Gravitational Lensing

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XII Зимняя Школа «Малочастичные системы:
теория и приложения

Дубна, ОИЯИ, 07 февраля 2014 года

General Relativity predicts that a light ray which passes by a spherical body of mass M with impact parameter b, is deflected by the “Einstein angle”:

$$\hat{\alpha} = \frac{4GM}{c^2 b} = \frac{2R_S}{b}$$

provided the impact parameter b is much larger than the corresponding Schwarzschild radius R_S:

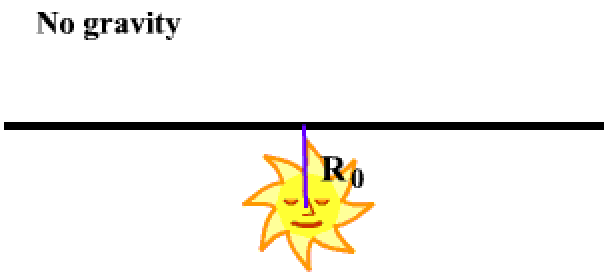
$$b \gg R_S = \frac{2GM}{c^2}$$

Historical remarks

- **<u>Before 1919</u>** :

  Newton:
  "Do not Bodies act upon Light at a distance, and by their action bend its Rays; and is not this action strongest at the least distance?"
- Cavendish, Soldner(1801), ... , Einstein(1911):
  "Newtonian" value

  α *= 2GM/v^2 b, v = c*



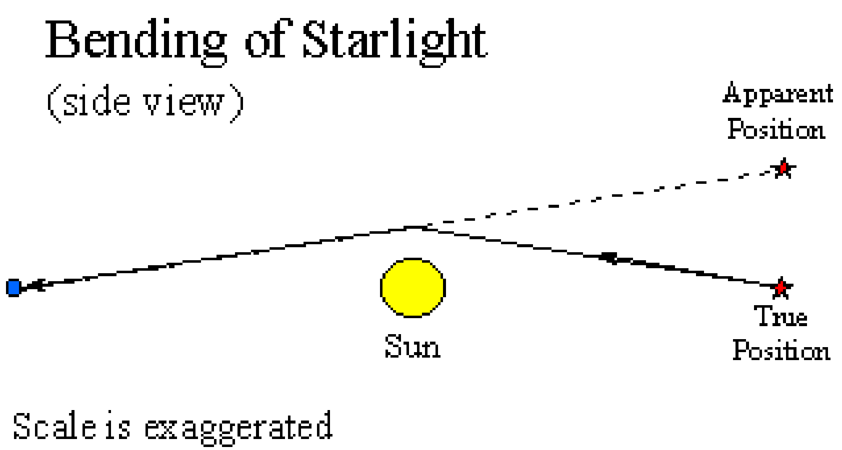
- Einstein(1915):
  correct result:

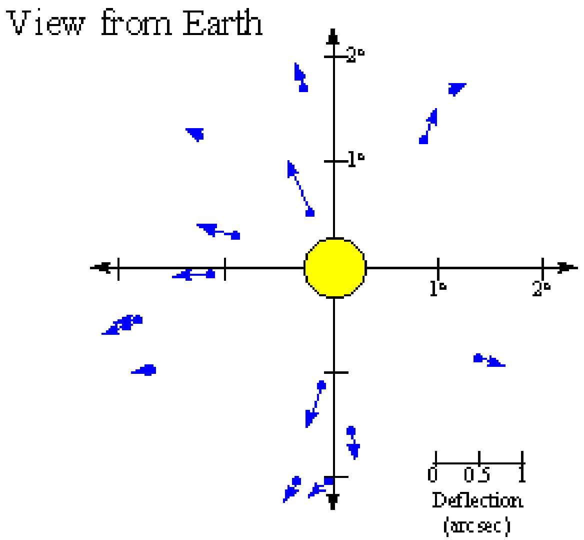
twice the Newtonian value

α *= 4GM/c^2 b = 2R_S/b*

Observations during Solar Eclipse

1915 – Einstein, angle=1.75”
1919 – 29 May, two expeditions for solar eclipse
Eddington and Dyson, stars in Giady
(Гиады)





Historical remarks

- **<u>The period 1919-1937</u> :**

  Lodge (1919) – *term* - lense

  Eddington (1920) – *multiple images,*

  Chwolson (1924) – *‘Rings’,*

  Einstein (1936) – *“the biggest cat gets all the milk”,*

  Tihov (1937) – *magnification in general case*

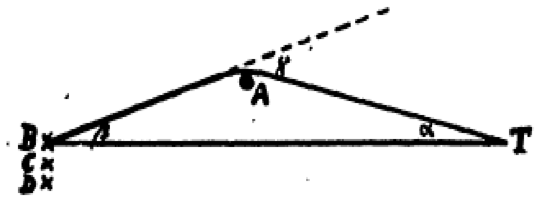
  Zwicky (1937) – *nebulas instead stars*

Über eine mögliche Form fiktiver Doppelsterne. Von *O. Chwolson.*

Es ist gegenwärtig wohl als höchst wahrscheinlich anzunehmen, daß ein Lichtstrahl, der in der Nähe der Oberfläche eines Sternes vorbeigeht, eine Ablenkung erfährt. Ist γ diese Ablenkung und γ_0 der Maximumwert an der Oberfläche, so ist $\gamma_0 \geqq \gamma \geqq 0$. Die Größe des Winkels ist bei der Sonne $\gamma_0 = 1''7$; es dürften aber wohl Sterne existieren, bei denen γ_0 gleich mehreren Bogensekunden ist; vielleicht auch noch mehr. Es sei A ein großer Stern (Gigant), T die Erde, B ein entfernter Stern; die Winkeldistanz zwischen A und B, von T aus gesehen, sei α, und der Winkel zwischen A und T, von B aus gesehen, sei β. Es ist dann

$$\gamma = \alpha + \beta\,.$$

Ist B sehr weit entfernt, so ist annähernd $\gamma = \alpha$. Es kann also α gleich mehreren Bogensekunden sein, und der Maximumwert von α wäre etwa gleich γ_0. Man sieht den Stern B von der Erde aus an zwei Stellen: direkt in der Richtung TB und außerdem nahe der Oberfläche von A, analog einem Spiegelbild. Haben wir mehrere Sterne B, C, D, so würden die Spiegelbilder umgekehrt gelegen sein wie in einem gewöhnlichen Spiegel, nämlich in der Reihenfolge D, C, B, wenn von A aus gerechnet wird (D wäre am nächsten zu A).



Der Stern A würde als fiktiver Doppelstern erscheinen. Teleskopisch wäre er selbstverständlich nicht zu trennen. Sein Spektrum bestände aus der Übereinanderlagerung zweier, vielleicht total verschiedenartiger Spektren. Nach der Interferenzmethode müßte er als Doppelstern erscheinen. Alle Sterne, die von der Erde aus gesehen rings um A in der Entfernung $\gamma_0 - \beta$ liegen, würden von dem Stern A gleichsam eingefangen werden. Sollte zufällig TAB eine gerade Linie sein, so würde, von der Erde aus gesehen, der Stern A von einem Ring umgeben erscheinen.

Ob der hier angegebene Fall eines fiktiven Doppelsternes auch wirklich vorkommt, kann ich nicht beurteilen.

O. Chwolson.

Petrograd, 1924 Jan. 28.

Astronomische Nachrichten, volume 221, p.329, 1924

- *Of course, there is not much hope of observing this phenomenon directly.*

Albert Einstein, 1936

- *The probability that nebulae which act as gravitational lenses will be found becomes practically a certainty.*

Fritz Zwicky, 1937

Historical remarks

- **The period 1963-1979** :
  discovery of QSO – ideal sources for gravitational lensing
  Klimov, Liebes, Refsdal ......
  Ingel, Byalko ......

- **Post-1979** :
  0957+561 – *first GL candidate*
  then arcs, rings
  1983 – the first international conference

Gravitational lensing in vacuum

Einstein's deflection law:

General Relativity predicts that a light ray which passes by a spherical body of mass M with impact parameter b, is deflected by the "Einstein angle":

$$\hat{\alpha} = \frac{4GM}{c^2 b} = \frac{2R_S}{b}$$

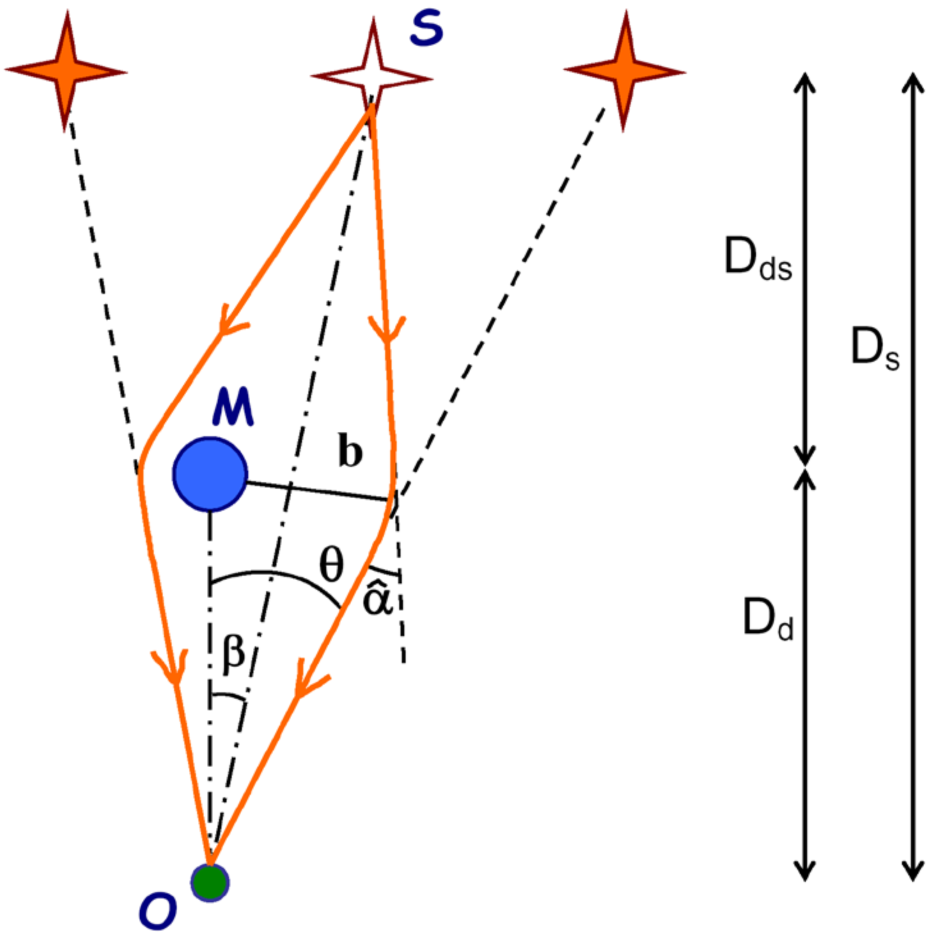
provided the impact parameter b is much larger than the corresponding Schwarzschild radius R_S:

$$b \gg R_S = \frac{2GM}{c^2}$$

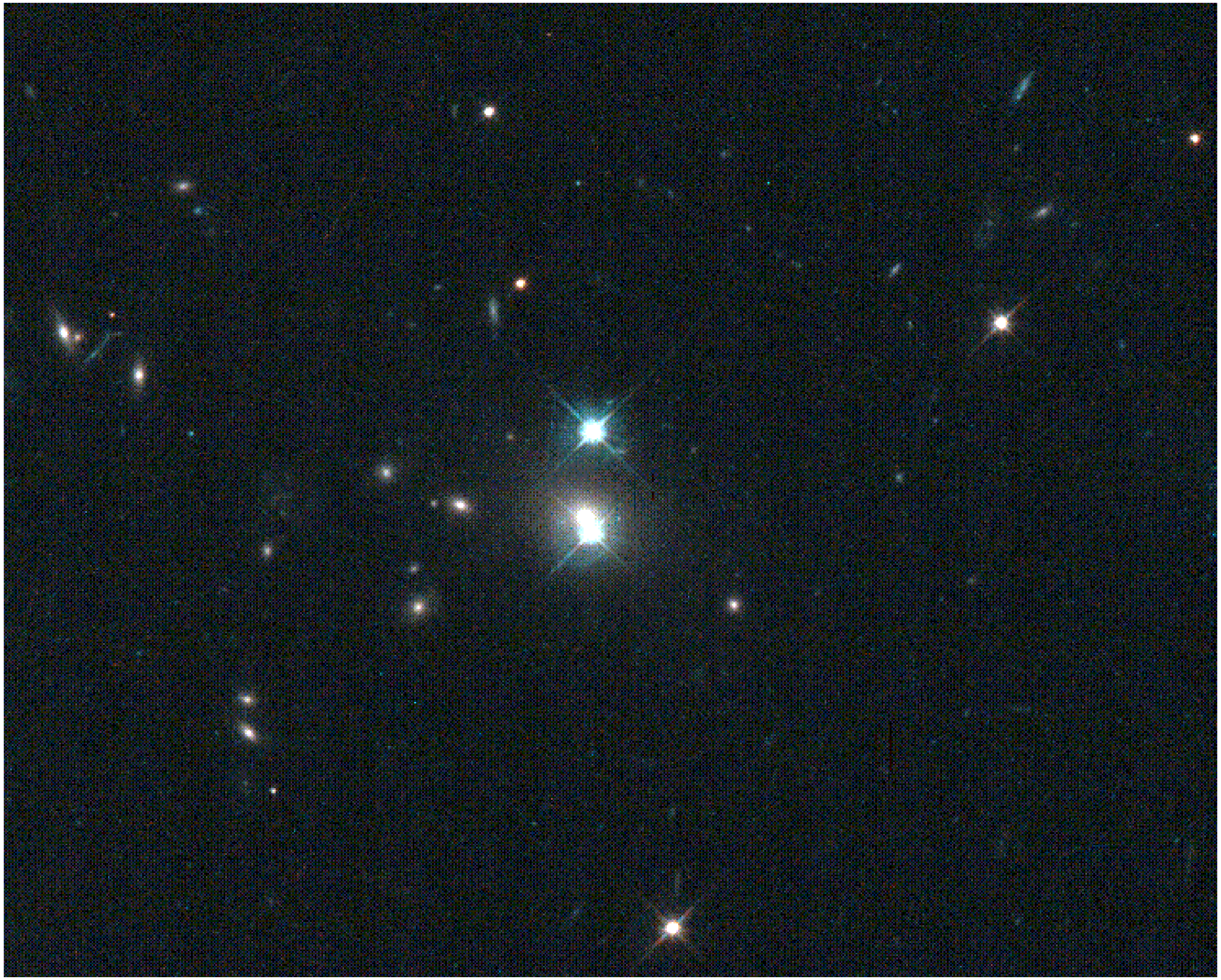
In the most astrophysical situations related with gravitational lensing approximation of weak deflection is well satisfied.

This angle does not depend on frequency of the photon

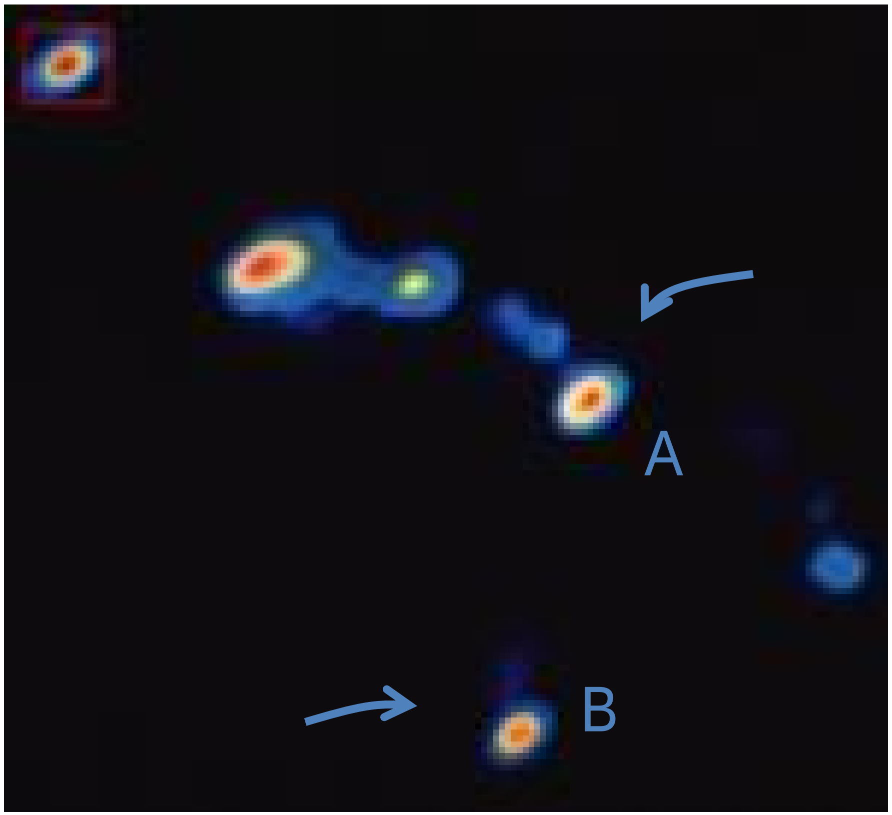
On basis of Einstein deflection angle ordinary GL theory is developed. At this picture there is the example of the simplest model of Schwarzschild point-mass lens which gives two images of source instead of one single real source.



The first observed lense (1981): QSO 0957+561, optics

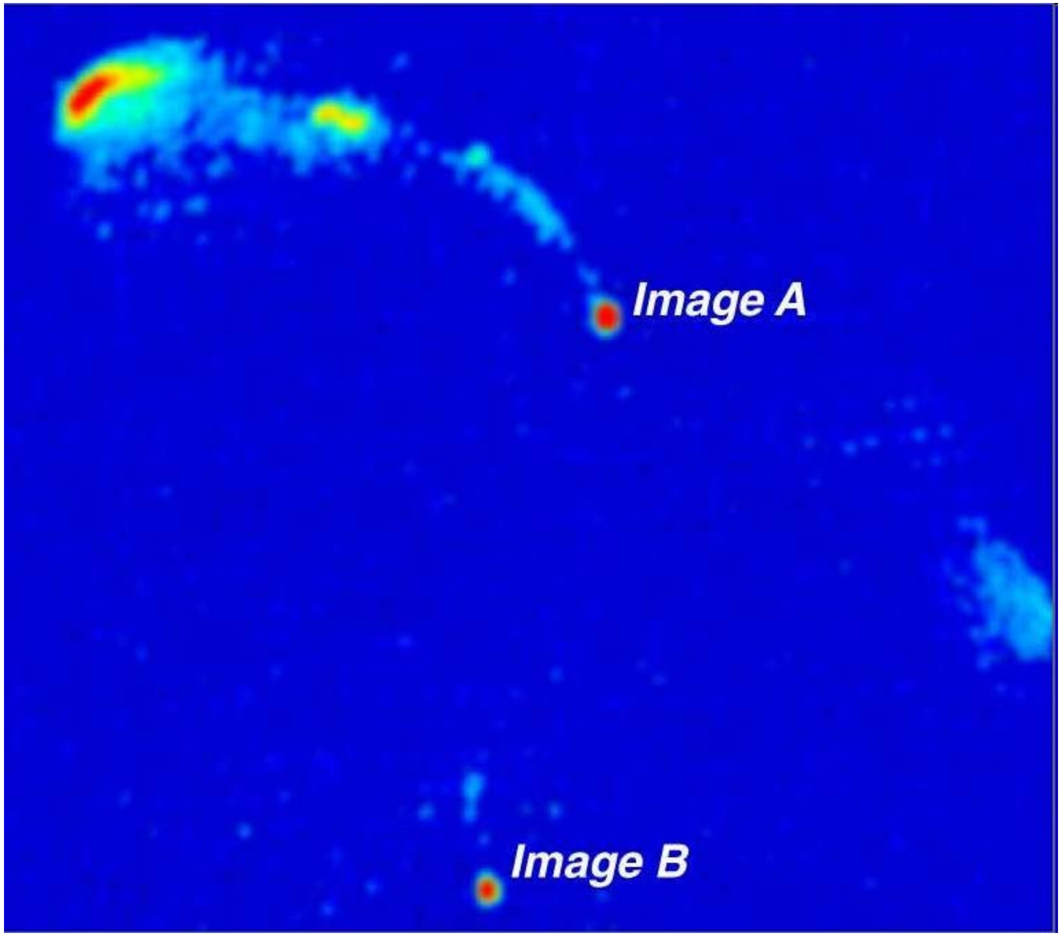


QSO 0957+561, radio VLA



- Maximum separation – 6.1 arcsec
- Image redshift – 1.41
- Lens redshift – 0.36
- B/A = 2/3

0957+561



- Merlin image of the Double Quasar 0597+561. The images of the core are 6 seconds of arc apart.

- ***e*-MERLIN**
- Operated by The University of Manchester on behalf of STFC, the Multi-Element Remotely Linked Interferometer Network is an array of **radio** telescopes distributed around Great Britain with a resolution greater than the Hubble Space Telescope.

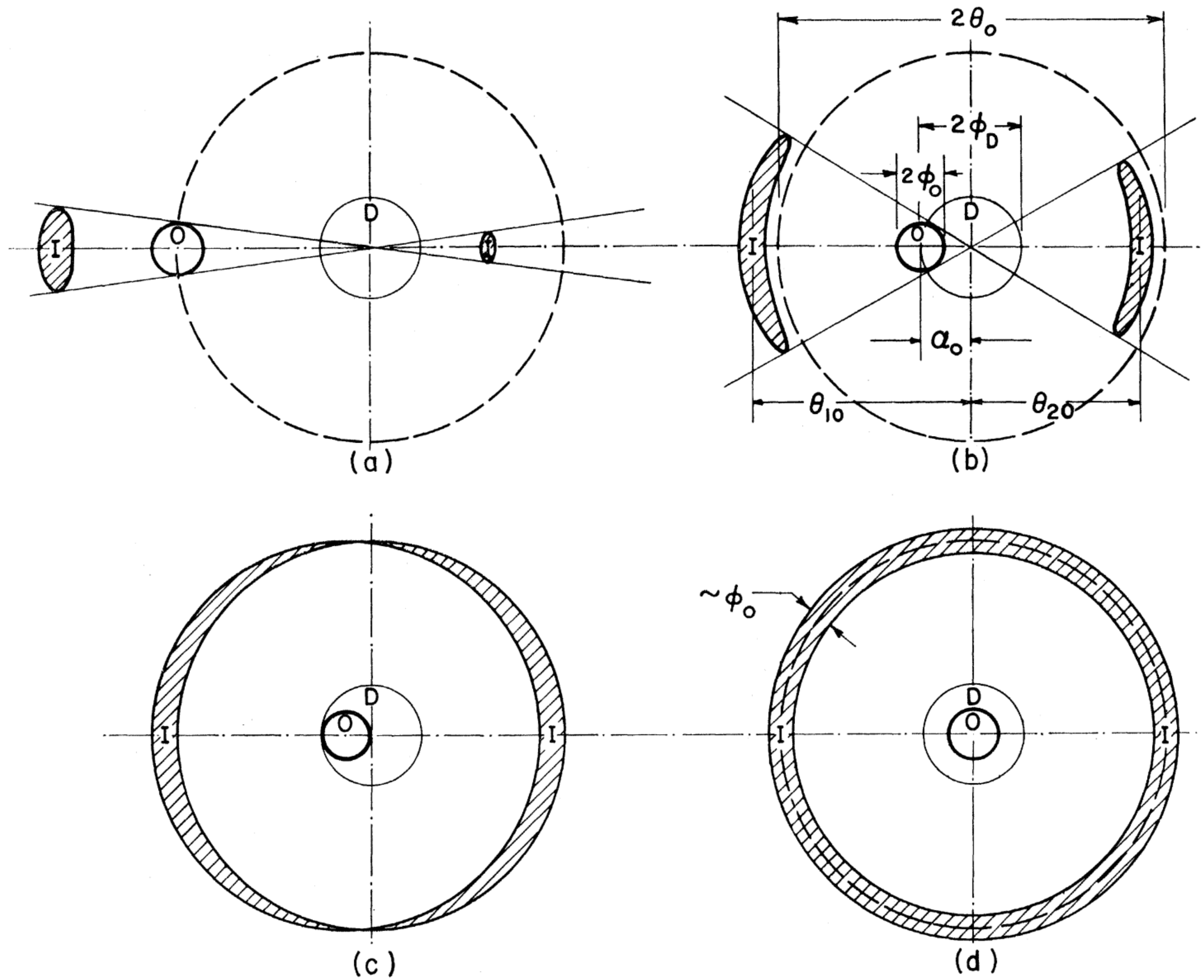
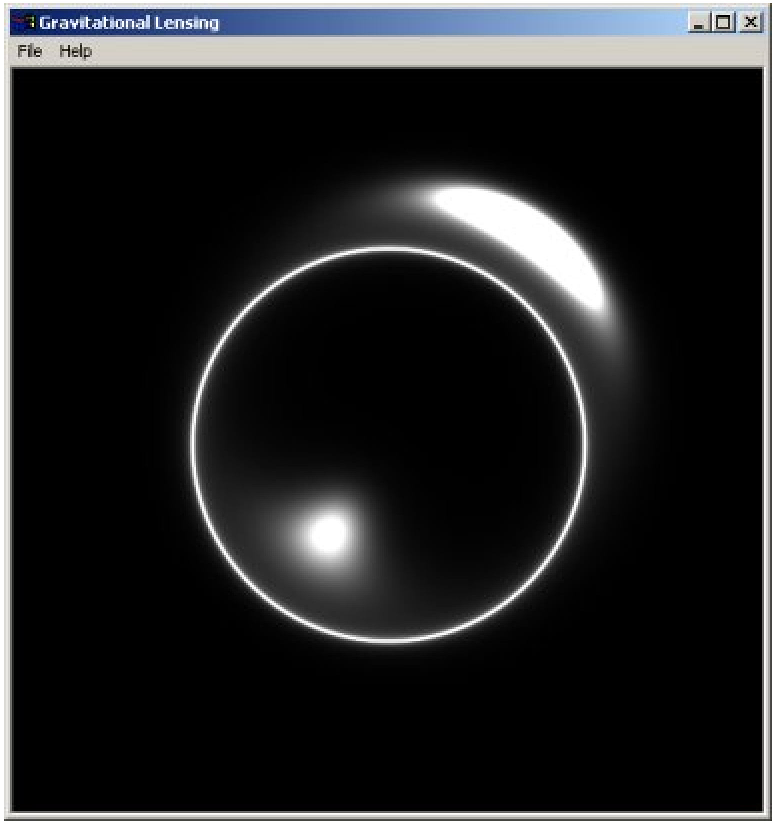


FIG. 2. Transformation of the image configuration *I* of the object *O* as the object moves from left to right behind the deflector *D*.

- Einstein's General Theory of Relativity demonstrates that a large mass can deform spacetime and bend the path of light. So, a very massive object, such as a cluster of galaxies can act as a gravitational lens. When light passes through the cluster from an object lying behind it, the light is bent and focused to produce an image or images of the source. The image may be magnified, distorted, or multiplied by the lens, depending upon the position of the source with respect to the lensing mass. The characteristics of the gravitationally lensed image depend upon the alignment of the observer, the lens and the background object. If the alignment is perfect, the resulting image is an Einstein Ring.



B1938+666

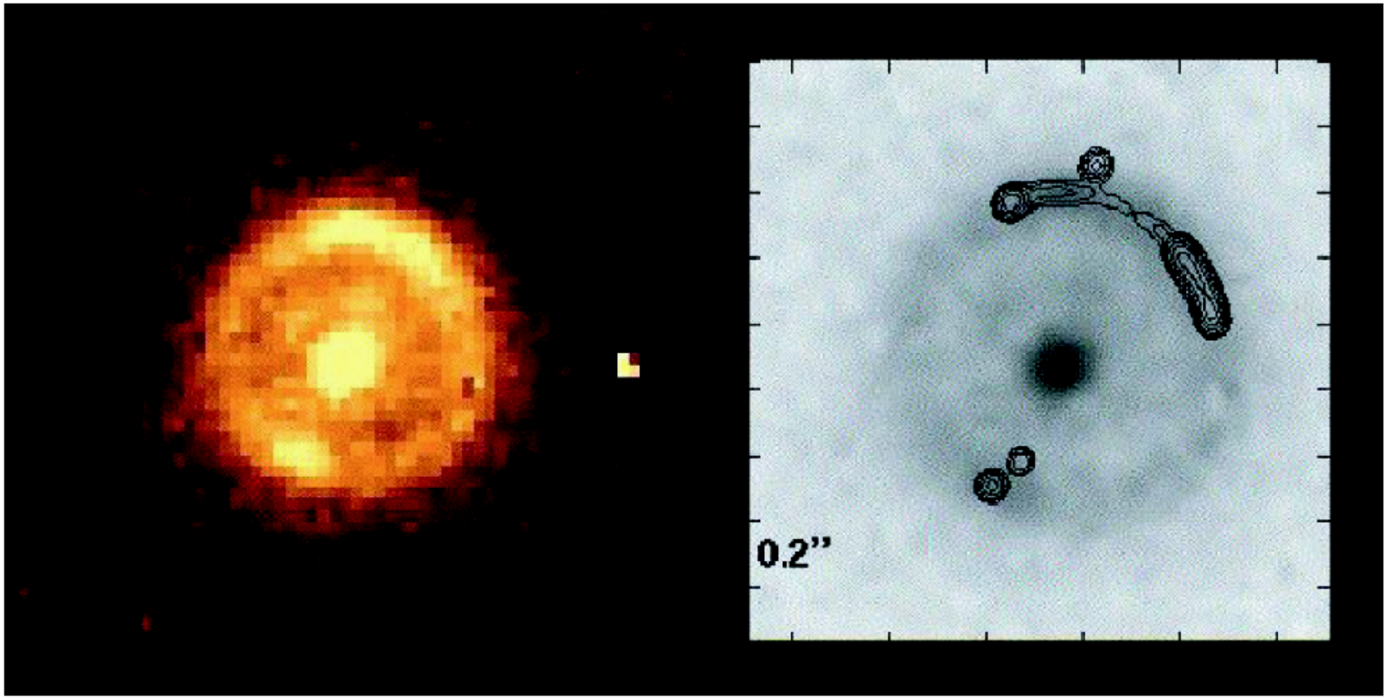
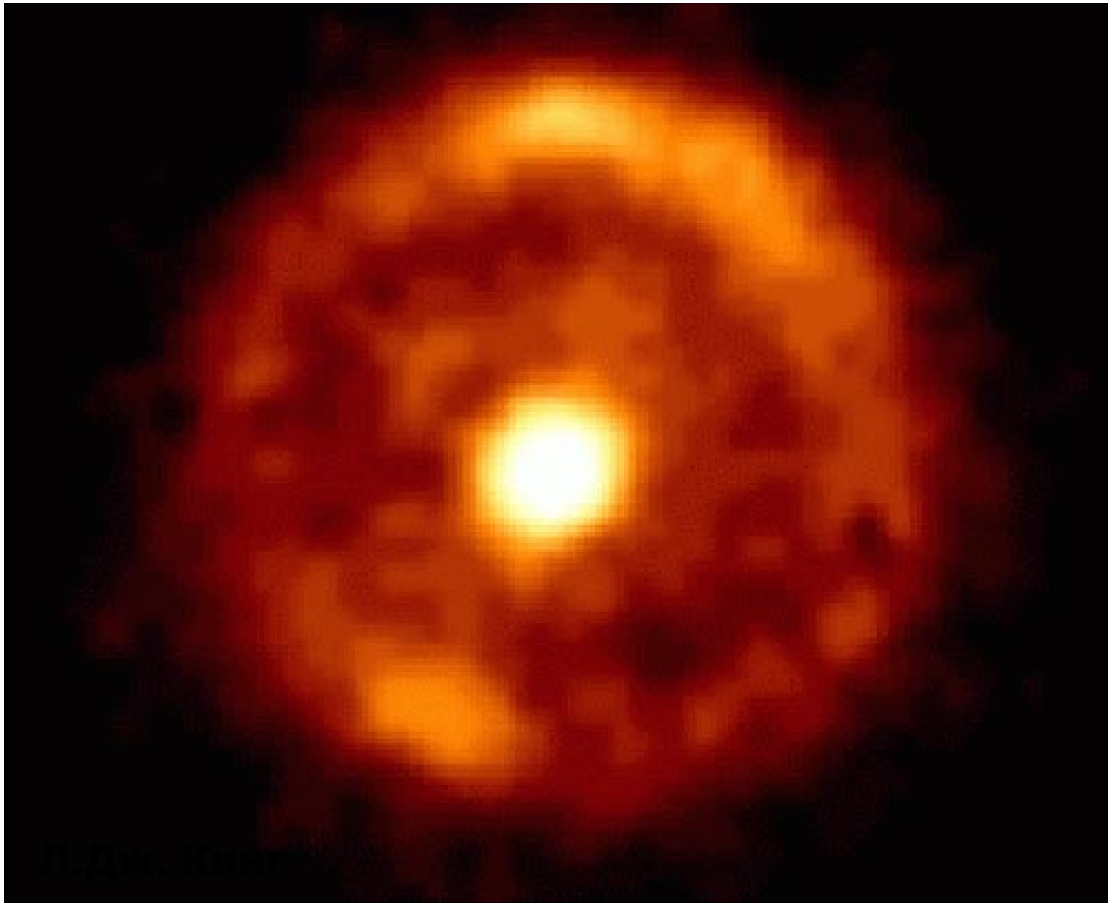


Fig. 8. The gravitational lens system B 1938+666. The *left panel* shows a NICMOS@HST image of the system, clearly showing a complete Einstein ring into which the Active Galaxy is mapped, together with the lens galaxy situated near the center of the ring. The *right panel* shows the NICMOS image as gray-scales, with the radio observations superposed as contours. The radio source is indeed a double, with one component being imaged twice (the two images just outside and just inside the Einstein ring), whereas the other source component has four images along the Einstein ring, with two of them close together (source: L.J. King, see King et al. 1998)

The Einstein Ring

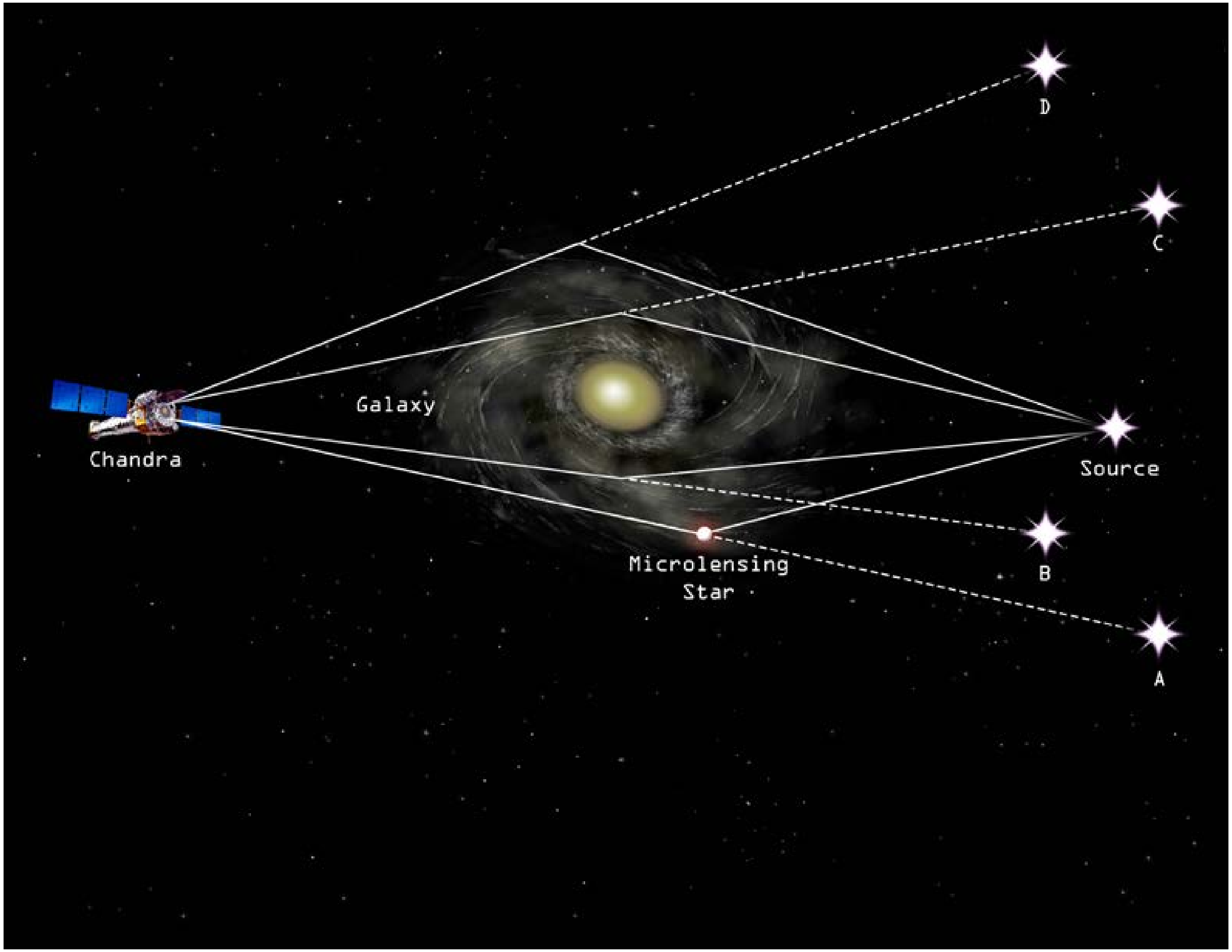


Angular diameter – 1 arc sec, liner diameter ~ 10^4 parsec

- B1938+666
- S and GL are galaxies

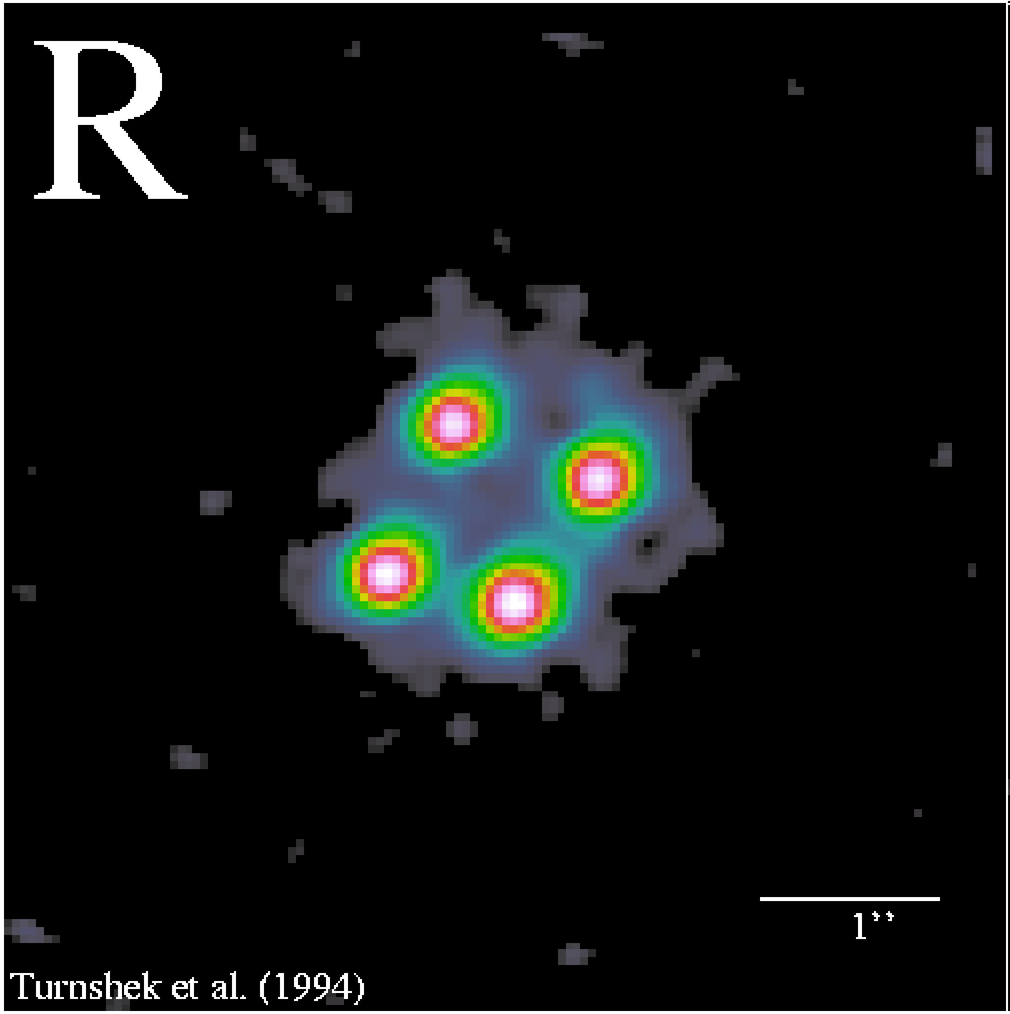
Cloverleaf Quasar (H1413+117)

Illustration of Gravitational Lensing Effect



http://chandra.harvard.edu/photo/2004/h1413/more.html

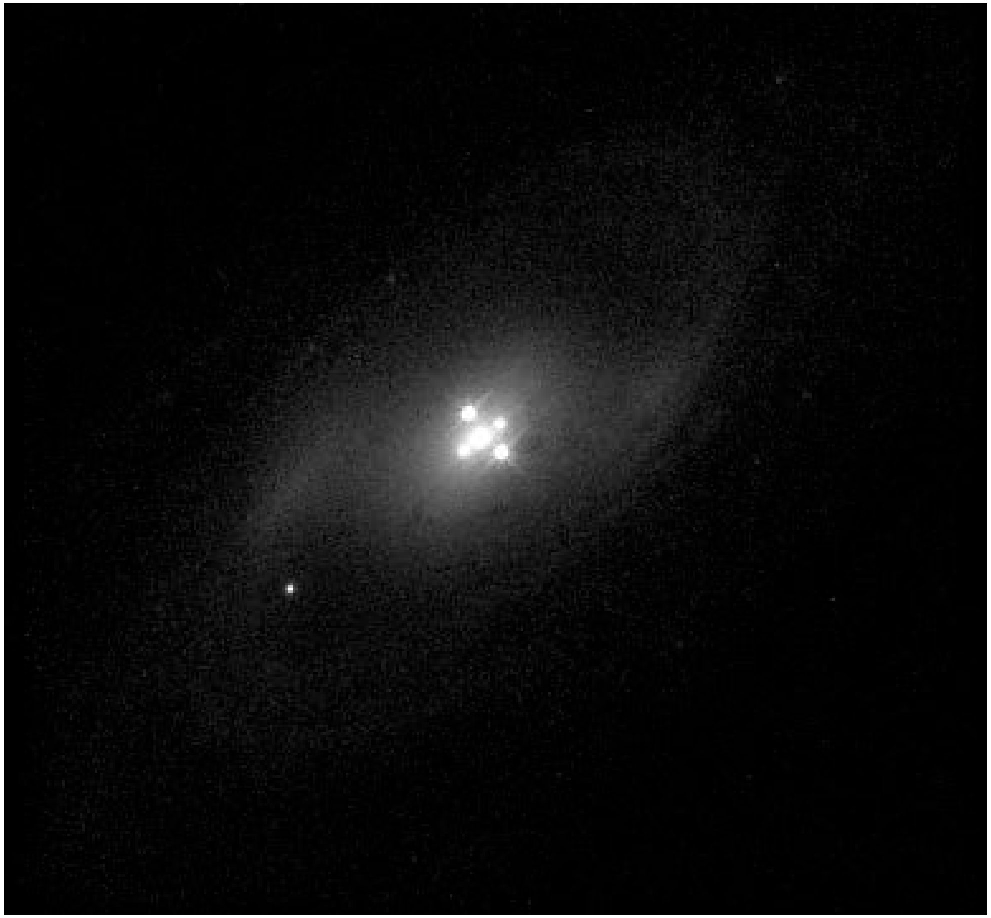
Cloverleaf Quasar (H1413+117)



- Hubble Optical Image of Cloverleaf Quasar

http://chandra.harvard.edu/photo/2004/h1413/more.html

The Einstein Cross

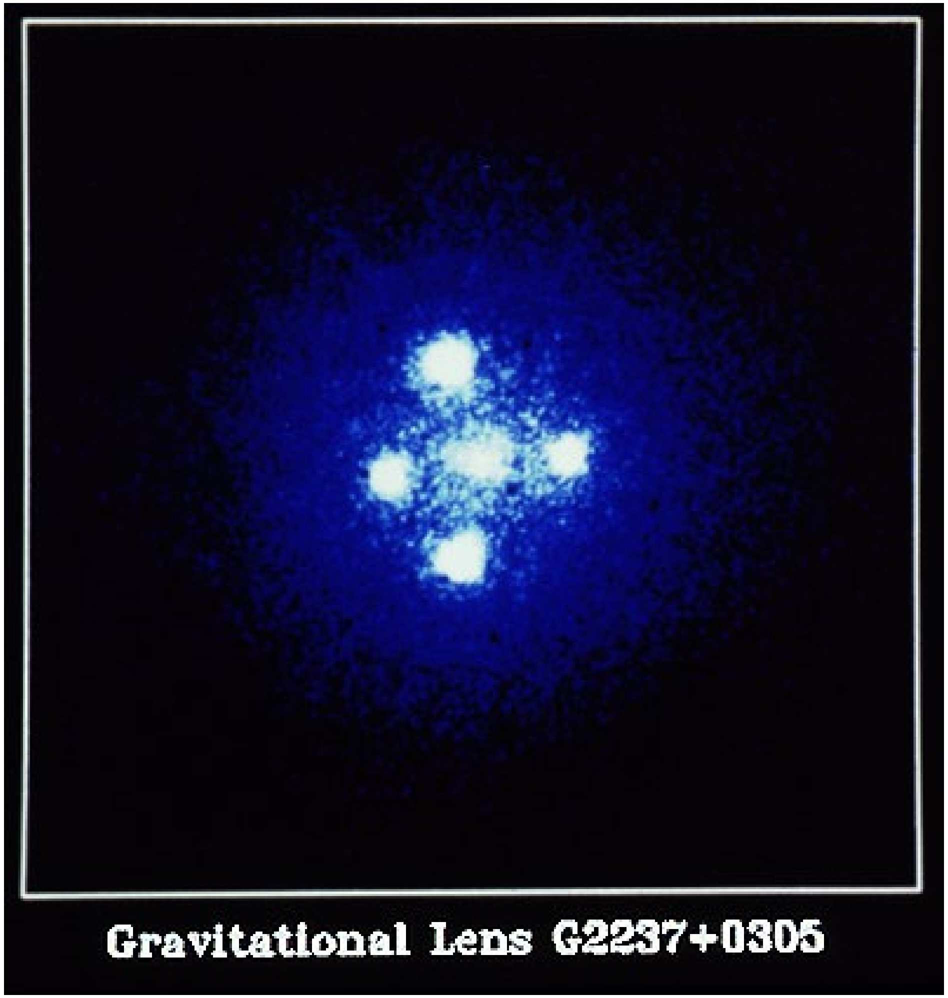


- Q2237+0305
- 1985, Huchra and others
- It is observed: four QSO images arrayed around the nucleus of the galaxy.
- The model : 1988, Schneider and others

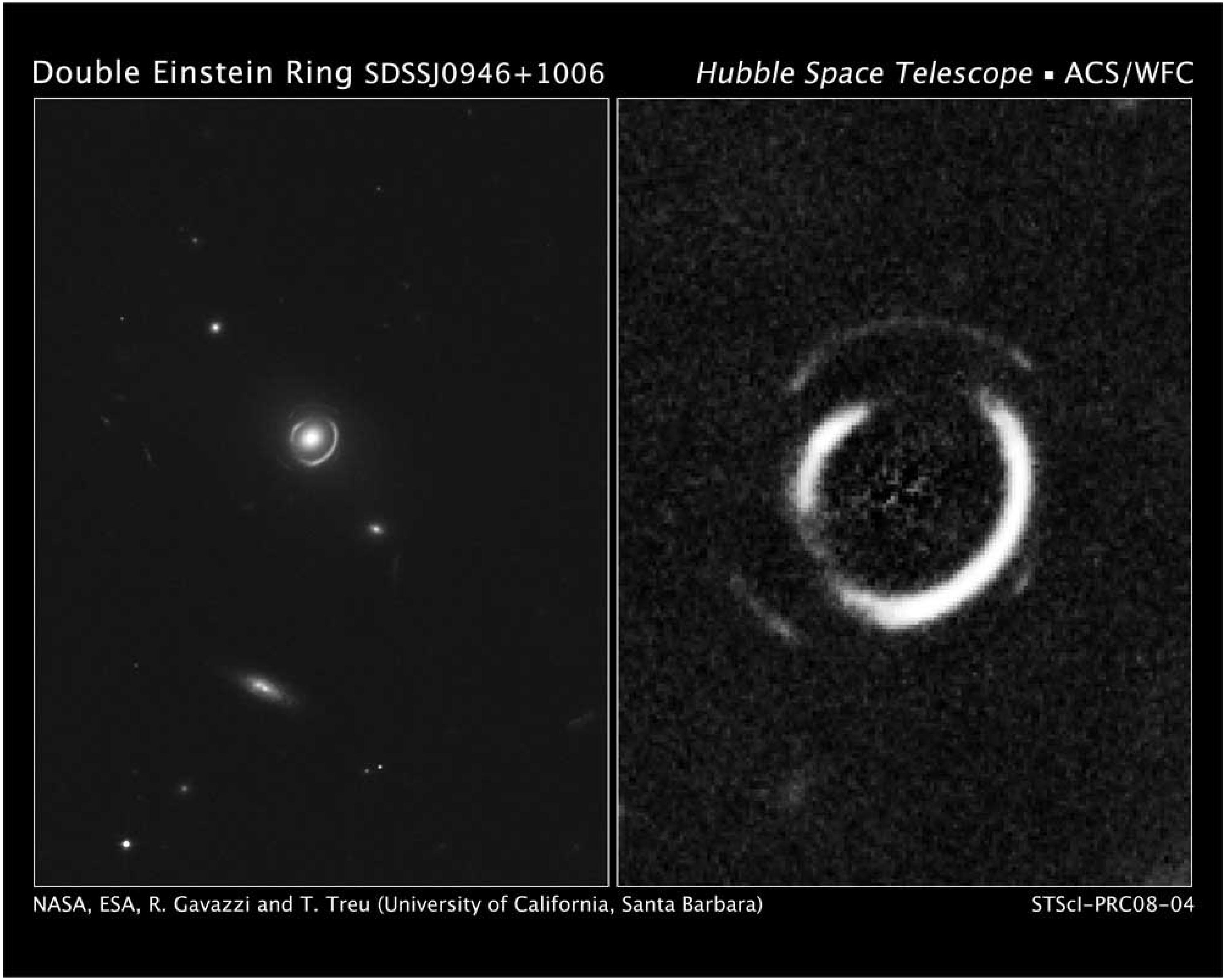
This picture of the gravitationally lensed quasar Q2237+0305 and the associated lensing spiral galaxy was taken by the 3.5-meter WIYN telescope, on the night of October 4, 1999.

http://antwrp.gsfc.nasa.gov/apod/ap070311.html

The Einstein Cross



- 2237+0305
- 1985, Huchra and others
- It is observed: four QSO images arrayed around the nucleus of the galaxy.
- The model : 1988, Schneider and others


Double Einstein Ring SDSSJ0946+1006
Hubble Space Telescope ▪ ACS/WFC
NASA, ESA, R. Gavazzi and T. Treu (University of California, Santa Barbara)
STScI-PRC08-04

The magnification factor

The surface brightness I for an image is identical to that of the source in the absence of the lens. The flux of an image of an infinitesimal source is the product of its surface brightness and the solid angle $\Delta\omega$ it subtends on the sky.

The magnification μ is the ratio of the flux of an image to the flux of the unlensed source:

$$\mu = \frac{\Delta\omega}{(\Delta\omega)_0}$$

- **“Strong lensing”:**

Multiple images, microlensing and arcs in clusters

- **Weak lensing:**

Weak distortions and small magnifications. Those cannot be identified in individual sources, but only in a statistical sense.

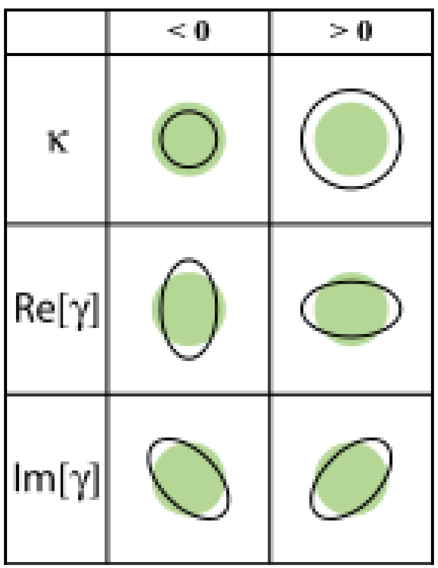
Mass reconstruction, dark matter

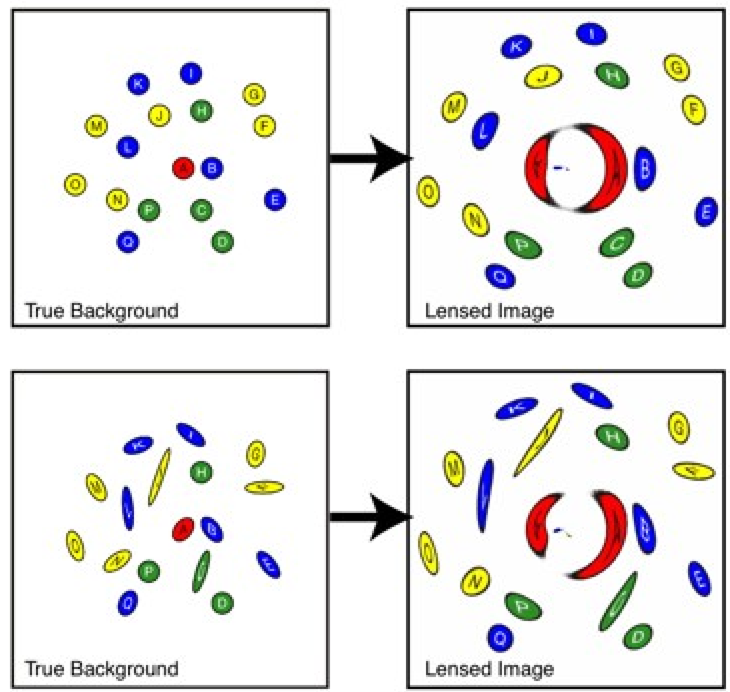
- **Microlensing:**

Version of strong lensing in which the image separation is too small to be resolved. Only change of flux.

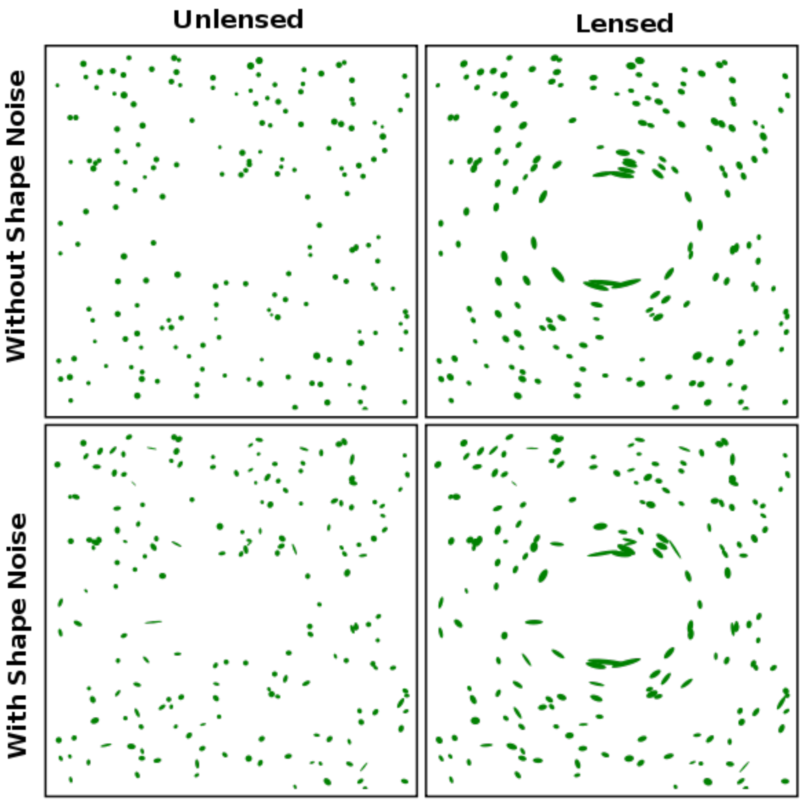
Weak lensing

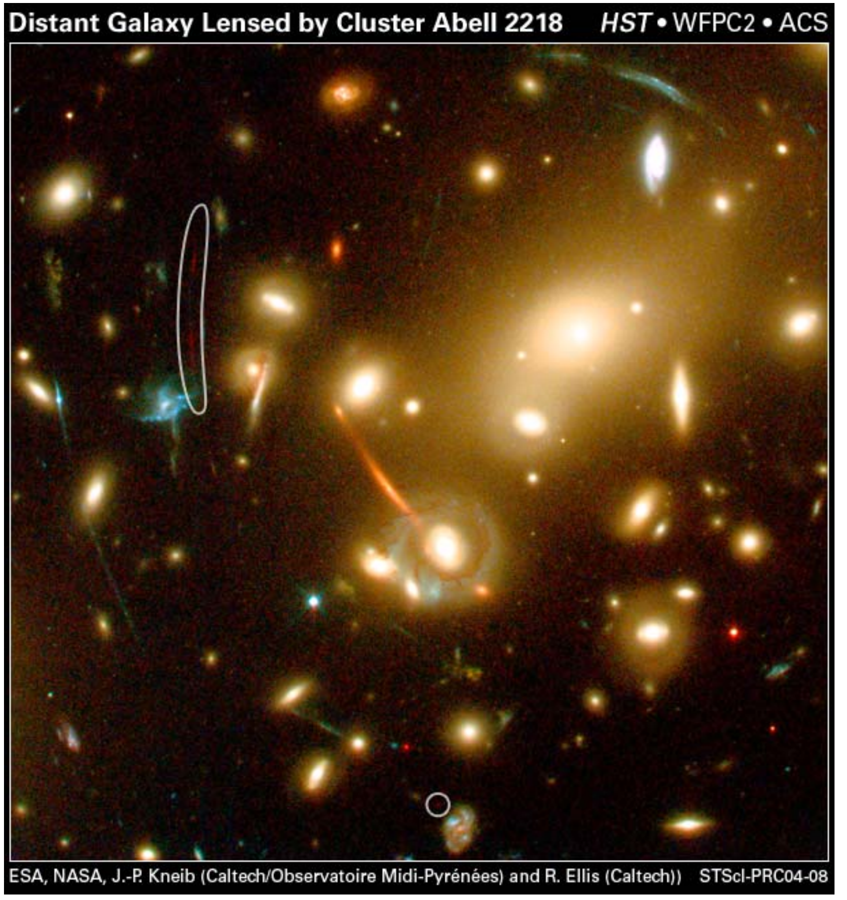
- Weak distortions and small magnifications. Those cannot be identified in individual sources, but only in a statistical sense.

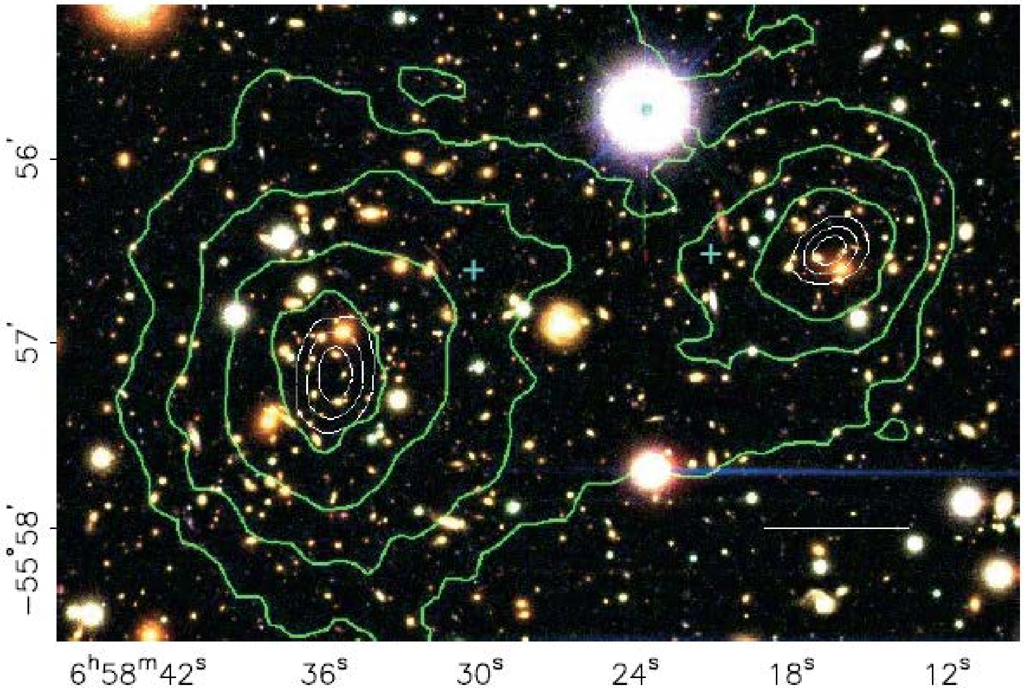




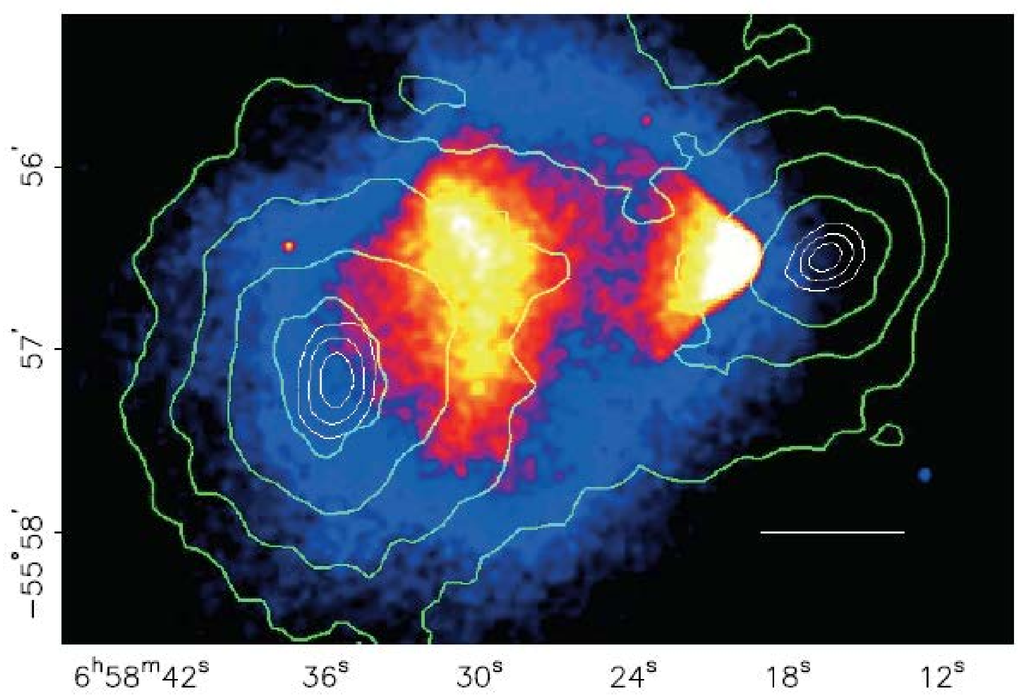
Weak lensing







56'
57'
−55°58'
6h58m42s
36s
30s
24s
18s
12s



56'
57'
−55°58'
6h58m42s
36s
30s
24s
18s
12s

FIG. 1.— Shown above in the top panel is a color image from the Magellan images of the merging cluster 1E0657−558, with the white bar indicating 200 kpc at the distance of the cluster. In the bottom panel is a 500 ks Chandra image of the cluster. Shown in green contours in both panels are the weak lensing κ reconstruction with the outer contour level at $\kappa = 0.16$ and increasing in steps of 0.07. The white contours show the errors on the positions of the κ peaks and correspond to 68.3%, 95.5%, and 99.7% confidence levels. The blue +s show the location of the centers used to measure the masses of the plasma clouds in Table 2.

We construct a map of the gravitational potential using weak gravitational lensing

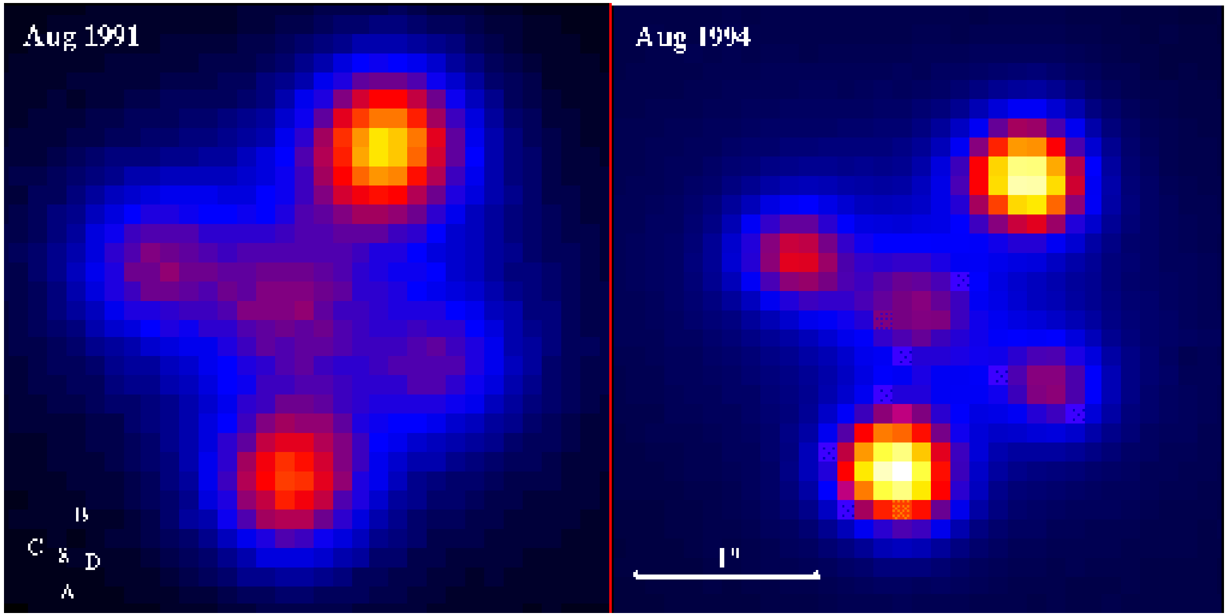
DIRECT EMPIRICAL PROOF OF EXISTENCE OF DARK MATTER

Douglas Clowe et al.

The Astrophysical Journal, 648:L109–L113, 2006 September 10

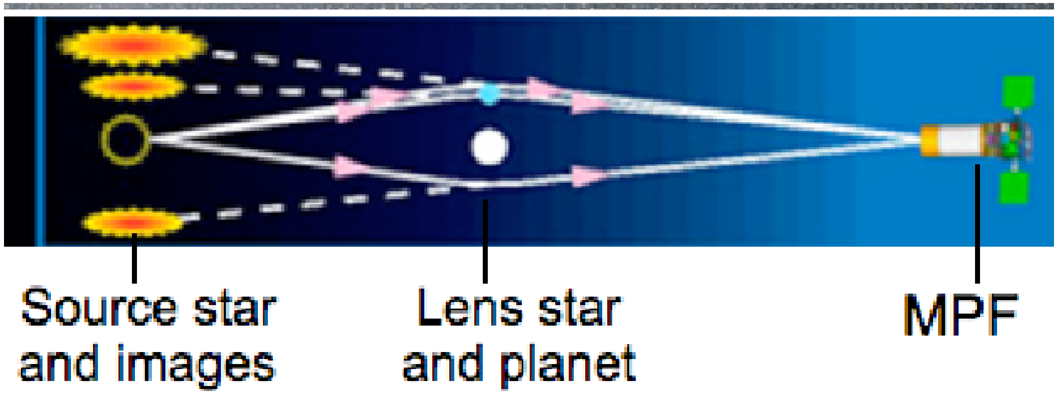
Microlensing

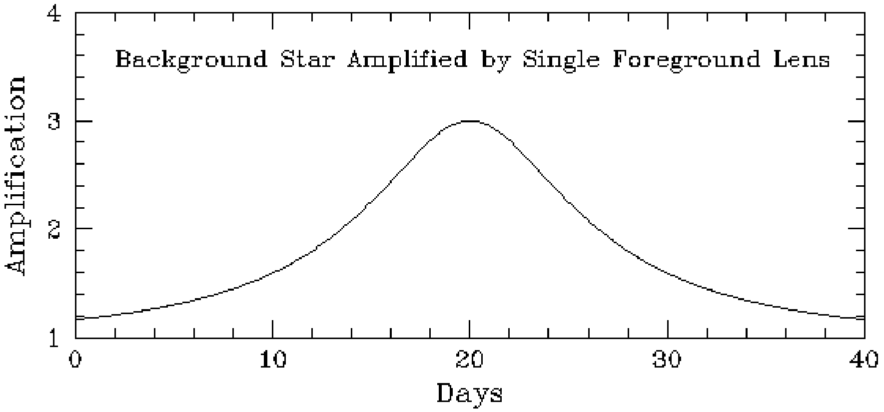
- The term microlensing (ML) arose from the fact that a single star of about solar mass can lead to images split into 'microimages' separated by microarcseconds. Multiple images due to stellar-mass objects can therefore not be resolved; the only observable effect ML is the alteration of the apparent luminosity of background sources.
- The images of a multiply imaged QSO are seen through a galaxy; since galaxies contain stars, stellar-mass objects can affect the brightness of these images.

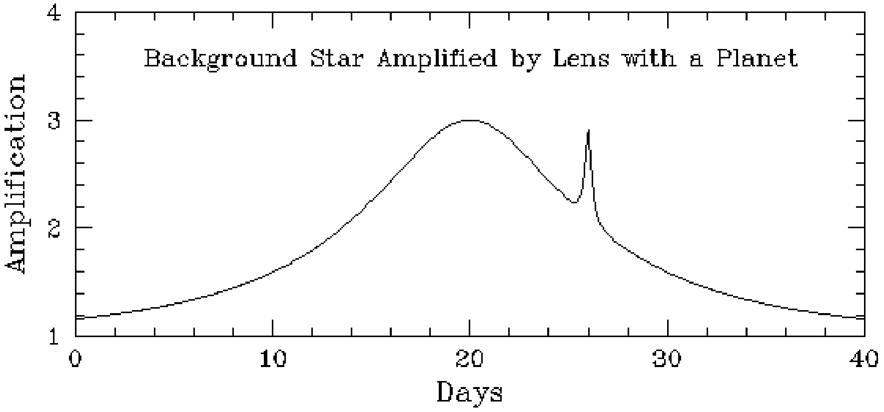


- Microlensing of the Einstein Cross

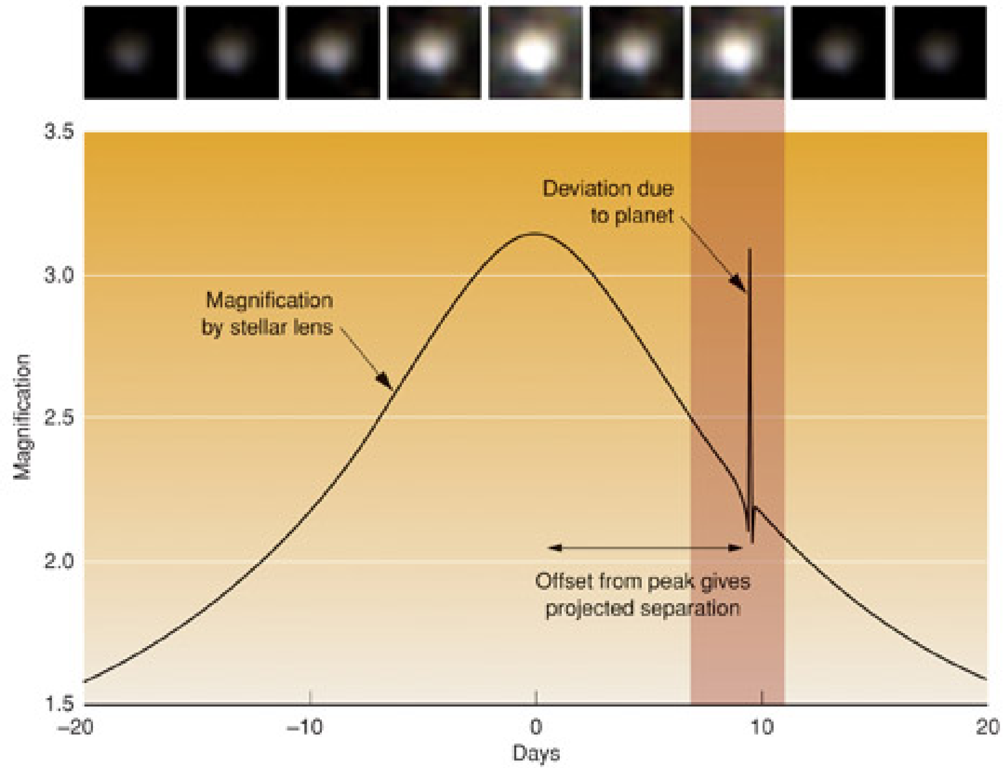
Microlensing: planets

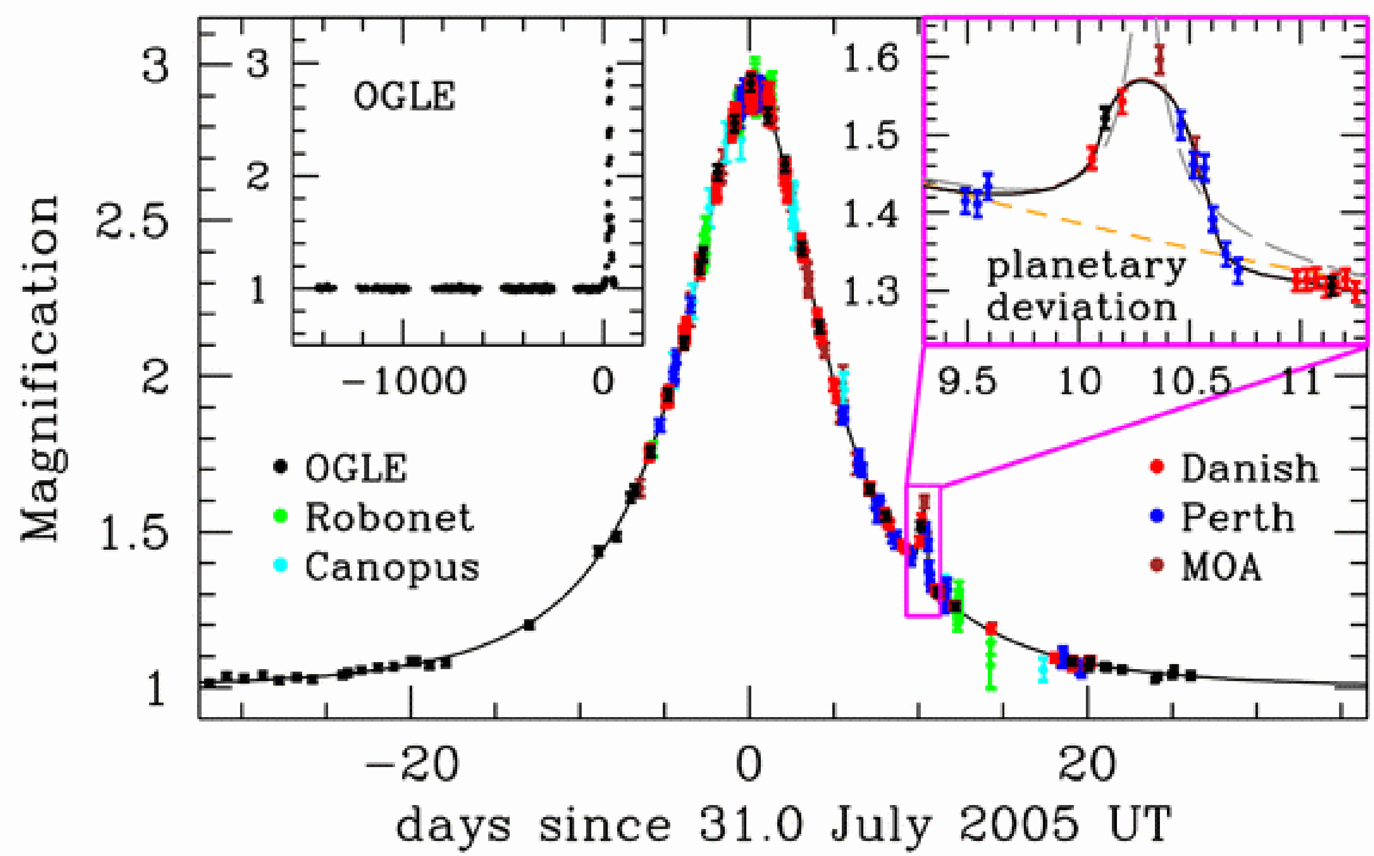






- Data from a microlensing event indicate a smooth, symmetric magnification curve as a lens star moves between a source star and an observatory on Earth. The short spike in magnification is caused by a planet orbiting the lens star.





Magnification
days since 31.0 July 2005 UT
OGLE
planetary deviation
OGLE
Robonet
Canopus
Danish
Perth
MOA

Types from the point of view of Deflection Angle

- **Weak deflection:**

Deflection angle is small

Einstein deflection angle:

General Relativity predicts that a light ray which passes by a spherical body of mass M with impact parameter b, is deflected by the “Einstein angle”:

$$\hat{\alpha} = \frac{4GM}{c^2 b} = \frac{2R_S}{b}$$

provided the impact parameter b is much larger than the corresponding Schwarzschild radius R_S:

$$b \gg R_S = \frac{2GM}{c^2}$$

Equation for the Schwarzschild lens

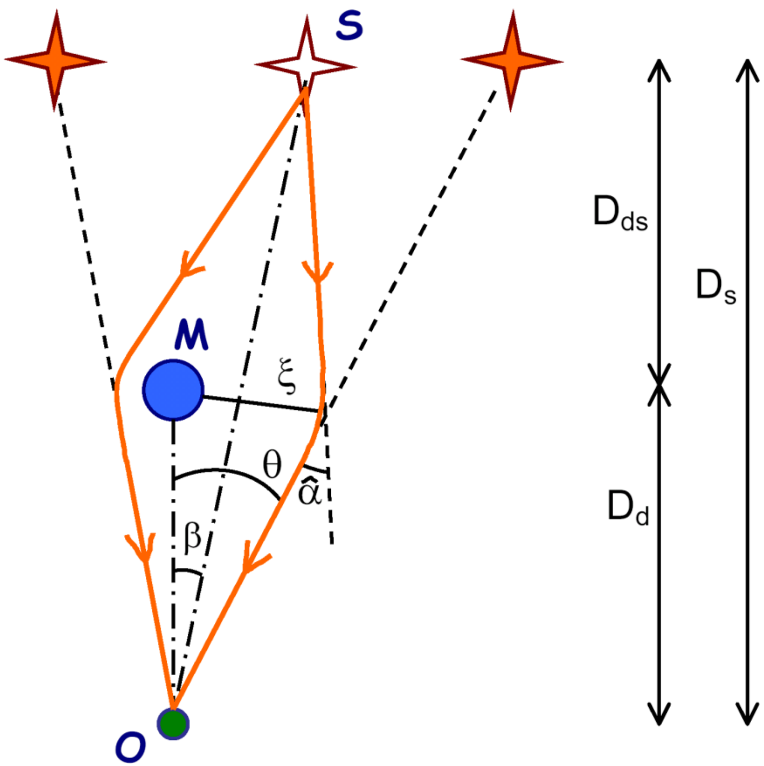
$$\beta = \theta - 2R_S \frac{D_{ds}}{D_s D_d} \frac{1}{\theta}, \quad \theta = \xi / D_d$$

Einstein angle

$$\alpha_0 = \sqrt{2R_S \frac{D_{ds}}{D_s D_d}}$$

Einstein radius радиус

$$\xi_0 = \sqrt{2R_S \frac{D_d D_{ds}}{D_s}} = \alpha_0 D_d$$



$$\beta = \theta - \frac{\alpha_0^2}{\theta}, \quad \theta^2 - \beta\theta - \alpha_0^2 = 0$$

$$\theta_{1,2} = \frac{1}{2}\left(\beta \pm \sqrt{4\alpha_0^2 + \beta^2}\right)$$

$$\Delta\theta = \theta_1 - \theta_2 = \sqrt{4\alpha_0^2 + \beta^2} \geq 2\alpha_0$$

Let us consider the motion of a photon in the neighborhood of a black hole with a Schwarzschild metric. We shall work with a system of units in which the Schwarzschild radius $R_S = 2M$ $(G=1,\ c=1)$, where M is the mass of the black hole. The Schwarzschild metric is given by

$$ds^2 = g_{ik}dx^i dx^k = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2M}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- Equations for the photon:

$$\left(\frac{dr}{d\lambda}\right)^2 + B^{-2}(r) = b^{-2},$$

$$\frac{d\varphi}{d\lambda} = \frac{1}{r^2},$$

$$\frac{dt}{d\lambda} = b^{-1}\left(1-\frac{2M}{r}\right)^{-1}.$$

Here $B^{-2}(r) = \left(1/r^2\right)\left(1-(2M/r)\right)$ is the effective potential and b is the impact parameter.

1. If $b < 3\sqrt{3}\,M$, then the photon falls to $R_S = 2M$ and is absorbed by the black hole.

2. If $b > 3\sqrt{3}\,M$, then the photon is deflected by an angle $\hat{\alpha}$ and flies off to infinity. Here there are two possibilities:

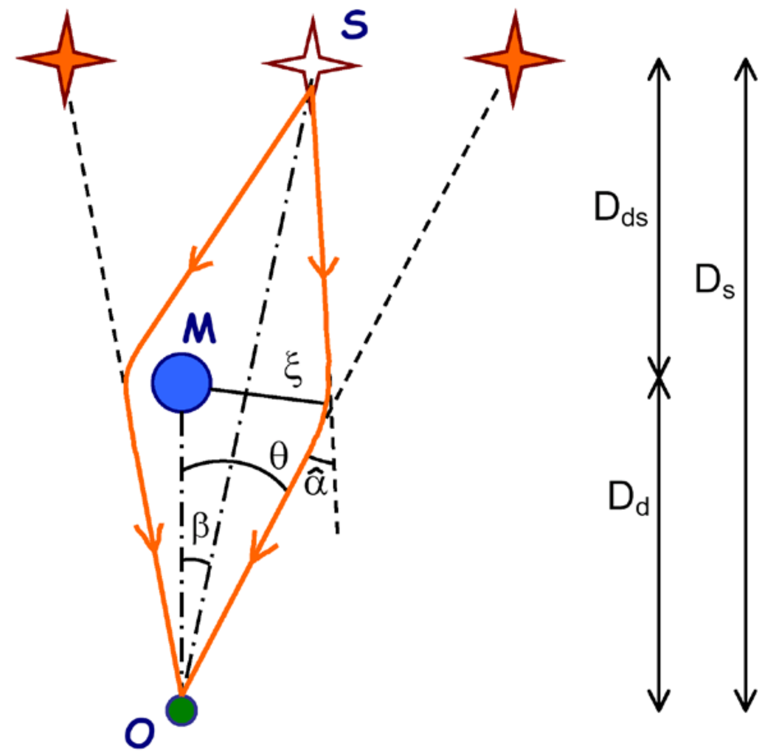
(a) If $b >> 3\sqrt{3}\,M$, then the orbit is almost a straight line with a small deflection by an angle $\hat{\alpha} = 4\,M/R$, where R is the distance of closest approach. This is the case customarily examined in the theory of weak gravitational lensing, when the impact parameter is much greater than the Schwarzschild radius of the lens.

(b) If $0 < b/M - 3\sqrt{3} << 1$, then the photon makes several turns around the black hole near a radius $r = 3M$ and flies off to infinity.

- Usual gravitational lensing is based on weak deflection limit:

$$\hat{\alpha} = 4\,M/R$$

- The Schwarzchild lens



Types from the point of view of Deflection Angle

- **Strong deflection:**

Deflection angle is not small

Exact deflection angle:

$$\widehat{\alpha} = 2\int_{R}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)}} - \pi$$

- R is the distance of the closest approach
- b is the impact parameter
- M is the mass of the black hole

The strong deflection limit

- If the value of the impact parameter is close to the critical value, we can also use approximate ***analytical*** formula for deflection angle (strong deflection limit). This formula is for case when photon incident from infinity undergoes one or more revolutions around the black hole and then escapes to infinity.

- Deflection angle as a function of distance of the closest approach

$$\hat{\alpha} = -2\ln\frac{R-3M}{36\left(2-\sqrt{3}\right)M} - \pi$$

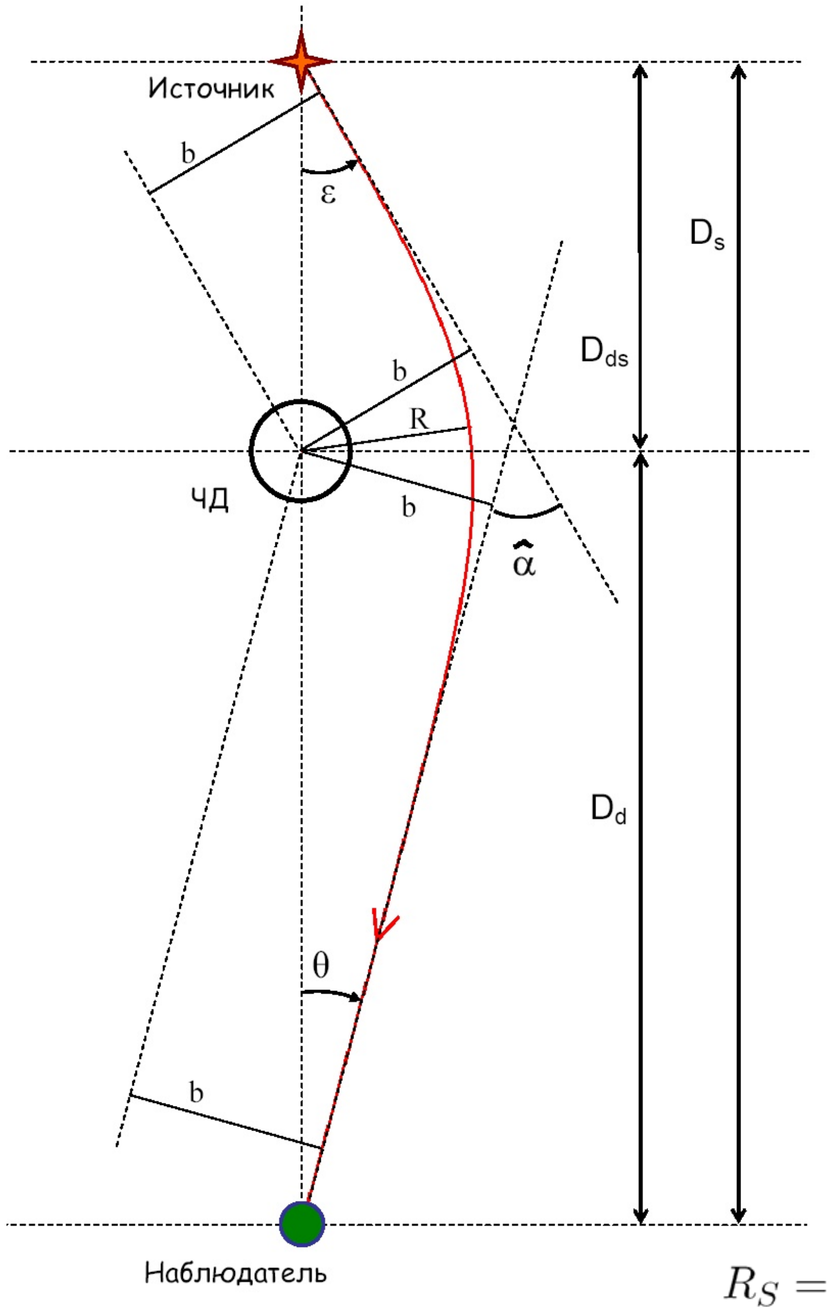
- Relation between impact parameter and distance of the closest approach in case of strong deflection

$$b = 3\sqrt{3}\,M + \frac{\sqrt{3}}{2}\frac{(R-3M)^2}{M}$$

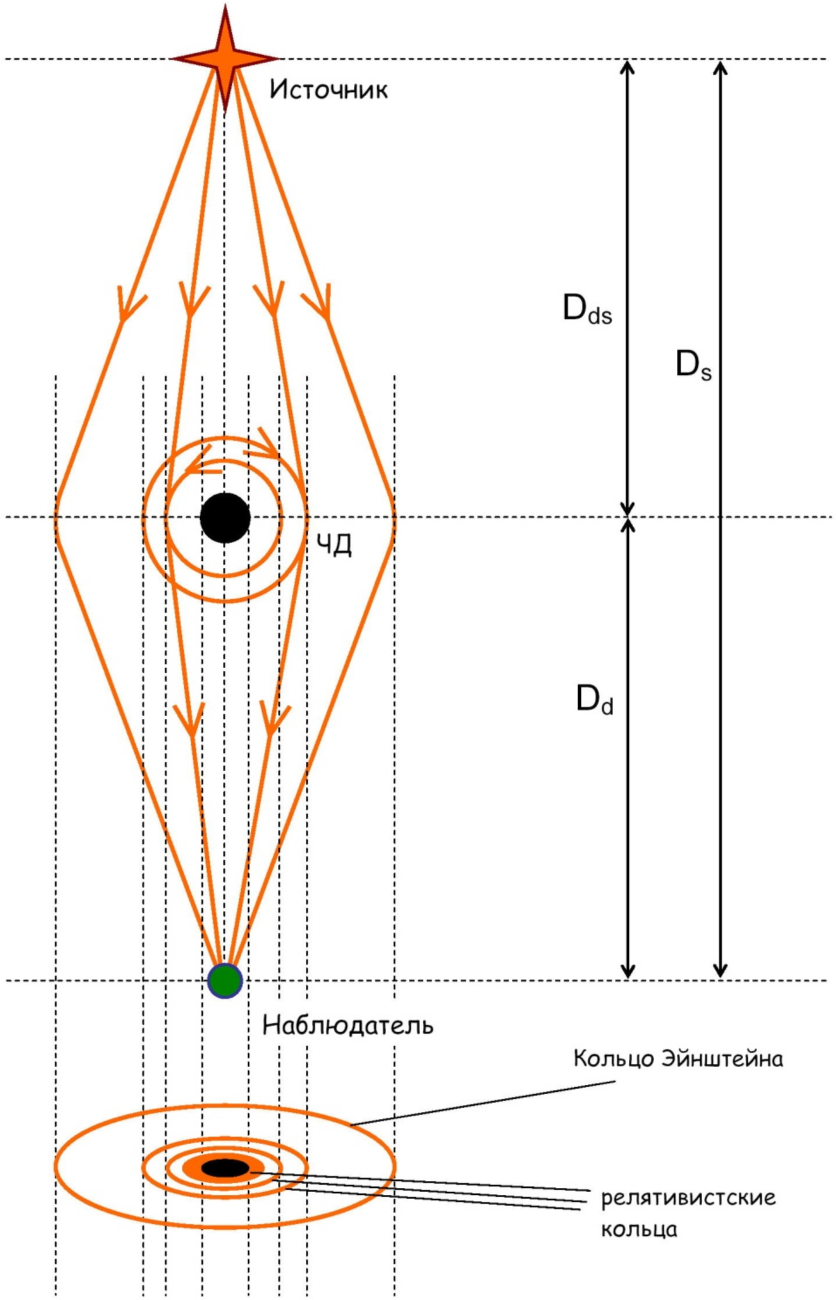
- Deflection angle as a function of distance the impact parameter

$$\hat{\alpha} = -\ln\left(b/M - 3\sqrt{3}\right) + \ln\left[648\left(7\sqrt{3}-12\right)\right] - \pi =$$

$$= -\ln\left(\frac{b}{b_{cr}} - 1\right) + \ln\left[216\left(7-4\sqrt{3}\right)\right] - \pi \simeq -\ln\left(\frac{b}{b_{cr}} - 1\right) - 0.40023$$



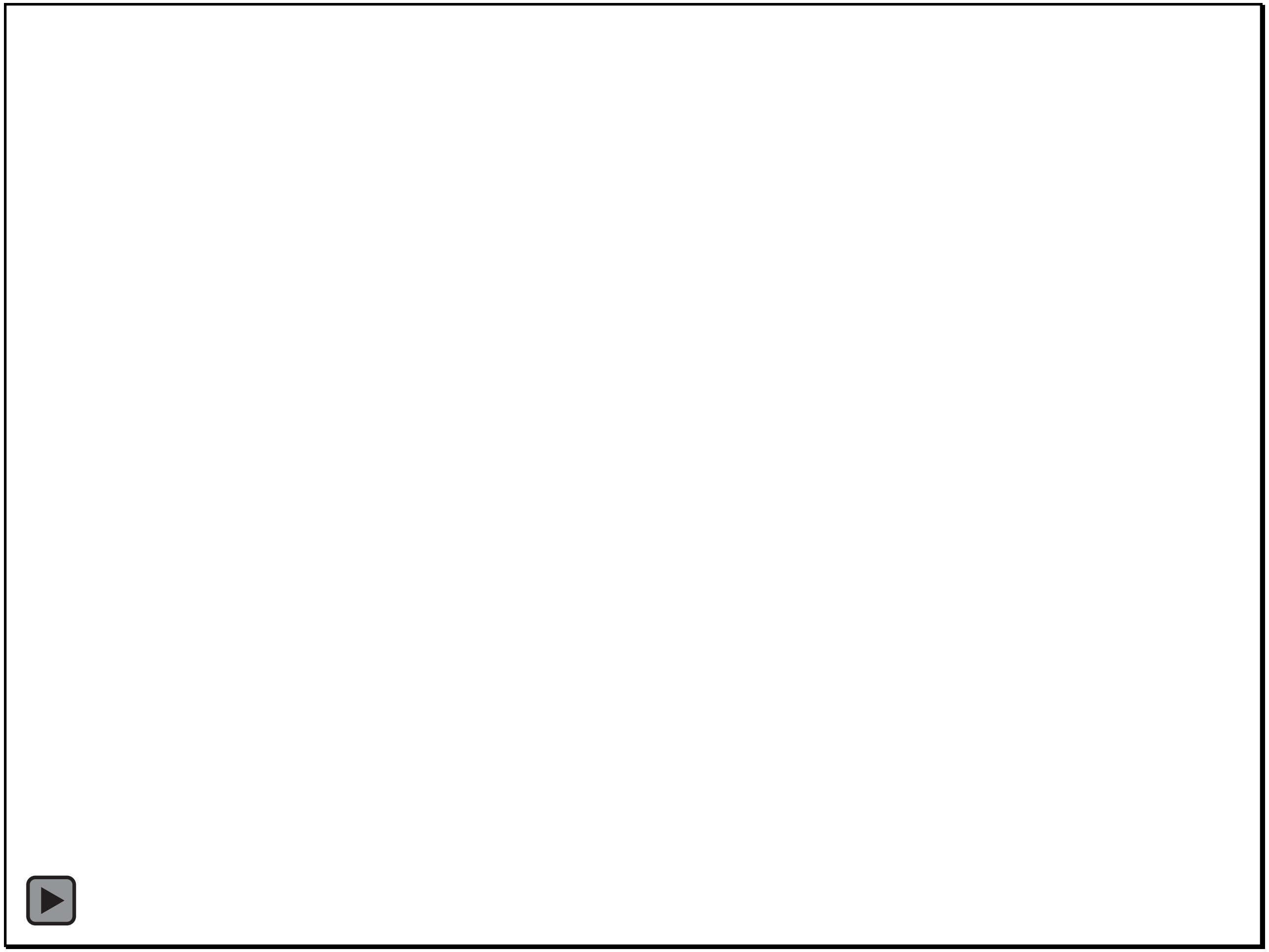
$$R_S = 2,\ R = 7,\ b \approx 8.3,\ D_d = 32,\ D_{ds} = 16.$$



- The angular radius of the Einstein ring

$$\theta_0 = \sqrt{4M\frac{D_{ds}}{D_d D_s}}$$

G.S. Bisnovatyi-Kogan,
O.Yu. Tsupko
Strong gravitational lensing by Schwarzschild black holes, Astrofizika, Vol. 51, No. 1, pp. 125–138 (February 2008).



- **Gravitational lensing in plasma**

Vacuum -> plasma

It is interesting to consider the gravitational lensing in a plasma, because in space the light rays mostly propagate through this medium. In an inhomogeneous plasma photons move along the curved trajectories, because a plasma is a dispersive medium with a permittivity tensor depending on the density. In the dispersive nonuniform medium the photon trajectory depends also on the photon frequency, and this effect has no relation to the gravity.

In inhomogeneous medium photons moves along curved trajectory, and if medium is dispersive the trajectory depends on frequency of the photon.

In medium the light rays move with the group velocity. For plasma with the index of refraction $n^2 = 1 - \omega_e^2/\omega^2$ the group velocity is $v_{gr} = cn$. The smaller frequency and bigger wavelength correspond to the smaller group velocity.

In plasma: $V_{phase} = c/n$, $V_{group} = cn$, $V_{phase} V_{group} = c^2$

Gravitational lensing in plasma, previous results:

The photon deflection in a non-homogeneous plasma, in presence of gravity, has been considered by:

D.O. Muhleman and I.D. Johnston, 1966; D.O. Muhleman, R.D. Ekers, and E.B. Fomalont, 1970.
A.P. Lightman, W.H. Press, R.H. Price, S.A. Teukolsky, Problem Book in Relativity and Gravitation, 1979.
P. V. Bliokh and A. A. Minakov, Gravitational Lenses (Naukova Dumka, Kiev, 1989), in Russian.

The consideration was done in a linear approximation, when the two effects: the vacuum deflection due to the gravitation, and the deflection due to the non-homogeneity of the medium, have been considered separately. The first effect is achromatic, the second one depends on the photon frequency if the medium is dispersive, but equals to zero if the medium is homogeneous.

Gravitational deflection of light in vacuum

It does not depend on frequency

+

Deflection of light in non-homogeneous medium
(non-relativistic effect)

It depends on frequency if the medium is dispersive, but is equal to zero if the medium is homogeneous

The new result:

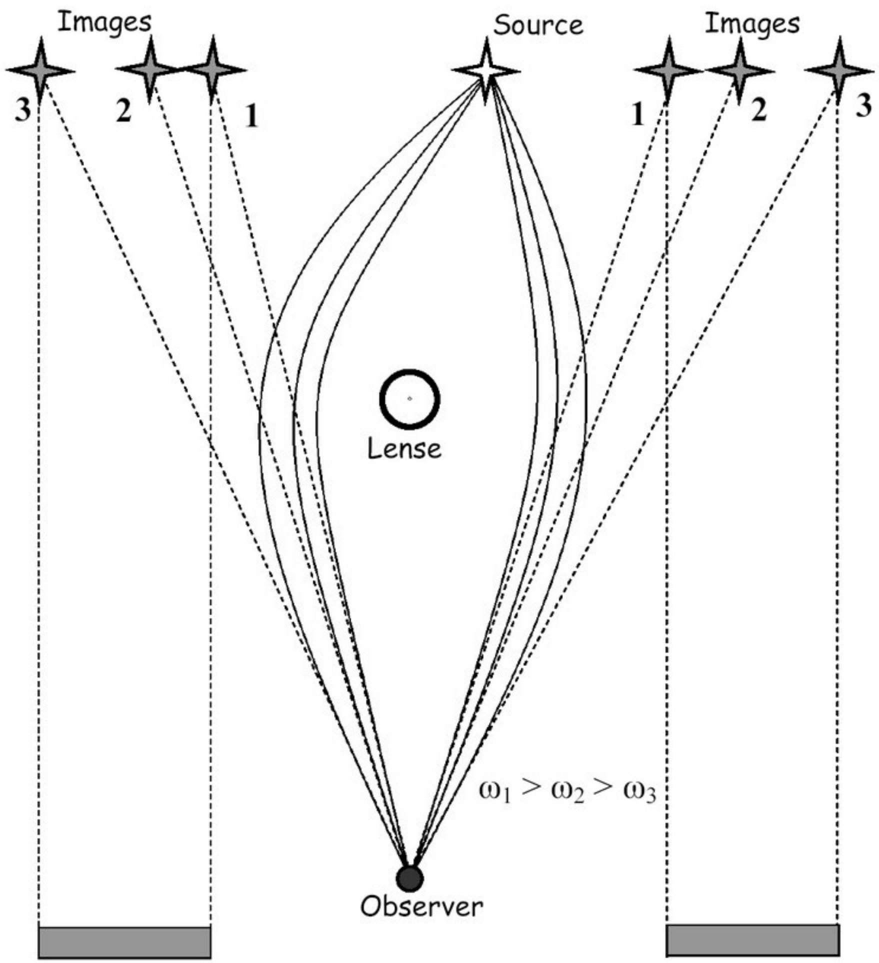
A general theory of the geometrical optic in the curved space-time, in arbitrary medium, is presented in the book of Synge (1960).
J.L. Synge, Relativity: the General Theory, North-Holland Publishing Company, Amsterdam, 1960.
On the basis of his general approach we have developed the model of gravitational lensing in plasma.

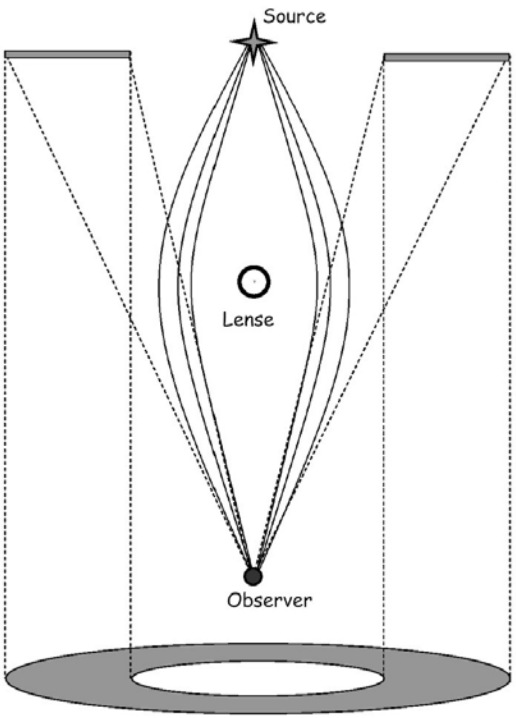
We have shown, that due to dispersive properties of plasma even in the homogeneous plasma gravitational deflection will differ from vacuum deflection angle, and gravitational deflection angle in plasma will depend on frequency of the photon (Bisnovatyi-Kogan, Tsupko, 2009).

$$\hat{\alpha} = \frac{2R_S}{b} = \frac{4GM}{c^2 b} \quad \text{-->} \quad \hat{\alpha} = \frac{R_S}{b}\left(1 + \frac{1}{1 - (\omega_e^2/\omega^2)}\right)$$

We derive the deflection angle for the photon moving in a weak gravitational field, in the Schwarzschild metric, in the arbitrary inhomogeneous plasma. Such approach is appropriate for the propagation in a cosmic plasma, because plasma densities changes significantly, from the density in the interstellar medium until the density in the neighborhood of a black hole. We consider here only the situation, when the whole deflection angle, from the combined plasma and gravity effects, remains small.

Gravitational radiospectrometer





$\lambda_1 < \lambda_2 < \lambda_3$

$v_{gr1} > v_{gr2} > v_{gr3}$

$\alpha_1 < \alpha_2 < \alpha_3$

Instead of two concentrated images with complicated spectra, we will have two line images, formed by the photons with different frequencies, which are deflected by different angles.

Gravitational lensing in homogeneous plasma: N_1=0

Deflection angle in a homogeneous plasma is:

$$\hat{\alpha} = \frac{R_S}{b} \left(1 + \frac{1}{1 - \omega_0^2/\omega^2} \right),$$

where $\omega_0^2 = \frac{4\pi e^2}{m} N_e$ is the electron plasma frequency, N_e is the electron concentration in homogeneous plasma, m is the mass of the electron, ω is the frequency of the photon (at infinity, $\sec^{-1}$).

Typical angular separation between images in homogeneous plasma is:

$$\alpha_0 = \sqrt{\left(1 + \frac{1}{1 - \omega_0^2/\omega^2} \right) R_S \frac{D_{ds}}{D_d D_s}}$$

Difference in the typical angular separation of images (between vacuum and plasma) is:

$$\frac{\delta\alpha_0}{\alpha_0} \simeq 2 \cdot 10^7 \, \frac{N_e}{\nu^2},$$

where ν is the photon frequency (Hz), $\omega = 2\pi\nu$.

RadioAstron

Frequency band (GHz) : 0.327 (P);
1.665 (L);
4.830 (C);
18.392-25.112 (K)

For the lowest frequency of RadioAstron $\nu = 327 \cdot 10^6$ Hz,

Angle difference between vacuum case and plasma case 0.00001 arcsec will be at N_e ~ 50 000

Amplification of the image depends on the lensing angle, therefore different images may have different spectra in the radio band, when the light propagates in regions with different plasma density.

For the Schwarzschild lense the amplification is proportional to the angular radius of the Einstein ring, which increases when the frequency approaches the plasma frequency.

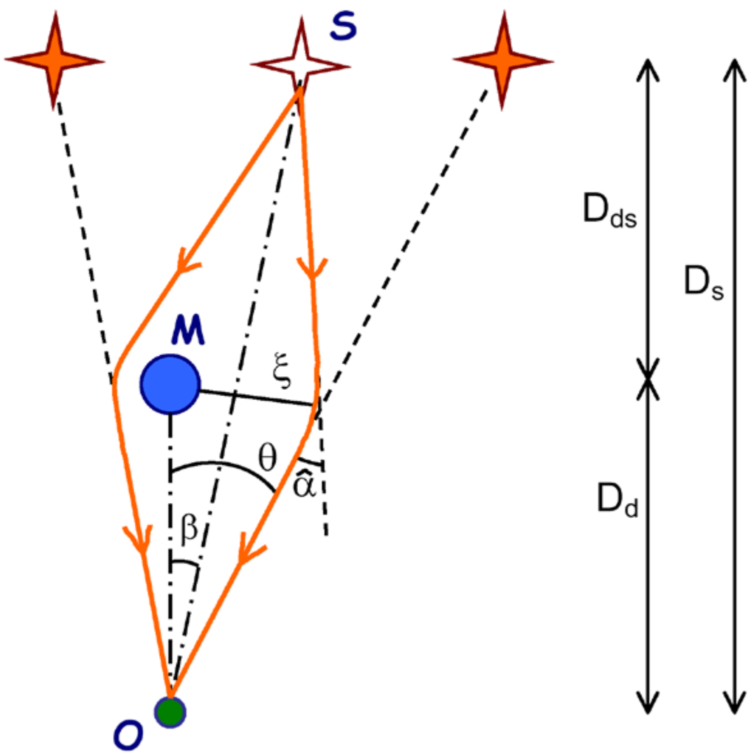
Amplification factors

$$\mu_+^{pl} = \frac{1}{4}\left[\frac{\tilde{y}}{\sqrt{\tilde{y}^2+4}} + \frac{\sqrt{\tilde{y}^2+4}}{\tilde{y}} + 2\right],$$

$$\mu_-^{pl} = \frac{1}{4}\left[\frac{\tilde{y}}{\sqrt{\tilde{y}^2+4}} + \frac{\sqrt{\tilde{y}^2+4}}{\tilde{y}} - 2\right],$$

$$\tilde{y} = \beta/\theta_0^{pl}$$

$$= y\left[\frac{1}{2}\left(1+\frac{1}{1-(\omega_e^2/\omega^2)}\right)\right]^{-1/2}.$$



$$\theta_0^{pl} = \sqrt{\left(1+\frac{1}{1-(\omega_e^2/\omega^2)}\right) R_S \frac{D_{ds}}{D_d D_s}} =$$

$$= \theta_0 \sqrt{\frac{1}{2}\left(1+\frac{1}{1-(\omega_e^2/\omega^2)}\right)},$$

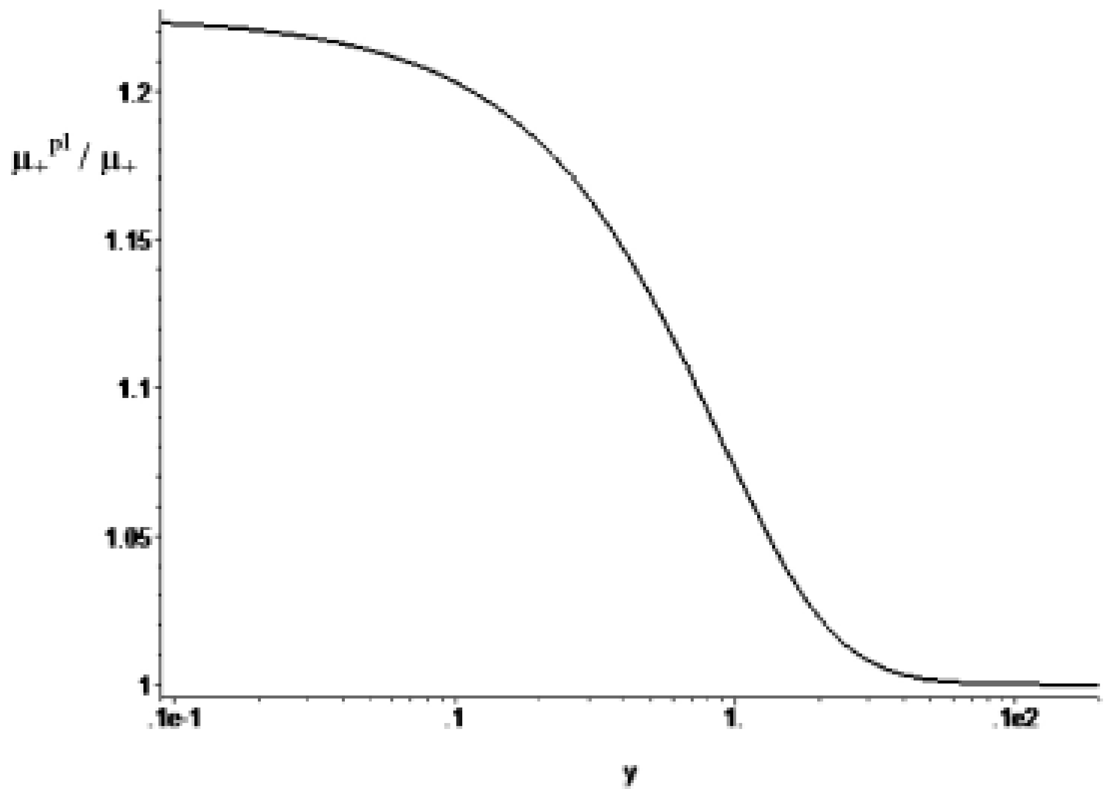


Figure 1. The ratio of the magnification of the primary image in presence of plasma, to the same value in vacuum.

for $\omega = \sqrt{2}\,\omega_e$.

$$g_{ik} = \eta_{ik} + h_{ik}, \quad h_{ik} \ll 1, \quad h_{ik} \to 0 \ \text{under} \ x^\alpha \to \infty .$$

Let us consider a static space-time assuming a small perturbation h_{ik} of flat metric. Let us consider, in this gravitational field, a static inhomogeneous plasma with a refraction index n, which depends on the space location x^α and the frequency of the photon $\omega(x^\alpha)$:

$$n^2 = 1 - \frac{\omega_e^2}{[\omega(x^\alpha)]^2}, \quad \omega_e^2 = \frac{4\pi e^2 N(x^\alpha)}{m}.$$

We denote $\omega(\infty) = \omega$, e is the charge of the electron, m is the mass of the electron, $N(x^\alpha)$ is the electron concentration in the inhomogeneous plasma, ω_e is the electron plasma frequency in the plasma.

Weak gravitational field in the arbitrary inhomogeneous plasma.

If the problem is axially symmetric, it is convenient to introduce the impact parameter b, and we obtain for the deflection angle of the photon moving along z-axis:

$$\hat{\alpha}_b = -\frac{R_S}{b} - \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{1}{1 - (\omega_e^2/\omega^2)} \frac{R_S\, b}{r^3} + \right.$$

$$\left. + \frac{K_e}{\omega^2 - \omega_e^2} \frac{b}{r} \frac{dN(r)}{dr} \right) dz .$$

To demonstrate the physical meaning of different terms in (32), we write this expression under condition $1 - n = \omega_e^2/\omega^2 \ll 1$. Carrying out the expansion of terms with the plasma frequency, we obtain:

$$\hat{\alpha}_b = -\frac{2R_S}{b} - \frac{1}{2}\frac{R_S\, b}{\omega^2}\int\limits_{-\infty}^{\infty}\frac{\omega_e^2}{r^3}\,dz -$$

$$-\frac{1}{2}\frac{K_e b}{\omega^2}\int\limits_{-\infty}^{\infty}\frac{1}{r}\frac{dN(r)}{dr}\,dz - \frac{1}{2}\frac{K_e b}{\omega^4}\int\limits_{-\infty}^{\infty}\frac{\omega_e^2}{r}\frac{dN(r)}{dr}\,dz. \qquad (33)$$

The first term is a vacuum gravitational deflection. The second term is an additive correction to the gravitational deflection, due to the presence of the plasma. This term is present in the deflection angle both in the inhomogeneous and in the homogeneous plasma, and depends on the photon frequency. The third term is a non-relativistic deflection due to the plasma inhomogeneity (the refraction). This term depends on the frequency, but it is absent if the plasma is homogeneous. The forth term is a small additive correction to the third term.

$$n^2 = 1 - \frac{\omega_e^2}{[\omega(x^\alpha)]^2}, \quad \omega_e^2 = \frac{4\pi e^2 N(x^\alpha)}{m} = K_e N(x^\alpha).$$

Non-uniform plasma

$$\hat{\alpha}_b = -\frac{2R_S}{b} - \frac{1}{2}\frac{R_S\, b}{\omega^2}\int_{-\infty}^{\infty}\frac{\omega_e^2}{r^3}\,dz - \frac{1}{2}\frac{K_e b}{\omega^2}\int_{-\infty}^{\infty}\frac{1}{r}\frac{dN(r)}{dr}\,dz$$ =1+2+3

1. Singular isothermal spher $\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$. $\left|\frac{\hat{\alpha}_2}{\hat{\alpha}_3}\right| = \frac{8}{3}\frac{\sigma_v^2}{c^2}$.

2. Non-singular isothermal gas spher $\rho(r) = \frac{\sigma_v^2}{2\pi G (r^2 + r_c^2)}$

$$\left|\frac{\hat{\alpha}_2}{\hat{\alpha}_3}\right| = \frac{3R_{sc}}{2r_c}\ (r_c \gg b), \quad \left|\frac{\hat{\alpha}_2}{\hat{\alpha}_3}\right| = \frac{2R_{sc}}{r_c}\ (r_c \ll b). \qquad R_{sc} = \frac{2GM_c}{c^2}$$

$$M_c = \frac{4\pi}{3}\rho_0 r_c^3, \qquad \rho_0 = \frac{\sigma_v^2}{2\pi G r_c^2},$$

b – impact parameter

3. Plasma sphere around a black hole $\left|\frac{\hat{\alpha}_2}{\hat{\alpha}_3}\right| = \frac{2\Re T}{c^2}$.

4. Plasma in a galaxy clusters

$$\rho_{gr}(r) = \frac{\sigma_v^2}{2\pi G r^2},$$

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-s}, \quad s = \frac{2\sigma_v^2}{\Re T}.$$

T – plasma temperature

$$\left|\frac{\hat{\alpha}_2}{\hat{\alpha}_3}\right| = \frac{2\Re T}{c^2}.$$

at

$$2\sigma_v^2 \ll \Re T;$$

The gravitational effect in plasma may be identified when plasma non-uniformity is not prevailing, what is possible in relativistic plasma, kT ~ m_e c^2

Conclusions (plasma)

1. In the homogeneous plasma gravitational deflection differs from vacuum (Einstein) deflection angle and depends on frequency of the photon. So gravitational lens in plasma acts as a **gravitational radiospectrometer**.

2. In the observations **the spectra of two images may be different** in the long wave side due to different plasma properties along the trajectory of the images.

3. The extended image may have different spectra in different parts of the image, with a maximum of the spectrum shifting to the long wave side in the regions with a larger deflection angle.

4. **The gravitational effect can be important in the case of a hot gas in the gravitational field of a galaxy cluster.**

1. Bisnovatyi-Kogan G.S., Tsupko O.Yu. Gravitational radiospectrometer // Gravitation and Cosmology. 2009. V.15. N.1. P.20. arXiv:0809.1021v2

2. Bisnovatyi-Kogan G.S., Tsupko O.Yu. Gravitational lensing in a non-uniform plasma, Mon. Not. R. Astron. Soc. 404, 1790–1800 (2010). arXiv:1006.2321v1