Gravitational lensing: Introduction

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Because of the analogy between the propagation of rays of light and the motion of bodies, I thought it not amiss to add the following Propositions for optical uses; not considering the nature of the rays of light, or inquiring whether they are bodies or not, but only determining the trajectories of bodies, which are extremely similar to the trajectories of rays. (Isaac Newton, Principia, 1687)

It is well-known that a propagation of light is describing by general relativity equations, however light deflection was discussed by Newton in 1704 in the "Optics". Newton wrote "Do not Bodies act upon Light at a distance, and by their action bend its Rays; and is not this action strongest at the least distance?"

$$\Theta = \frac{2GM}{c^2R}, \tag{1}$$

or more precisely, $\Theta \sim R^{-1}$, since at Newton's time dimensional quantities were not written (it was no generally accepted system of units like SI or SGS). Moreover, in Principia, Newton proved that if

$$\vec{a} = \ddot{\vec{r}} \sim \frac{\vec{R}}{R^3}, \tag{2}$$

in this case a motion may be only conical sections such as parabola, ellipse or hyperbola. Since Newton calculated semi-axis of hyperbola, he knew the angle between its asymptotes.

Therefore, clearly that Newton knew Eq. (2), moreover dimensional constants were known as well, since Giovanni Domenico Cassini, fr. Jean-Dominique Cassini (8 June 1625 14 September 1712, born in Perinaldo, near San-Remo, at that time in the Republic of Genoa) sent his colleague Jean Richer to Cayenne, French Guiana to measure simultaneously the Mars position (the parallax technique), thus Cassini evaluated a distance between Sun and the Earth (AU). The Cassini estimate was $a = AU = 146 * 10^6 \text{ km}$, but the modern quantity is $a = AU = 149,6 * 10^6\mathrm{km}$, so he improved a precision in about 10 times.

Problem. Evaluate parallax angle for these measurements.

Roemer estimated that light would take about 22 minutes to travel a distance equal to the diameter of Earth's orbit around the Sun: this is equivalent to about 220,000 kilometers per second in modern units, about 26% lower than the true value. While the exact details of Roemer's calculations have been lost, the error is probably due to to an error in the orbital elements of Jupiter, leading Roemer to believe that Jupiter was closer to the Sun than is actually the case. Roemer's theory was controversial at the time he announced it, and he never convinced the director of the Royal Observatory, Giovanni Domenico Cassini, to fully accept it. However, it quickly gained support among other natural philosophers of the period, such as Christian Huygens and Isaac Newton.

Figure 1: Giovanni Domenico Cassini

Figure 2:

Figure 3: Ole Roemer (16441710), depicted here some time after his discovery of the speed of light (1676), at a time when he was already a statesman in his native Denmark. The engraving is probably posthumous.

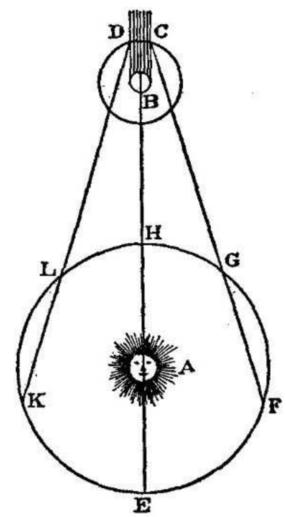


Fig. 70.

Figure 4: A redrawn version of the illustration from the 1676 news report. Roemer compared the apparent duration of Io's orbits as Earth moved towards Jupiter (F to G) and as Earth moved away from Jupiter (L to K).

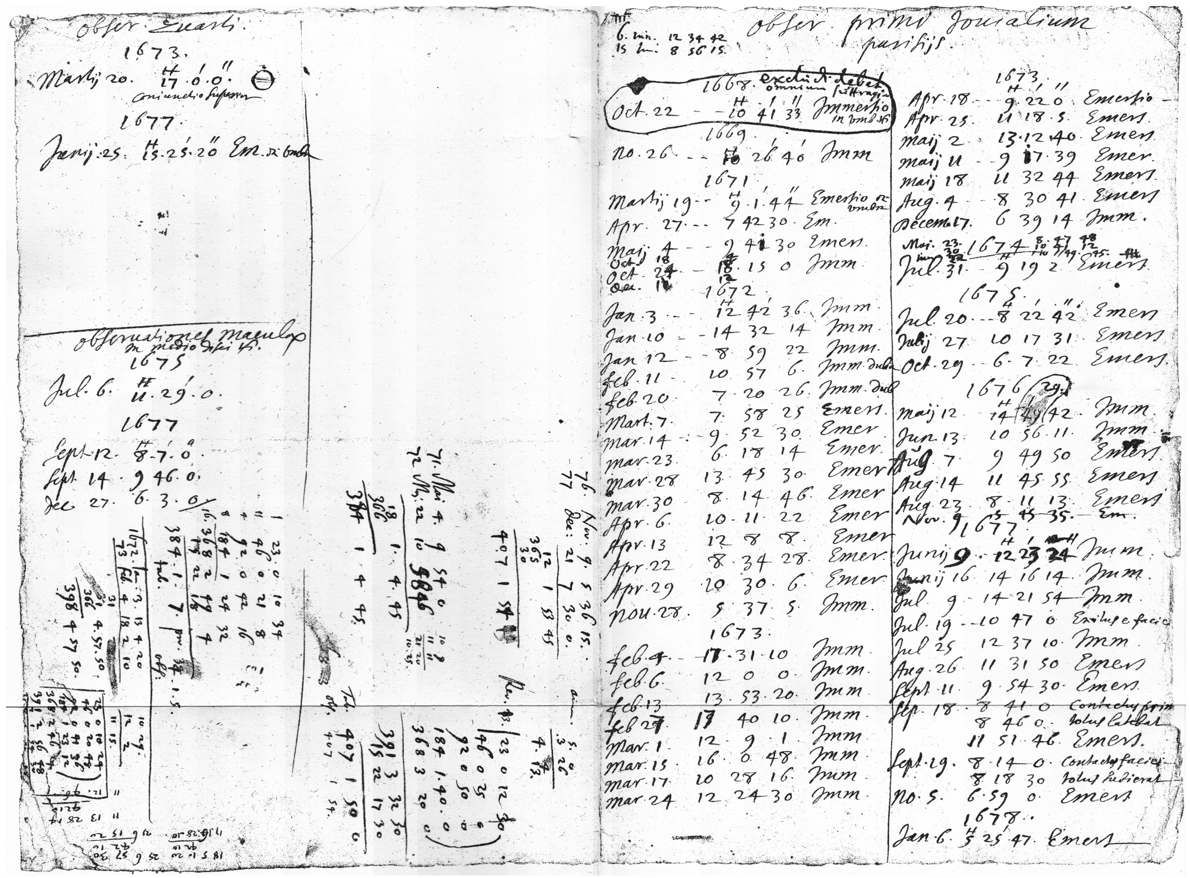


Figure 5: Roemer's notes, written at some point after January 1678 and rediscovered in 1913. The timings of eclipses of Io appear on the right hand side of this image, which would have been "page one" of the folded sheet.

Since the orbital period for the Earth was know with a high precision even in times of Tycho Brahe (born Tyge Ottesen Brahe (de Knudstrup) (14 December 1546 24 October 1601)), and if we know a distance between the the Sun and the Earth (AU), one can evaluate $GM_\odot$, since from the third Kepler law we have $T^2/a^3 = 4\pi^2/(GM_\odot)$.

Roemer's results were published in 1676 and 1677 (English translation), there in 1687 constants in Eq. (2) were known and it was an opportunity to evaluate the deflection angle for solar radius and mass. If we assume $R = R_\odot$ and $M = M_\odot$ then $\Theta = 0.''87$ (the correct GR estimate is $\Theta = 1.''75$).

But if we use values of these numbers in Newton's times ($c = 2.2 * 10^{10}$ cm/sec) surprisingly we have the deflection angle $\Theta = 1.''61$ (it is very close to predictions of GR).

The first derivation of Eq. (1) was obtained by Cavendish (Will, 1988).

These studies were triggered by a correspondence with his friend J. Michell.

The first published derivation of Eq. (1) was obtained by Johann von Soldner in 1804 (submitted in 1801).

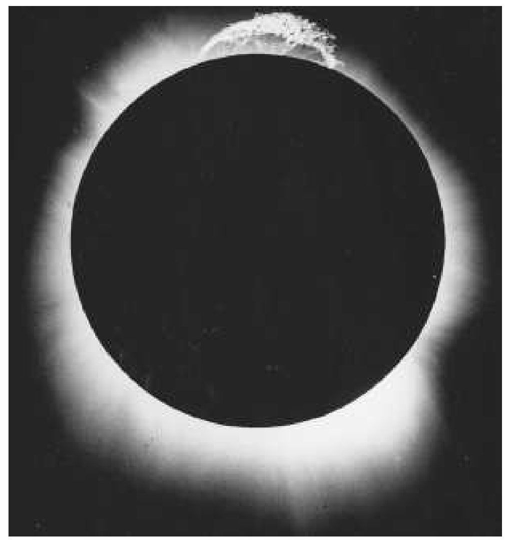
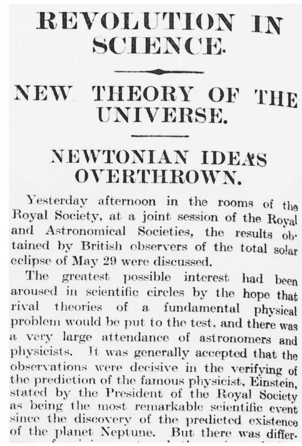


Figure 6: One of the photographs taken by Eddington at Principe. No stars are visible here, but the image does capture a spectacular solar prominence.



REVOLUTION IN SCIENCE.

NEW THEORY OF THE UNIVERSE.

NEWTONIAN IDEAS OVERTHROWN.

Yesterday afternoon in the rooms of the Royal Society, at a joint session of the Royal and Astronomical Societies, the results obtained by British observers of the total solar eclipse of May 29 were discussed.

The greatest possible interest had been aroused in scientific circles by the hope that rival theories of a fundamental physical problem would be put to the test, and there was a very large attendance of astronomers and physicists. It was generally accepted that the observations were decisive in the verifying of the prediction of the famous physicist, Einstein, stated by the President of the Royal Society as being the most remarkable scientific event since the discovery of the predicted existence of the planet Neptune. But there was differ-

Figure 7: The headline in the Times reporting the meeting at which Eddington and Dyson presented the results of the expedition. Copyright Times, 6 November 1919.

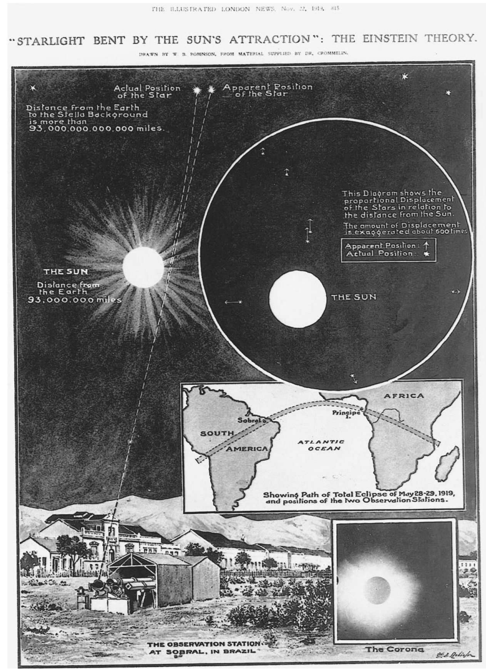


Figure 8: The Illustrated London News presentation of the expedition and its results.

In the beginning of XX century people had possibilities to measure a deflection of light. In 1838 Bessel announced that 61 Cygni had a parallax of

0.314 arcseconds; which, given the diameter of the Earth's orbit, indicated that the star was about 3 parsecs (9.8 light years) away.

Einstein obtained the same relation in the framework of SR in 1911 and suggested to measure it in 1914 during the next solar eclipse. His assistant Freundlich came to Russia to measure the deflection but World War I started and German astronomers were interned and (fortunately for Einstein the measurements were not performed (Brecher)).

In 1919 Eddington and Dyson decided to check the prediction of GR during solar eclipse in the Southern Hemisphere. Observations were made simultaneously in the cities of Sobral (Brazil) and in island Principe on the west coast of Africa.

In the framework of GR in 1915 (and 1916) Einstein obtained the following expression of light deflection

$$\Theta = \frac{4GM}{c^2R}. \tag{3}$$

In 1919 Eddington and Dyson decided to check the prediction of GR during solar eclipse in the Southern Hemisphere. Observations were made simultaneously in the cities of Sobral (Brazil) and in island Principe on the west coast of Africa.

They have to select one from three options:

1) there is no light deflection by a solar gravitational field;

2) a prediction of Newtonian theory is correct $0.''87$;

3) GR prediction is correct $1.''75$.

Sobral's results gave $1.''98 \pm 0.''12$,

Principe's $1.''61 \pm 0.''30.$

However, Sobrale' results $0.''93$ were rejected.

In his 1979 paper Harvey compares the 1919 results with those he recovered using modern techniques. Harvey comments (p. 198)

For the 4-inch plates there is no great difference between the value obtained by Dyson et al. and that from the new measurements, but the error has been considerably reduced. For the Astrographic plates, however, a significant improvement has been achieved by the new measurements. Where the previous reduction yielded a value of $0.''93$ with an unspecified, large error, the new determination is $1.''55 \pm 0.''34$. This is still a weak result, but does provide support for that from the 4-inch plates. Combining the two fresh determinations, weighted according to their standard errors, gives $1.''87 \pm 0.''13$, a result which is just within one standard error of the predicted value.

Table 1: Gravitational displacement at the Suns limb in seconds of arc.

Determination	Displacement
Predicted from Einsteins Theory	$1.''75$
4-inch plates reduced by Dyson et al.	$1.''98 \pm 0.''18$
4-inch plates measured on the Zeiss	$1.''90 \pm 0.''11$
Astrographic plates reduced by Dyson et al.	$0.''93$
Astrographic plates measured on the Zeiss	$1.''55 \pm 0.''34$

Soldner calculations

We assume that a light was emitted from point A in a direction which is perpendicular to vector $\vec{CA}$. After an instant t a light will be in point M. We introduce the following notations $CM = r, CP = x, MP = y, MCP = \phi$.

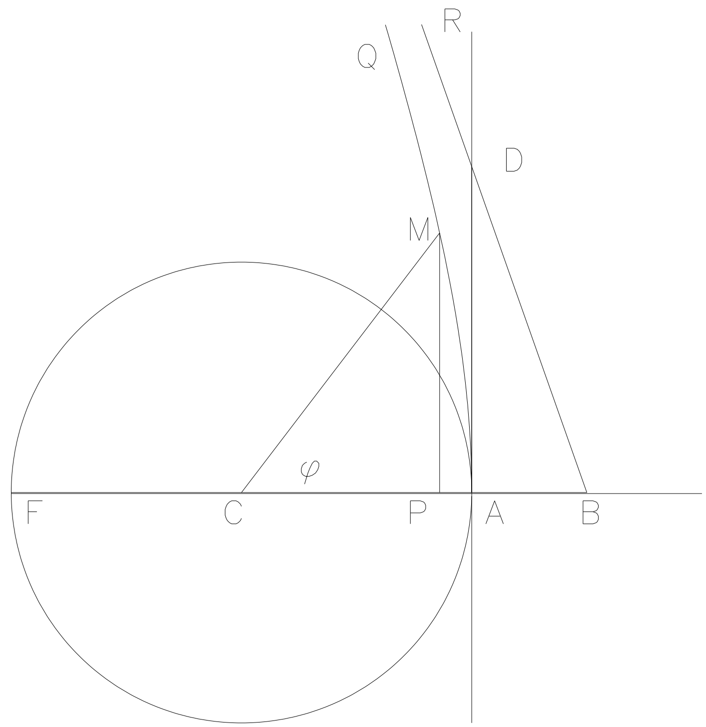


Figure 9: Reproduction of Soldner picture for his derivation of light deflection. A circle with a center at illustrate a spherically symmetric gravitating body. A light emitted from point A, is moving in a tangent direction AD. A current position of light is M and an angle ϕ. AMQ is a light trajectory (a piece of hyperbola).

The gravitational force acting on light creates an acceleration in a radial direction GM/r^2. Thus,

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^2}\cos\phi, \tag{4}$$

$$\frac{d^2y}{dt^2} = -\frac{GM}{r^2}\sin\phi, \tag{5}$$

If we multiply Eq. (4) with $-\sin\phi$, Eq. (5) with $\cos\phi$ and adding them, we obtain

$$-\frac{d^2x}{dt^2}\sin\phi + \frac{d^2y}{dt^2}\cos\phi = 0, \tag{6}$$

If we multiply Eq. (4) with $\cos\phi$, Eq. (5) with $\sin\phi$ and adding them, we obtain

$$\frac{d^2x}{dt^2}\cos\phi + \frac{d^2y}{dt^2}\sin\phi = -\frac{GM}{r^2}, \tag{7}$$

if we introduce polar coordinates

$$x = r\cos\phi, \tag{8}$$

$$y = r\sin\phi, \tag{9}$$

then

$$\dot{x} = \dot{r}\cos\phi - r\sin\phi\dot{\phi}, \tag{10}$$

$$\dot{y} = \dot{r}\sin\phi + r\cos\phi\dot{\phi}, \tag{11}$$

and

$$\ddot{x} = \ddot{r}\cos\phi - 2\dot{r}\dot{\phi}\sin\phi - r\phi^2\cos\phi - r\ddot{\phi}\sin\phi,$$

$$\ddot{y} = \ddot{r}\sin\phi + 2\dot{r}\dot{\phi}\cos\phi - r\phi^2\sin\phi + r\ddot{\phi}\cos\phi, \tag{12}$$

So, we obtain

$$-\ddot{x}\sin\phi + \ddot{y}\cos\phi = 2\dot{r}\dot{\phi} + r\ddot{\phi} = 0. \tag{13}$$

Similarly,

$$\ddot{x}\cos\phi + \ddot{y}\sin\phi = \ddot{r} - r\dot{\phi}^2 = 0. \tag{14}$$

It is easy to see that $r^2\dot{\phi}$ is the integral of motion.

Really,

$$\frac{d(r^2\dot{\phi})}{dt} = 2r\dot{r}\dot{\phi} + r^2\ddot{\phi} = 0. \tag{15}$$

So, we obtain a well-known fact, if a force depends only on distance gravitating body and a moving particle, we have an integral of motion we call it integral of areas, since, relation $r^2\phi$ determines a double area of triangle covering by moving particle (it is a moving photon in our case). Clearly that the area is determined by

$$C = AC \cdot v. \tag{16}$$

If we select units that the radius of gravitating body is 1 (or we measure distances in units of the radius), we have $C = v$. Therefore,

$$\dot{\phi} = \frac{v}{r^2}. \tag{17}$$

Substituting ϕ, we obtain

$$\ddot{r} - \frac{v^2}{r^3} = -\frac{GM}{r^2}. \tag{18}$$

It is easy to see that the last equation has the first integral (the energy integral)

$$\dot{r}^2 + \frac{v^2}{r^2} - \frac{2GM}{r} = D. \tag{19}$$

Therefore

$$dt = \frac{dr}{\sqrt{D + 2GM/r - v^2/r^2}}. \tag{20}$$

Changing variable t with ϕ, we obtain

$$d\phi = \frac{vdr}{\sqrt{[D + 2GM/r] - v^2/r^2}}. \tag{21}$$

To integrate the last equation we introduce a new variable

$$z = \frac{v}{r} - \frac{GM}{r}. \tag{22}$$

so,

$$dz = -\frac{vdr}{r^2}. \tag{23}$$

and

$$d\phi = -\frac{dz}{\sqrt{[D + (GM)^2/v^2] - z^2}}. \tag{24}$$

Therefore,

$$\phi = arccos\frac{z}{\sqrt{D + (GM)^2/v^2}} + \alpha. \tag{25}$$

or

$$\cos(\phi - \alpha) = \frac{z}{\sqrt{D + (GM)^2/v^2}}. \tag{26}$$

Coming back to radial variable r, we obtain

$$\cos(\phi - \alpha) = \frac{v^2 - (GM)r}{r\sqrt{v^2 D + (GM)^2}}. \tag{27}$$

Assuming that the origin for angle ϕ is selected as $\phi = 0, r = AC = 1$ (or $\alpha = 0$)

$$\cos(\phi) = \frac{v^2 - (GM)r}{r\sqrt{v^2 D + (GM)^2}}. \tag{28}$$

We have from a choice of an origin of ϕ, we obtain D

$$\cos(\phi) = \frac{v^2 - (GM)r}{r(v^2 - GM)}. \quad (29)$$

Moreover, we assume $v^2 > GM$. Introducing Cartesian coordinates again

$$x = 1 - r\cos\phi, y = r\sin\phi \quad (30)$$

We have

$$y^2 = \frac{v^2(v^2 - 2GM)}{(GM)^2}(1-x)^2 - \frac{2v^2}{(GM)^2}(v^2 - GM)(1-x) + \frac{v^4}{(GM)^2}. \quad (31)$$

On the other hand,

$$y^2 = \frac{2v^2}{GM}x + \frac{v^2(v^2 - 2GM)}{(GM)^2}x^2. \tag{32}$$

Clearly, the last equation is determined a conci section, moreover, if

$$v^2 > 2GM, \tag{33}$$

a trajectory is hyperbola, if

$$v^2 = 2GM, \tag{34}$$

a trajectory is parabola, if

$$v^2 < 2GM, \tag{35}$$

a trajectory is ellipse, if

$$v^2 = GM, \tag{36}$$

a trajectory is circumference.

Soldner knew Laplace paper about "dark stars" where the inequality (33) is violated but Soldner consider only the case when the relation (33) is correct (33), so

$$y^2 = \frac{2b^2}{a}x + \frac{b^2}{a^2}x^2, \tag{37}$$

where a is the major semi-axis, b is the minor semi-axis

$$\tan\omega = \frac{AB}{AD} = \frac{a}{b}, \tag{38}$$

or

$$\tan\omega = \frac{GM}{v\sqrt{v^2 - 2GM}}. \tag{39}$$

Since $v^2 \gg 2GM$, the deflection angle is very small

$$\omega = \frac{GM}{v^2}. \tag{40}$$

A total deflection angle is $\Theta = 2\omega$, since ω is an angle between the hyperbola asymptote and y-axis and reminding that a radius of gravitating body is 1, we obtain (2).

Substituting solar parameters $R_\odot = 6.96 \cdot 10^{10} cm, M_\odot = 1.989 \cdot 10^{33} g, G = 6.673 \cdot 10^{-8} c^3/(g \cdot sec^2), c = 2.997 \cdot 10^{10} cm/sec$, we obtain $\Theta = 0.''875$. $\Theta = 0.''84$ (due to uncertainties in constants).

In Soldner's times it was very hard to measure the deflection angle in observations.

Problem. If a force is radial, we have a planar motion. Prove it for both cases: Newtonian theory and General Relativity.

A shorter derivation of a light deflection in Newtonian gravity

First we consider a photon motion in the framework of Newtonian gravitational theory. We suppose that a photon is a particle having a mass $m = \frac{h\nu}{c^2}$.

Let us consider a photon motion near a star having a mass M_*. If a photon is emitted by a source S then we denote an impact distance of the photon motion by p . If we use Cartesian coordinate frame Oxy then the equation of motion of a light ray has the following form:

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{GmM_*}{|\vec{r}|^3} \cdot \vec{r}. \tag{41}$$

As follows from Eqn (41), a photon mass is shortened, so there is a light bending effect even in the framework of Newtonian theory.

It is well-known from the analysis of the Newtonian equations of motion that a test particle trajectory may be a hyperbola, a parabola or an ellipse.

The quantitative criteria of different types of a particle trajectory consist of a comparison of a potential energy of particle in the gravitational field ($U = \frac{GM_* h\nu}{c^2 p}$ for the case) and a kinetic energy ($E = h\nu$ for the case). Since the test particle is a photon for our case, so the criterion is the fraction of a gravitational potential of a body and a square of speed of light $\frac{GM_*}{c^2 p}$. The fraction is much less unity for considered astronomical models, so the trajectory is a hyperbola and kinetic energy of a photon is much greater than its potential energy.

Below we will analyse a light ray displacement along the axis Oy, which is perpendicular to an original velocity of a photon. Since a light ray moves practically along the axis Ox, in zeroth order in the parameter $\frac{GM_*}{c^2p}$ we have the following equation of motion $x = ct$. If we express t via x and substitute it in Eqn (41), then we obtain the equation in the parametric form $y(x)$

$$\frac{d^2y}{dx^2} = -\frac{GM_*y}{c^2(x^2+y^2)^{3/2}}. \tag{42}$$

We suppose that the displacement is a very small one, thus we assume $y \approx p$ in the right hand side of the Eqn (42). So it is possible to calculate an integral of the right hand side of the Eqn (42). Really, using the substitution

$x = p\,\mathrm{tg}\phi$, we obtain

$$-p \int_{-\infty}^{x} \frac{dx}{(x^2 + p^2)^{3/2}} = -p^{-1}(\sin\phi + 1). \tag{43}$$

We note that $\frac{dy}{dx}$ is a tangent line for the photon trajectory, the difference of the values $\frac{dy}{dx}$ for $+\infty$ and $-\infty$ is equal to the bending angle of a photon in the gravitational field of a star M_*:

$$\Delta\varphi = \left.\frac{dy}{dx}\right|_{x=-\infty} - \left.\frac{dy}{dx}\right|_{x=+\infty} = \left.\frac{dy}{dx}\right|_{\phi=-\pi/2} - \left.\frac{dy}{dx}\right|_{\phi=+\pi/2} = -\frac{2GM_*}{c^2 p}. \tag{44}$$

We obtain the bending angle which is equal to a half of a correct value of bending angle. The difference is connected with an usage of non-relativistic approximation, but a photon is a relativistic particle moving with the limiting speed (the speed of light).

A derivation of Einstein relation for deflection of light

It is one of the classical tests of GR.

We will follow the derivation from the book (C. Moller, The Theory of Relativity (Clarendon, Oxford, 1969)).

Assume that an impact parameter is Δ. Therefore, a dependence of a radial coordinate r as a function of angle ϕ is (Landau & Lifschitz, Theory of classical fields)

$$\left(\frac{dr}{d\phi}\right)^2 = r^4\left(\frac{1}{\Delta^2} - \frac{1}{r^2}\left(1 - \frac{r_g}{r}\right)\right)^2, \tag{45}$$

where $r_g = 2GM/c^2$. Changing variable $u = 1/r$, therefore, ignoring small quantity $r_g u$, we obtain

$$\phi = \int_0^u \Delta du[1 - (\Delta u)^2]^{-1/2} = \arcsin(\Delta u). \tag{46}$$

Thus, $u = \sin\phi/\Delta$ or $r = \Delta/\sin\phi$. So, an approximate solution of Eq. (46) is a straight line at a distance from the origin Δ, moreover, the minimal approach of a photon is for $\phi = \pi/2$ and a photon is going to infinity for $\phi \to \pi$. Eq. (46) may be written the following form

$$\left(\frac{du}{d\phi}\right) = \left[\frac{1}{\Delta^2} - u^2(1 - r_g u)\right]^{1/2}, \tag{47}$$

If we introduce new variable

$$\sigma = \Delta u(1 - r_g u)^{1/2} = \Delta u\left(1 - \frac{r_g u}{2}\right). \tag{48}$$

Therefore,

$$\frac{du}{d\phi} = \frac{1}{\Delta}(1-\sigma^2)^{1/2}, \tag{49}$$

Since $r_g u$ is a small quantity, we have

$$\Delta u = \sigma(1-r_g u)^{-1/2} \approx \sigma\left(1+\frac{r_g u}{2}\right) \approx \sigma\left(1+\frac{r_g \sigma}{2\Delta}\right), \tag{50}$$

Therefore,

$$\Delta du = d\sigma\left(1+\frac{r_g \sigma}{\Delta}\right), \tag{51}$$

From Eq. (49) we obtain that

$$\phi = \int_0^u \Delta du(1-\sigma^2)^{-1/2} = \int_0^\sigma \frac{(1+r_g\sigma/\Delta)d\sigma}{\sqrt{1-\sigma^2}} = \arcsin(\sigma) - \frac{r_g}{\Delta}(1-\sigma^2)^{1/2} + \frac{r_g}{\Delta}.$$

According to Eq. (49) the maximal u_m (minimal r_m) (pericenter) corresponds to $\sigma = 1$. The angle corresponding to the pericenter is determined from the integral

$$\phi_m = \frac{\pi}{2} + \frac{r_g}{\Delta} \quad (52)$$

Therefore, a light deflection is $\Theta = 2(\phi_m - \pi/2)$, or $\Theta = 2r_g/\Delta$. Substituting $\sigma = 1$ in Eq. (48), we obtain u_m, corresponding to the pericenter

$$u_m = \frac{1}{\Delta}(1 + \frac{r_g}{2\Delta}) \quad (53)$$

Ignoring the small term in the last relation we obtain Einstein expression for a light deflection in framework of GR

$$\Theta \approx 2r_g u_m = \frac{4GM}{c^2 r_m} \quad (54)$$

Keep in mind that we obtained the last relation assuming $r_g \ll \Delta$.

Problem. **Explain quantitatively factor 2 in Newtonian and GR results.**