Nonlinear Waves in Disordered Media: Localization and Delocalization

S. Flach, MPIPKS Dresden

Two lectures:

- Obtaining Anderson localization
- Destruction of Anderson localization
Lecture I:

- Obtaining Anderson localization
- Destruction of Anderson localization
Glossary

**Particles:** zero/full transmission below/above barrier, no interference, phase does not matter

**Waves:** partial transmission below/above barrier, interference, phase matters

**Quantum / classical waves:** Identical description for single qm particle / linear case

**Quantum many body waves:** linear equations in VERY high-dimensional Hilbert (vector) space

**Classical nonlinear waves:** nonlinear equations, e.g. from mean field approximation for MANY quantum particles

**Nonlinearity:** wave-wave (mode-mode) interactions

**Localization:** waves start to travel, but never get away
I was cited for work both in the field of magnetism and in that of disordered systems, and I would like to describe here one development in each held which was specifically mentioned in that citation. The two theories I will discuss differed sharply in some ways. The theory of local moments in metals was, in a sense, easy: it was the condensation into a simple mathematical model of ideas which were very much in the air at the time, and it had rapid and permanent acceptance because of its timeliness and its relative simplicity. What mathematical difficulty it contained has been almost fully cleared up within the past few years.

Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.
As usually, any new result has been obtained already before,  
And of course by others …

“Well, in my country,” said Alice, still panting a little, “you would generally get  
to somewhere else, if you ran very fast for a long time, as we’ve been doing”.  
“A slow sort of country!”, said the queen. “Now here, it takes all the running you  
can do, to stay in the same place.”
Experimental Evidence for Anderson Localization

waves in disordered media – Anderson localization for: electrons, phonons, photons, BEC, ...

**Electrons:** in: Akkermans et al 2006

**Ultrasound:** Weaver 1990

**Microwaves:** Dalichaoush et al 1991, Chabanov/Pradhan/ et al 2000


**BEC:** Billy et al 2008, Roati et al 2008

*Figure 1* | Observation of exponential localization. a, A small BEC

*Figure 2* | Experimental results for propagation in disordered lattices.
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

\( f = 3.04 \text{ GHz} \)

\( f = 7.33 \text{ GHz} \)

Localized State
Anderson Insulator

Extended State
Anderson Metal
1\textsuperscript{st} stage: laser cooling

Laser cooling
use laser light to cool and stop atoms.

Trapping of atoms
combination of optical and magnetic traps

T~\mu\textsuperscript{K}

2\textsuperscript{nd} stage: evaporative cooling

Ramping down the potential depth

N↓ and T↓ phase space density↑

T~nK \Rightarrow Bose-Einstein condensation

Steven Chu (b.1948)
Claude Cohen-Tannoudji (b.1933)
William D. Phillips (b.1948)
multicomponent BEC

low-D BEC

Bose-Fermi mixtures

BEC in optical potentials

dilute gas BEC

dipolar BEC

molecular BEC

1995

Eric Cornell  Wolfgang Ketterle  Carl E. Wieman
(b.1961)  (b.1957)  (b.1948)
Direct observation of Anderson localization of matter waves in a controlled disorder

Juliette Billy¹, Vincent Josse¹, Zhan-Chun Zhu¹, Alain Bernard¹, Ben Hambrecht¹, Pierre Lugan¹, David Clément¹, Laurent Sanchez-Palencia¹, Philippe Bouchet² & Alain Aspect¹

Observing single-particle Anderson localization with Bose-Einstein condensates
Observation of the signature of AL

BEC parameters: \(N = 1.7 \times 10^4\) atoms, \((\mu_{in} = 220\text{Hz})\)
Weak disorder: \(V_R / \mu_{in} = 0.12 \ll 1\)

\[k_{\text{max}} / k_c = 0.63 \pm 0.09\]

\[
\exp(-2|z|/L_{\text{loc}})
\]

⇒ Exponential decay of the density in the wings: \(L_{\text{loc}} = 530 \pm 80\ \mu\text{m}\)
An optical one-dimensional waveguide lattice (Silberberg et al ’08)

- Evanescent coupling between waveguides
- Light coherently tunnels between neighboring waveguides
- Dynamics is described by the Tight-Binding model

\[ i \frac{\partial U_n}{\partial z} = \beta_n U_n + C_{n,n\pm1} \left[ U_{n+1} + U_{n-1} \right] \]

\( \beta_n \) – waveguide’s refraction index /width
\( C_{n,n\pm1} \) – separation between waveguides
Injecting a narrow beam (~3 sites) at different locations across the lattice

(a) Periodic array – expansion
(b) Disordered array - expansion
(c) Disordered array - localization
SOME FACTS, THOUGHTS AND IDEAS
Anderson Model

- Lattice - tight binding model
- Onsite energies $\epsilon_i$ - random
- Hopping matrix elements $t_{ij}$

$-W/2 < \epsilon_i < W/2$
uniformly distributed

$$t_{ij} = \begin{cases} t & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$t < t_c$
Insulator
All eigenstates are localized

$t > t_c$
Metal
There appear states extended all over the whole system
Anderson Transition

DoS

$t > t_c$

$t < t_c$

$\epsilon$

DoS

$E^2_c$

$E^1_c$

extended

all states are localized

$E_c$ - mobility edges (one particle)
Localization of single-particle wave-functions. Continuous limit:

\[
\begin{bmatrix}
-\frac{\nabla^2}{2m} + U(r) - \epsilon_F \\
\end{bmatrix}
\psi_\alpha(r) = \xi_\alpha \psi_\alpha(r)
\]

- **\(d=1\):** All states are localized
- **\(d=2\):** All states are localized
- **\(d > 2\):** Anderson transition
The one-dimensional tight-binding model

- The periodic Lattice (Bloch, 1928)

\[-i \frac{\partial \psi_n}{\partial t} = E \psi_n + T [\psi_{n+1} + \psi_{n-1}]\]

Eigenfunctions extend over entire lattice (Bloch functions)

- The disordered lattice (Anderson, 1958)

\[-i \frac{\partial \psi_n}{\partial t} = E_n \psi_n + T_{n,n\pm1} [\psi_{n+1} + \psi_{n-1}]\]
Wave packet evolution

- Exciting a *single* site as an initial condition

Ordered lattice  Disordered lattice  Disordered lattice - averaged
Properties of disordered states in the 1d Anderson model:

Stationary states: \[ \lambda A_l = \epsilon_l A_l - A_{l-1} - A_{l+1} \]

Normal mode (NM) eigenvectors: \[ A_{\nu,l} \left( \sum_l A^2_{\nu,l} = 1 \right) \]

Eigenvalues: \[ \lambda_\nu \in \left[ -2 - \frac{W}{2}, 2 + \frac{W}{2} \right] \]

Width of EV spectrum: \[ \Delta_D = W + 4 \]

Asymptotic decay: \[ A_{\nu,l} \sim e^{-l/\xi(\lambda_\nu)} \]

Localization length: \[ \xi(\lambda_\nu) \leq \xi(0) \approx 100/W^2 \]

Localization volume of NM: \[ V \]

\[ V(W < 4) \approx 3\xi \quad V(W > 10) \approx 1 \]
What happens when we add nonlinear terms to the equations of motion?

- Eigenmodes of the linear equations can be continued as periodic orbits, however there are infinitely many resonances; still many modes stay localized, with frequencies inside or outside the spectrum of the lin. equations (S. Aubry, ’00, ’01)

- Finite sets of eigenmodes can be continued as quasiperiodic orbits as well, with similar properties as for periodic orbits (Wang/Bourgain ’08)

- All these statements are about manifolds of zero measure in phase space. What about the rest?

- Linear wave equations correspond to integrable dynamical systems

- Nonlinear terms will in general destroy integrability

- Will they also destroy localization?
Kolmogorov – Arnold – Moser (KAM) theory


Integrable classical Hamiltonian \( \hat{H}_0, d > 1 \):

Separation of variables: \( d \) sets of action-angle variables

\[ I_1, \theta_1 = 2\pi \omega_1 t; \ldots; I_2, \theta_2 = 2\pi \omega_2 t; \ldots \]

Quasiperiodic motion: set of the frequencies, \( \omega_1, \omega_2, \ldots, \omega_d \) which are in general incommensurate

Actions \( I_i \) are integrals of motion \( \partial I_i / \partial t = 0 \)

\[ \theta_1 \quad \times \quad \theta_2 \quad \times \ldots \Rightarrow \]

Q: Will an arbitrary weak perturbation \( V \) of the integrable Hamiltonian \( \hat{H}_0 \) destroy the tori and make the motion ergodic (when each point at the energy shell will be reached sooner or later)?

A: Most of the tori survive weak and smooth enough perturbations

KAM theorem
Each point in the space of the integrals of motion corresponds to a torus and vice versa

Finite motion. Localization in the space of the integrals of motion?

- KAM applies to finite systems
- Does it apply to waves in infinite systems?
- How are KAM thresholds scaling with number of degrees of freedom?
- Will nonlinear waves observe KAM regime?
- If they do – then localization remains
- If they do not – waves can delocalize

Some answers will be obvious from the next lecture …