Oscillating Solitons of the Parametrically Driven Damped Nonlinear Schrödinger Equation

E. V. Zemlyanaya
Joint Institute for Nuclear Research, Dubna, Russia

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**Equation under study:**

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi. \]

\( \gamma > 0 \) is the damping coefficient, 
\( h \) is the amplitude of the parametric driver.

We are looking for periodic solutions by solving the NLS equation as a **boundary-value problem** on a two-dimensional domain \(( -\infty, \infty ) \times (0, T) \).

**Boundary conditions:**

\[ \psi(x, t) = 0 \quad \text{as} \quad x \to \pm \infty, \quad \psi(x, t + T) = \psi(x, t). \]
Results of direct numerical simulation

\[ \psi(x) = A e^{-i\theta} \text{sech}(Ax), \]

\[ A = \sqrt{1 + \sqrt{h^2 - \gamma^2}}, \]

\[ \theta = \frac{\text{arcsin}(\gamma/h)}{2} \]

Numerical continuation of stationary multi-soliton complexes

The Hopf bifurcation point:
γ = 0.565;
h = 0.9435

I. Barashenkov and E. Zemlyanaya,
Phys Rev Lett 83 (1999) 2568
Method of numerical study. New variables

New variables $\tau$ and $\Psi$:

$$\tau t = t; \quad 0 \leq \tau \leq 1; \quad \Psi(x,\tau) = \psi(x,t).$$

Modified equation with respect of unknown $\Psi$ and $T$:

$$i\Psi_{\tau} + T \cdot \Phi(\Psi(x, \tau), h, \gamma) = 0 \quad \text{where}$$

$$\Phi \equiv \Psi_{xx} + 2|\Psi|^2 \cdot \Psi - \Psi - h\Psi* + i\gamma \Psi.$$

Boundary conditions:

$$\Psi(-L,\tau) = \Psi(+L,\tau) = 0; \quad \Psi(x,0) = \Psi(x,1);$$

Additional equation (phase condition)

$$R \equiv \text{Re}[\Phi(\Psi(x*,t*), h, \gamma)] = 0; \quad x^* = t^* = 0.$$
Method of numerical study.  
Newtonian scheme (1)

\[ \Psi_{k+1} = \Psi_k + \xi_k v_k; \quad T_{k+1} = T_k + \xi_k \mu_k; \]

\( k \) – number of iteration;

0\(<\xi_k\leq1 \) parameter of the Newtonian scheme;

\[ v_k = v^{(1)} + v^{(2)} \mu_k; \]

(1) \[ i v^{(1)}(\tau) + T_k v^{(1)}_{xx} + A_k v^{(1)} + B_k v^{(1)*} = -\Phi_k \]

(2) \[ i v^{(2)}(\tau) + T_k v^{(2)}_{xx} + A_k v^{(2)} + B_k v^{(2)*} = -C_k \]

BCs: \[ v^{(1)}(\pm L, \tau) = -\Psi_k(\pm L, \tau); \quad v^{(2)}(\pm L, \tau) = 0; \]

\[ v^{(1,2)}(x, 0) - v^{(1,2)}(x, 1) = - [\Psi^{(1,2)}(x, 0) - \Psi^{(1,2)}(x, 1)] \]

\[ A_k = 4T_k \Psi_k (\Psi_k)^* - T_k - i\gamma T_k; \quad B_k = 2T_k (\Psi_k)^2 - hT_k; \]

\[ C_k = \Psi_{xx} + 2\Psi_k^*(\Psi_k)^2 - \Psi_k - h(\Psi_k)^* - i\gamma \Psi_k; \]
$\mu_k$ is calculated at each iteration as follows

$$
F = [V_R^{(2)}]_{xx} + 6\Psi_R^2 V_R^{(2)} + 4\Psi_I \Phi_R V_I^{(2)} + 2\Psi_I^2 V_R^{(2)} - V_R^{(2)} - hV_R^{(2)} - \gamma V_I^{(2)}
$$

$$
G = [V_R^{(1)}]_{xx} + 6\Psi_R^2 V_R^{(1)} + 4\Psi_I \Psi_R V_I^{(1)} + 2\Psi_I^2 V_R^{(1)} - V_R^{(1)} - hV_R^{(1)} - \gamma V_I^{(1)}
$$

$$
R = [\Psi_R]_{xx} + 2\Psi_R^3 + 2\Psi_I^2 \Psi_R - \Psi_R - h\Psi_R - \gamma \Psi_I
$$

$$
\Psi_R = \text{Re} \Psi(x^*, 0); \quad \Psi_I = \text{Im} \Psi(x^*, 0)
$$

$$
V_R^{(1,2)} = \text{Re}V^{(1,2)}(x^*, 0); \quad V_I^{(1,2)} = \text{Im}V^{(1,2)}(x^*, 0)
$$

Spatial stepsize 0.05; stepsize in time 0.01; interval [-50,50]
Stability analysis
E.Zemlyanaya, I.Barashenkov, N.Alexeeva.
Springer Lecture Notes in Computer Sciences 5434 (2009) 139

Periodic solution is linearized in small perturbation $u+iv$:

$$\psi(x, t) = \psi_0(x, t) + u(x, t) + iv(x, t)$$

After expansion $u$ and $v$ in the Fourier series on the interval $(-L, L)$, according to the Floquet theory we obtain:

$$\psi_0 = \begin{pmatrix} R(x, t) \\ I(x, t) \end{pmatrix}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad q_n = \pi n / L, \quad w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

The system is solved numerically with initial condition

$$u_n(0) = \delta_{n\alpha}, \quad v_n(0) = 0 \quad (n = -N, \ldots, N), \quad \alpha = -N, \ldots, N$$

The monodromy matrix is constructed. Its eigenvalues allow us to make conclusions about stability properties of periodic solitons.
Numerical continuation of stationary two-soliton solutions

(a) $\gamma=0.01$

(b) $\gamma=0.4$

Numerical continuation of stationary two-soliton solutions

Hopf bifurcations points of stationary solitons at the \((h,\gamma)\)-plane

Numerical continuation in \(h\) for the fixed \(\gamma\).

Weak damping:
- \(\gamma = 0.1\)
- \(\gamma = 0.2\)
- \(\gamma = 0.265\)

Moderate damping:
- \(\gamma = 0.3\)
- \(\gamma = 0.35\)
- \(\gamma = 0.38\)
- \(\gamma = 0.41\)

Strong damping:
- \(\gamma = 0.565\)
Numerical results (1)
Weak damping: $\gamma=0.265; 0.2; 0.1$

Numerical results (2)
Moderate damping: $\gamma = 0.3, 0.35$

For $\gamma = 0.3$:
- $h = 0.55$, $T = 4.356$
- $h = 0.876$, $T = 2.743$

For $\gamma = 0.35$:
- $h = 0.64$, $T = 2.967$
Numerical results (3)
One-periodic two-soliton solutions
Moderate damping: $\gamma=0.35, 0.38$.

- Start $h=0.801$
- Period doubling
- $h=0.741, T=15.88$
- $h=0.95, T=2.5$
- Quasiperiodic solitons have a rise
Numerical results (4)
One-periodic two-soliton solutions
Moderate damping: $\gamma=0.41$ (the case of 4 HBs)

A time-periodic two-soliton complex oscillating out of phase with each other. $\gamma = 0.41$, $h = 1.049$, $T = 1.991$
Numerical results (5)
One-periodic two-soliton solutions
Strong damping: $\gamma=0.565$

Numerical results (6)
Stability diagram of stationary and time-periodic solitons at the \((h, \gamma)\)-plane
We decompose $\psi$ as

$$
\psi = A_+ \left[ U(\bar{x}, \bar{t}) + iV(\bar{x}, \bar{t}) \right] e^{-i\theta^+},
$$

This casts the NLS in the form:

$$
\begin{align*}
-V_t - 2\Gamma V &= -U_{xx} + U - 2(U^2 + V^2)U, \\
+U_t + 2HV &= -V_{xx} + V - 2(U^2 + V^2)V.
\end{align*}
$$

We expand $\psi(x; t)$ as:

- $u$ and $v$ are real;
- $A$ and $B$ are complex

Resulting system of equations:

$$
\begin{align*}
\begin{align*}
&\quad u_{xx} - u + 2(u^2 + v^2)u + 4(3|A|^2 + |B|^2)u + 4(AB^* + A^*B)v - 2\Gamma v = 0, \\
&\quad v_{xx} - v + 2(u^2 + v^2)v + 4(|A|^2 + 3|B|^2)v + 4(AB^* + A^*B)u + 2Hv = 0, \\
&\quad A_{xx} - A + 2(3u^2 + v^2)A \\
&\quad \quad + 2(3|A|^2 + 2|B|^2)A + 2(2uv + A^*B)B - 2\Gamma B - i\Omega B = 0, \\
&\quad B_{xx} - B + 2(u^2 + 3v^2)B \\
&\quad \quad + 2(2|A|^2 + 3|B|^2)B + 2(2uv + B^*A)A + 2HB + i\Omega A = 0.
\end{align*}
\end{align*}
$$

where $\Omega = 2\pi/(A_+^2 \cdot T)$,

$$
\Gamma = \gamma/A_+^2 \quad \text{and} \quad H = \sqrt{\hbar^2 - \gamma^2/A_+^2}.
$$

3- and 5-mode approximation (2)

Full NLS

3-mode approximation

5-mode approximation

$\gamma = 0.3$

$\gamma = 0.35$
Summary

• A boundary-value problem and stability problem have been formulated for numerical investigation of temporally periodic solitons of parametrically driven damped NLS.
• Transformations of temporally periodic solitons have been numerically studied; interconnection between coexisting branches of stable and unstable solutions has been analyzed.
• New temporally periodic solitons have been found.
• Stability diagram of stationary and oscillating two-soliton complexes has been constructed at the \((h,\gamma)\)-plane.
• Shown that the bifurcation diagram can be reproduced a three- and five-mode approximation.

THANK YOU!