Massless QED

$$L = \bar{\psi}_0 i \not{D} \psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}$$

$$D_\mu \psi_0 = \left(\partial_\mu - i e_0 A_{0\mu}\right) \psi_0 \qquad F_{0\mu\nu} = \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}$$

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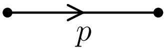
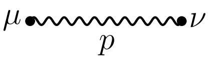
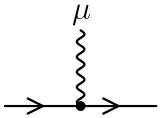


$$= i S_0(p) \quad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$$



$$= -i D^0_{\mu\nu}(p)$$

$$D^0_{\mu\nu}(p) = \frac{1}{p^2}\left[g_{\mu\nu} - (1 - a_0)\frac{p_\mu p_\nu}{p^2}\right]$$



$$= i e_0 \gamma^\mu$$

Renormalization

Renormalized quantities

$$\psi_0 = Z_\psi^{1/2}\psi \qquad A_0 = Z_A^{1/2}A \qquad a_0 = Z_A a \qquad e_0 = Z_\alpha^{1/2}e$$

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Minimal renormalization constants

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon}\frac{\alpha}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon}\right)\left(\frac{\alpha}{4\pi}\right)^2 + \cdots$$

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Dimensionalities: $[L] = d$, $[A_0] = 1 - \varepsilon$, $[\psi_0] = 3/2 - \varepsilon$, $[e_0] = \varepsilon$. Exactly dimensionless

$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon}\frac{e^2}{(4\pi)^{d/2}}e^{-\gamma\varepsilon} \qquad \frac{e_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon}\frac{\alpha(\mu)}{4\pi}Z_\alpha(\alpha(\mu))e^{\gamma\varepsilon}$$

Photon propagator

$$-iD_{\mu\nu}(p) = -iD^0_{\mu\nu}(p) + (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\nu}(p)$$
$$+ (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\gamma}(p)i\Pi^{\gamma\delta}(p)(-i)D^0_{\gamma\nu}(p) + \cdots$$

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$$D_{\mu\nu}(p) = D^0_{\mu\nu}(p) + D^0_{\mu\alpha}(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$

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$$\left.\begin{aligned} A_{\mu\nu} &= A_\perp \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + A_{||}\frac{p_\mu p_\nu}{p^2} \\ A^{-1}_{\mu\nu} &= A^{-1}_\perp \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + A^{-1}_{||}\frac{p_\mu p_\nu}{p^2} \end{aligned}\right\} \quad A^{-1}_{\mu\lambda}A^{\lambda\nu} = \delta^\nu_\mu$$

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$$D^{-1}_{\mu\nu}(p) = (D^0)^{-1}_{\mu\nu}(p) - \Pi_{\mu\nu}(p)$$

Photon propagator

Ward identity $\Pi_{\mu\nu}(p)p^\nu = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

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Renormalized propagator $D_{\mu\nu}(p) = Z_A(\alpha(\mu)) D^r_{\mu\nu}(p;\mu)$

$$D^r_{\mu\nu}(p;\mu) = D^r_\perp(p^2;\mu)\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + a(\mu)\frac{p_\mu p_\nu}{(p^2)^2}$$

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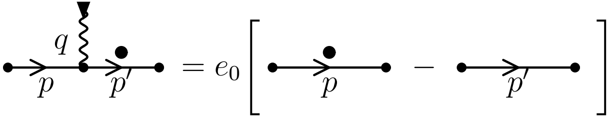
$Z_A(\alpha)$ is constructed to make

$$D^r_\perp(p^2;\mu) = Z_A^{-1}(\alpha(\mu))\frac{1}{p^2(1-\Pi(p^2))}$$

finite at $\varepsilon \to 0$. But the longitudinal part must be finite too:

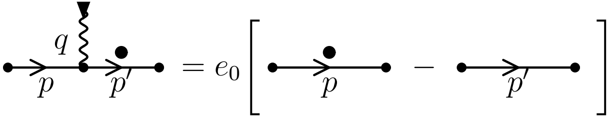
$$a(\mu) = Z_A^{-1}(\alpha(\mu))a_0$$

Ward identity



$$iS_0(p')\; ie_0 \not{q}\; iS_0(p) = ie_0 \left[S_0(p') - S_0(p) \right]$$

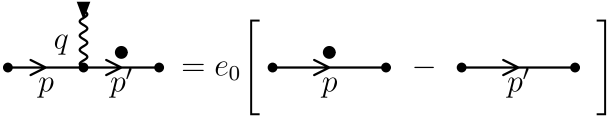
Ward identity



$$iS_0(p')\, ie_0 \not{q}\, iS_0(p) = ie_0 \left[S_0(p') - S_0(p)\right]$$

$$\frac{\partial S_0(p)}{\partial p^\mu} = -S_0(p)\gamma_\mu S_0(p)$$

Ward identity



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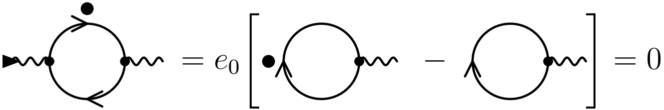
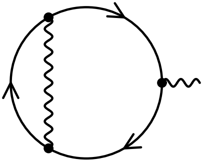
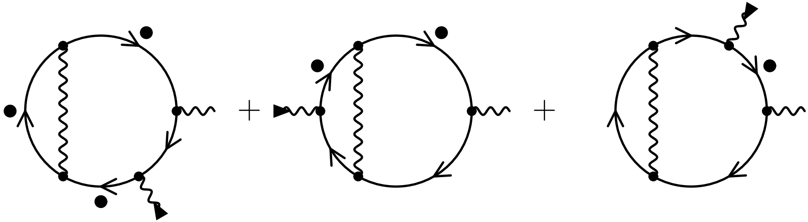
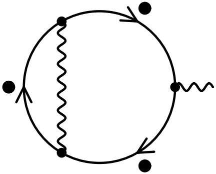
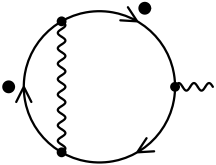
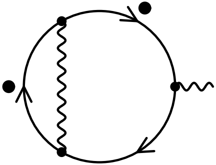
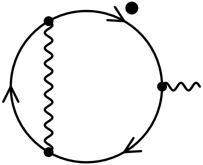
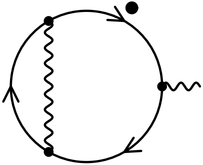
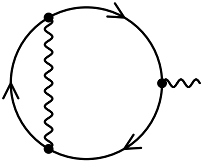




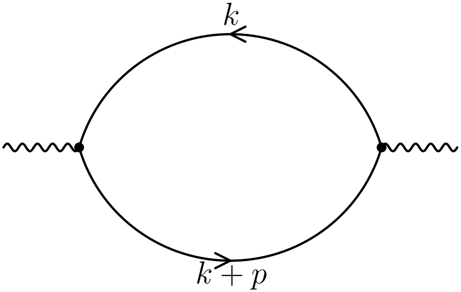


$= e_0 \Big[$  $-$ 

$+$  $-$ 

$+$  $-$  $\Big] = 0$

Photon self-energy

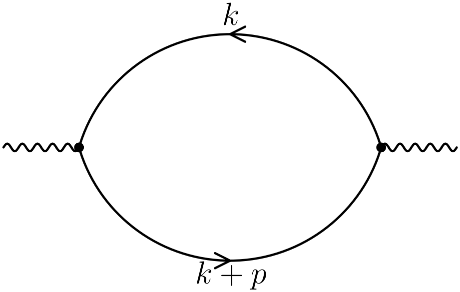


Fermion loop -1

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2) =$$

$$-\int \frac{d^d k}{(2\pi)^d} \,\mathrm{Tr}\, ie_0\gamma_\mu i \frac{\not{k} + \not{p}}{(k+p)^2} ie_0 \gamma_\nu \frac{\not{k}}{k^2}$$

Photon self-energy



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$$\Pi(p^2) = \frac{-ie_0^2}{(d-1)(-p^2)} \int \frac{d^d k}{(2\pi)^d} \frac{\operatorname{Tr}\gamma_\mu(\not{k}+\not{p})\gamma^\mu \not{k}}{[-(k+p)^2]\,(-k^2)}$$

Photon self-energy

$$\Pi(p^2) = \frac{d-2}{d-1}\frac{ie_0^2}{-p^2}\int \frac{d^d k}{(2\pi)^d}\frac{4(k+p)\cdot k}{\left[-(k+p)^2\right]\left(-k^2\right)}$$

Photon self-energy

$$\Pi(p^2) = \frac{d-2}{d-1}\frac{ie_0^2}{-p^2}\int \frac{d^dk}{(2\pi)^d}\frac{4(k+p)\cdot k}{\left[-(k+p)^2\right]\left(-k^2\right)}$$

Set $-p^2 = 1$

$$D_1 = -(k+p)^2 \qquad D_2 = -k^2$$

Photon self-energy

$$\Pi(p^2) = \frac{d-2}{d-1} \frac{ie_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{4(k+p)\cdot k}{[-(k+p)^2]\,(-k^2)}$$

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Multiplication table

$$p^2 = -1 \qquad k^2 = -D_2 \qquad p\cdot k = \frac{1}{2}(1 + D_2 - D_1)$$

Photon self-energy

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$$\Pi(p^2) = 2\frac{d-2}{d-1}ie_0^2\int\frac{d^dk}{(2\pi)^d}\frac{-2D_2+1+D_2-D_1}{D_1D_2}$$

Photon self-energy

Restoring $-p^2$ by dimensionality

$$\Pi(p^2) = -\frac{e_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 2\frac{d-2}{d-1}G_1$$
$$= \frac{e_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4\frac{d-2}{(d-1)(d-3)(d-4)}g_1$$

Photon field renormalization

Transverse propagator

$$p^2 D_\perp(p^2) = \frac{1}{1 - \Pi(p^2)} = 1 + \frac{e_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4 \frac{d-2}{(d-1)(d-3)(d-4)} g_1$$

Photon field renormalization

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Re-expressing via $\alpha(\mu)$

$$p^2 D_\perp(p^2) = 1 + \frac{\alpha(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 \, 4 \frac{d-2}{(d-1)(d-3)(d-4)}$$

$$L = \log \frac{-p^2}{\mu^2}$$

Photon field renormalization

$$e^{\gamma\varepsilon} g_1 = 1 + \mathcal{O}(\varepsilon^2)$$

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$$e^{\gamma\varepsilon} g_1 = 1 + \mathcal{O}(\varepsilon^2)$$

$$p^2 D_\perp(p^2) = 1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon}\left[1 - \left(L - \frac{5}{3}\right)\varepsilon + \cdots\right]$$

Photon field renormalization

$$e^{\gamma\varepsilon} g_1 = 1 + \mathcal{O}(\varepsilon^2)$$

$$p^2 D_\perp(p^2) = 1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon}\left[1 - \left(L - \frac{5}{3}\right)\varepsilon + \cdots\right]$$

This should be $Z_A(\alpha(\mu)) p^2 D^r_\perp(p^2;\mu)$:

$$Z_A(\alpha) = 1 - \frac{4}{3}\frac{\alpha}{4\pi\varepsilon}$$

$$p^2 D^r_\perp(p^2;\mu) = 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi}\left(L - \frac{5}{3}\right)$$

RG equation

$D_\perp(p^2) = Z_A(\alpha(\mu)) D_\perp^r(p^2;\mu)$ does not depend on μ:

$$\frac{\partial D_\perp^r(p^2;\mu)}{\partial \log \mu} + \gamma_A(\alpha(\mu)) D_\perp^r(p^2;\mu) = 0$$

$$\gamma_A(\alpha(\mu)) = \frac{d \log Z_A(\alpha(\mu))}{d \log \mu}$$

RG equation

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For a minimal renormalization constant using

$$\frac{d\log\alpha(\mu)}{d\log\mu} = -2\varepsilon + \cdots$$

we obtain

$$\gamma(\alpha) = \gamma_0 \frac{\alpha}{4\pi} + \cdots = -2 z_1 \frac{\alpha}{4\pi} + \cdots$$

$$Z(\alpha) = 1 - \frac{\gamma_0}{2} \frac{\alpha}{4\pi\varepsilon} + \cdots$$

Anomalous dimension

$$\gamma_A(\alpha) = \frac{8}{3}\frac{\alpha}{4\pi}$$

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RG equation

$$\frac{\partial p^2 D_\perp^r}{\partial L} = \frac{\gamma_A}{2} p^2 D_\perp^r$$

with the initial condition

$$p^2 D_\perp^r(p^2; \mu^2 = -p^2) = 1 - \frac{20}{9}\frac{\alpha(\mu)}{4\pi}$$

is enough to reconstruct the result

Anomalous dimension

$$\gamma_A(\alpha) = \frac{8}{3}\frac{\alpha}{4\pi}$$

RG equation

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with the initial condition

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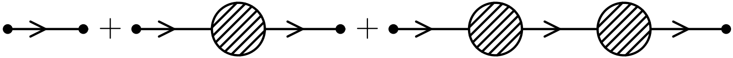
is enough to reconstruct the result

$a_0 = Z_A(\alpha(\mu))a(\mu)$ does not depend on μ:

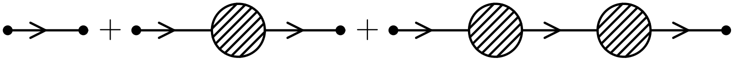
$$\frac{da(\mu)}{d\log\mu} + \gamma_A(\alpha(\mu))a(\mu) = 0$$

Electron propagator



$$iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p)$$
$$+ iS_0(p)(-i)\Sigma(p)iS_0(p)(-i)\Sigma(p)iS_0(p) + \cdots$$

Electron propagator



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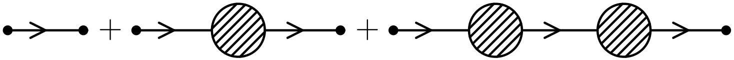
$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

Electron propagator



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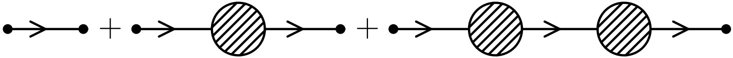
$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

Electron propagator



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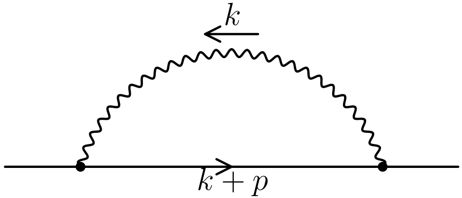
$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

Massless case: $\Sigma(p) = \not{p}\Sigma_V(p^2)$ (helicity conservation)

$$S(p) = \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\not{p}}$$

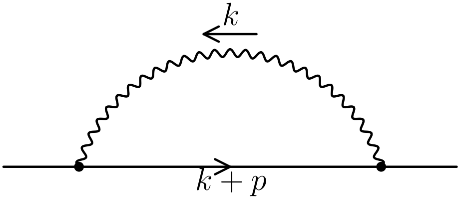
Electron self-energy



$$-i\not{p}\Sigma_V(p^2) = \int \frac{d^dk}{(2\pi)^d} ie_0\gamma^\mu i\frac{\not{k}+\not{p}}{(k+p)^2} ie_0\gamma^\nu \frac{-i}{k^2}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$$

where $\xi = 1 - a_0$.

Electron self-energy



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where $\xi = 1 - a_0$. Taking $\frac{1}{4}\,\mathrm{Tr}\,\not{p}$

$$\Sigma_V(p^2) = \frac{ie_0^2}{-p^2}\int \frac{d^dk}{(2\pi)^d}\frac{N}{D_1 D_2}$$

$$N = \frac{1}{4}\,\mathrm{Tr}\,\not{p}\gamma^\mu(\not{k}+\not{p})\gamma^\nu\left(g_{\mu\nu} + \xi\frac{k_\mu k_\nu}{D_2}\right)$$

Electron self-energy

Using the multiplication table

$$
\begin{aligned}
N &= \frac{1}{4}\,\mathrm{Tr}\,\not{p}\gamma_\mu(\not{k}+\not{p})\gamma^\mu + \frac{\xi}{D_2}\,\frac{1}{4}\,\mathrm{Tr}\,\not{p}\not{k}(\not{k}+\not{p})\not{k} \\
&= -(d-2)(p^2+p\cdot k) + \frac{\xi}{D_2}\left[k^2 p\cdot k + 2(p\cdot k)^2 - p^2k^2\right] \\
&= \frac{1}{2}\left[d-2+\xi\left(\frac{1}{D_2}-1\right)\right]
\end{aligned}
$$

Electron self-energy

Using the multiplication table

$$N = \frac{1}{4} \operatorname{Tr} \not{p} \gamma_\mu (\not{k} + \not{p}) \gamma^\mu + \frac{\xi}{D_2} \frac{1}{4} \operatorname{Tr} \not{p} \not{k} (\not{k} + \not{p}) \not{k}$$

$$= -(d-2)(p^2 + p \cdot k) + \frac{\xi}{D_2} \left[k^2 p \cdot k + 2 (p \cdot k)^2 - p^2 k^2 \right]$$

$$= \frac{1}{2} \left[d - 2 + \xi \left(\frac{1}{D_2} - 1 \right) \right]$$

$$\Sigma_V(p^2) = -\frac{e_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{1}{2} \left[(d - 2 - \xi) G(1,1) + \xi G(1,2) \right]$$

Electron self-energy

$$\frac{G(1,2)}{G(1,1)} = -(d-3)$$

Electron self-energy

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$$\begin{aligned}\Sigma_V(p^2) &= -\frac{e_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}}\frac{d-2}{2}a_0 G_1 \\ &= \frac{e_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}}\frac{d-2}{(d-3)(d-4)}a_0 g_1\end{aligned}$$

Electron field renormalization

$$\not{p} S(p) = \frac{1}{1 - \Sigma_V(p^2)}$$

expressed via the renormalized quantities

$$\begin{aligned}
\not{p} S(p) &= 1 + \frac{\alpha(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 \, a(\mu) \frac{d-2}{(d-3)(d-4)} \\
&= 1 - \frac{\alpha(\mu)}{4\pi\varepsilon} a(\mu) e^{-L\varepsilon} (1 + \varepsilon + \cdots)
\end{aligned}$$

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$$= 1 - \frac{\alpha(\mu)}{4\pi\varepsilon} a(\mu) e^{-L\varepsilon} (1 + \varepsilon + \cdots)$$

should be $Z_\psi(\alpha(\mu), a(\mu)) \not{p}S_r(p;\mu)$:

$$Z_\psi(\alpha, a) = 1 - a\frac{\alpha}{4\pi\varepsilon}$$

$$\not{p}S_r(p) = 1 + a(\mu)(L-1)\frac{\alpha(\mu)}{4\pi}$$

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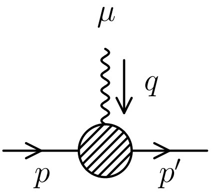
$$\gamma_\psi(\alpha, a) = 2a\frac{\alpha}{4\pi}$$

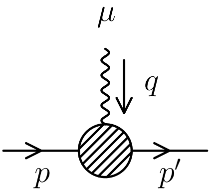
Vertex



$$= ie_0 \Gamma^\mu(p, p')$$

$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

Vertex



$$= ie_0\Gamma^\mu(p, p')$$

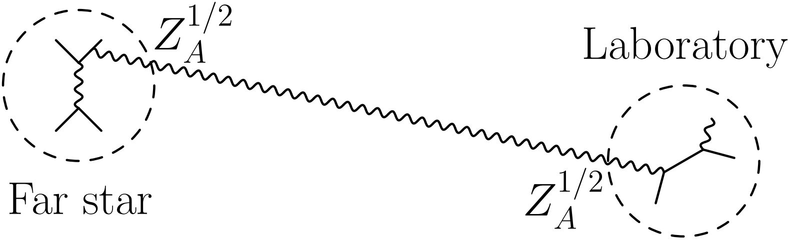
$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

When expressed via renormalized quantities,

$$\Gamma^\mu = Z_\Gamma \Gamma^\mu_r$$

Matrix element

S-matrix element $=$ vertex $\times$ $Z_i^{1/2}$ for each i



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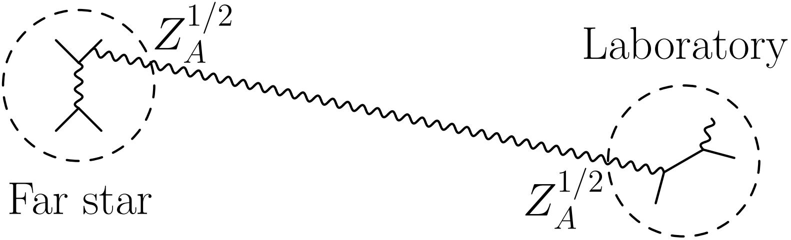

The physical matrix element
$e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite

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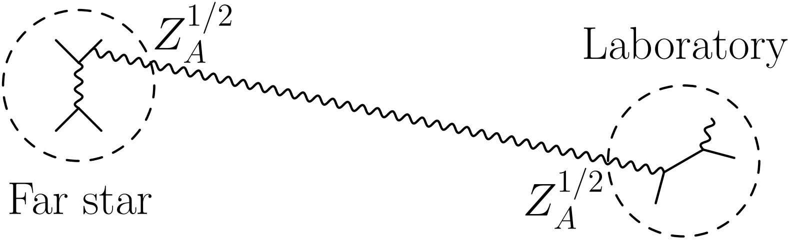

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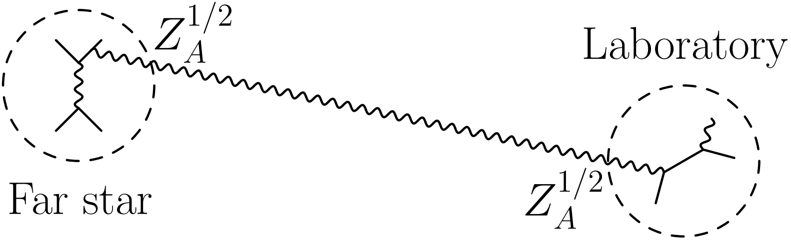

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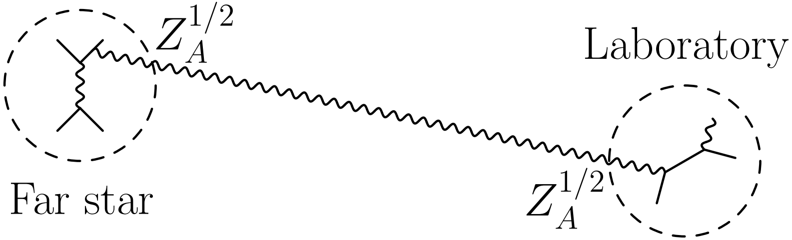

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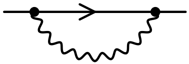
$$Z_\alpha = (Z_\Gamma Z_\psi)^{-2} Z_A^{-1}$$

Ward identity



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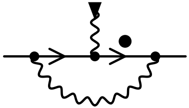

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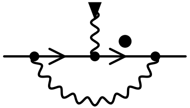
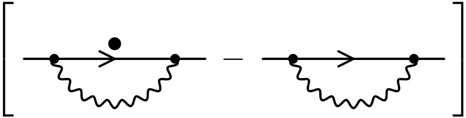
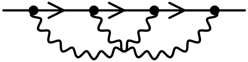


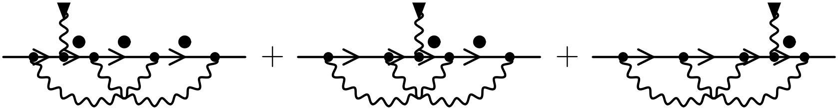
$= e_0$



Ward identity



Ward identity



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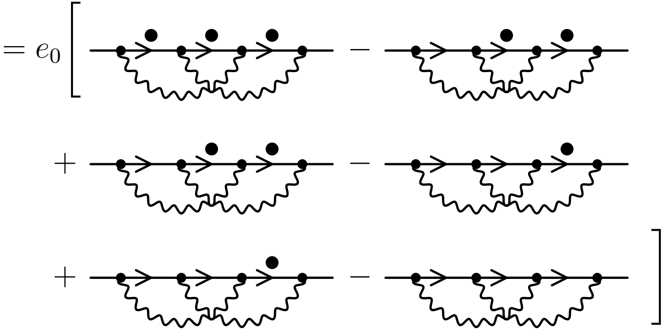

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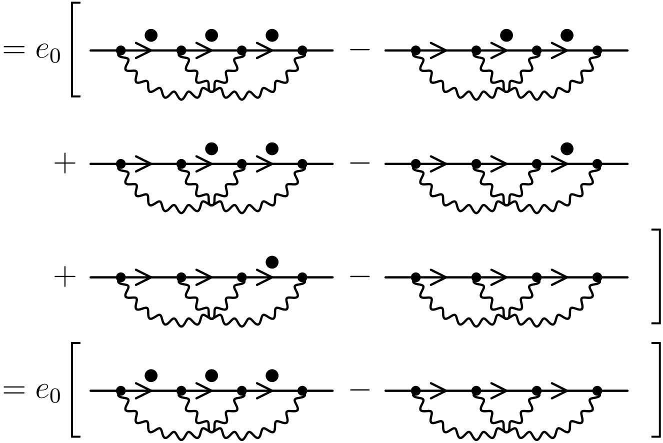

Ward identity

$$\Lambda^\mu(p,p')q_\mu = \Sigma(p) - \Sigma(p') \qquad \Gamma^\mu(p,p')q_\mu = S^{-1}(p') - S^{-1}(p)$$

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$$Z_\Gamma \Gamma_r^\mu q_\mu = Z_\psi^{-1}\left[S_r^{-1}(p') - S_r^{-1}(p)\right]$$

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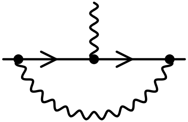
$$Z_\psi Z_\Gamma = 1$$

$$Z_\alpha = Z_A^{-1}$$

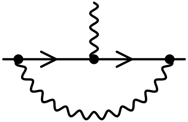
Direct calculation



$$ie_0\Lambda^\alpha = \int \frac{d^d k}{(2\pi)^d} ie_0\gamma^\mu i\frac{\not{k}}{k^2} ie_0\gamma^\alpha i\frac{\not{k}}{k^2} ie_0\gamma^\nu \times \frac{-i}{k^2}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu \not{k}\gamma^\alpha \not{k}\gamma^\mu - \xi k^2\gamma^\alpha}{(k^2)^2}$$

Direct calculation



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Averaging $\not{k}\gamma^\alpha\not{k} \to (k^2/d)\gamma_\nu\gamma^\alpha\gamma^\nu$

$$\Lambda^\alpha = -ie_0^2 a_0\gamma^\alpha \int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2)^2}$$

Direct calculation

$$\Gamma^\alpha = \gamma^\alpha \left[1 + a(\mu)\frac{\alpha(\mu)}{4\pi\varepsilon}\right]$$

$$Z_\Gamma = 1 + a\frac{\alpha}{4\pi\varepsilon}$$

agrees with Z_ψ

Charge renormalization

e_0^2 does not depend on μ:

$$\frac{d\log\alpha(\mu)}{d\log\mu} = -2\varepsilon - 2\beta(\alpha(\mu))$$

$$\beta(\alpha_s(\mu)) = \frac{1}{2}\frac{d\log Z_\alpha(\alpha_s(\mu))}{d\log\mu}$$

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For a minimal renormalization constant

$$Z_\alpha(\alpha) = 1 + z_1\frac{\alpha}{4\pi\varepsilon} + \cdots$$

we obtain

$$\beta(\alpha) = \beta_0\frac{\alpha}{4\pi} + \cdots = -z_1\frac{\alpha}{4\pi} + \cdots$$

$$Z_\alpha(\alpha) = 1 - \beta_0\frac{\alpha}{4\pi\varepsilon} + \cdots$$

Charge renormalization

$$\beta(\alpha) = \beta_0 \frac{\alpha}{4\pi} = -\frac{4}{3}\frac{\alpha}{4\pi}$$

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$$\beta(\alpha) = \beta_0 \frac{\alpha}{4\pi} = -\frac{4}{3}\frac{\alpha}{4\pi}$$

$$\frac{d}{d\log\mu}\frac{\alpha(\mu)}{4\pi} = -2\beta_0 \left(\frac{\alpha(\mu)}{4\pi}\right)^2$$

$$\frac{d}{d\log\mu}\frac{4\pi}{\alpha(\mu)} = 2\beta_0$$

$$\frac{4\pi}{\alpha(\mu')} - \frac{4\pi}{\alpha(\mu)} = 2\beta_0 \log\frac{\mu'}{\mu}$$

$$\alpha(\mu') = \frac{\alpha(\mu)}{1 + 2\beta_0 \frac{\alpha(\mu)}{4\pi} \log\frac{\mu'}{\mu}}$$