Gravitational lensing: Schwarzschild lens

Alexander F. Zakharov

February 9, 2009

Basic notations of gravitational lens theory. Point like lens case

D_s is a distance between a source and observer, D_d is a distance between a lens and observer D_{ds}. A plane passing through a gravitational lens (in perpendicular direction to a vector between an observer and a lens) is called the gravitational lens plane, similarly a parallel plane passing through source is called as a the source plane. Thus we have gravitational lens equation

$$\vec{\eta} = D_s\vec{\xi}/D_d - D_{ds}\vec{\Theta}(\vec{\xi}), \tag{1}$$

where vectors $\boldsymbol{\eta}, \boldsymbol{\xi}$ are determined coordinates in source and lens plane respectively but an angle is determined by the following relation for the point-like lens model (the Schwarzschild lens case)

$$\vec{\Theta}(\vec{\xi}) = 4GM\vec{\xi}/c^2\xi^2. \tag{2}$$

Vanishing the r.h.s (1) we obtain conditions when a source, lens and an observer are on one straight line ($\boldsymbol{\eta}= \mathbf{0}$). We have a corresponding length in the lens plane $\xi_0 = \sqrt{4GMD_dD_{ds}/(c^2D_s)}$. ξ_0 is called Einstein–Chwolson radius. One can calculate the Einstein–Chwolson angle which is $\theta_0 = \xi_0/D_d$. Let us calculate θ_0 values for some standard cases. If we assume $D_s \gg D_d$, then

$$\theta_0 \approx 2'' \times 10^{-3} \left(\frac{GM}{M_\odot}\right)^{1/2} \left(\frac{kpc}{D_d}\right)^{-1/2}.$$

If a lens is one of the closest galaxies with a mass $M = 10^{12}M_\odot$, at a distance $D_d = 100$ kpc, then $\theta_0 \approx 200''$. If a lens is at a distance 1 kpc from observer in our Galaxy and a lens has a mass $M = M_\odot$, then $\theta_0 \approx 2'' \times 10^{-3}$.

In dimensionless variables we have

$$\vec{x} = \vec{\xi}/\xi_0, \quad \vec{y} = D_s\vec{\eta}/(\xi_0 D_d),$$
$$\vec{\alpha} = \vec{\Theta} D_{ds} D_d/(D_s \xi_0) \tag{3}$$

then a gravitational lens equation has a form

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}) \tag{4}$$

or

$$\vec{y} = \vec{x} - \vec{x}/x^2 \tag{5}$$

Solving it in respect to $\boldsymbol{x}$, we obtain

$$\vec{x^{\pm}} = \vec{y}[1/2 \pm \sqrt{1/4 + 1/y^2}]. \tag{6}$$

Therefore a distance between images may be easy obtained.

$$x^{+} = y\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{y^2}}\right], \tag{7}$$

$$x^{-} = y\left[-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{y^2}}\right], \tag{8}$$

$$l = x^{+} + x^{-} = 2y\sqrt{\frac{1}{4} + \frac{1}{y^2}}, \tag{9}$$

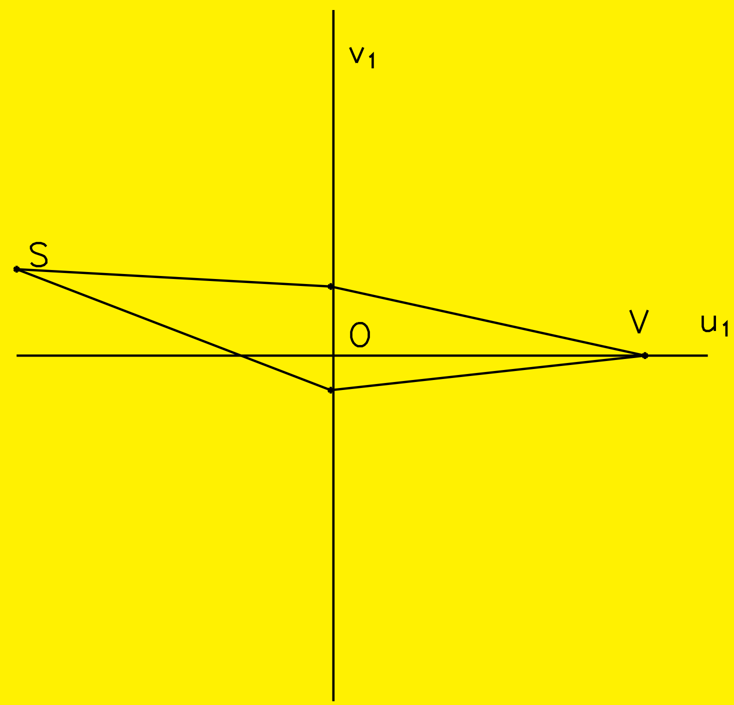

Figure 1: A point like lens is located in the origin (0,0). A point S is a source position, V is an observer position.

It is easy to understand why we have two images for $\Theta \sim 1/\Delta$ (Δ is an impact parameter).

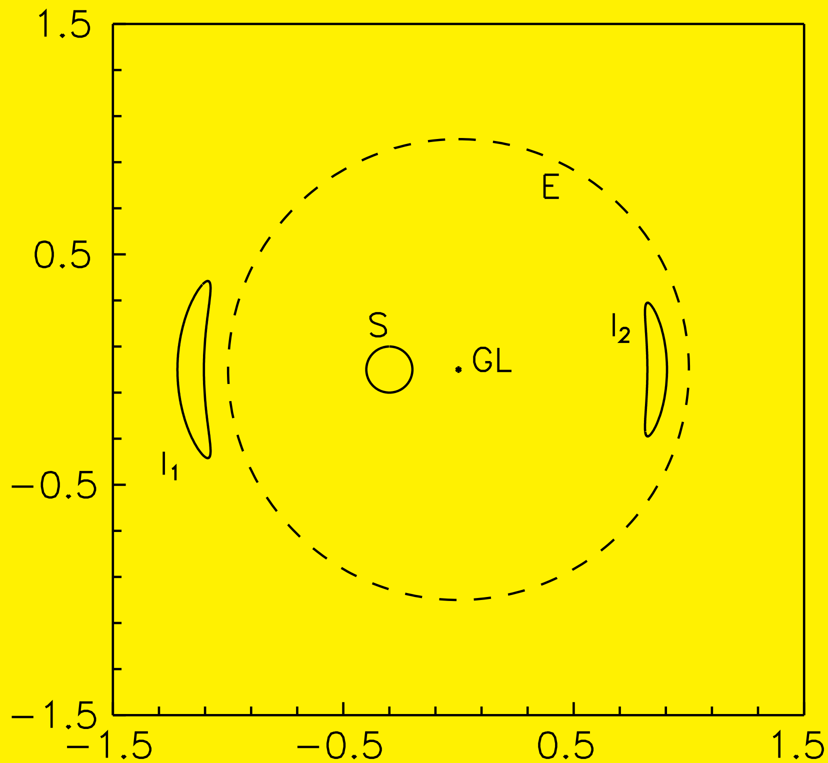

Figure 2: A circular source and its images generated by a Schwarzschild lens. An unit circumference with a center at (0,0) is a Schwarzschild —Chwolson ring. A source radius is $r = 0.1$, a distance between the center of a source and a lens position is $d = 0.3$.

If $y \ll 1$, then $l \approx 2$, so a distance between images is about the Schwarzschild —Chwolson diameter. Assume we have a small source near the origin $y \ll 1$. We have two images (x_+ is outside the ring, x_- is inside). Images are stretching (in roughly $1/y$ rimes).

Let us study changes of source size in radial direction if source is located at $x-$axis.

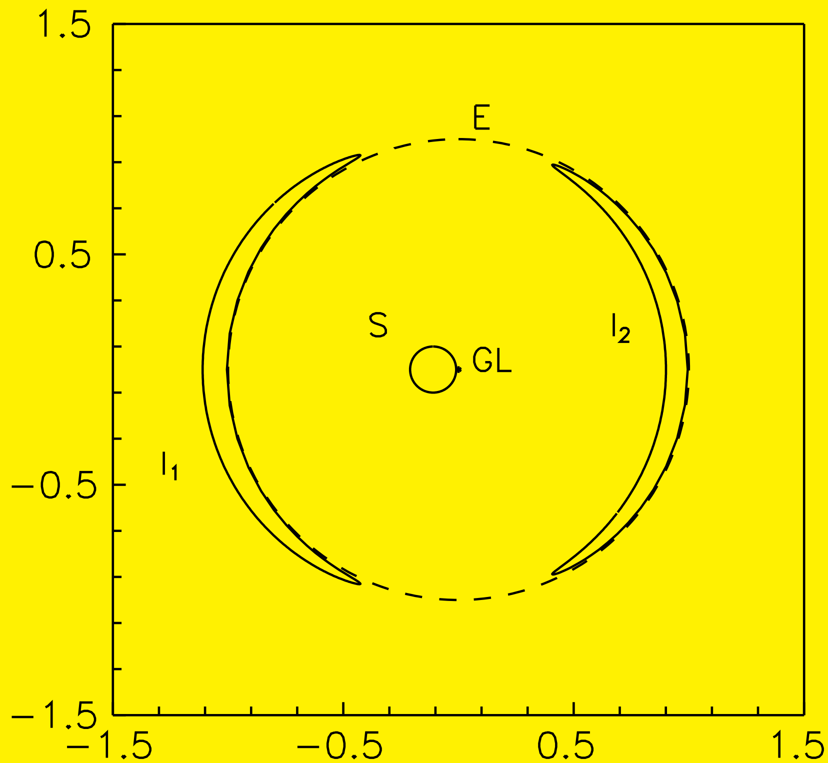

Figure 3: $d = 0.11$.

For $y \to 0$

$$\frac{dx^{\pm}}{dy} = \frac{1}{2} \pm \frac{y}{\sqrt{4+y^2}} = \frac{1}{2} \pm \frac{y}{2} + O(y^2), \tag{10}$$

so images are squeezing roughly in two times in comparison with a source.

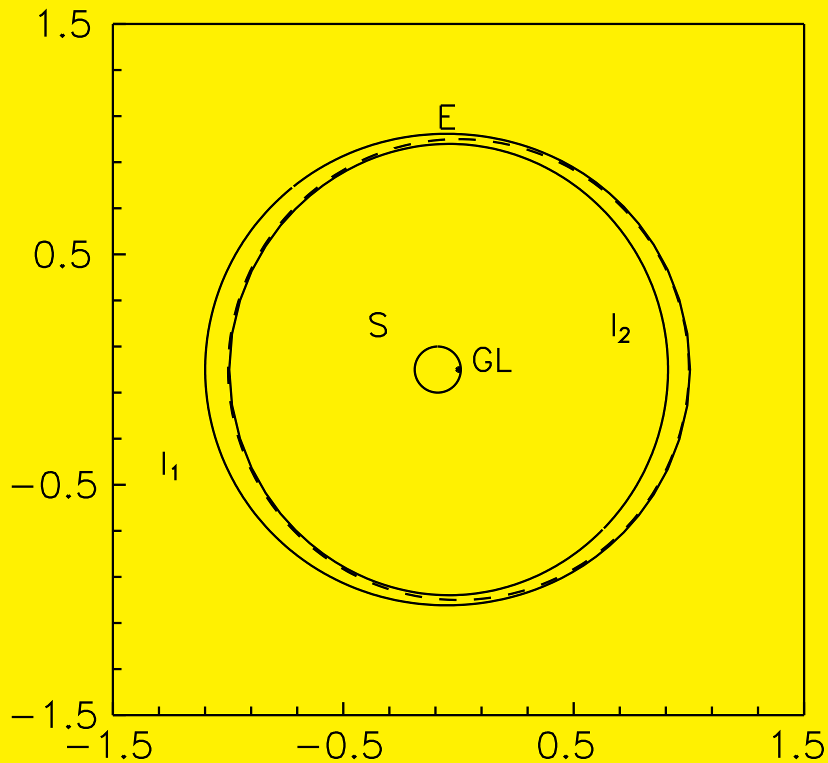

Figure 4: $d = 0.09$.

Let us evaluate amplification for the gravitational lens model. If the A is more than 1 we are speaking about gravitational lensing (focusing). Let us introduce angular variables, namely $\boldsymbol{\theta} = \boldsymbol{\xi}/D_0, \quad \boldsymbol{\beta} = \boldsymbol{\eta}/D_s,$ therefore a ratio between a solid angle of images and sources is amplification factor.

$\Delta\omega_0$ is a solid angle for a source, $\Delta\omega$ is a solid angle for images

$$\mu = \frac{\Delta\omega}{\Delta\omega_0} = |\det \frac{d\vec{\beta}}{d\vec{\theta}}|^{-1} = |\det \frac{d\vec{y}}{d\vec{x}}|^{-1} \tag{11}$$

since $\quad \boldsymbol{y} = \boldsymbol{\beta}/\theta_0, \quad \boldsymbol{x} = \boldsymbol{\theta}/\theta_0.$

We choose coordinates that $x_2 = y_2 = 0$.

Evaluating Jacobians $\boldsymbol{x}_\pm \mapsto \boldsymbol{y}$ (in the point $\boldsymbol{y} = (y_1, 0)$),

we will denote such as $\mu_\pm = |\det d\boldsymbol{x}^\pm/d\,\boldsymbol{y}\,|$

In this case only diagonal terms are non-vanishing, namely,

$$\frac{\partial {x_1}^{\pm}}{\partial y_1}\,|_{(y_1,0)} = \frac{1}{2}\left(1 \pm \frac{y_1}{\sqrt{4+{y_1}^2}}\right), \tag{12}$$

$$\frac{\partial {x_2}^{\pm}}{\partial y_2}\,|_{(y_1,0)} = \frac{1}{2}\left(1 \pm \frac{\sqrt{4+{y_1}^2}}{y_1}\right), \tag{13}$$

So,

$$\mu_{\pm} = \frac{1}{4}\left(\frac{y_1}{\sqrt{4+{y_1}^2}} + \frac{\sqrt{4+{y_1}^2}}{y_1} \pm 2\right), \tag{14}$$

$$\tag{15}$$

Clearly, that a ratio of amplifications is determined by the relation

$$\frac{\mu_+}{\mu_-} = \left(\frac{\sqrt{4 + {y_1}^2} + y_1}{\sqrt{4 + {y_1}^2} - y_1} \right)^2 , \tag{16}$$

(17)

Let us consider $\mu_\pm$ and their ratios for $y_1 \to 0$

$$\mu_+ = \frac{1}{2y_1} + \frac{1}{2} + O(y_1), \tag{18}$$

$$\mu_- = \frac{1}{2y_1} - \frac{1}{2} + O(y_1), \tag{19}$$

$$\frac{\mu_+}{\mu_-} = 1 + y_1 + O({y_1}^2). \tag{20}$$

In another limit $y_1 \to \infty$ we have

$$\mu_+ = 1 + {y_1}^{-4} + O({y_1}^{-6}), \tag{21}$$

$$\mu_- = {y_1}^{-4} + O({y_1}^{-6}), \tag{22}$$

$$\frac{\mu_+}{\mu_-} = {y_1}^{4} + O({y_1}^{0}). \tag{23}$$

Warning: One has to use the Schwarzschild model in a careful way, since in lensing by stars radius of star has to be smaller than x_-, otherwise the fainter image will disappear.

For gravitational lensing with galaxies transparent gravitational lens models are better.

A more simple explanation

It is possible to evade formal and cumbersome calculations if we use the following assumption for a simplification of our analysis. We suppose that the shape of any image is an ellipse (really the shapes of the images are more complex figures, for example, for great amplifications the shapes remind "Moon's crescents", although behind a gravitational lens the source image is a circle). Both I_1 and I_2 are slightly squeezed along a line which connects $I_1\,D\,I_2$ and the images are extended in the perpendicular direction. The ellipse square is determined by the following expression $\Omega = \pi ab$, where a and b are the major and minor semiaxises respectively. Using proportionality relations we find semiaxises a and b through parameters of a lens and a source. So, the square of the first image is determined (in steradians) by the following expression

$$\Omega_1 = \pi\varphi_s^2 \cdot \left(\frac{1}{2} + \frac{u}{4} + \frac{1}{4u}\right)$$

and a square of the second image by the expression

$$\Omega_2 = \pi\varphi_s^2 \cdot \left(-\frac{1}{2} + \frac{u}{4} + \frac{1}{4u}\right),$$

where $u = \sqrt{1 + 4 \cdot \theta_0^2\theta^{-2}}$ is an auxiliary variable. Sizes and luminosities of the images are different, but their total luminosity is greater than the luminosity of an unlensed source S, actually

$$\Omega_1 + \Omega_2 = \frac{1}{2}\pi\varphi_s^2(u + \frac{1}{u}) > \pi\varphi_s^2. \tag{24}$$

Really, the discovery of microlensing effects is based on the usage of the property. Since photons belonging to different spectral bands are deflected identically in the gravitational field of a body D, therefore the amplification is the same for different spectral bands.

The sum of luminosity of two images divided by undistorted luminosity is called amplification factor and usually is denoted as A, so

$$A = \frac{\Omega_1 + \Omega_2}{\pi \varphi_s^2}.$$

If the impact parameter for a source S is small that is defined formally as $\theta << \theta_0$, then the parameter u is great and an amplification factor is much greater than unity, or $A = \dfrac{\theta_0}{\theta} >> 1$.

If $\theta \to 0$ or the source centre S, an observer O and a gravitational lens D lie on a straight line, a total luminosity formally tend to infinity as $\Omega_1 + \Omega_2 \to \infty$. We obtain the incorrect conclusion since we use the incorrect assumption about shapes of images for small impact parameters and approximate calculations of the semi-axis a and b.

If $\theta \to 0$ or the source centre S, an observer O and a gravitational lens D lie on a straight line, the gravitational lens will form an image as a luminous ring or so-called "Einstein's ring". It has a radius θ_0, a length $2\pi\theta_0$ and a depth:

$$2\phi_s \left.\frac{d\theta_1}{d\theta}\right|_{\theta_0} = \phi_s$$

which is equal to a source radius. In other words, a solid angle Ω corresponding to the ring on the celestial sphere is equal to $2\pi\theta_0\phi_s$. The amplification factor is equal to $2\pi\frac{\theta_0}{\phi_s}$ for this case.