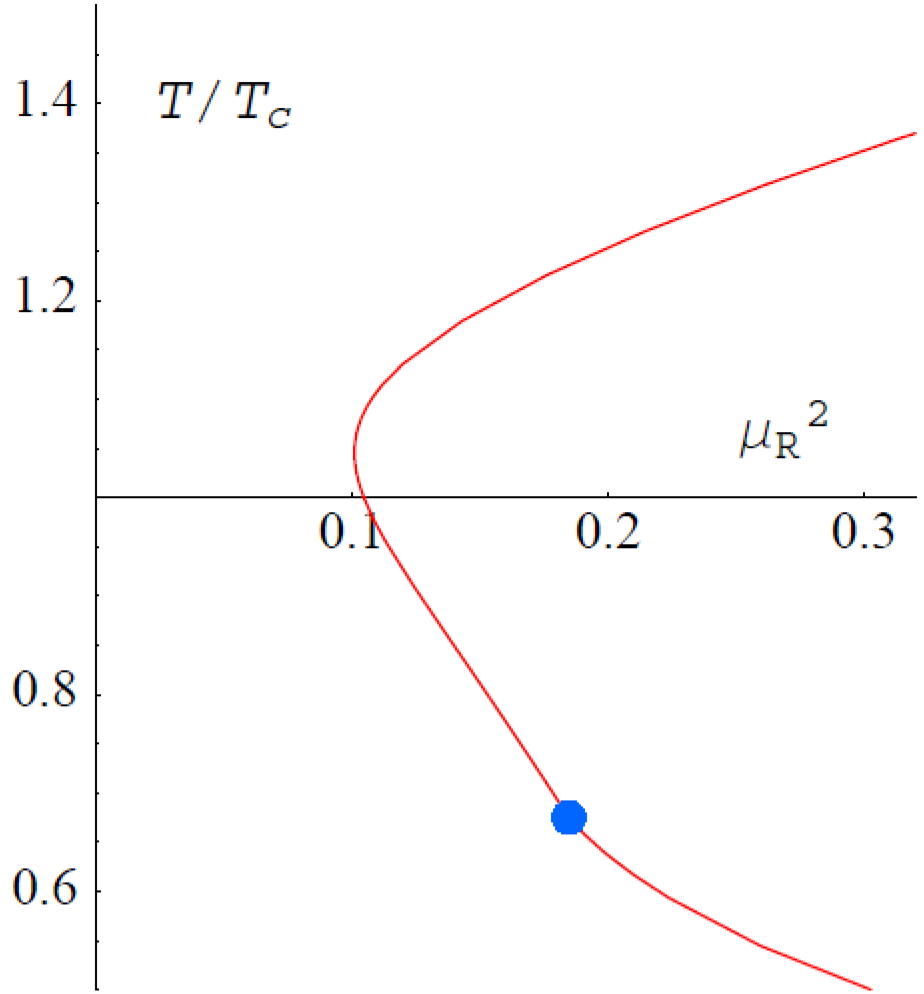
Better control over the endpoint?

QCD Critical Point and Complex Chemical Potential Singularities

M. A. Stephanov

Singularities limit the radius of convergence

This can be computed from the Taylor expansion and observed at purely imaginary μ



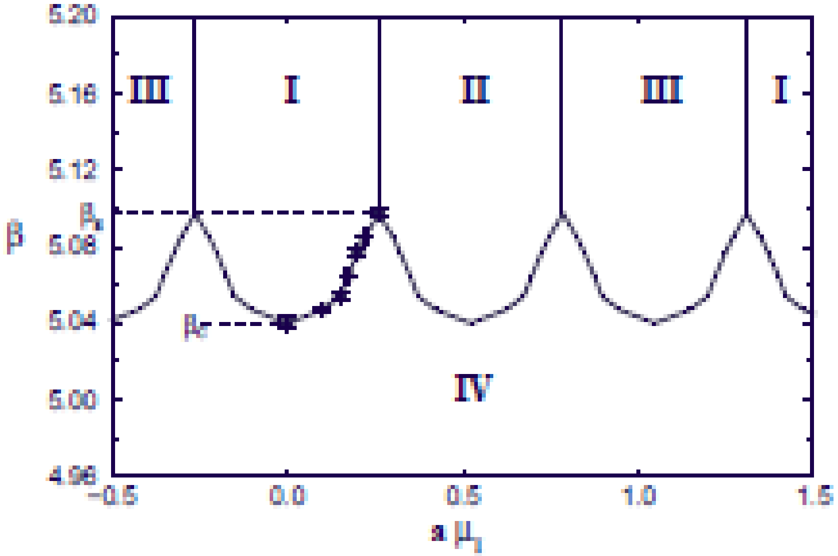
THE PHASE DIAGRAM IN THE IMAGINARY μ – T SPACE

$$Z(\mu_I/T) \equiv Z(V, T, i\mu_I/T) = \mathrm{Tr}\left(e^{i\mu_I N/T} e^{-\frac{H_{\mathrm{QCD}}}{T}}\right)$$

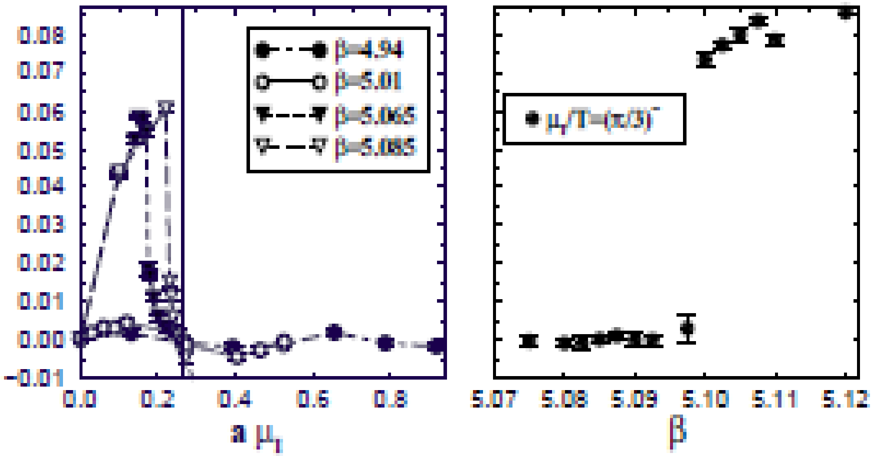
- N is a number operator: $Z(\mu_I/T)$ periodic in μ_I with period $2T\pi$; moreover a period $2T\pi/3$ is expected in the confined phase, where only physical states with N multiple of 3 are present.
- Observation (Roberge and Weiss) : $Z(\mu_I)$ is always periodic $2T\pi/3$, for any physical temperature!
- Low T : smooth periodicity
- High T : non-analytic behaviour with discontinuities at

  $$\theta = 2\pi/3(k + 1/2)$$

  corresponding to phase transitions from one Z_3 sector to the other.
- $P(\vec{x})e^{i\mu_I/T}$, instead of $P(\vec{x})$: μ_I/T fixes the preferred vacuum.



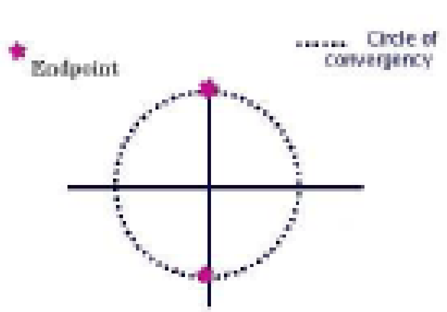
Sketch of the phase diagram in the μ_I–β plane.



Imaginary part of the barion density as a function of μ_I for different values of β (left–hand side), and as a function of β at $\mu_I/T = \frac{\pi}{3}^-$ (right-hand side).

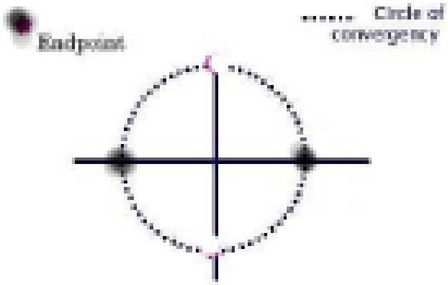
M. D'Elia, MpL, 2001

Singularities for complex μ



Endpoint of the RW Transition

$$T_R > T_c$$

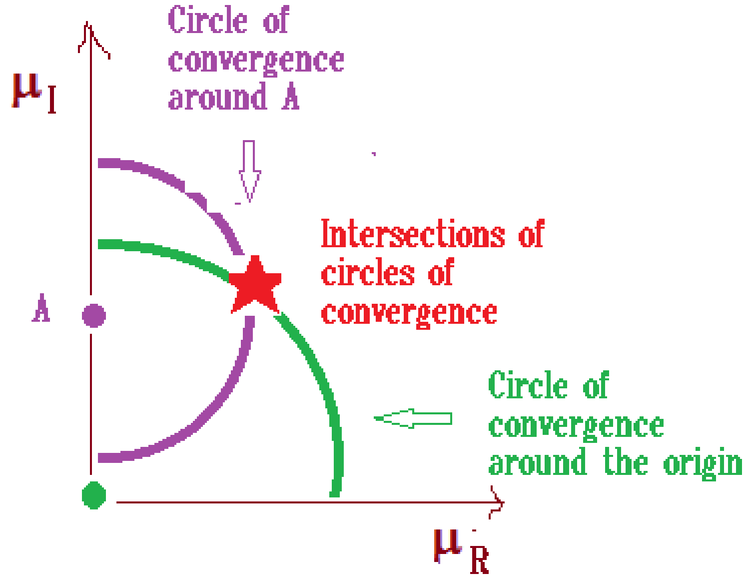


Endpoint of the Chiral Transition

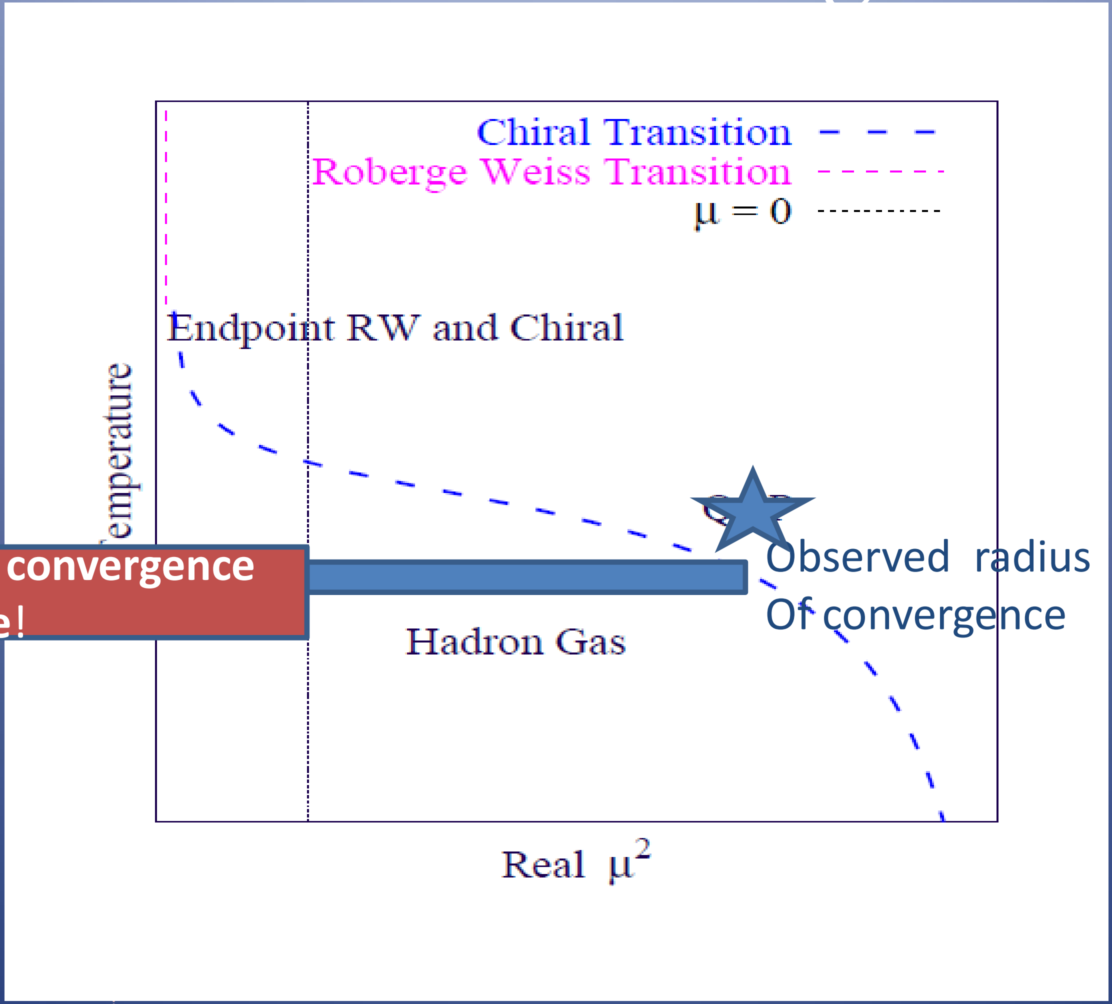
$$T_\chi < T_c$$

Taylor expansion around μI

Falcone, Laermann, MpL



Use imaginary μ to validate radius of convergence



Check radius of convergence here!

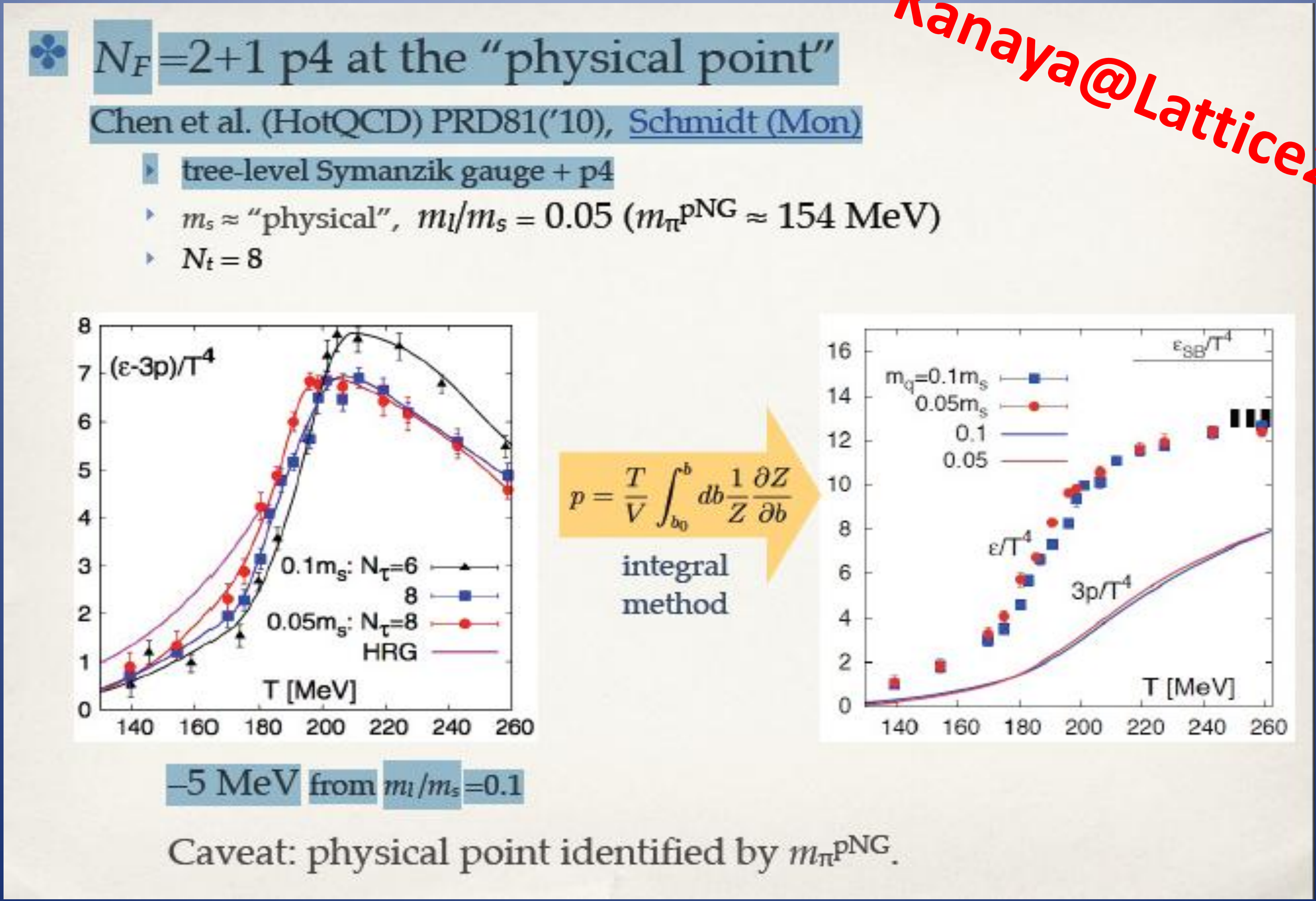
Observed radius
Of convergence

l
o
n

sQGP, thermodynamics, and the phase diagram for a complex chemical potential

Equation of state

Kanaya@Lattice2010



✤ N_F=2+1 p4 at the "physical point"

Chen et al. (HotQCD) PRD81('10), Schmidt (Mon)

- tree-level Symanzik gauge + p4
- $m_s \approx$ "physical", $m_l/m_s = 0.05$ ($m_\pi^{\mathrm{PNG}} \approx 154$ MeV)
- $N_t = 8$

$$p = \frac{T}{V}\int_{b_0}^{b} db \frac{1}{Z}\frac{\partial Z}{\partial b}$$

integral method

–5 MeV from m_l/m_s=0.1

Caveat: physical point identified by m_π^{PNG}.

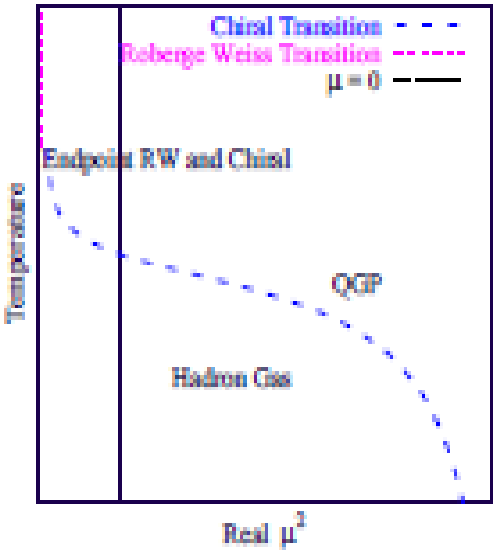
Nature of the sQGP phase

- **Quarkonium Spectrum and Spectral Functions**: many contributions in parallel sessions at Lattice2010.
- **Charmonium:** more popular so far
- **Bottomonium:** Important at the LHC
- **Trasport Coefficients**

More theoretical input needed **!**

THERMODYNAMICS AND CRITICAL LINES in THE T μ^2 PLANE

Three regimes for thermodynamics:



- Low Temperature,

  away from critical lines:

  Hadron Gas

  $n(T,\mu) = K(T)\sinh(N_c\mu/T)$

- In the sQGP region:

  $p(T,\mu)$

  $= b(T)|t + a(T)(\mu^2 - \mu_c^2)|^{(2-\alpha)}$

  implying

  $n(T,\mu) =$

  $A(T)\mu(\mu^{c2} - \mu^2)^{(2-\alpha)}$

- High Temperature,

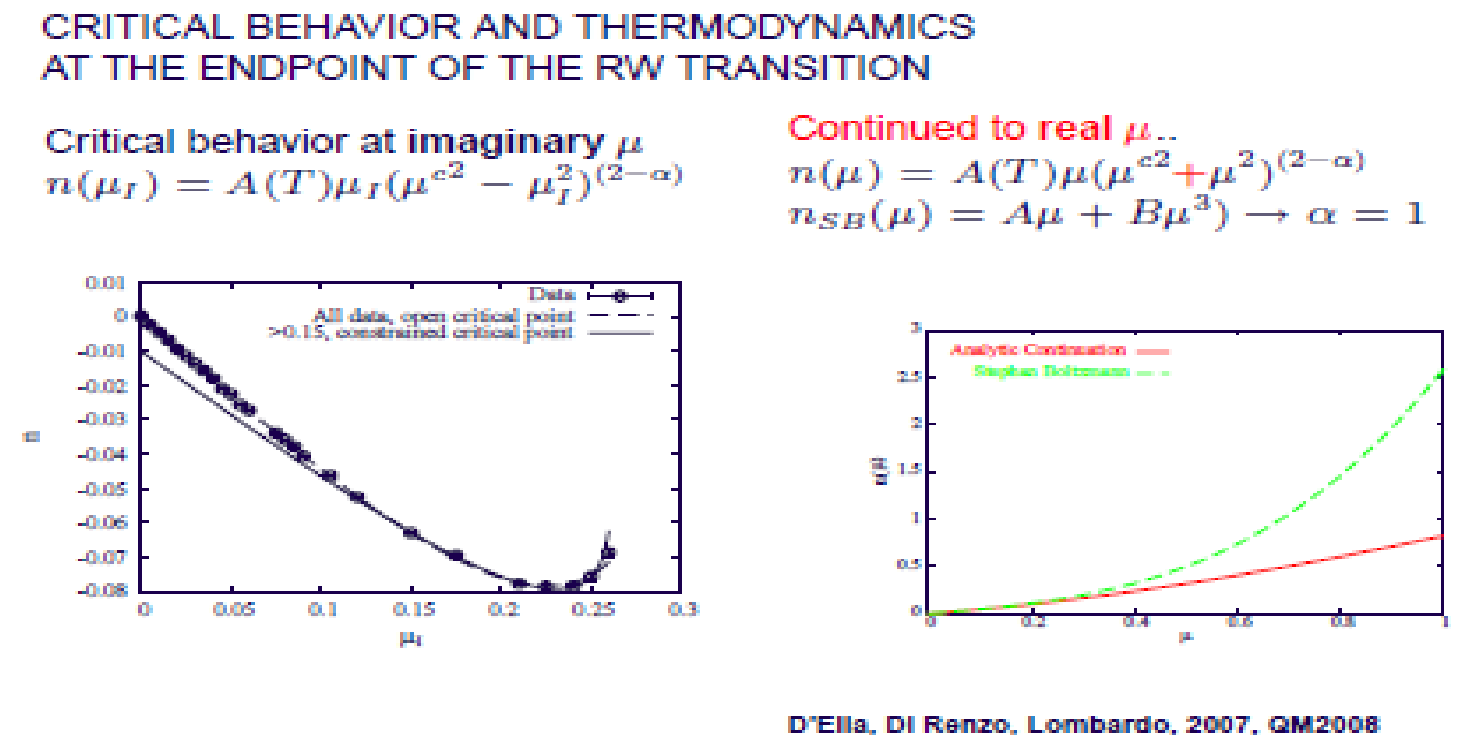
  away from critical line

  Approach to Free Field

  $n(T,\mu) \rightarrow n_{SB}(T,\mu)$

Singular behaviour at the End point of the RW transition

Nf=4 (D,Elia, di Renzo, Lombardo (2007))
Nf=2 (D'Elia, Sanfilippo(2009))
Nf=3 (de Forcrand, Philipsen (2009))



CRITICAL BEHAVIOR AND THERMODYNAMICS AT THE ENDPOINT OF THE RW TRANSITION

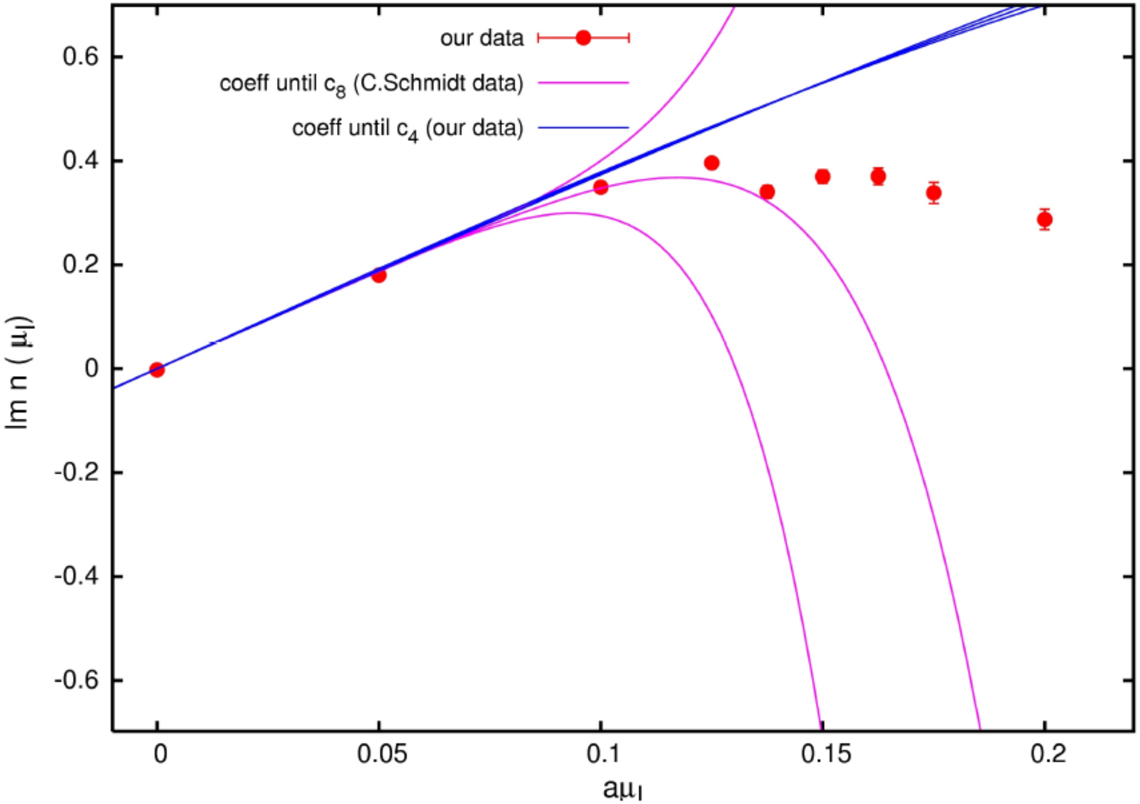
Critical behavior at imaginary μ
$n(\mu_I) = A(T)\mu_I(\mu^{c2} - \mu_I^2)^{(2-\alpha)}$

Continued to real μ..
$n(\mu) = A(T)\mu(\mu^{c2}+\mu^2)^{(2-\alpha)}$
$n_{SB}(\mu) = A\mu + B\mu^3) \rightarrow \alpha = 1$

D'Elia, Di Renzo, Lombardo, 2007, QM2008

Imaginary chemical potential and Taylor expansion

$$Im(n(\mu_I)) = 2N_\tau c_{200}\mu_I - 4N_\tau^3 c_{400}\mu_I^3 + 6N_\tau^5 c_{600}\mu_I^5 - 8N_\tau^7 c_{800}\mu_I^7 + O(\mu_I^9)$$



Falcone, Laermann, MpL

FREEZOUT, AND THE LATTICE

JOHANN WOLFGANG GOETHE
UNIVERSITÄT
FRANKFURT AM MAIN

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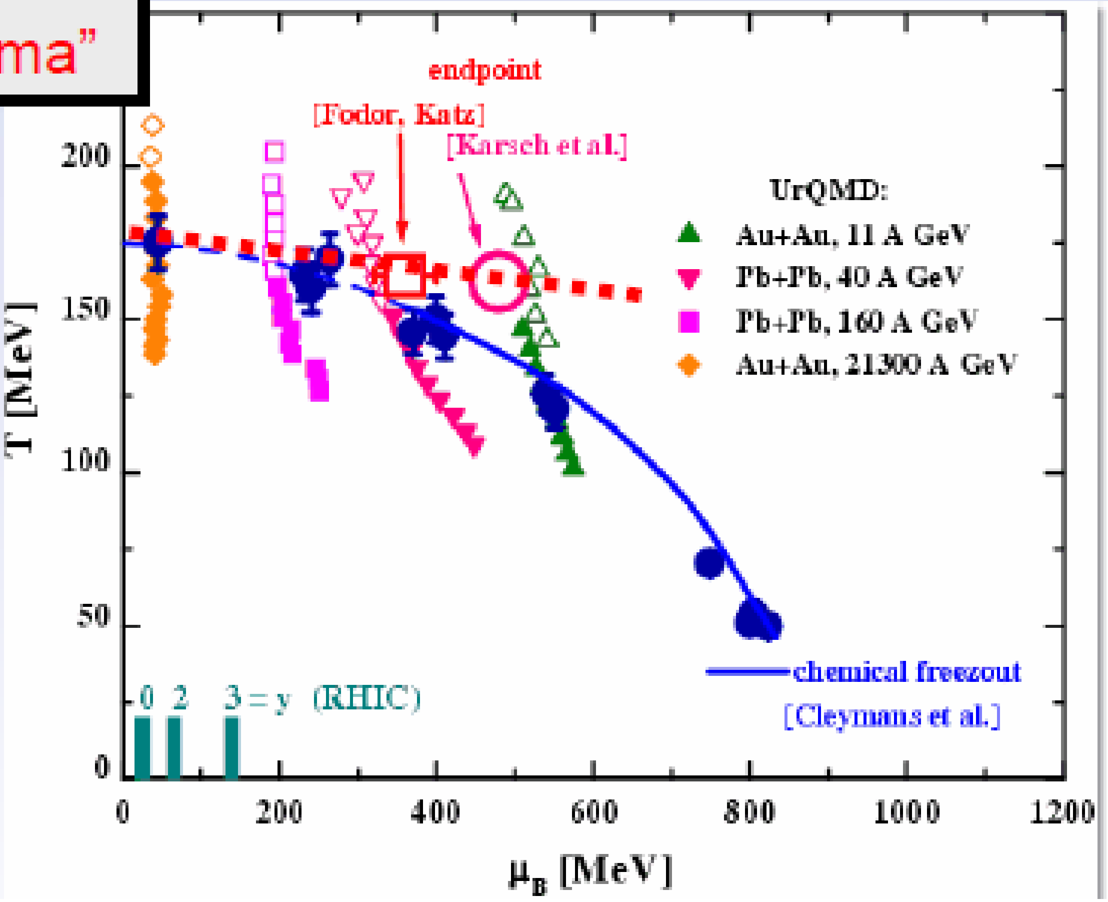
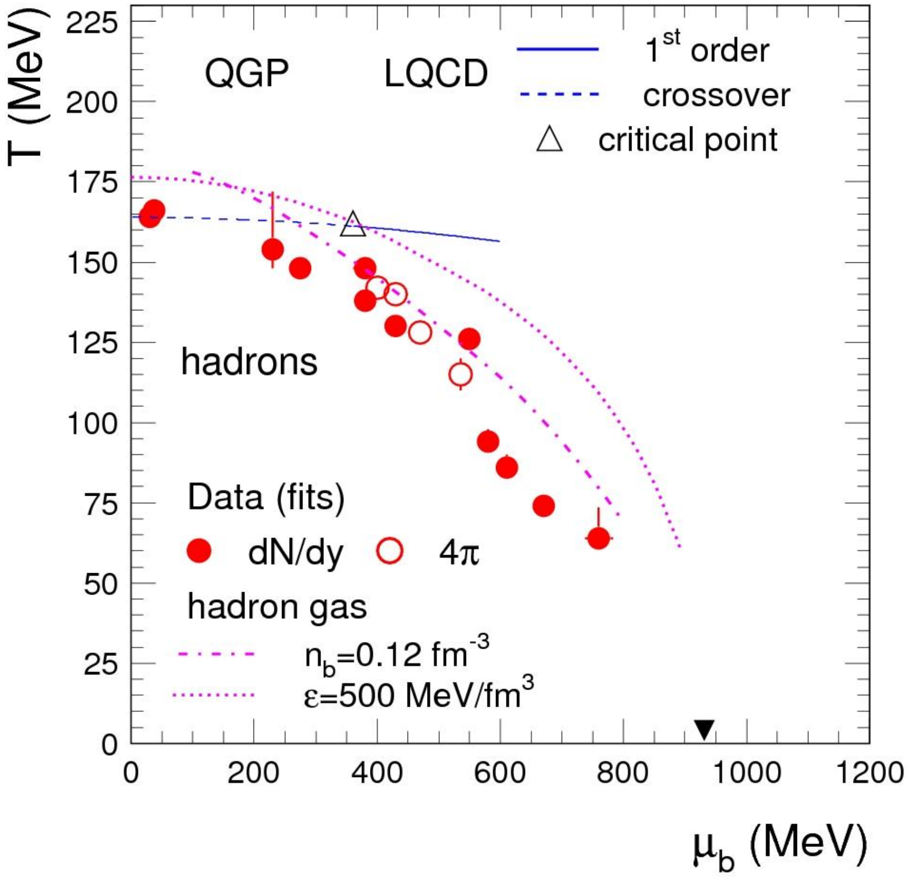


Freezout

Andronic, Braun-Munzinger, Stachel 2009 – Courtesy of the Authors



FREEZOUT

Values of μ_q^F/T at freezout for the temperatures used in the lattice simulations.

Table 1: Freezout parameters

T/T_c	μ_B^F (GeV)	μ_q^F/T
0.81	0.48	1.16
0.87	0.38	0.85
0.90	0.3	0.65
0.96	0.15	0.30

Previous analysis have shown that for this range of temperatures the Hadron Gas parametrization is satisfied by the first coefficents.

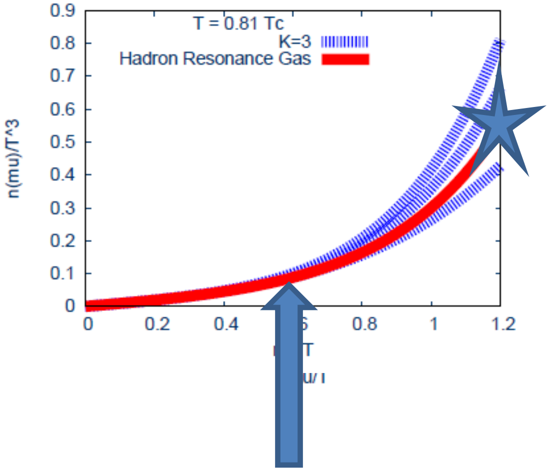
Then, to assess the extent of the convergence, we can directly contrast $n_q^3(T,\mu_{qI})/T^3$ and $n_q^{HG}(T,\mu_{qI})/T^3$, with $F(T)=\frac{2}{3}c_2$.

Lower bound on the radius of convergence And freezout point

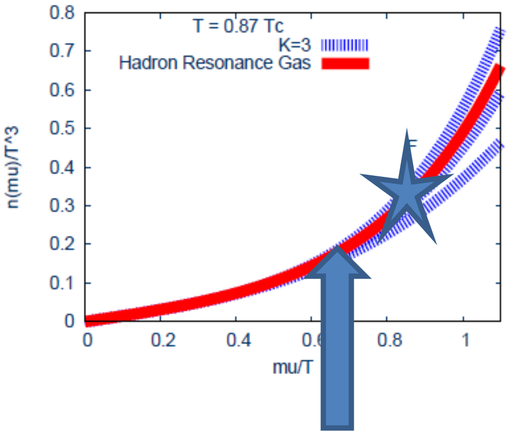
Data from RBC Collaboration
Courtesy E. Laermann and C. Schmidt. C. Ratti and MpL QM09

★ Freezout point

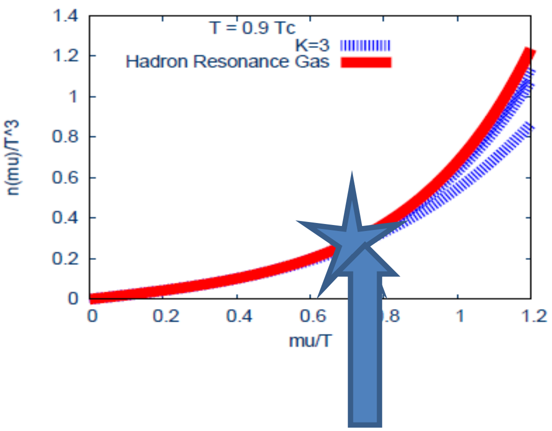
T = 0.81 Tc



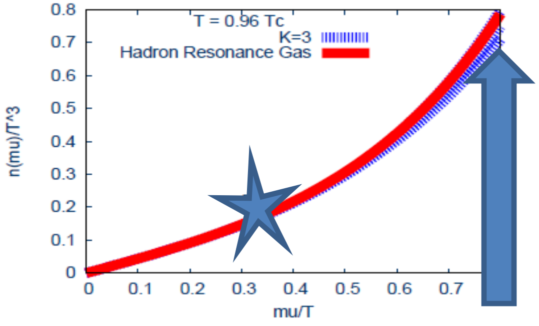
T = 0.87 Tc



T = 0.90 Tc

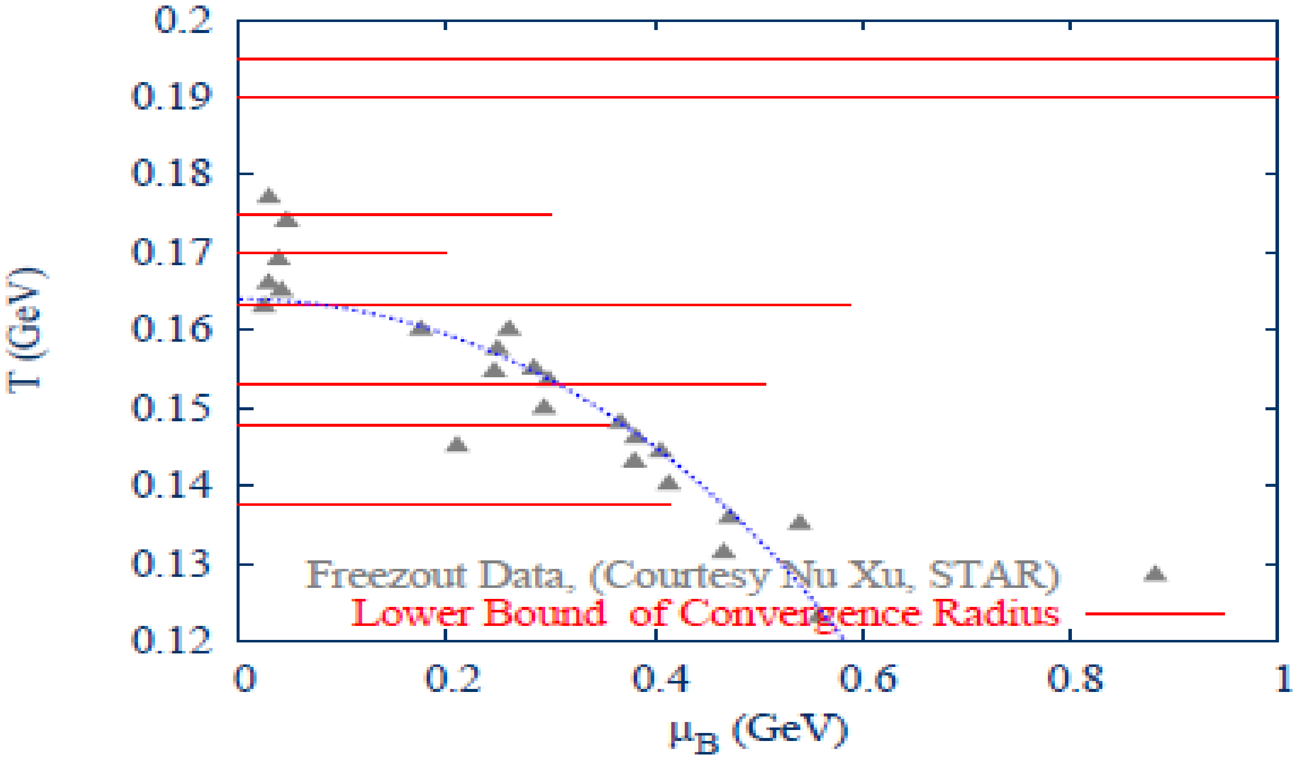


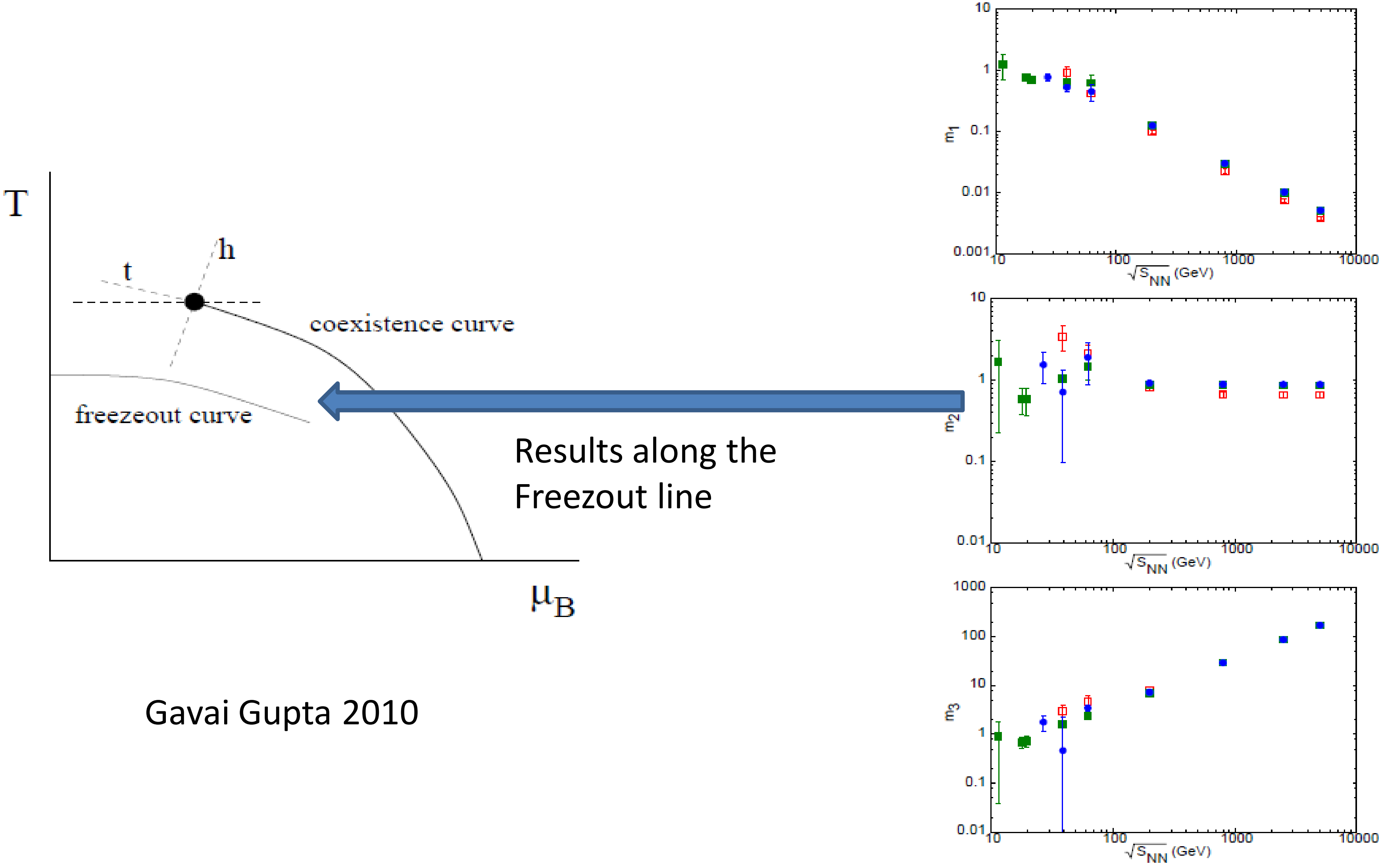
T = 0.96 Tc



Freezout line might well be amenable to a lattice study

Lattice data from RBC-Bielefeld Collaboration – C. Ratti MpL QM09





Results along the Freezout line

Gavai Gupta 2010

Application IV

THE QUARKYONIC PHASE

The quarkyonic phase

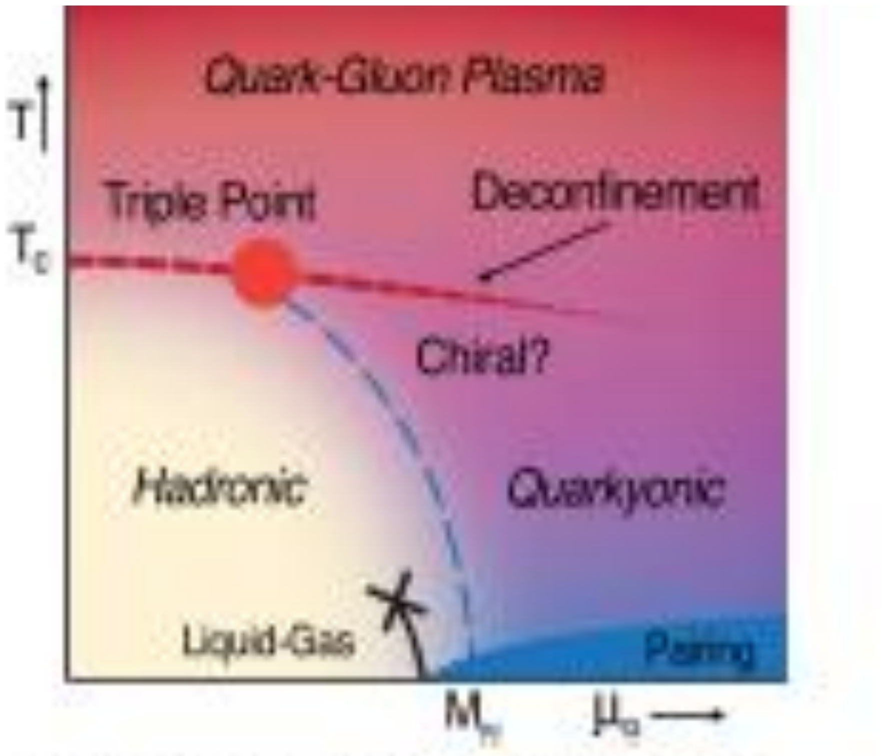
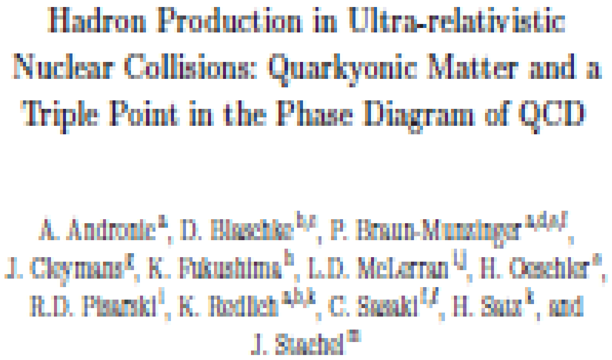


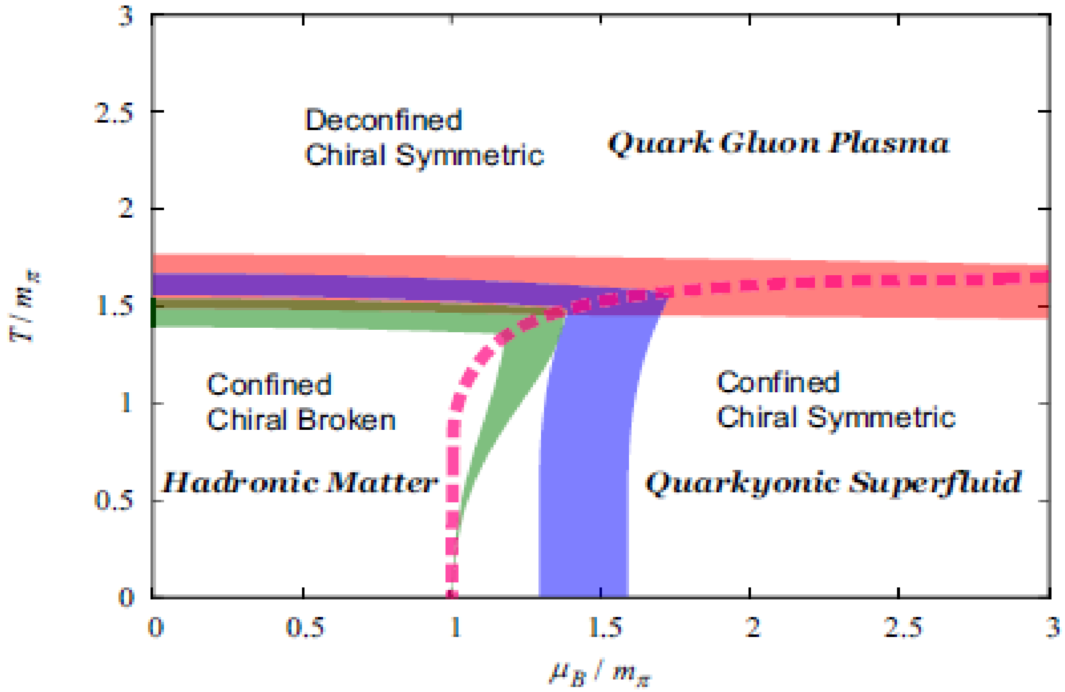
Fig. 5. The phase diagram of strongly interacting matter.



Hadron Production in Ultra-relativistic
Nuclear Collisions: Quarkyonic Matter and a
Triple Point in the Phase Diagram of QCD

A. Andronic[a], D. Blaschke[b,c], P. Braun-Munzinger[a,d,e,f],
J. Cleymans[g], K. Fukushima[h], L.D. McLerran[i,j], H. Oeschler[e],
R.D. Pisarski[i], K. Redlich[a,b,k], C. Sasaki[l,f], H. Satz[k], and
J. Stachel[m]

Quarkyonic phase – Two color



Brauner, Fukushima, Hidaka 2009

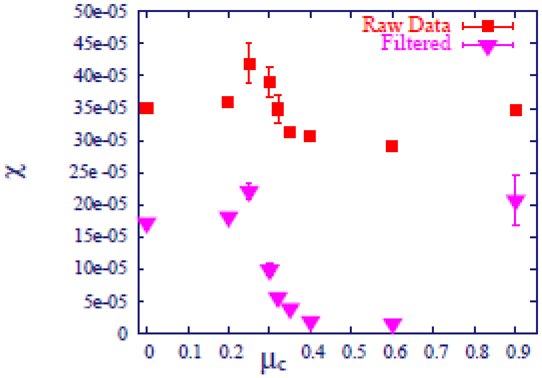
Superfluid phase still confined

GLUONIC OBSERVABLES IN THE BEC PHASE of QC_2D

0^{++} **Glueball : lighter in the BEC phase**

Susceptibility: $\chi = < P^2 > - < P >^2$ **peaks at** μ_c

Normal Phase	
m_π/m_ρ	m_0^{++}/m_ρ
0.40	1.07
0.42	1.26
BEC	
0.64	0.80
0.80	0.23



Lombardo,Petrarca,Paciello,Taglienti,2007

A Quarkyonic Phase in Dense Two Color Matter

Simon Hands
Department of Physics, Swansea University, Singleton Park, Swansea SA2 8PP, U.K.

Seyong Kim
Department of Physics, Sejong University, Seoul 143-747, Korea.

Jon-Ivar Skullerud
Department of Mathematical Physics, National University of Ireland Maynooth, Maynooth, County Kildare, Ireland.

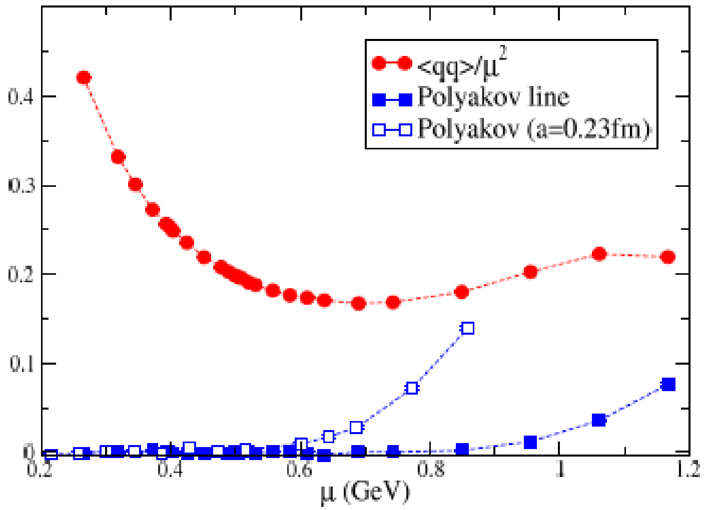
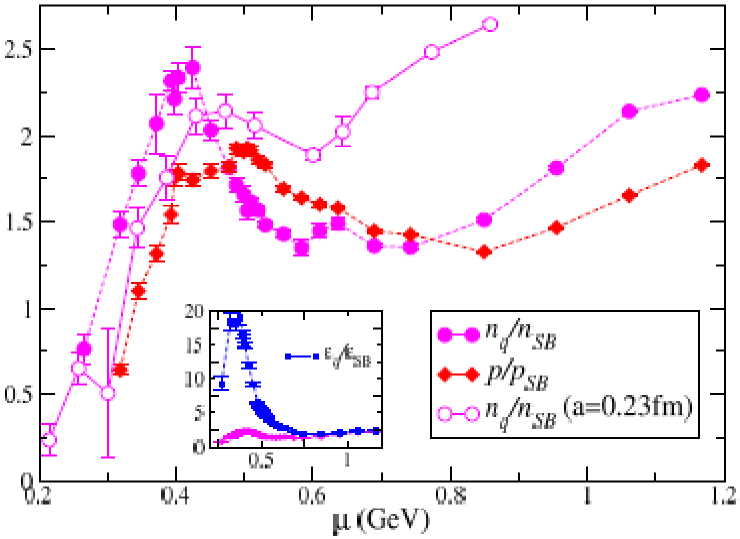


FIG. 4: (Color online) Superfluid order parameter $\langle qq\rangle/\mu^2$ and Polyakov line versus μ.



IG. 1: (Color online) n_q/n_{SB} and p/p_{SB} vs. μ for QC$_2$D. nset shows $\varepsilon_q/\varepsilon_{SB}$ for comparison.

MESOSCOPIC ANALYSIS OF THE PHASE DIAGRAM

TOWARDS
THE REAL SOLUTION

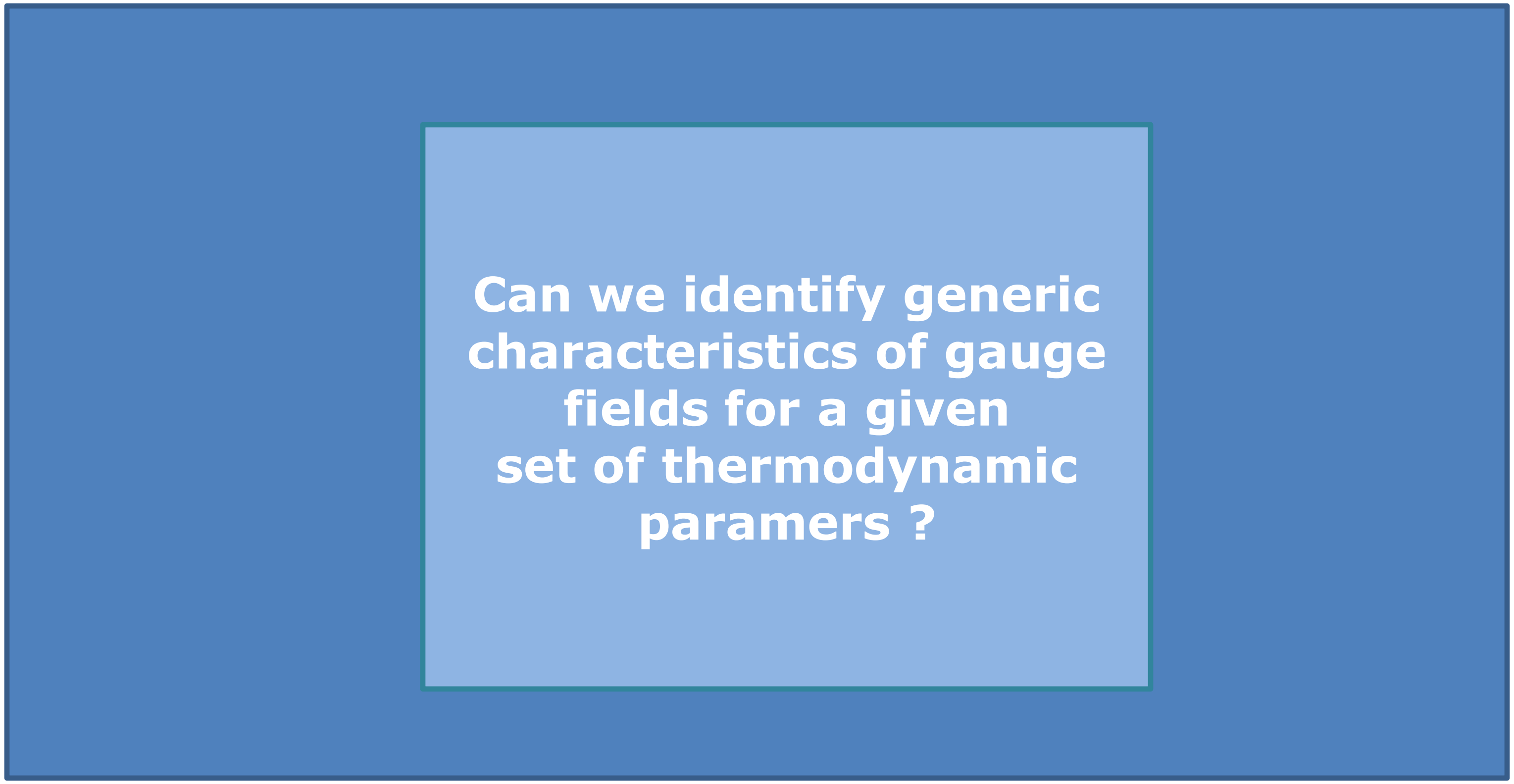
COMPLEX LANGEVIN?

DENSITY OF STATE METHODS?

NEW ALGORITHMS?

LEARNING FROM SIMPLER SYSTEMS?

Gauge Fields and T, μ_B, μ_I, …

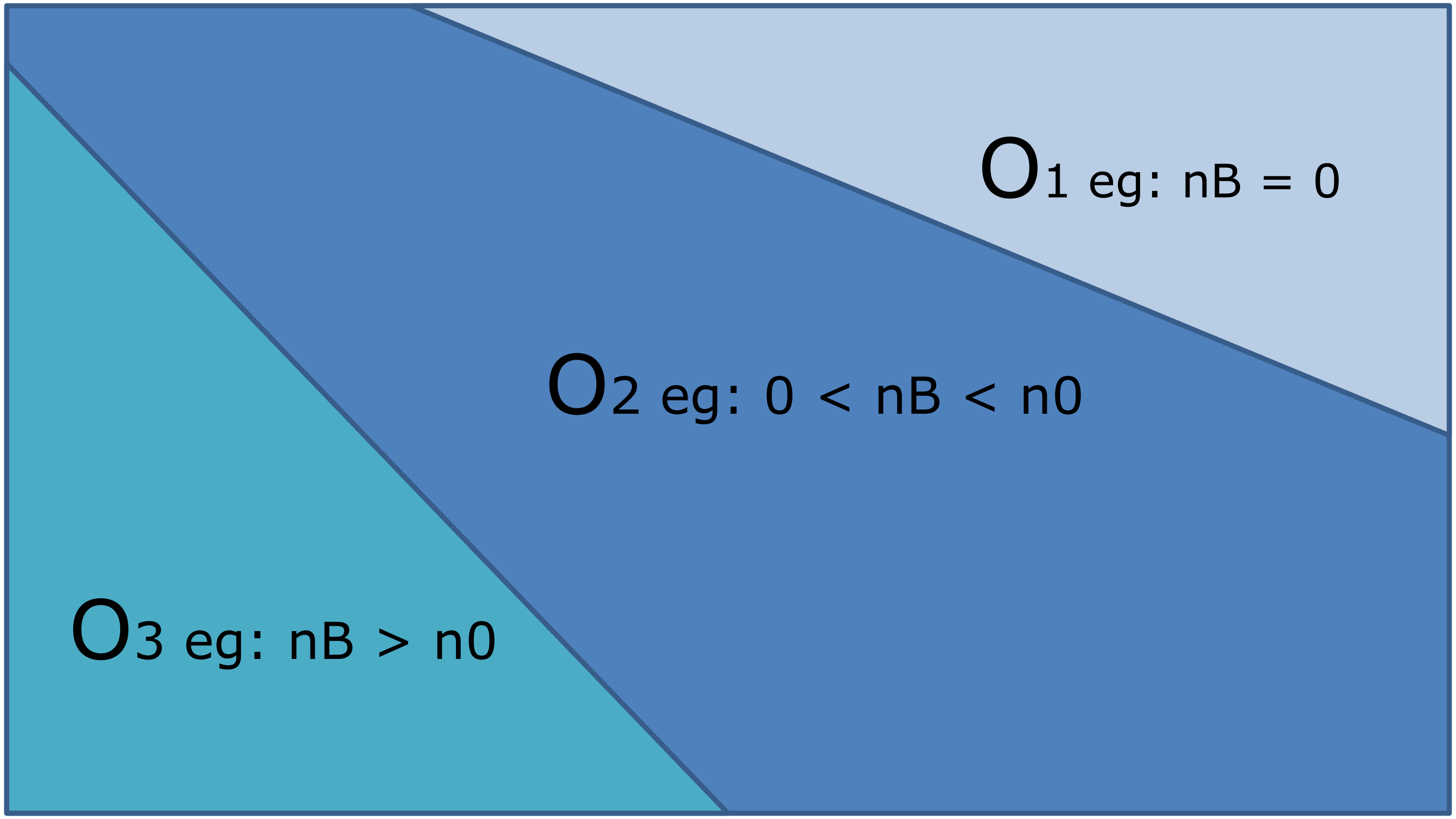


Can we identify generic characteristics of gauge fields for a given set of thermodynamic paramers ?

{U}

Gauge Fields and Observables

Label regions of phase space according to value of macroscopic O: which peculiarities of U’s for a given O?



O_1 eg: nB = 0

O_2 eg: 0 < nB < n0

O_3 eg: nB > n0

{U}

Distributions: mesoscopic Probes of gauge dynamics

$$\langle \delta(\theta - \theta') \rangle$$

- The θ distribution $\langle \delta(\theta - \theta') \rangle_{N_f} d\theta$: assesses overlap between simulation and target ensemble.

$$\langle \mathcal{O}\ \delta(\theta - \theta') \rangle .$$

- The constrained distribution $\langle \mathcal{O}\ \delta(\theta - \theta') \rangle$ shows how averages are built up in the spirit of the Density of States Method $< \mathcal{O} >= \int \langle \mathcal{O} \delta(\theta - \theta') \rangle_{N_f} d\theta$

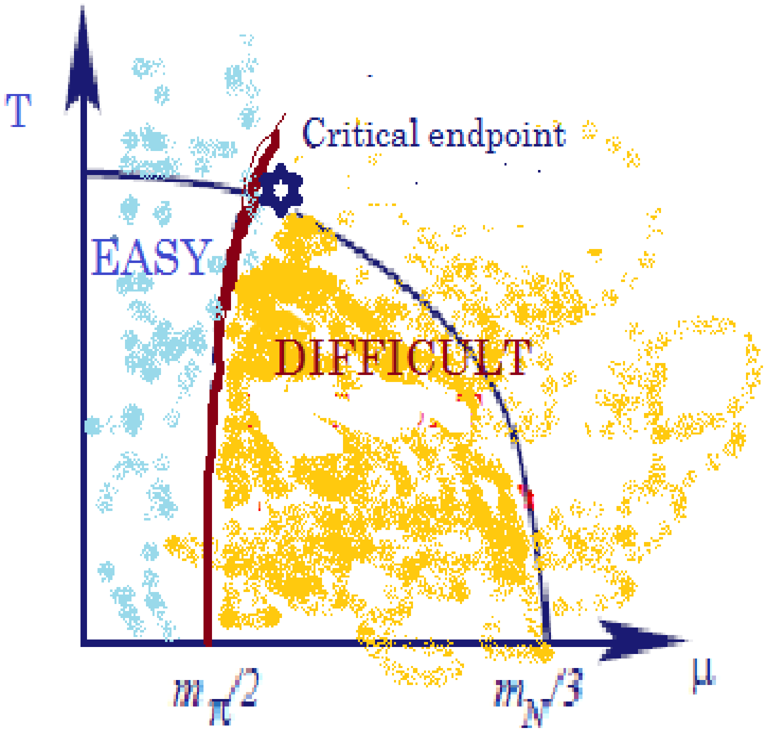
$$1 \quad = \int \langle \delta(\theta - \theta') \rangle \ d\theta$$

$$\langle \mathcal{O} \rangle \quad = \int \langle \mathcal{O}\ \delta(\theta - \theta') \rangle . \ d\theta$$

Why?

- **Physics: Microscopic Structure, Instanton configurations, etc.**
- **Lattice Simulations: Overlap Issues, Density of States, Sign Problem, ..**

A lattice-minded view of the phase diagram



Consider Θ = phase of the determinant

- $\langle \delta(\theta - \theta') \rangle_{N_f} d\theta = \frac{\int dA |\det(D+\mu\gamma_0+m)|^{N_f} e^{iN_f\theta'} \delta(\theta-\theta') e^{-S_{\mathrm{YM}}}}{\int dA |\det(D+\mu\gamma_0+m)|^{N_f} e^{iN_f\theta'} e^{-S_{\mathrm{YM}}}} d\theta.$

- Factorization of the θ-distribution:
  $\langle \delta(\theta - \theta') \rangle_{N_f} = e^{i\theta N_f} \frac{Z_{|N_f|}}{Z_{N_f}} \langle \delta(\theta - \theta') \rangle_{|N_f|}.$

- $N_f = 2$
  $\langle \delta(\theta - \theta') \rangle_{1+1} = e^{2i\theta} \frac{Z_{1+1^*}}{Z_{1+1}} \langle \delta(\theta - \theta') \rangle_{1+1^*},$

Recap of ChPT: baryonless theory

$$G_0(\mu,-\tilde{\mu}) = \frac{V m_\pi^2 T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}\cosh(\frac{(\mu+\tilde{\mu})n}{T}).$$

G0 can only depend on isospin chemical potential

$$\nu_I \equiv \frac{d}{d\mu_1}\Delta G_0(\mu_1,-\mu)\Big|_{\mu_1=\mu},$$

$$\chi^B_{ud} \equiv \frac{d^2}{d\mu_1 d\mu_2}\Delta G_0(\mu_1,\mu_2)\Big|_{\mu_1=\mu_2=\mu},$$

Non trivial μ dependence in cross derivatives

	$\mathcal{E} = 1+1$	$\mathcal{E} = PQ$
$\langle n\rangle_{\mathcal{E}}$	0	ν_I
$\langle n^2\rangle_{\mathcal{E}}$	χ^B_{ud}	$\nu_I^2 + \chi^B_{ud}$

Gaussian Distributions

$\mu < m_\pi/2$, **ChPT**

θ **distribution, and** n_B **distribution with** θ

$$\langle \delta(\theta - \theta') \rangle_{1+1} = \frac{e^{2i\theta}}{\sqrt{\pi \Delta G_0}} e^{-\theta^2/\Delta G_0 + \Delta G_0}, \qquad \theta \in [-\infty, \infty].$$

$G_0 =$ Energy difference between neutral and charged pions

$$\langle n_B \, \delta(\theta - \theta') \rangle_{1+1} = \left(\lim_{\tilde{\mu} \to \mu} \frac{d}{d\tilde{\mu}} \Delta G_0(-\mu, \tilde{\mu}) \right) \sum_{n=-\infty}^{\infty} (1 + i\frac{\theta + 2\pi n}{\Delta G_0}) \frac{e^{2i\theta}}{\sqrt{\pi \Delta G_0}} e^{-(\theta + 2\pi n)^2/}$$

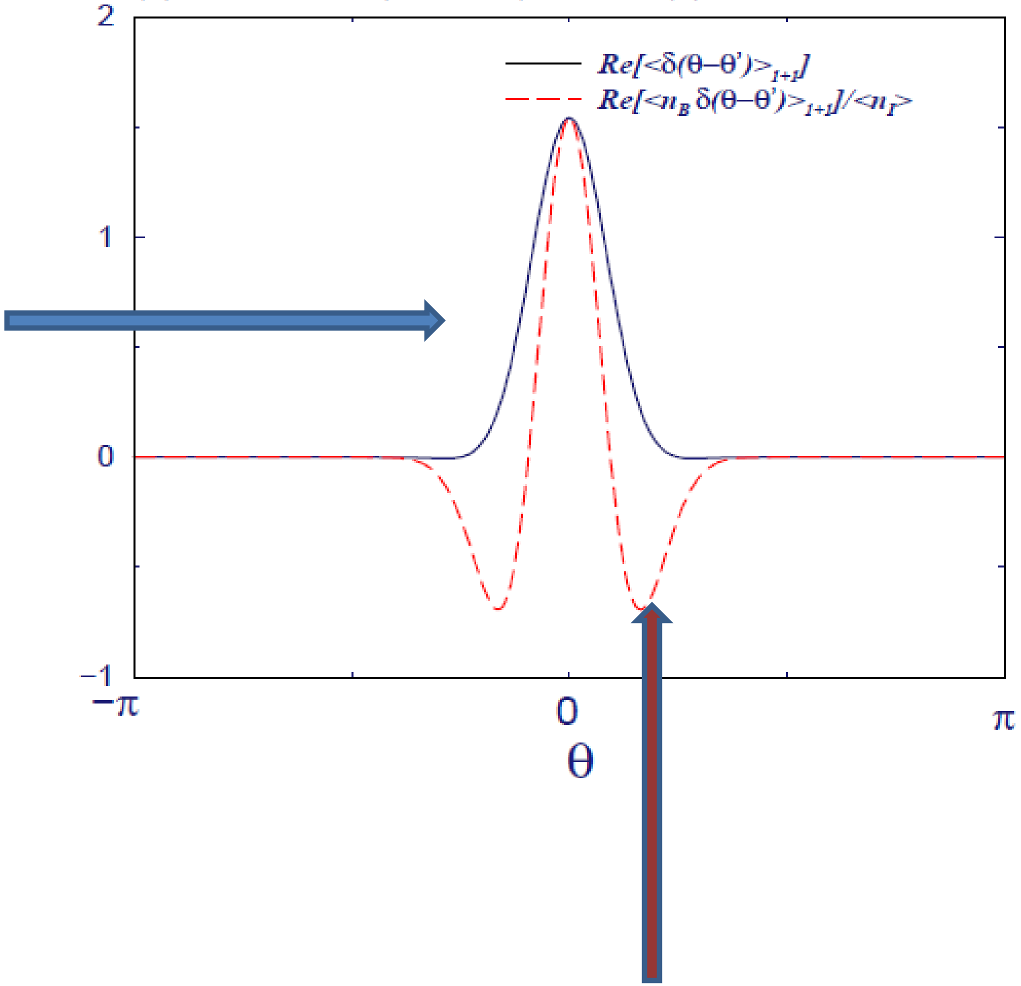
N.B.

$$\langle n_B \rangle = 0$$

in ChPT

Small ΔG_0

$\langle\ \delta(\theta-\theta')\rangle_{1+1}$ and $\langle n_B\ \delta(\theta-\theta')\rangle_{1+1}$



$\int = 1$

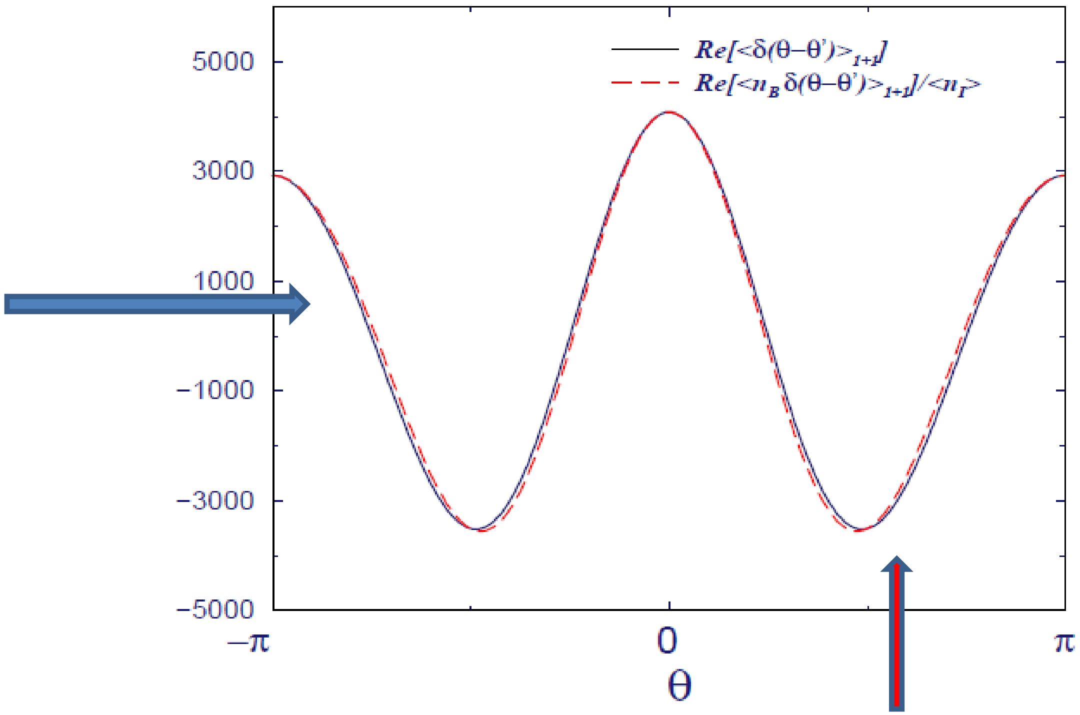
$\int = 0$

Low isospin density, easy

Large ΔG_0

$\langle\, \delta(\theta - \theta')\rangle_{1+1}$ and $\langle n_B\, \delta(\theta - \theta')\rangle_{1+1}$

High isospin density, difficult



$\int = 1$

$\int = 0$

The distribution of the quark number operator: Real and Imaginary

$$P^{1+1}_{\mathrm{Re}[n]}(x) \equiv \left\langle \delta\left(x - \frac{1}{2}(n(\mu) - n(-\mu))\right)\right\rangle_{1+1}$$

$$P^{1+1}_{\mathrm{Re}[n]}(x) \;=\; \frac{1}{\sqrt{\pi(\chi^B_{ud} + \chi^I_{ud})}} e^{-(x-\nu_I)^2/(\chi^B_{ud}+\chi^I_{ud})}.$$

$$\langle \mathrm{Re}[n]\rangle_{1+1} = \int_{-\infty}^{\infty} dx\; x P^{1+1}_{\mathrm{Re}[n]}(x) = \nu_I,$$

$$\langle (\mathrm{Re}[n])^2\rangle_{1+1} = \nu_I^2 + \frac{1}{2}(\chi^B_{ud} + \chi^I_{ud}).$$

$$P^{1+1}_{\mathrm{Im}[n]}(y) \equiv \left\langle \delta\left(y + i\frac{1}{2}(n(\mu) + n(-\mu))\right)\right\rangle_{1+1}$$

$$P^{1+1}_{\mathrm{Im}[n]}(y) \;=\; \frac{1}{\sqrt{\pi(\chi^I_{ud} - \chi^B_{ud})}} e^{(iy+\nu_I)^2/(\chi^I_{ud}-\chi^B_{ud})}$$

$$\langle \mathrm{Im}[n]\rangle_{1+1} = \int_{-\infty}^{\infty} dy\; y P^{1+1}_{\mathrm{Im}[n]}(y) = i\nu_I.$$

$$\langle (\mathrm{Im}[n])^2\rangle_{1+1} = \quad -\nu_I^2 + \frac{1}{2}(\chi^I_{ud} - \chi^B_{ud}).$$

The distribution of the quark number

$$P_n^{1+1}(x,y) \;=\; P_{\mathrm{Re}[n]}^{1+1}(x)\,P_{\mathrm{Im}[n]}^{1+1}(y)$$

Factorization non trivial !

$$\langle n\rangle_{1+1} = \int dxdy\,(x+iy)P_n^{1+1}(x,y) = \int dx\,xP_{\mathrm{Re}[n]}^{1+1}(x) + i\int dy\,yP_{\mathrm{Im}[n]}^{1+1}(y) = \nu_I + ii\nu_I = 0.$$

$$\langle n^2\rangle_{1+1} = \chi_{ud}^B.$$

The fluctuations of the quark number operator follow the offdiagonal susceptibility

What we can compute on the Lattice: The Partial Quenched distribution

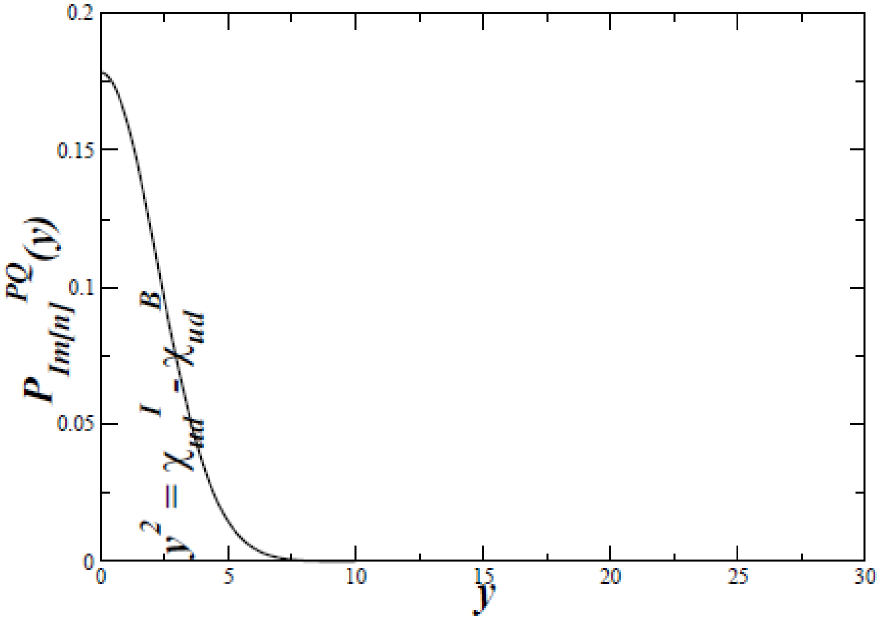
$$P^{PQ}_{\mathrm{Re}[n]}(x) = P^{1+1}_{\mathrm{Re}[n]}(x).$$

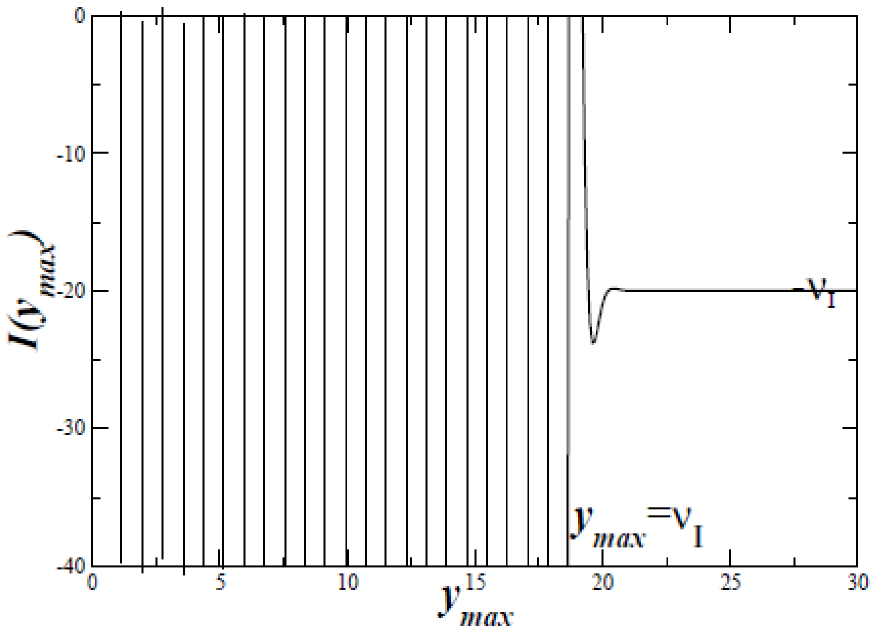
$$P^{PQ}_{\mathrm{Im}[n]}(y) = \frac{1}{\sqrt{\pi(\chi^I_{ud} - \chi^B_{ud})}} e^{-y^2/(\chi^I_{ud} - \chi^B_{ud})}$$

$$\langle n^{PQ} \rangle_{1+1^*} = \nu_I$$

$$\langle (n^{PQ})^2 \rangle_{1+1^*} = \chi^B_{ud} + \nu_I^2 ,$$

Can we reweight PQ -> Full QCD ?





$$\int_{-y_{max}}^{y_{max}} dy\ iyP^{1+1}_{\mathrm{Im}[n]}(y) \sim -\nu_I .$$

$$y^2_{max} - \nu^2_I \gg \chi^I_{ud} - \chi^B_{ud}$$

Might have to sample the extreme tail of the PQ distribution

Lorentzian distributions

The θ-distribution for $\mu > m_\pi/2$

- From the momenta

  $$\langle \delta(2\theta - 2\theta') \rangle = \frac{1}{\pi} \sum_{p=-\infty}^{\infty} e^{-2ip\theta} \langle e^{2ip\theta'} \rangle .$$

- Quenched result

  $$\langle \delta(2\theta - 2\theta') \rangle = \frac{1}{\pi} \frac{\sinh(VL_B)}{\cosh(VL_B) - \cos(2\theta)} .$$

  This is a compactified Lorentzian, centered at zero!

- Unquenched result

  $$\langle \delta(2\theta - 2\theta') \rangle_{1+1} = e^{2i\theta} \frac{e^{VL_B}}{\pi} \frac{\sinh(VL_B)}{\cosh(VL_B) - \cos(2\theta)} .$$

  Again a compactified Lorentzian, centered , times $e^{2i\theta}$

QCD IN ONE EUCLIDEAN DIMENSION

$\det M = 2^{-nN_c} \det[e^{n\mu_c} + e^{-n\mu_c} + e^{n\mu} U + e^{-n\mu} U^\dagger], U \in U(N_c).$

$$Z_{N_f}(\mu_c, \mu) = \int_{U(N_c)} dU \det M.$$

- No baryons in $U(N_c)$: no μ dependence
- However: the quenched model has a phase transition at $\mu_c = \sinh^{-1} m$

  Verbaarschot Ravagli; Bilic Demeterfi and Petersson
- Good guidance for 'baryonless' QCD
- Moments again

  $\langle e^{2ip\theta'} \rangle = \int_{U(N_c)} dU \frac{\det^p M}{\det^p M^\dagger}.$

$$\langle e^{2ip\theta'} \rangle = \int_{U(N_c)} dU \frac{\det^p(1 - U e^{n\mu - n\mu_c}) \det^p(1 - U^\dagger e^{-n\mu - n\mu_c})}{\det^p(1 - U e^{-n\mu - n\mu_c}) \det^p(1 - U^\dagger e^{n\mu - n\mu_c})}.$$

1 dimensional QCD : θ distributions

Gaussian

- $\underline{\mu < \mu_c}$

$$\langle e^{2ip\theta'} \rangle = \langle e^{2i\theta'} \rangle^{p^2} = \left(1 - \frac{\mu^2}{\mu_c^2}\right)^{p^2} .$$

$$\langle \delta(\theta - \theta') \rangle = \frac{1}{\sqrt{\pi\Omega}} e^{-\theta^2/\Omega} \qquad \text{for} \qquad \mu < \mu_\mathrm{c},\ \mathrm{N_c} \to \infty,$$

$$\Omega \equiv -\log(1 - \mu^2/\mu_c^2).$$

Lorentzian

- $\underline{\mu > \mu_c}$

$$\langle e^{2ip\theta'} \rangle = e^{-2n|p|N_c\mu},$$

$$\langle \delta(2\theta - 2\theta') \rangle = \frac{1}{\pi} \frac{\sinh(2nN_c\mu)}{\cosh(2nN_c\mu) - \cos(2\theta)} \qquad \text{for} \qquad \mu > \mu_c ,$$

1 dimensional QCD : θ distributions

Gaussian

- $\underline{\mu < \mu_c}$

$$\langle e^{2ip\theta'} \rangle = \langle e^{2i\theta'} \rangle^{p^2} = \left(1 - \frac{\mu^2}{\mu_c^2}\right)^{p^2} .$$

$$\langle \delta(\theta - \theta') \rangle = \frac{1}{\sqrt{\pi\Omega}} e^{-\theta^2/\Omega} \qquad \text{for} \qquad \mu < \mu_{\rm c}, \ \mathrm{N_c} \longrightarrow \infty,$$

$$\Omega \equiv -\log(1 - \mu^2/\mu_c^2).$$

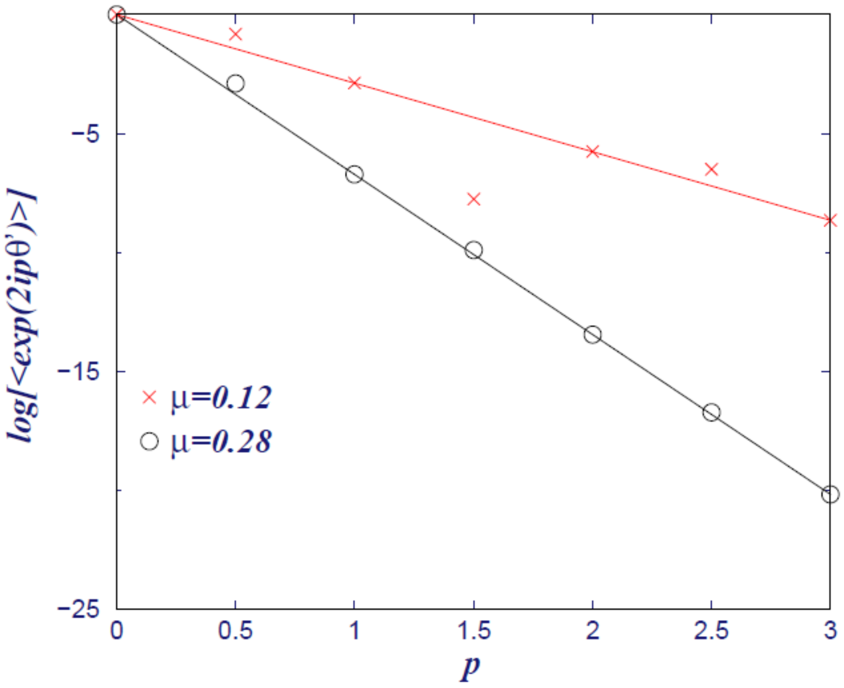
Lorentzian

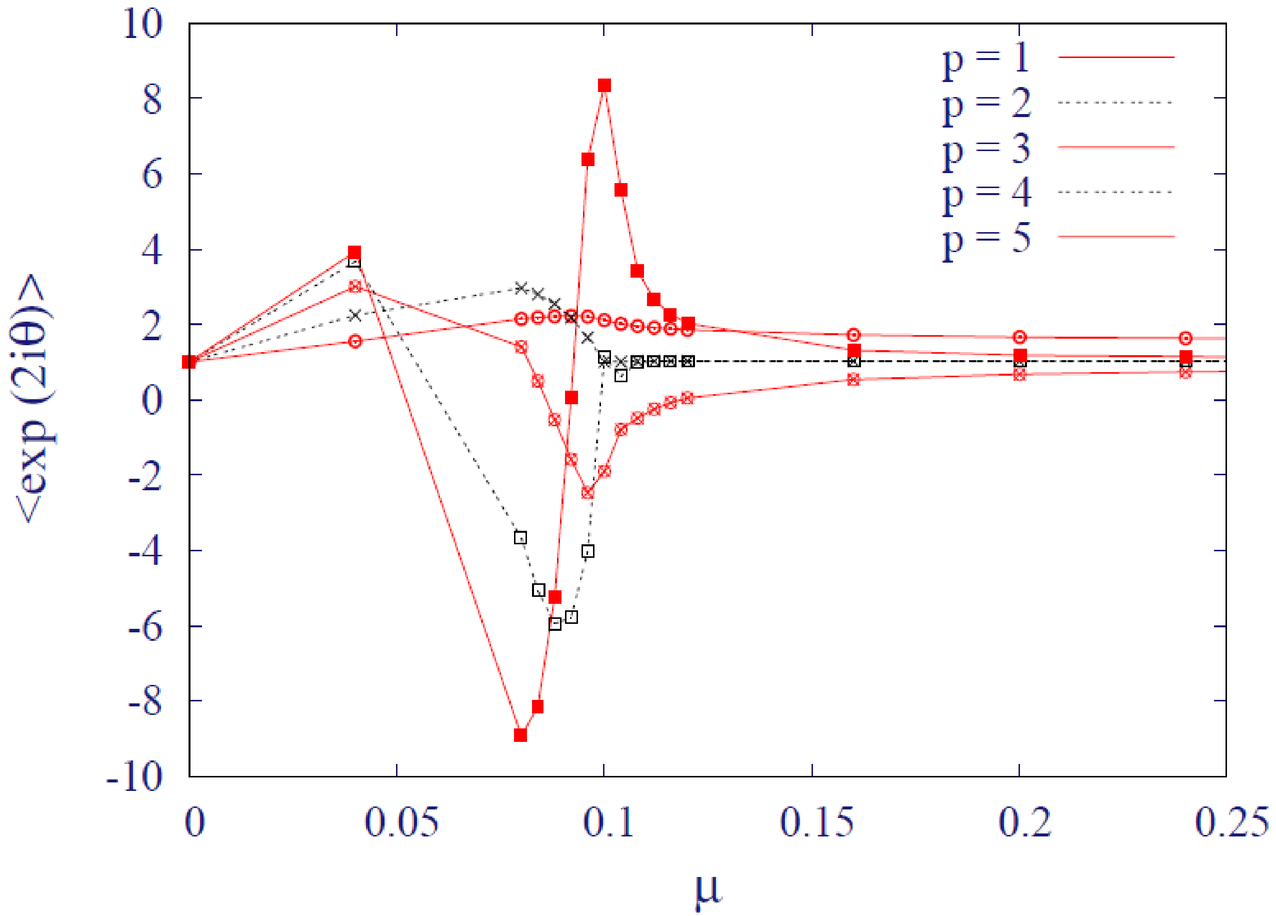
- $\underline{\mu > \mu_c}$

$$\langle e^{2ip\theta'} \rangle = e^{-2n|p|N_c\mu},$$

$$\langle \delta(2\theta - 2\theta') \rangle = \frac{1}{\pi} \frac{\sinh(2nN_c\mu)}{\cosh(2nN_c\mu) - \cos(2\theta)} \qquad \text{for} \qquad \mu > \mu_c ,$$

Numerical Results $\mu > \mu_c$




p = 1
p = 2
p = 3
p = 4
p = 5
<exp (2iθ)>
μ
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
0.05
0.1
0.15
0.2
0.25

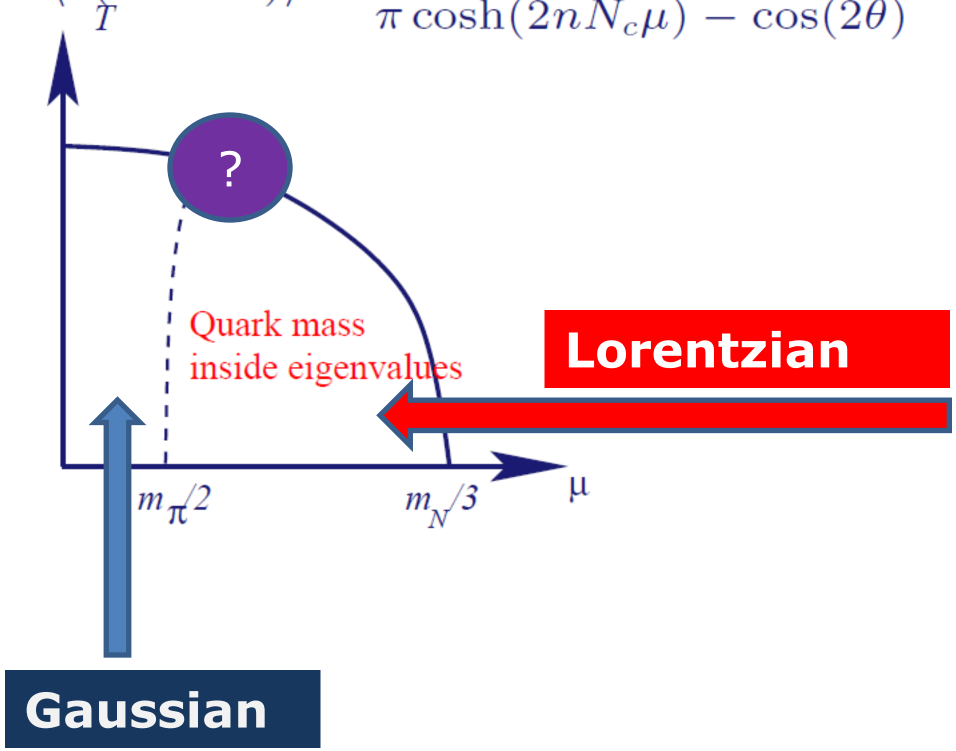
Summarizing:

- Quark mass outside eigenvalues: Gaussian

$$\langle\delta(\theta-\theta')\rangle = \frac{1}{\sqrt{\pi\Omega}}e^{-\theta^2/\Omega} \qquad \text{for} \qquad \mu < \mu_c,\ \mathrm{N_c} \to \infty,$$

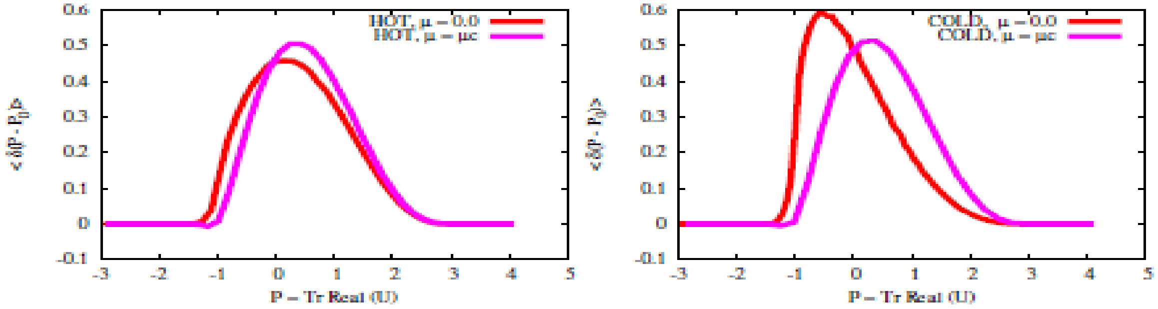
- Quark mass inside eigenvalues: Lorentzian

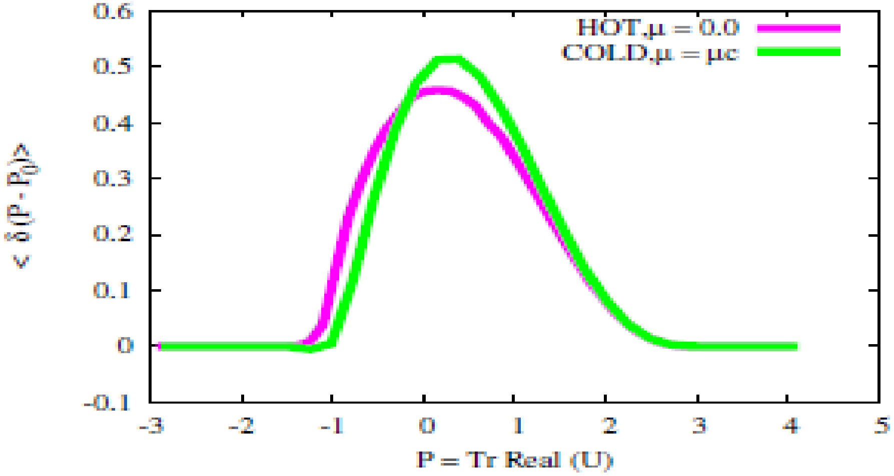
$$\langle\delta(2\theta-2\theta')\rangle = \frac{1}{\pi}\frac{\sinh(2nN_c\mu)}{\cosh(2nN_c\mu)-\cos(2\theta)} \qquad \text{for} \qquad \mu > \mu_c\,,\ 2\theta \in [-\pi,\pi].$$



Finite density easier at finite T

Hot ensembles
overlap better
among themselves





.. and might even
overlap with cold ones

THE ROLE OF BARYONS

- Average phase factor from Taylor expansion:

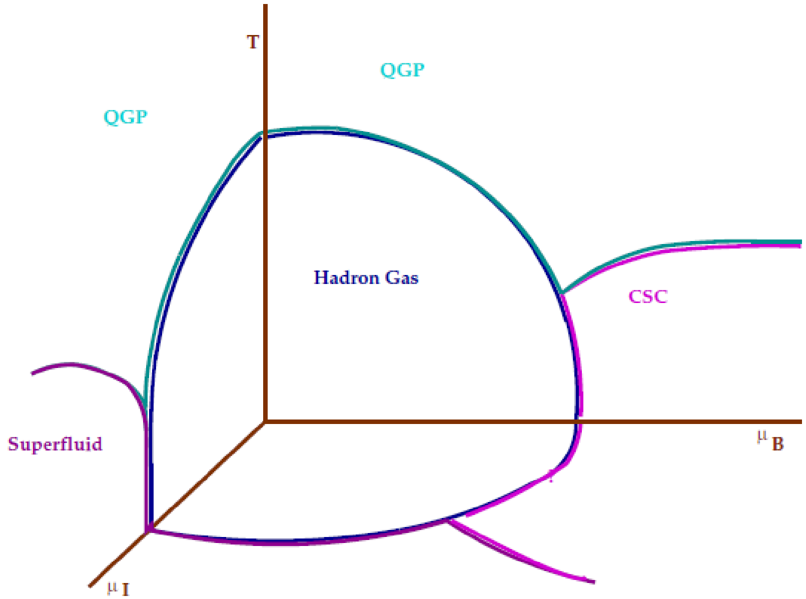
$$\langle e^{2i\theta}\rangle_{1+1^*} = e^{L^3 T(c_2 - c_2^I)\mu^2}$$

Bielefed-Swansea Phys Rev D 71 (2005)

- In general,

$$\frac{\partial}{\partial\mu}\log\langle e^{2i\theta}\rangle_{1+1^*} = \frac{\partial}{\partial\mu}\log Z_{1+1} - \frac{\partial}{\partial\mu}\log Z_{1+1^*} \propto (n_B(\mu) - n_I(\mu))$$

- $n_B(\mu) - n_I(\mu) = 0$, no sign problem while a sign problem becoming increasingly more severe corresponds to $n_B(\mu) - n_I(\mu) < 0$



Physis of high isospin density and high baryon density become similar at high T

Mesons and Baryons: sign problem easier when densities become comparable

Thermal baryons lessen the sign problem

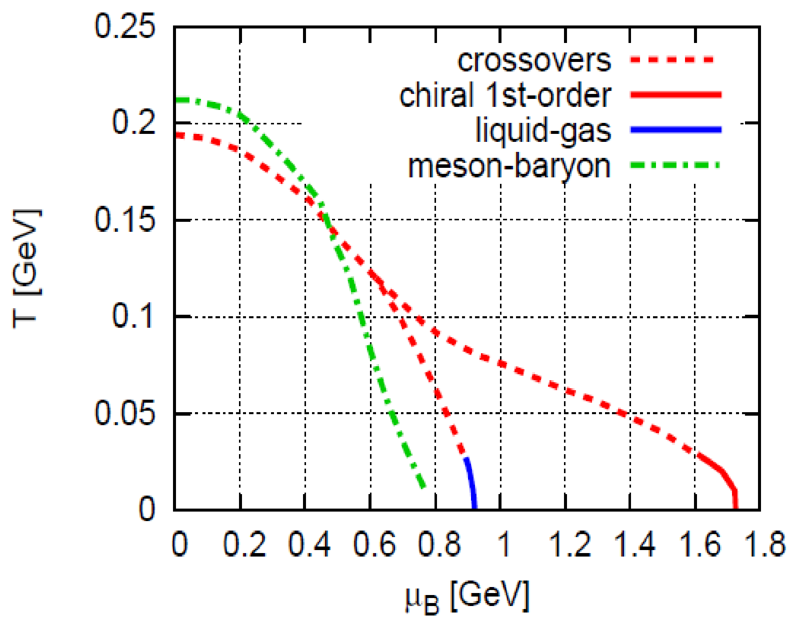
M. D'Elia and F. Sanfilippo 2009

Criterium?

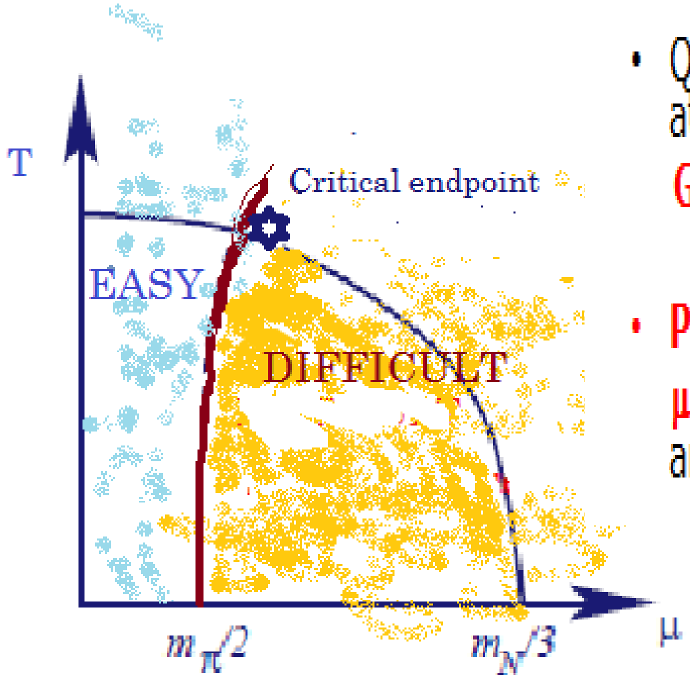
Calculation of 'equivalence point' baryons-mesons

C. Sasaki@nfqcd2010,Kyoto

- **phase diagram in PDM**: $m_{N\_} = 1.5$ GeV



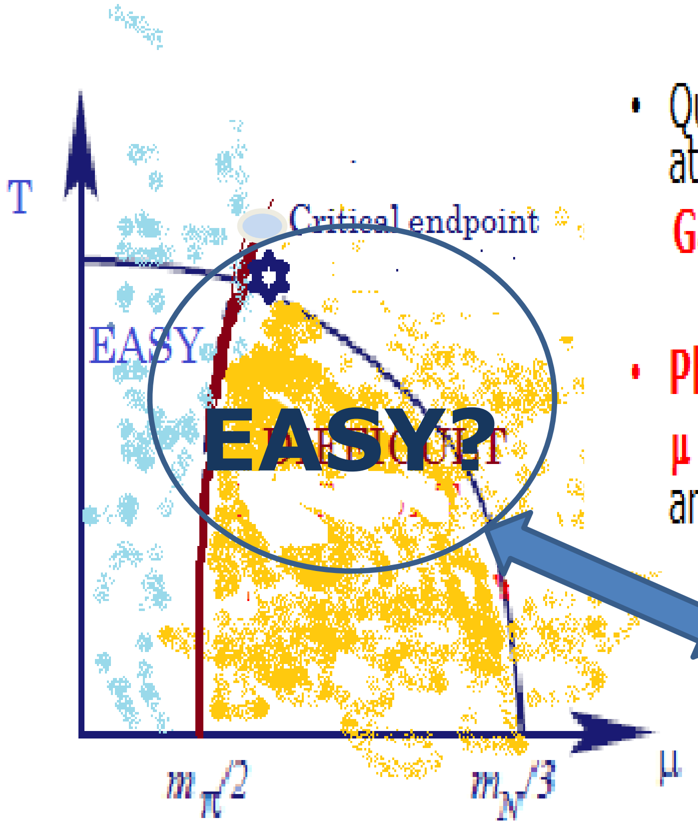
Summary



- Qualitative differences in the mesoscopic physics at high density:
  Gaussian -> Lorentzian distribution

- Phase of the determinant important even for $\mu < m_\pi / 2$-subtle cancellations between real and imaginary components

Summary



- Qualitative differences in the mesoscopic physics at high density:

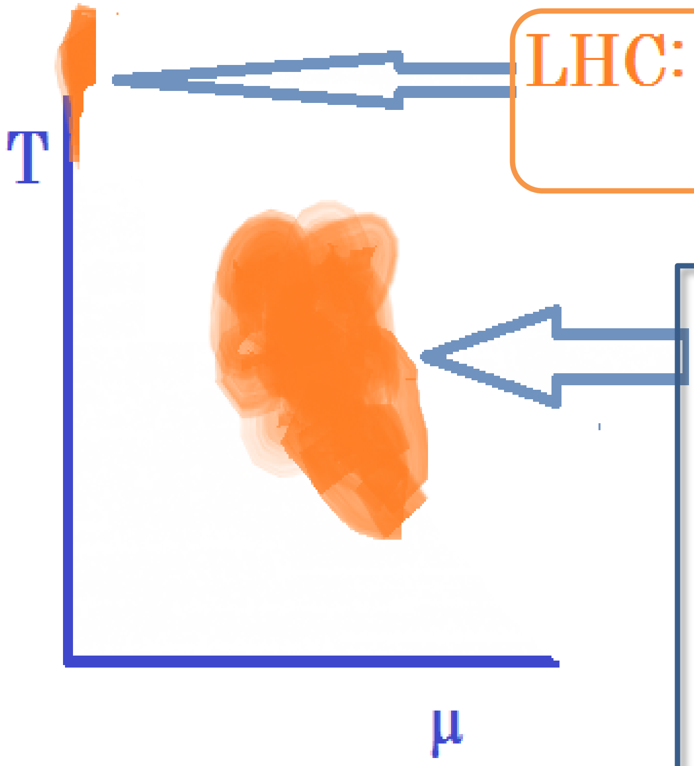
  Gaussian -> Lorentzian distribution

- Phase of the determinant important even for $\mu < m \pi / 2$-subtle cancellations between real and imaginary components

Thermally activated baryons might ‘wash out’ the dangerous threshold.

PHASES OF QCD AND CRITICAL POINT FROM THE LATTICE

OUTLOOK



LHC: Higher T, zero μ

Entering the precision era

NICA, RHICII, FAIR

smaller T, larger μ

Expect Experimental And lattice results on: Critical Endpoint, Freezout region, and (maybe) Exotic phases