Gravitational lensing: Models

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Basic notations

Gravitational lens equation in a dimensionless form

Remind GL equation

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\hat{\vec{\alpha}}(\vec{\xi}), \tag{1}$$

where $\vec{\eta}$ is a position of source, $\vec{\xi}$ is a position of image in the lens plane, D_d is a distance between an observer and lens, D_s is a distance between an observer and a source, D_{ds} is a distance between a source and lens.

If distances are much greater than lens sizes we use flat GL approximation, projecting bulk mass density onto the lens plane and as a result we have a surface mass density $\Sigma(\vec{\xi})$. Therefore we have the following relation for deflection angle

$$\hat{\vec{\alpha}} = \int_{R^2} \frac{4G\Sigma(\vec{\xi'})}{c^2} \frac{\vec{\xi} - \vec{\xi'}}{|\vec{\xi} - \vec{\xi'}|^2} d^2\xi', \tag{2}$$

where we integrate in the lens plane. Therefore, deflection angle is a superposition of deflection angle for mass elements $dm = \Sigma(\vec{\xi'})d^2\xi'$.

Rewrite Eqs.(1), (2) in dimensionless form. Denote characteristic distance in the lens plane ξ_0 and corresponding length in the source plane $\eta_0 = \xi_0 D_s / D_d$. Let us introduce dimensionless vectors $\vec{x} = \vec{\xi}/\xi_0$, $\vec{\xi} = \vec{\eta}/\eta_0$, and a dimensionless mass density

$$k(\vec{x}) = \frac{\Sigma(\xi_0 \vec{x})}{\Sigma_{cr}}, \tag{3}$$

where the critical mass density is

$$\Sigma_{cr} = \frac{c^2 D_s}{4\pi G D_d D_{ds}}. \tag{4}$$

Taking into account the notations we can rewrite gravitational lens equation in the form

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}), \tag{5}$$

where

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi}\int_{R^2} k(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^2} = \frac{D_d D_{ds}}{\xi_0 D_s}\hat{\vec{\alpha}}(\xi_0 \vec{x}). \tag{6}$$

Potential functions

It is easy to see that deflection angle may be represented as a gradient of a new function in respect to $\vec{x}$

$$\vec{\alpha} = \nabla \psi, \tag{7}$$

where

$$\psi(\vec{x}) = \frac{1}{\pi} \int_{R^2} \frac{4G\Sigma(\vec{\xi'})}{c^2} \frac{\vec{\xi} - \vec{\xi}}{|\vec{\xi} - \vec{\xi'}|^2} \tag{8}$$

is a logarithm potential associated with a surface mass density $k(\vec{x})$.

So the mapping $\vec{x} \mapsto \vec{y}$ is gradient one

$$\vec{y} = \nabla \left(\frac{1}{2} \vec{x}^2 - \psi(\vec{x}) \right) \tag{9}$$

(singularities of these mappings were completely classified by Arnold (1972,1974))

or if we introduce scalar function

$$\phi(\vec{x}, \vec{y}) = \frac{1}{2} (\vec{x} - \vec{y})^2 - \psi(\vec{x}), \tag{10}$$

$$\nabla\phi(\vec{x},\vec{y}) = 0, \tag{11}$$

where a gradient is taking in respect to variable $\vec{x}$. One can see that there is Laplace equation connecting functions ψ and k,

$$\Delta\psi = 2k, \tag{12}$$

where Laplace operator is taking in respect to $\vec{x}$.

Magnification, convergence and shear

Jacobian matrix

$$A(\vec{x}) = \frac{\partial \vec{y}}{\partial \vec{x}}, \quad A_{ij} = \frac{\partial y_i}{\partial x_j}, \tag{13}$$

Magnification

$$\mu(\vec{x}) = 1/\det\ A(\vec{x}). \tag{14}$$

So, an image of a distant point like source at a position $\vec{x}$ will be amplified (or demagnified) in $|\mu(\vec{x})|$ times. Magnification may be positive or negative and in this case corresponding images have positive or negative parity.

For some values $\vec{x}$ the determinant $\det A(\vec{x})$ may be vanishing (therefore $\mu(\vec{x})$ is infinity), and we call these points as critical points. An image of the critical set with a gravitational lens mapping is called as caustics. Clearly, that a point like approximation is not acceptable for these cases.

From Eqs. (13) and (10) we have

$$A_{ij} = \phi_{ij} = \delta_{ij} - \psi_{ij}, \tag{15}$$

where a partial derivative in respect to variable x_i is denoted with index

i. From Eq. (15) we have that the matrix A is symmetrical one.

Using Eq. (12), we obtain that the Jacobian matrix may be written in the following form

$$A = \begin{pmatrix} 1 - k - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - k + \gamma_1 \end{pmatrix}, \tag{16}$$

where

$$\gamma_1 = (\psi_{11} - \psi_{22})/2, \; \gamma_2 = \psi_{12} = \psi_{21}. \tag{17}$$

Therefore, we have the following expressions for a determinant and trace

A:

$$\det A = (1-k)^2 - \gamma^2, \tag{18}$$

$$\mathrm{tr}\ A = 2(1-k). \tag{19}$$

For eigenvalues of matrix A we have

$$a_{1,2} = 1 - k \pm \gamma. \tag{20}$$

In the last relation (20) γ means

$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}. \tag{21}$$

$1 - k$ is called a convergence or Ricci-focusing, γ is a shear.

General properties of symmetric lenses

Deflection angle

Let us consider a family of circular symmetric mass density distributions $\Sigma(\vec{\xi}) = \Sigma(|\vec{\xi}|)$.

Therefore, we have for a deflection angle

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi} \int_{R^2} d^2x' k(\vec{x}) \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^2}, \tag{22}$$

where $k(\vec{x}) = \sigma(\xi_0 \vec{x})/\Sigma_{cr}, \Sigma_{cr} = c^2 D_s/(4\pi G D_d D_{ds})$.

Since we have a circular symmetrical case we can select a positive direction for axis x_1, and $\vec{x} = (x, 0), x \geq 0$.

Introducing polar coordinates we have $\vec{x}' = x'(\sin\varphi, \cos\varphi)$, and therefore $k(x') := k(\vec{x}')$. Using an evident relation for the Jacobian for the transformation $d^2x' = x'dx'd\varphi$,

we obtain

$$\alpha_1(x) = \frac{1}{\pi}\int_0^\infty x'dx'k(x')\int_0^{2\pi} d\varphi\frac{x - x'\cos\varphi}{x^2 + {x'}^2 - 2xx'\cos\varphi}, \tag{23}$$

$$\alpha_2(x) = \frac{1}{\pi}\int_0^\infty x'dx'k(x')\int_0^{2\pi} d\varphi\frac{-x'\sin\varphi}{x^2 + {x'}^2 - 2xx'\cos\varphi}. \tag{24}$$

One can see

$$\int_0^{2\pi} d\varphi \frac{-x' \sin\varphi}{x^2 + x'^2 - 2xx' \cos\varphi} = 0. \tag{25}$$

Therefore $\vec{\alpha}$ is parallel to $\vec{x}$. Therefore vector $\vec{y}$, which determines a source position is also parallel to vector $\vec{x}$. The integral may be evaluated with a complex analysis technique (residues)

$$I = \int_0^{2\pi} R(\cos\varphi, \sin\varphi) d\varphi, \tag{26}$$

for instance, (for $|a| < 1$)

$$\int_0^{2\pi} \frac{d\varphi}{1 + a\cos\varphi} = \frac{2\pi}{\sqrt{1 - a^2}} \tag{27}$$

Therefore, one can see

$$\int_0^{2\pi} d\varphi \frac{x - x'\cos\varphi}{x^2 + x'^2 - 2xx'\cos\varphi} = \begin{cases} 0, & for \quad x' > x, \\ 2\pi/x, & for \quad x' < x. \end{cases} \tag{28}$$

Therefore from Eq. (23) we obtain

$$\alpha(x) := \alpha_1(x) = \frac{2}{x}\int_0^x x'dx'k(x') = \frac{m(x)}{x}, \tag{29}$$

where we introduce a definition

$$m(x) := 2\int_0^x x'dx'k(x'),$$

it means dimensionless mass inside a circle with a radius x.

Remind that we have the following relation between normalized and non-normalized vectors $\vec{\alpha}$ and $\hat{\vec{\alpha}}$

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{\xi_0 D_s}{D_d D_{ds}} \vec{\alpha} \left(\frac{\vec{\xi}}{\xi_0} \right). \tag{30}$$

Therefore using relation (29), we have

$$\hat{\alpha}(\xi) = \frac{1}{\xi} \frac{4G}{c^2} 2\pi \int_0^{\xi} \xi' d\xi' \Sigma(\xi') = \frac{4GM(\xi)}{c^2 \xi}, \tag{31}$$

where we introduce the following definition for mass inside circle with a radius ξ:

$$M(\xi) := 2\pi \int_0^{\xi} \xi' d\xi' \Sigma(\xi').$$

From Eq. (31) we can see that the deflection angle coincides with the Einstein angle for mass $M(\xi)$ inside a circle with a radius ξ.

Therefore, we obtain a scalar gravitational lens equation for circular symmetrical case

$$y = x - \alpha(x) = x - m(x)/x, \tag{32}$$

where $x \in R$, $m(x) := m(|x|)$.

Taking into account the symmetry we can restrict our consideration with a region $y \geq 0$. Since $m(x) \geq 0$, from Eq. (32) we obtain $x \geq y$ for any positive solution x, but for any negative solution x one has to have inequality $-m(x)/x > y$.

Deflection potential ψ and Fermat's potential

For a defection potential we have assuming that $x \geq 0$

$$\psi(x) = \frac{1}{\pi} \int_0^\infty x' dx' k(x') \int_0^{2\pi} d\varphi \ln \sqrt{x^2 + x'^2 - 2xx' \cos\varphi}. \qquad (33)$$

We can obtain

$$\psi(x) = 2 \int_0^x x' dx' k(x') \ln\left(\frac{x}{x'}\right). \qquad (34)$$

Differentiating (34) in respect to x we obtain $\alpha(x) = d\psi(x)/dx$

For Fermat's potential $\phi(x, y)$ we have

$$\phi(x, y) = \frac{1}{2}(x - y)^2 - \psi(x), \tag{35}$$

therefore GL equation is equivalent to the following equation

$$\partial\phi/\partial x = 0. \tag{36}$$

Schwarzschild lens (reminding)

A point like mass M is located in the origin $\vec{\xi} = 0$. Therefore, surface mass density

$$\Sigma(\vec{\xi}) = M\delta^2(\vec{\xi}).$$

A natural length scale is EC radius

$$\xi_0 = \sqrt{2R_S \frac{D_s D_{ds}}{D_d}}.$$

Therefore $m(x) = 1$, and GL equation has the following form

$$y = x - 1/x,$$

which has two solutions:

$$x_{1,2} = \left(y \pm \sqrt{y^2 + 4} \right) /2, \tag{37}$$

so, there two images from both sides of the lens.

Magnification for an image at point x,

$$\mu = \left(1 - 1/x^4 \right)^{-1}.$$

Substituting the solutions of GL Eq. we obtain

$$\mu_{1,2} = \pm\frac{1}{4}\left(\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} \pm 2\right). \tag{38}$$

A total magnification is

$$\mu_p = \mu_1 - \mu_2 = \frac{y^2+4}{y\sqrt{y^2+4}}. \tag{39}$$

From Eq.(33) we have

$$\psi = \ln x, \tag{40}$$

Therefore a time delay for these images

$$\Delta t = \frac{4GM}{c^2}(1 + z_d)\tau(y), \tag{41}$$

where

$$\tau(y) = \frac{1}{2}y\sqrt{y^2 + 4} + \ln \frac{\sqrt{y^2 + 4} + y}{\sqrt{y^2 + 4} - y}. \tag{42}$$

Since two images have comparable brightness only if $y \leq 1$, , so $\tau(1) \approx 2.08$, so the time delay (41) is about a time to intersect EC ring.

For a source with an uniform surface density with a radius R we have that a maximal amplification factor is

$$\mu_{max} = \sqrt{4 + R^2}/R. \tag{43}$$

Singular Isothermal Sphere (SIS)

For mass distribution in galaxies and galactic clusters people use Singular Isothermal Sphere (SIS). (The model fits flat rotation curves).

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}, \tag{44}$$

where σ_v^2 is velocity dispersion. Therefore, a surface mass density is

$$\Sigma(\xi) = 2\int_0^{+\infty} \rho(\sqrt{\xi^2 + h^2})dh = \frac{\sigma_v^2}{2G\xi}, \tag{45}$$

and a deflection angle is

$$\hat{\alpha} = 4\pi\sigma_v^2/c^2. \tag{46}$$

The model has two features which have to be taken into account. First, the infinite density at $\xi = 0$, but a mass is finite in any finite volume.

Second, a total mass is infinity, but if we consider images with impact parameters $|\xi| < R$, then axial symmetrical distribution of mass with $|\xi| > R$ may be ignored.

Choosing the length scale factor

$$\xi_0 = 4\pi \frac{\sigma_v^2}{c^2} \frac{D_{ds}}{D_s}, \tag{47}$$

we have

$$k(x) = 1/2x, \quad \alpha(x) = x/|x|. \tag{48}$$

In this case GL Eq. has the following form

$$y = x - x/|x|. \tag{49}$$

We can take $y > 0$ (without losing the generality) because we are free to choose a suitable coordinate system: for $y < 1$ we have two solution $x = y + 1 \;\; x = y - 1$, therefore these solutions are located from an opposite sides of GL, for $y > 1$ we have only one solution $x = y + 1$.

The magnification at a point x is determined by

$$\mu = |x|/(|x| - 1). \tag{50}$$

The circumference $|x| = 1$ is a tangent critical curve. From the relation a shear we have $\gamma(x) = k(x) = 1/(2x)$, therefore a relative stretching in a tangent direction is $|\mu|$, meanwhile it is no stretching (or squeezing in radial direction).

A total amplification is

$$\mu_p = \begin{cases} 2/y & y \leq 1 \\ (1+y)/y & y \geq 1 \end{cases} . \tag{51}$$

For $y \to 1$ the second image started to be fainter.

For a deflection potential we have $\psi(x) = |x|$, and time delay is

$$c\Delta t = \left[4\pi \left(\frac{\sigma_v}{c}\right)^2\right] \frac{D_d D_{ds}}{D_s}(1 + z_d)2y. \tag{52}$$

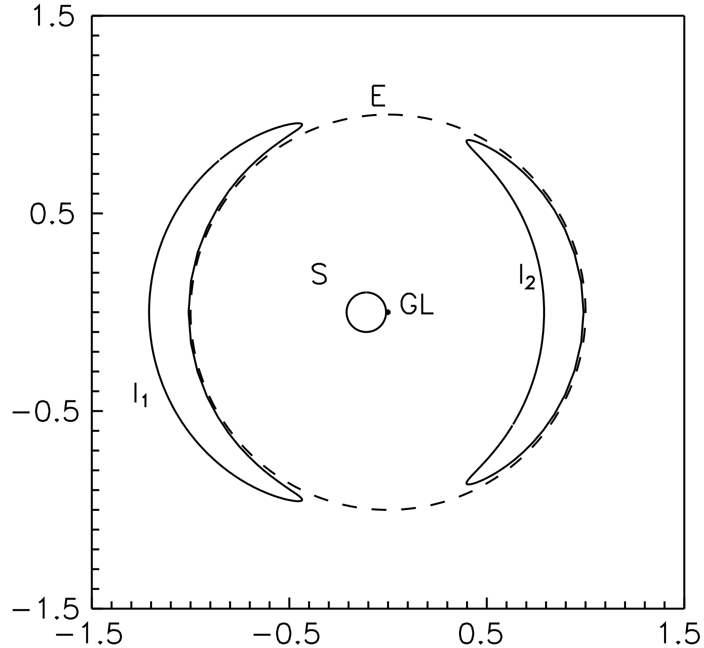

Figure 1: Image of a circular source for the transparent lens. Radius of source $r = 0.1$, impact parameter $y = 0.11$. It is clear that radius of source is the same as widths of images in radial direction.

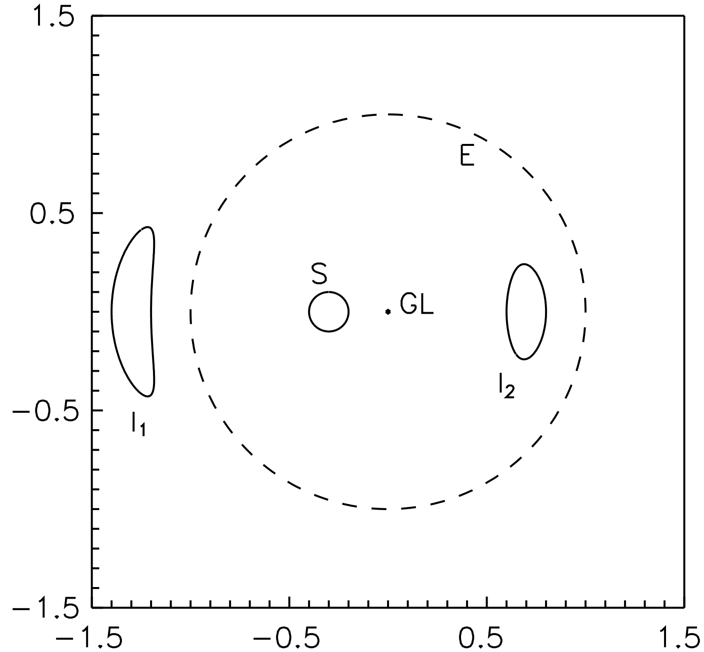

Figure 2: Image of a circular source for the transparent lens. Radius of source $r = 0.1$, impact parameter $y = 0.3$. It is also clear that radius of source is the same as widths of images in radial direction.

Softened isothermal sphere or isothermal sphere with a core (ISC).

For this model we have no infinite density at the origin since we have a core with a radius r_c, and this model is more realistic one.

$$\rho(r) = \frac{\sigma_v^2}{2\pi G(r^2 + r_c^2)}, \tag{53}$$

so the model (53) coincides with SIS (44) for $r_c = 0$ or $r \gg r_c$. Therefore, a surface mass density and a total mass are determined by

$$\Sigma(\xi) = \frac{\sigma_v^2}{2\pi G\sqrt{\xi^2 + r_c^2}}, \tag{54}$$

$$m(\xi) = \frac{\sigma_v^2}{G}(\sqrt{\xi^2 + r_c^2} - r_c). \quad (55)$$

Introducing variables $x = \xi/r_c, y = (\eta/r_c)(D_d/D_s)$, we have GL equation

$$y = x - D(\sqrt{1+x} - 1)/x. \quad (56)$$

A parameter $D := (4\pi\sigma_v^2/c^2)(D_d D_{ds}/r_c D_s)$, defines a number of solutions.

So for $D \leq 2$ we have only one solution, for $D > 2$ we have three solutions if y is relatively small.

Solutions may be found as intersections of line $y =$const with a cure $y = x - D(\sqrt{1+x^2} - 1)/x$.

For a magnification

$$\mu = \left| \left(1 - D\frac{\sqrt{1+x^2}-1}{x^2} \right) \left(1 + D\frac{\sqrt{1+x^2}-1}{x^2} - D\frac{1}{\sqrt{1+x^2}} \right) \right|^{-1} . \tag{57}$$

A qualitative analysis of the gravitational lens equation

We will show that gravitational lens equation has only one solution if $D < 2$ and have three solutions if $D > 2$ and $y > y_{cr}$ (we consider gravitational lens equation for $y > 0$), where y_{cr} is a local maximal value of right hand of Eq. (56). It is possible to show that we determine the value x_{cr} which corresponds to y_{cr} using the following expression

$$x_{cr}^2 = \frac{2D - 1 - \sqrt{4D+1}}{2}, \tag{58}$$

It is easy to see that according to (58) $x_{cr}^2 > 0$ if and only if $D > 2$.

$$y_{cr} = x_{cr} - D\frac{\sqrt{1 + x_{cr}^2} - 1}{x_{cr}}, \tag{59}$$

If we choose $x_{cr} < 0$ then $y_{cr} > 0$. So, if $D \le 2$ then gravitational lens equation has only one solution for ($y > 0$), if $D > 2$ then gravitational lens equation has one solution (if $y > y_{cr}$), three distinct solutions (if $y < y_{cr}$), one single solution and one double solution (if $y = y_{cr}$).

It is possible to show that gravitational lens equation is equivalent to the following equation

$$x^3 - 2yx^2 - (D^2 - y^2 - 2D)x - 2yD = 0, \tag{60}$$

jointly with the inequality

$$x^2 - yx + D > 0. \tag{61}$$

Thus it is possible to obtain the analytical solutions of gravitational lens equation by the well-known way. We perform $z = x - \frac{2y}{3}$ and obtain incomplete equation of third degree

$$z^3 + pz + q = 0, \tag{62}$$

where $p = 2D - D^2 - \frac{y^2}{3}$ and $q = \frac{2y}{3}\left(\frac{y^2}{9} - D(D+1)\right)$, so we have the following expression for the discriminant

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = \frac{D^2}{27}\left[-y^4 + y^2(2D^2 + 10D - 1) + D(2-D)^3\right]. \tag{63}$$

If $Q \geq 0$ then Eq. (62) has unique real solution (therefore the gravitational lens equation (56) has unique real solution). We use Cardan expression for the solution

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} + \sqrt[3]{-\frac{q}{2} - \sqrt{Q}} + \frac{2y}{3}. \tag{64}$$

We suppose the case $D > 2$. If $y > y_{cr}$ then the gravitational lens equation has unique solution. If $Q \geq 0$ then we use the expression (64) for the solution. If $Q < 0$ then we have the following expression

$$x = 2\sqrt{-\frac{p}{3}}\cos\frac{\alpha + 2k\pi}{3} + \frac{2y}{3}, \quad (k = 0, 1, 2) \tag{65}$$

where

$$\cos\alpha = -\frac{q}{2\sqrt{-\left(\frac{p}{3}\right)^3}}, \tag{66}$$

and we select only one solution which corresponds to the inequality (61) which corresponds to $k = 0$ in (65) because if the gravitational lens equation has only one solution then we have a positive solution x for a positive value of impact parameter y therefore there is the inequality $x > y$ which is easy to see from (59). It is possible to check that maximal solution of (60) corresponds to $k = 0$ therefore the solution is the solution of (59).

If $y < y_{cr}$ then the gravitational lens equation has three distinct solutions and we use the Eqs. (65-66) to obtain the solutions.

We consider now the case $D < 2$. We know that the gravitational lens equation has unique solution for the case. If $Q \geq 0$ then we use the expression (64) for the solution. If $Q < 0$ then we have the following expressions (65-66) and we select only one solution which corresponds to the inequality (61) which also corresponds to $k = 0$ as in the previous case.

It is known that magnification for gravitational lens solution x_k is defined by the following expression

$$\mu_k = \left| \left(1 - \frac{D(\sqrt{1+x^2} - 1)}{x} \right) \left(1 + D\frac{\sqrt{1x^2} - 1}{x^2} - D\frac{1}{\sqrt{1+x^2}} \right) \right|, \quad (67)$$

so the total magnification is equal

$$\mu_{\rm tot}(y) = \sum \mu_k, \tag{68}$$

where the summation is taken over all solutions of gravitational lens equation for a fixed value y.

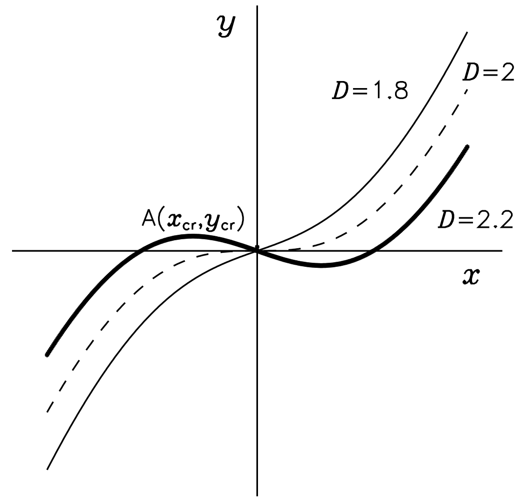

Figure 3: The right hand side of the gravitational lens equation for different values of the parameters $D = 1.8, 2, 2.2$.

Singularities

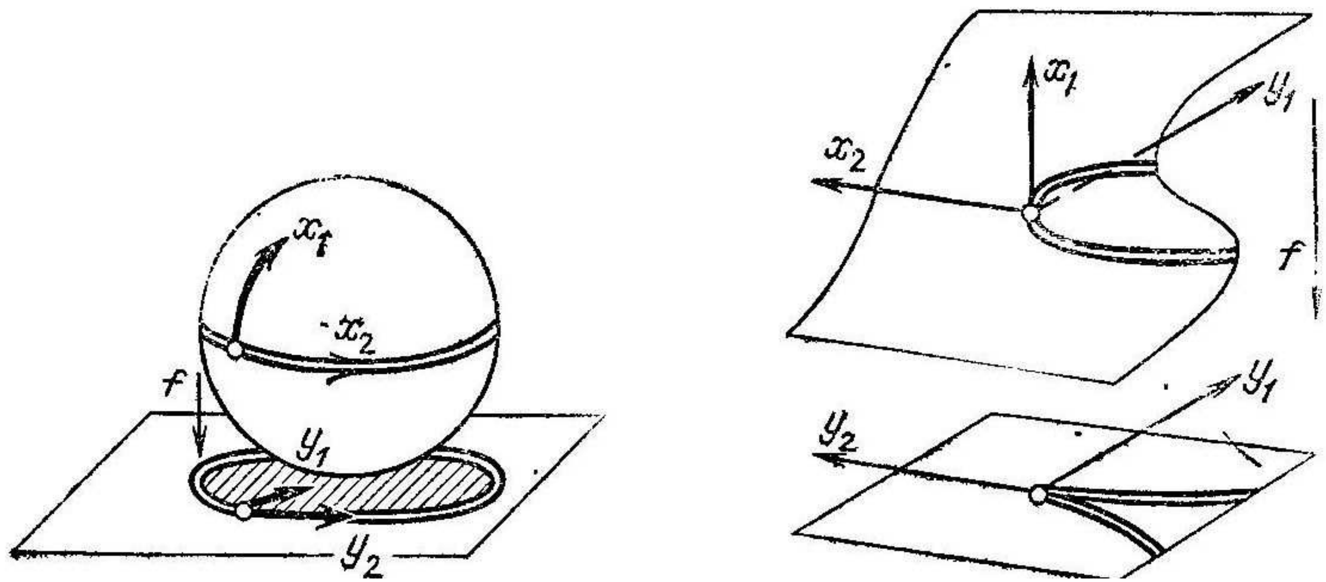

Figure 4: Singularities. Fold (left) and Cusp (right) . V. I. Arnold's drawing.

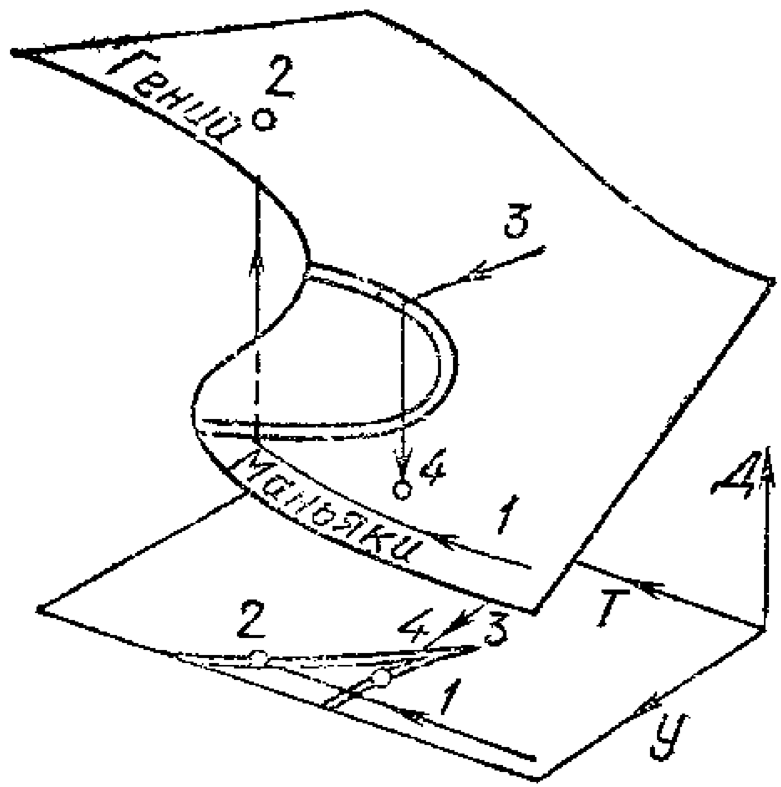

Figure 5: Zeeman's sociological model (the catastrophe machine).

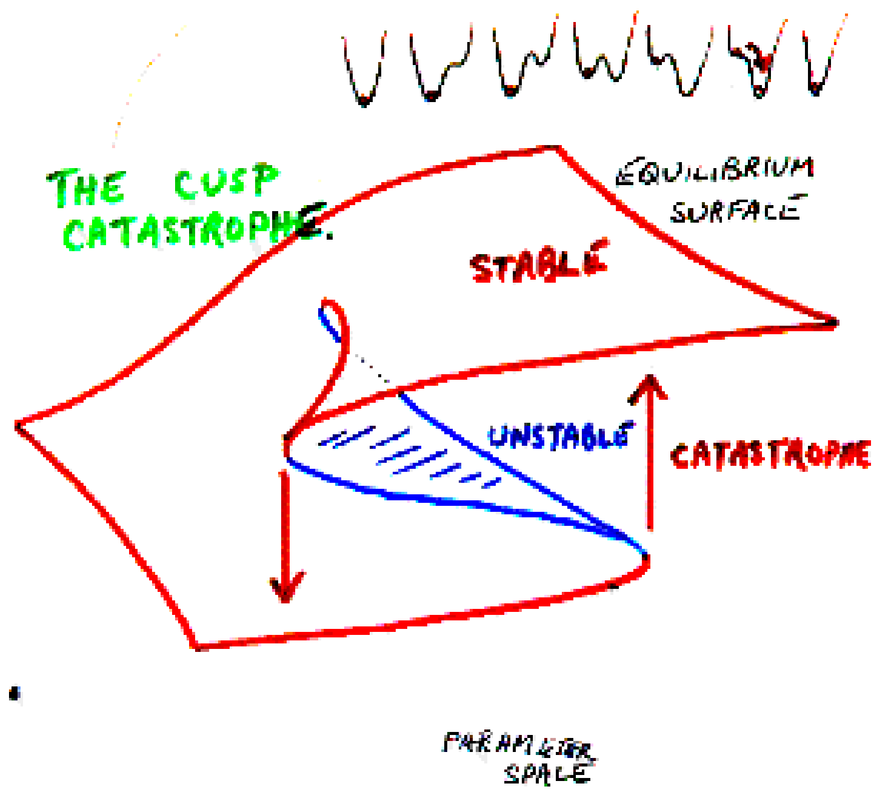

Figure 6: Cusp type singularity. Rene Thom (1923-) drawing.