

# Massless QED

$$L = \bar{\psi}_0 i \not{D} \psi_0 - \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}$$

$$D_\mu \psi_0 = (\partial_\mu - ie_0 A_{0\mu}) \psi_0$$

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$$\begin{array}{l}
 \bullet \xrightarrow{p} \bullet \quad = iS_0(p) \quad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2} \\
 \bullet \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \bullet \quad = -iD_{\mu\nu}^0(p)
 \end{array}$$

$$D_{\mu\nu}^0(p) = \frac{1}{p^2} \left[ g_{\mu\nu} - (1 - a_0) \frac{p_\mu p_\nu}{p^2} \right]$$

$$\begin{array}{l}
 \mu \\
 | \\
 \bullet \text{---} \text{---} \text{---} \text{---} \text{---} \bullet \\
 | \\
 \bullet \xrightarrow{\quad} \bullet \xrightarrow{\quad} \bullet
 \end{array}
 = ie_0 \gamma^\mu$$

# Renormalization

Renormalized quantities

$$\psi_0 = Z_\psi^{1/2} \psi \quad A_0 = Z_A^{1/2} A \quad a_0 = Z_A a \quad e_0 = Z_\alpha^{1/2} e$$

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Minimal renormalization constants

$$Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon} \frac{\alpha}{4\pi} + \left( \frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \left( \frac{\alpha}{4\pi} \right)^2 + \dots$$

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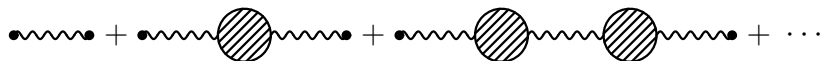
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Dimensionalities:  $[L] = d$ ,  $[A_0] = 1 - \varepsilon$ ,  $[\psi_0] = 3/2 - \varepsilon$ ,  
 $[e_0] = \varepsilon$ . Exactly dimensionless

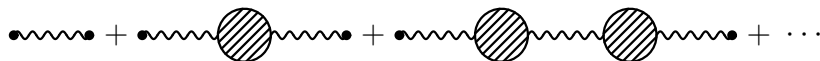
$$\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon} \frac{e^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} \quad \frac{e_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\alpha(\mu)}{4\pi} Z_\alpha(\alpha(\mu)) e^{\gamma\varepsilon}$$

# Photon propagator



$$\begin{aligned} -iD_{\mu\nu}(p) &= -iD_{\mu\nu}^0(p) + (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\nu}^0(p) \\ &+ (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\gamma}^0(p)i\Pi^{\gamma\delta}(p)(-i)D_{\gamma\nu}^0(p) + \dots \end{aligned}$$

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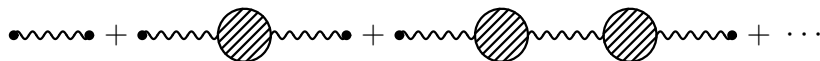


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$$D_{\mu\nu}(p) = D_{\mu\nu}^0(p) + D_{\mu\alpha}^0(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$



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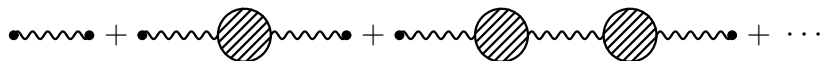


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$$\left. \begin{aligned} A_{\mu\nu} &= A_{\perp} \left[ g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right] + A_{\parallel} \frac{p_{\mu}p_{\nu}}{p^2} \\ A_{\mu\nu}^{-1} &= A_{\perp}^{-1} \left[ g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right] + A_{\parallel}^{-1} \frac{p_{\mu}p_{\nu}}{p^2} \end{aligned} \right\} A_{\mu\lambda}^{-1}A^{\lambda\nu} = \delta_{\mu}^{\nu}$$

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$$D_{\mu\nu}^{-1}(p) = (D^0)_{\mu\nu}^{-1}(p) - \Pi_{\mu\nu}(p)$$

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Ward identity  $\Pi_{\mu\nu}(p)p^\nu = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

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Renormalized propagator  $D_{\mu\nu}(p) = Z_A(\alpha(\mu)) D_{\mu\nu}^r(p; \mu)$

$$D_{\mu\nu}^r(p; \mu) = D_{\perp}^r(p^2; \mu) \left[ g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + a(\mu) \frac{p_\mu p_\nu}{(p^2)^2}$$

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$Z_A(\alpha)$  is constructed to make

$$D_{\perp}^r(p^2; \mu) = Z_A^{-1}(\alpha(\mu)) \frac{1}{p^2(1 - \Pi(p^2))}$$

finite at  $\varepsilon \rightarrow 0$ . But the longitudinal part must be finite too:

$$a(\mu) = Z_A^{-1}(\alpha(\mu)) a_0$$

# Ward identity

$$iS_0(p') ie_0 \not{q} iS_0(p) = ie_0 [S_0(p') - S_0(p)]$$

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$$\frac{\partial S_0(p)}{\partial p^\mu} = -S_0(p) \gamma_\mu S_0(p)$$



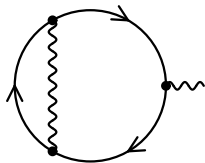
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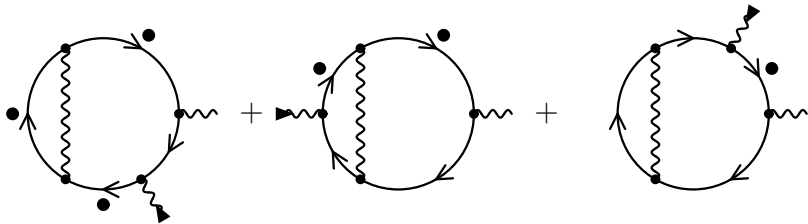
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$$\text{Diagram} = e_0 \left[ \text{Diagram 1} - \text{Diagram 2} \right] = 0$$



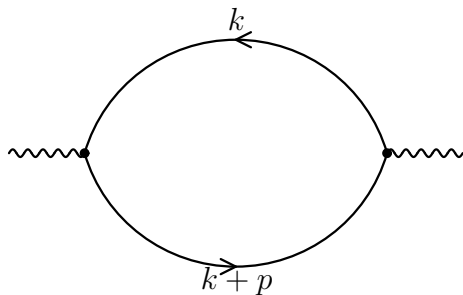


$$\begin{aligned}
&= e_0 \left[ \text{Diagram 1} - \text{Diagram 2} \right. \\
&+ \text{Diagram 3} - \text{Diagram 4} \\
&+ \left. \text{Diagram 5} - \text{Diagram 6} \right] = 0
\end{aligned}$$

The diagrammatic equation shows a sum of six terms, each represented by a circular loop with a wavy line and an external wavy line. The terms are arranged in three rows, with the first row enclosed in a large bracket and the entire sum enclosed in a final large bracket. The terms are:

- Diagram 1:** A circle with a wavy line on the left and an external wavy line on the right. Four dots are on the circle: one on the left, one at the top, one at the bottom, and one on the right. Arrows on the circle indicate a clockwise flow.
- Diagram 2:** Similar to Diagram 1, but the external wavy line is on the left and the dot on the right is absent.
- Diagram 3:** Similar to Diagram 1, but the external wavy line is on the right and the dot on the left is absent.
- Diagram 4:** Similar to Diagram 1, but the external wavy line is on the left and the dot on the right is absent.
- Diagram 5:** Similar to Diagram 1, but the external wavy line is on the right and the dot on the left is absent.
- Diagram 6:** Similar to Diagram 1, but the external wavy line is on the left and the dot on the right is absent.

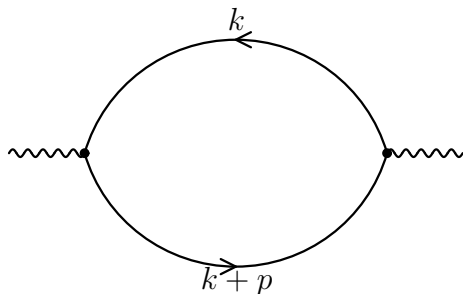
# Photon self-energy



Fermion loop  $-1$

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2) =$$
$$- \int \frac{d^d k}{(2\pi)^d} \text{Tr} i e_0 \gamma_\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i e_0 \gamma_\nu \frac{\not{k}}{k^2}$$

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$$\Pi(p^2) = \frac{-ie_0^2}{(d-1)(-p^2)} \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr} \gamma_\mu (\not{k} + \not{p}) \gamma_\nu \not{k}}{[-(k+p)^2] (-k^2)}$$

# Photon self-energy

$$\Pi(p^2) = \frac{d-2}{d-1} \frac{ie_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{4(k+p) \cdot k}{[-(k+p)^2](-k^2)}$$

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Set  $-p^2 = 1$

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$$\Pi(p^2) = 2 \frac{d-2}{d-1} ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{-2D_2 + 1 + D_2 - D_1}{D_1 D_2}$$

# Photon self-energy

Restoring  $-p^2$  by dimensionality

$$\begin{aligned}\Pi(p^2) &= -\frac{e_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 2\frac{d-2}{d-1}G_1 \\ &= \frac{e_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4\frac{d-2}{(d-1)(d-3)(d-4)}g_1\end{aligned}$$

# Photon field renormalization

Transverse propagator

$$p^2 D_{\perp}(p^2) = \frac{1}{1 - \Pi(p^2)} = 1 + \frac{e_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4 \frac{d-2}{(d-1)(d-3)(d-4)} g_1$$

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Re-expressing via  $\alpha(\mu)$

$$p^2 D_{\perp}(p^2) = 1 + \frac{\alpha(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 4 \frac{d-2}{(d-1)(d-3)(d-4)}$$

$$L = \log \frac{-p^2}{\mu^2}$$

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This should be  $Z_A(\alpha(\mu)) p^2 D_\perp^r(p^2; \mu)$ :

$$Z_A(\alpha) = 1 - \frac{4}{3} \frac{\alpha}{4\pi\varepsilon}$$
$$p^2 D_\perp^r(p^2; \mu) = 1 + \frac{4}{3} \frac{\alpha(\mu)}{4\pi} \left( L - \frac{5}{3} \right)$$



## RG equation

$D_{\perp}(p^2) = Z_A(\alpha(\mu))D_{\perp}^r(p^2; \mu)$  does not depend on  $\mu$ :

$$\frac{\partial D_{\perp}^r(p^2; \mu)}{\partial \log \mu} + \gamma_A(\alpha(\mu))D_{\perp}^r(p^2; \mu) = 0$$

$$\gamma_A(\alpha(\mu)) = \frac{d \log Z_A(\alpha(\mu))}{d \log \mu}$$

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For a minimal renormalization constant using

$$\frac{d \log \alpha(\mu)}{d \log \mu} = -2\varepsilon + \dots$$

we obtain

$$\gamma(\alpha) = \gamma_0 \frac{\alpha}{4\pi} + \dots = -2z_1 \frac{\alpha}{4\pi} + \dots$$

$$Z(\alpha) = 1 - \frac{\gamma_0}{2} \frac{\alpha}{4\pi\varepsilon} + \dots$$

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with the initial condition

$$p^2 D_{\perp}^r(p^2; \mu^2 = -p^2) = 1 - \frac{20}{9} \frac{\alpha(\mu)}{4\pi}$$

is enough to reconstruct the result

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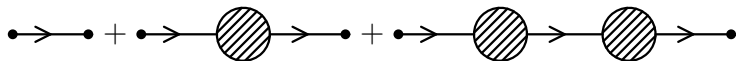
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is enough to reconstruct the result

$a_0 = Z_A(\alpha(\mu))a(\mu)$  does not depend on  $\mu$ :

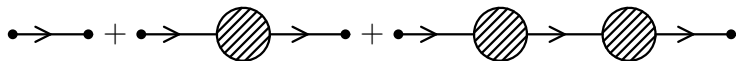
$$\frac{da(\mu)}{d \log \mu} + \gamma_A(\alpha(\mu))a(\mu) = 0$$

# Electron propagator



$$iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) \\ + iS_0(p)(-i)\Sigma(p)iS_0(p)(-i)\Sigma(p)iS_0(p) + \dots$$

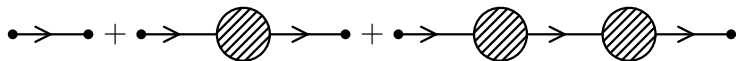
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$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

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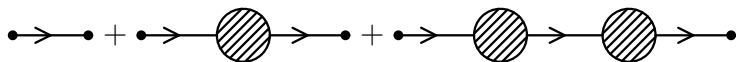
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$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$



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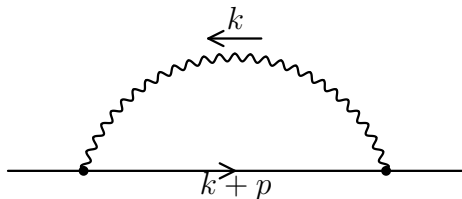
$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

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Massless case:  $\Sigma(p) = \not{p}\Sigma_V(p^2)$  (helicity conservation)

$$S(p) = \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\not{p}}$$

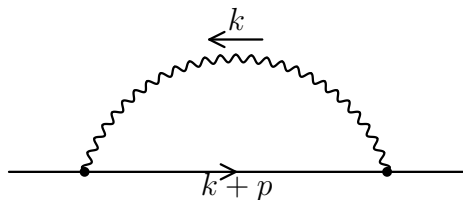
# Electron self-energy



$$-i\not{p}\Sigma_V(p^2) = \int \frac{d^d k}{(2\pi)^d} i e_0 \gamma^\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i e_0 \gamma^\nu \frac{-i}{k^2} \left( g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

where  $\xi = 1 - a_0$ .

# Electron self-energy



$$-i\not{p}\Sigma_V(p^2) = \int \frac{d^d k}{(2\pi)^d} i e_0 \gamma^\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i e_0 \gamma^\nu \frac{-i}{k^2} \left( g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

where  $\xi = 1 - a_0$ . Taking  $\frac{1}{4} \text{Tr}$

$$\Sigma_V(p^2) = \frac{i e_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{N}{D_1 D_2}$$

$$N = \frac{1}{4} \text{Tr} \not{p} \gamma^\mu (\not{k} + \not{p}) \gamma^\nu \left( g_{\mu\nu} + \xi \frac{k_\mu k_\nu}{D_2} \right)$$

# Electron self-energy

Using the multiplication table

$$\begin{aligned} N &= \frac{1}{4} \text{Tr } \not{p} \gamma_\mu (\not{k} + \not{p}) \gamma^\mu + \frac{\xi}{D_2} \frac{1}{4} \text{Tr } \not{p} \not{k} (\not{k} + \not{p}) \not{k} \\ &= -(d-2)(p^2 + p \cdot k) + \frac{\xi}{D_2} [k^2 p \cdot k + 2(p \cdot k)^2 - p^2 k^2] \\ &= \frac{1}{2} \left[ d - 2 + \xi \left( \frac{1}{D_2} - 1 \right) \right] \end{aligned}$$

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$$\Sigma_V(p^2) = -\frac{e_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{1}{2} [(d-2-\xi)G(1,1) + \xi G(1,2)]$$

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$$\frac{G(1,2)}{G(1,1)} = -(d-3)$$

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# Electron field renormalization

$$\not{p}S(p) = \frac{1}{1 - \Sigma_V(p^2)}$$

expressed via the renormalized quantities

$$\begin{aligned}\not{p}S(p) &= 1 + \frac{\alpha(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 a(\mu) \frac{d-2}{(d-3)(d-4)} \\ &= 1 - \frac{\alpha(\mu)}{4\pi\varepsilon} a(\mu) e^{-L\varepsilon} (1 + \varepsilon + \dots)\end{aligned}$$



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should be  $Z_\psi(\alpha(\mu), a(\mu))\not{p}S_r(p; \mu)$ :

$$Z_\psi(\alpha, a) = 1 - a \frac{\alpha}{4\pi\varepsilon}$$

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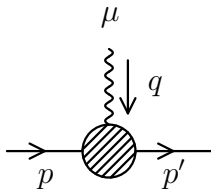
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$$\gamma_\psi(\alpha, a) = 2a \frac{\alpha}{4\pi}$$

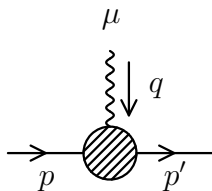
# Vertex



$$= ie_0 \Gamma^\mu(p, p')$$

$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

# Vertex



A Feynman diagram representing a vertex correction. It consists of a central shaded circle with diagonal hatching. Two horizontal lines with arrows pointing to the right enter and exit the circle, labeled with momenta  $p$  and  $p'$  respectively. A vertical wavy line with an arrow pointing downwards enters the top of the circle, labeled with index  $\mu$  and momentum  $q$ .

$$= ie_0 \Gamma^\mu(p, p')$$

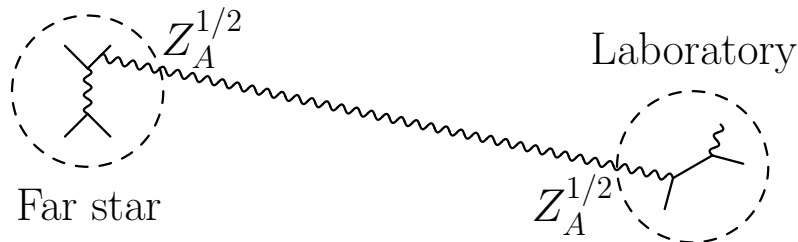
$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

When expressed via renormalized quantities,

$$\Gamma^\mu = Z_\Gamma \Gamma_r^\mu$$

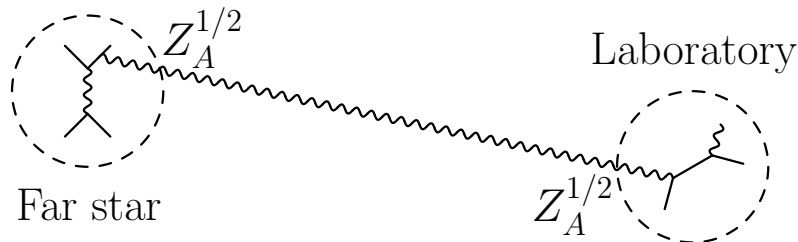
# Matrix element

$S$ -matrix element = vertex  $\times Z_i^{1/2}$  for each  $i$



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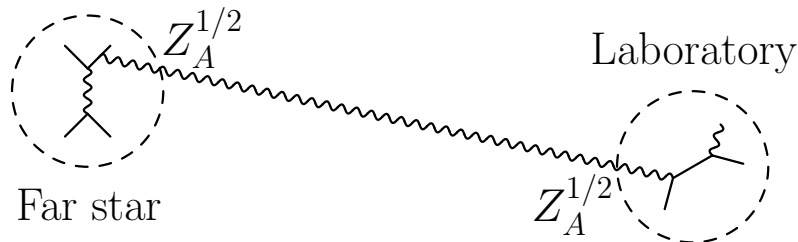
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The physical matrix element  
 $e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$  must be finite

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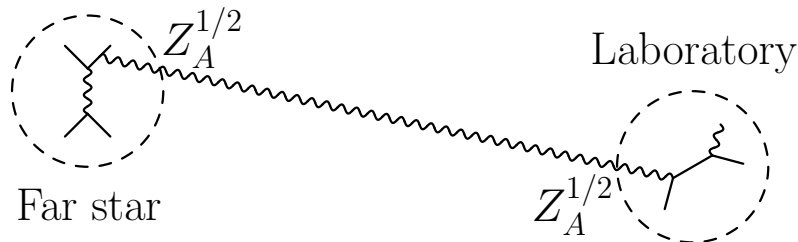
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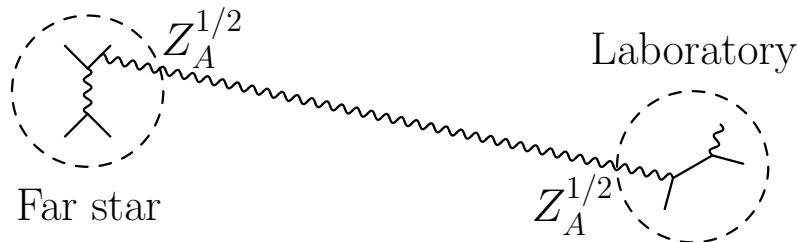


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# Matrix element

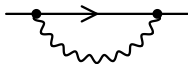
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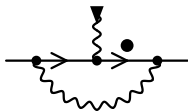
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$$Z_\alpha = (Z_\Gamma Z_\psi)^{-2} Z_A^{-1}$$

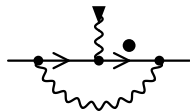
# Ward identity



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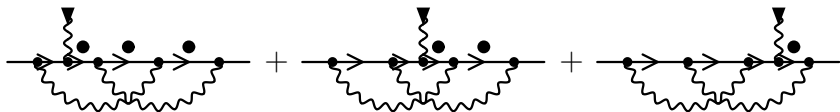
$$= e_0 \left[ \text{Diagram 1} - \text{Diagram 2} \right]$$

The equation shows the Ward identity for the vertex correction. The left-hand side is the vertex correction diagram. The right-hand side is the difference of two diagrams, multiplied by the coupling constant  $e_0$ . The first diagram in the brackets is the vertex correction diagram with a dot on the fermion line between the second and third vertices. The second diagram is the tree-level vertex correction diagram.

# Ward identity



# Ward identity



# Ward identity

$$= e_0 \left[ \begin{array}{c} \text{Diagram 1} - \text{Diagram 2} \\ + \\ \text{Diagram 3} - \text{Diagram 4} \\ + \\ \text{Diagram 5} - \text{Diagram 6} \end{array} \right]$$

The diagram shows a sequence of six Feynman diagrams arranged in three rows. Each diagram consists of a horizontal fermion line with four vertices and two wavy boson lines. The vertices are marked with black dots. The diagrams are connected by minus signs between columns and plus signs between rows. The diagrams represent different ways of inserting a boson into a fermion line, illustrating the Ward identity.

# Ward identity

$$= e_0 \left[ \begin{array}{c} \text{Diagram 1} - \text{Diagram 2} \\ + \\ \text{Diagram 3} - \text{Diagram 4} \\ + \\ \text{Diagram 5} - \text{Diagram 6} \end{array} \right] = e_0 \left[ \begin{array}{c} \text{Diagram 7} - \text{Diagram 8} \end{array} \right]$$

The diagram shows a sequence of Feynman diagrams representing a Ward identity. It consists of two main parts, each enclosed in large square brackets and preceded by an equals sign and the symbol  $e_0$ . The first part contains three rows of diagrams, each row with a plus sign on the left and a minus sign on the right. The second part contains one row of diagrams with a plus sign on the left and a minus sign on the right. Each diagram features a horizontal fermion line with four vertices marked by black dots and arrows pointing to the right. Wavy lines connect the vertices in a chain-like structure. In the first part, the wavy lines are connected to the vertices in a specific way, and the diagrams are subtracted. In the second part, the wavy lines are connected differently, and the diagrams are subtracted. The overall expression shows that the sum of the first part is equal to the second part.



# Ward identity

$$\Lambda^\mu(p, p')q_\mu = \Sigma(p) - \Sigma(p') \quad \Gamma^\mu(p, p')q_\mu = S^{-1}(p') - S^{-1}(p)$$

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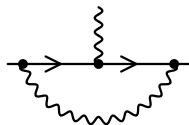
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$$Z_\psi Z_\Gamma = 1$$

$$Z_\alpha = Z_A^{-1}$$

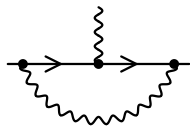
# Direct calculation



$$ie_0\Lambda^\alpha = \int \frac{d^d k}{(2\pi)^d} ie_0\gamma^\mu i \frac{\not{k}}{k^2} ie_0\gamma^\alpha i \frac{\not{k}}{k^2} ie_0\gamma^\nu$$
$$\times \frac{-i}{k^2} \left( g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu \not{k} \gamma^\alpha \not{k} \gamma^\mu - \xi k^2 \gamma^\alpha}{(k^2)^2}$$

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Averaging  $\not{k} \gamma^\alpha \not{k} \rightarrow (k^2/d) \gamma_\nu \gamma^\alpha \gamma^\nu$

$$\Lambda^\alpha = -ie_0^2 a_0 \gamma^\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2}$$

## Direct calculation

$$\Gamma^\alpha = \gamma^\alpha \left[ 1 + a(\mu) \frac{\alpha(\mu)}{4\pi\varepsilon} \right]$$

$$Z_\Gamma = 1 + a \frac{\alpha}{4\pi\varepsilon}$$

agrees with  $Z_\psi$



# Charge renormalization

$e_0^2$  does not depend on  $\mu$ :

$$\frac{d \log \alpha(\mu)}{d \log \mu} = -2\varepsilon - 2\beta(\alpha(\mu))$$

$$\beta(\alpha_s(\mu)) = \frac{1}{2} \frac{d \log Z_\alpha(\alpha_s(\mu))}{d \log \mu}$$

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For a minimal renormalization constant

$$Z_\alpha(\alpha) = 1 + z_1 \frac{\alpha}{4\pi\varepsilon} + \dots$$

we obtain

$$\beta(\alpha) = \beta_0 \frac{\alpha}{4\pi} + \dots = -z_1 \frac{\alpha}{4\pi} + \dots$$

$$Z_\alpha(\alpha) = 1 - \beta_0 \frac{\alpha}{4\pi\varepsilon} + \dots$$

# Charge renormalization

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$$\frac{d}{d \log \mu} \frac{\alpha(\mu)}{4\pi} = -2\beta_0 \left( \frac{\alpha(\mu)}{4\pi} \right)^2$$

$$\frac{d}{d \log \mu} \frac{4\pi}{\alpha(\mu)} = 2\beta_0$$

$$\frac{4\pi}{\alpha(\mu')} - \frac{4\pi}{\alpha(\mu)} = 2\beta_0 \log \frac{\mu'}{\mu}$$

$$\alpha(\mu') = \frac{\alpha(\mu)}{1 + 2\beta_0 \frac{\alpha(\mu)}{4\pi} \log \frac{\mu'}{\mu}}$$